New Inference for Constant-Stress Accelerated Life Tests With Weibull Distribution and Progressively Type-II Censoring

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***Abstract*—Constant-stress procedures based on parametric lifetime distributions and models are often used for accelerated life testing in product reliability experiments. Maximum likelihood estimation (MLE) is the typical statistical inference method. This paper presents a new inference method, named the random variable transformation (RVT) method, for Weibull constant-stress accelerated life tests with progressively Type-II right censoring (including ordinary Type-II right censoring). A two-parameter Weibull life distribution with a scale parameter that is a log-linear function of stress is used. RVT inference life distribution parameters and the log-linear function coefficients are provided. Exact confidence intervals for these parameters are also explored. Numerical comparisons of RVT-based estimates to MLE show that the proposed RVT inference is promising, in particular for small sample sizes.**

***Index Terms*—Accelerated life-testing, censored data, confidence interval, maximum likelihood estimation, progressively censoring, random variable transformation, Weibull distribution.**

Acronyms and Abbreviations

ALT	accelerated life test
CSALT	constant-stress ALT
SSALT	step-stress ALT
cdf	cumulative distribution function
MLE	maximum likelihood estimation
MSE	mean squared error
CI	confidence interval
RVT	random variable transformation
CP	coverage probability

Manuscript received May 12, 2013; revised September 20, 2013 and December 03, 2013; accepted January 12, 2014. Date of publication April 03, 2014; date of current version August 28, 2014. This work was supported in part by the National Natural Science Foundation of China under contract numbers 11371322 and 11261048, by the Foundation of Ministry of Education of China under Grant 12YJA910005, and by the Zhejiang Industrial Development Policy Key Research Centre of Philosophy and Social Science of Zhejiang Province and Zhejiang Provincial Key Research Base of Management Science and Engineering. Associate Editor: R. H. Yeh.

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Digital Object Identifier 10.1109/TR.2014.2313804

Notation

β	shape parameter of Weibull distribution
$\hat{\beta}_M$	MLE of β
$\hat{\beta}$	new estimator of β from RVT method
α_0, α_1	parameters for log-linear stress-level model
$\hat{\alpha}_{0,M}, \hat{\alpha}_{1,M}$	MLE of α_0, α_1
$\hat{\alpha}_0, \hat{\alpha}_1$	new estimators of α_0, α_1 from RVT method
k	the total number of stress levels
x_0	the designed stress level
x_i	ith accelerated stress level
θ_i	scale parameter at stress level x_i
n_i	the number of test units placed at level x_i
$R_{i,j}$	jth progressive censoring scheme at level x_i
r_i	the total number of failure units under n_i and x_i
$t_{i,j}$	jth failure time at level x_i

I. Introduction

In many industrial fields, it is required for lots of products to operate for a long period of time. In support, it is important to improve reliability in relation to the required lifetime of products. Fortunately, accelerated life testing (ALT) can quickly yield information about the lifetime distributions of products by inducing early failure with stronger stress than normal. The results obtained at the accelerated conditions are analyzed in terms of a model to relate life length to stress; they are extrapolated to the design stress to estimate the life distribution. The constant-stress ALT (CSALT) and the step-stress ALT (SSALT) are two important methods for ALT. The problem of modeling data from CSALT and SSALT, and making inferences from such data, have been studied by many authors. For CSALT, Wiel and Meeker [1] studied accuracy of approximate confidence bounds for a Weibull CSALT model. Yang [2] considered optimum 4-level CSALT plans under a location-scale family of distributions. Watkins [3] discussed the likelihood method for fitting Weibull CSALT models. Barbosa *et al.* [4] proposed the piecewise exponential model, and gave the estimation procedure based on generalized linear models. Wang and Kececioglu [5] further studied this issue, and gave an efficient algorithm to fit the Weibull CSALT model. Tang *et al.* [6] discussed an optimum CSALT plan for a two-parameter exponential distribution. Dorp and Mazzuchi [7] discussed Bayes inference for ALT. León *et al.* [8] discussed Bayesian modeling of CSALT with random effects. Watkins and John [9] discussed maximum likelihood estimates for CSALT terminated by Type-II censoring at one of the stress levels. Pascual [10] studied the planning of CSALT in the presence of competing risks under Weibull distributions. Ma and Meeker [11] discussed strategies for planning CSALT with small sample sizes. Liu and Tang [12] considered CSALT for repairable systems with multiple s-independent risks, and derived accelerated life test plans. Tang and Liu [13] proposed a sequential CSALT, and discussed its inference procedure and test plan. Monroe *et al.* [14] considered the design of the CSALT experiments based on a generalized linear model approach. Yu and Chang [15] applied a Bayesian model to average quantile estimation for CSALT. Liu [16] discussed the model and plan for CSALT with s-dependent failure modes. For SSALT, DeGroot and Goel [17] proposed the tampered random variable model. Nelson [18] proposed the cumulative exposure model. Bhattacharyya and Soejoeti [19] proposed the tampered failure rate model. It is worth mentioning that Wang [20] gave a necessary condition to decide whether or not a given model such as the cumulative exposure model is rational. Miller and Nelson [21], as well as Bai *et al.* [22], discussed optimum plans for simple SSALT. Khamis and Higgins [23] obtained the optimum 3-step SSALT plans. Dorp *et al.* [24] developed a Bayes model for SSALT. Teng and Yeo [25] used the method of least squares to estimate the life-stress relationship in SSALT. Balakrishnan *et al.* [26] obtained point and interval estimations for the exponential simple step-stress model. Fan and Wang [27] considered a SSALT model for Weibull series systems with masked data. Nelson [28], and Bagdonavicius and Nikulin [29] provided some excellent information on past and current developments in the area.

Progressive censoring is a generalized form of censoring which includes conventional right censoring as a special case. Compared to conventional censoring, however, it provides higher flexibility to the experimenter in the design stage by allowing the removal of test units at non-terminal time points, and thus it proves to be highly efficient and effective in utilizing the available resources (Montanari and Cacciari [30], Balakrishnan and Aggarwala [31]). Another advantage of progressive censoring is that the degeneration-related information of the test units is obtained from those removed units (Balasooriya *et al.* [32]). For these reasons, we consider a more general censoring scheme called progressive Type-II censoring. Progressive Type-II censoring is a method which enables an efficient exploitation of the available resources by continual removal of a pre-specified number of surviving test units at each failure time. Montanari and Cacciari [30] gave an interesting application of progressive censoring on an aging study carried out on XLPE-insulated cable models. Gouno *et al.* [33], and Balakrishnan and Han [35] discussed the optimal step-stress ALT plans under progressive Type-I censoring. Fan *et al.* [34] considered exponential progressive SSALT based on Box-Cox transformation. Wang and Yu [36] discussed the optimal step-stress ALT plans under progressive Type-II censoring. Wang [37] derived interval estimation for exponential progressive Type-II censored step-stress ALT. A book dedicated completely to progressive censoring was published by Balakrishnan and Aggarwala [31]. Moreover, Balakrishnan [38] gave an excellent, extensive review of the progressive censoring methodology.

Under a combination of CSALT and progressive Type-II censoring, the sample size is typically not large, so that large-sample based inference methods such as MLE-based asymptotic unbiased estimates and asymptotic normal confidence intervals (CI) may not be suitable, and can even be misleading. In this paper, we consider CSALT with progressive Type-II censoring, and provide RVT inference for parameter estimation and CIs.

The Weibull CSALT model considered is under the following two assumptions.

A1. For any stress level x_i, the lifetime distribution of a test unit is Weibull with cumulative distribution function (cdf)

$$F_i(t) = 1 - \exp\left(-(t/\theta_i)^{\beta}\right), \quad t > 0, \tag{1}$$

where $\beta > 0$ is the shape parameter, and $\theta_i > 0$ is the scale parameter.

A2. The stress-life relationship is given by

$$\log(\theta_i) = \alpha_0 + \alpha_1 x_i. \tag{2}$$

where α_0 and α_1 are unknown parameters.

The log-linear model above for the scale parameter includes the exponential life distribution as a special case which was widely studied in the literature.

Under the CSALT model and progressively censored scheme, Section II outlines the MLE of the Weibull CSALT model. Sections III details the RVT inference method and properties. Section IV focuses on exact CIs for unknown parameters and their functions. Section V evaluates the numerical performance of the RVT-based estimators, and provides a comparison with MLE. Furthermore, both methods are applied to a real-data example, and the results are discussed in Section VI. Section VII concludes.

II. MLE

The CSALT under a progressively censored scheme is set as follows.

Let x_0 be the designed stress level, and let $x_1 < x_2 < \ldots < x_k$ be the k accelerated stress levels. Suppose that n_i test units are placed at stress level x_i. Prior to the experiment, a number $r_i(\leq n_i)$ is fixed, and the progressive censoring scheme $R_i = (R_{i,1}, R_{i,2}, \ldots, R_{i,r_i})$ with $R_{i,j} \geq 0$ and $\sum_{j=1}^{r_i} R_{i,j} + r_i = n_i$ is specified. At the first failure time $T_{i,1}$, $R_{i,1}$ units are randomly removed from the remaining $n_i - 1$ surviving units. At the second failure time $T_{i,2}$, $R_{i,2}$ units are randomly removed from the remaining $n_i - 2 - R_{i,1}$. The test continues until the r_ith failure time T_{i,r_i}. At failure time T_{i,r_i}, all remaining units are removed. When $R_{i,j} = 0, i = 1, \ldots, k, j = 1, \ldots, r_i - 1$, then $R_{i,r_i} = n - r_i$, which corresponds to the conventional CSALT with a Type-II censoring scheme. In total, let $t = \{t_{i,j} :$

$i = 1, \ldots, k; j = 1, \ldots, r_i\}$ be the observed values of lifetime $T = \{T_{i,1}, \ldots, T_{i,r_i}\}_{i=1}^{k}$.

Therefore, based on the likelihood function

$$L(\beta, \alpha_0, \alpha_1 | t) = \prod_{i=1}^{k} \prod_{j=1}^{r_i} \frac{\beta}{\theta_i^{\beta}} t_{i,j}^{\beta-1} \times \exp\left(-\sum_{i=1}^{k}\sum_{i=1}^{r_i}(R_{i,j}+1)\left(\frac{t_{i,j}}{\theta_i}\right)^{\beta}\right),$$

with $\log(\theta_i) = \alpha_0 + \alpha_1 x_i$, and $\sum_{j=1}^{r_i} R_{i,j} + r_i = n_i$, we have the log-likelihood function as

$$\begin{aligned} l(\beta, \alpha_0, \alpha_1) \propto & \sum_{i=1}^{k} r_i \log(\beta) + (\beta - 1) \sum_{i=1}^{k}\sum_{j=1}^{r_i} \log(t_{i,j}) \\ & - \alpha_0 \beta \sum_{i=1}^{k} r_i - \alpha_1 \beta \sum_{i=1}^{k} r_i x_i \\ & - \sum_{i=1}^{k}\sum_{j=1}^{r_i} (R_{i,j}+1) t_{i,j}^{\beta} \exp(-\alpha_0\beta - \alpha_1 \beta x_i). \end{aligned}$$

Hence the MLEs $\hat{\beta}_M, \hat{\alpha}_{0,M}, \hat{\alpha}_{1,M}$ of the parameters $\beta, \alpha_0, \alpha_1$ are the solutions of the equations

$$\begin{aligned} & \frac{1}{\beta} + \frac{\sum_{i=1}^{k}\sum_{j=1}^{r_i}\log(t_{i,j})}{\sum_{i=1}^{k} r_i} - \\ & \frac{\sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)t_{i,j}^{\beta}\log(t_{i,j})\exp(-\alpha_1\beta x_i)}{\sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)t_{i,j}^{\beta}\exp(-\alpha_1\beta x_i)} = 0, \\ & \sum_{i=1}^{k} r_i x_i \sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)t_{i,j}^{\beta}\exp(-\alpha_1\beta x_i) - \\ & \quad \sum_{i=1}^{k} r_i \sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)x_i t_{i,j}^{\beta}\exp(-\alpha_1\beta x_i) = 0, \\ & \alpha_0 = \frac{1}{\beta}\log\left(\sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)t_{i,j}^{\beta}\exp(-\alpha_1\beta x_i)\right) \\ & \qquad - \frac{1}{\beta}\log\left(\sum_{i=1}^{k} r_i\right). \end{aligned}$$

Note that

$$\begin{aligned} \frac{\partial^2 l}{\partial \beta^2} = & -\beta^{-2}\left(\sum_{i=1}^{k} r_i + \sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)x_i\left(\frac{t_{i,j}}{\theta_i}\right)^{\beta} \times \left[\log\left(\frac{t_{i,j}}{\theta_i}\right)\right]^2\right), \\ \frac{\partial^2 l}{\partial \alpha_0^2} = & -\beta^2\sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)\left(\frac{t_{i,j}}{\theta_i}\right)^{\beta}, \\ \frac{\partial^2 l}{\partial \alpha_1^2} = & -\beta^2\sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)x_i^2\left(\frac{t_{i,j}}{\theta_i}\right)^{\beta}, \\ \frac{\partial^2 l}{\partial\beta\partial\alpha_0} = & -\sum_{i=1}^{k} r_i + \sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)\left(\frac{t_{i,j}}{\theta_i}\right)^{\beta} \\ & + \sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)\left(\frac{t_{i,j}}{\theta_i}\right)^{\beta}\log\left(\frac{t_{i,j}}{\theta_i}\right)^{\beta}, \\ \frac{\partial^2 l}{\partial\beta\partial\alpha_1} = & -\sum_{i=1}^{k} r_i x_i + \sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)x_i\left(\frac{t_{i,j}}{\theta_i}\right)^{\beta} \\ & + \sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)x_i\left(\frac{t_{i,j}}{\theta_i}\right)^{\beta}\log\left(\frac{t_{i,j}}{\theta_i}\right)^{\beta}, \\ \frac{\partial^2 l}{\partial\alpha_0\partial\alpha_1} = & -\beta^2\sum_{i=1}^{k}\sum_{j=1}^{r_i}(R_{i,j}+1)x_i\left(\frac{t_{i,j}}{\theta_i}\right)^{\beta}. \end{aligned}$$

Numerical solutions of these estimators will be studied in Section V. The Fisher-information matrix is often used to calculate the covariance matrices associated with MLE. Here, the observed Fisher-information matrix for $(\beta, \alpha_0, \alpha_1)$ is given by

$$I(\hat{\beta}_M, \hat{\alpha}_{0,M}, \hat{\alpha}_{1,M}) = \begin{pmatrix} -\frac{\partial^2 l}{\partial\beta^2} & -\frac{\partial^2 l}{\partial\beta\partial\alpha_0} & -\frac{\partial^2 l}{\partial\beta\partial\alpha_1} \\ -\frac{\partial^2 l}{\partial\beta\partial\alpha_0} & -\frac{\partial^2 l}{\partial\alpha_0^2} & -\frac{\partial^2 l}{\partial\alpha_0\partial\alpha_1} \\ -\frac{\partial^2 l}{\partial\beta\partial\alpha_1} & -\frac{\partial^2 l}{\partial\alpha_0\partial\alpha_1} & -\frac{\partial^2 l}{\partial\alpha_1^2} \end{pmatrix}_{(\hat{\beta}_M, \hat{\alpha}_{0,M}, \hat{\alpha}_{1,M})}.$$

III. RVT Inference

We first consider the case with the known shape parameter β, and propose new estimators for parameters α_0, α_1, and $\theta_0 = \exp(\alpha_0 + \alpha_1 x_i)$, then extend the estimation for unknown β.

A. The Known Shape Parameter Case

When parameter β is known, let

$$S_i = \sum_{j=1}^{r_i}(R_{i,j}+1)T_{i,j}^{\beta}, \quad i = 1, 2, \ldots, k.$$

Then it is well known that $2S_i/\theta_i^{\beta}$ follows the χ^2 distribution with $2r_i$ degrees of freedom.

According to the property of the log-Gamma distribution (for example, see Lawless [39]), the log-transformation of S_i satisfies

$$E[\log(S_i) - \beta\log(\theta_i)] = \psi(r_i),$$
$$Var[\log(S_i) - \beta\log(\theta_i)] = \psi'(r_i),$$

where $\psi(x) = d\log(\Gamma(x))/dx$, $\psi'(x) = d^2\log(\Gamma(x))/dx^2$.

Therefore, we consider the following regression model.

$$E(U_i/\beta) = \log(\theta_i) = \alpha_0 + \alpha_1 x_i, \quad Var(U_i/\beta) = \frac{\psi'(r_i)}{\beta^2},$$

where $U_i = \log(S_i) - \psi(r_i)$.

According to the Gauss-Markov theorem (for example, see Rao [40]), the unbiased estimators of (α_0, α_1) are respectively given by

$$\tilde{\alpha}_0 = \frac{GH - IM}{\beta(FG - I^2)}, \quad \tilde{\alpha}_1 = \frac{FM - IH}{\beta(FG - I^2)}, \tag{3}$$

where $F = \sum_{i=1}^{k}[\psi'(r_i)]^{-1}$, $I = \sum_{i=1}^{k}[\psi'(r_i)]^{-1}x_i$, $G = \sum_{i=1}^{k}[\psi'(r_i)]^{-1}x_i^2$, $H = \sum_{i=1}^{k}[\psi'(r_i)]^{-1}U_i$, $M = \sum_{i=1}^{k}[\psi'(r_i)]^{-1}x_iU_i$.

Further, we have

$$Var(\tilde{\alpha}_0) = \frac{G}{\beta^2(FG-I^2)},$$
$$Var(\tilde{\alpha}_1) = \frac{F}{\beta^2(FG-I^2)},$$
$$Cov(\tilde{\alpha}_0,\tilde{\alpha}_1) = -\frac{I}{\beta^2(FG-I^2)}.$$

Therefore, the scale parameter θ_0 at designed stress level x_0 could be estimated by $\tilde{\theta}_0 = \exp(\tilde{\alpha}_0 + \tilde{\alpha}_1 x_0)$.

Along the same line as Wang and Yu [36], we obtain the following results for the estimation of θ_0.

Theorem 1: Let $\tilde{\alpha}_0$, and $\tilde{\alpha}_1$ defined in (3) be the unbiased estimators of α_0, and α_1 respectively, and $D_i = [G-(x_0+x_i)I+x_0x_iF]/[\beta\psi'(r_i)(FG-I^2)]$. Then, we have the following.

1) If $r_i + D_i > 0$ $(i = 1,2,\ldots,k)$, then the expectation of $\tilde{\theta}_0$ exists, but $\tilde{\theta}_0$ is a biased estimator of θ_0. However, an unbiased estimator of θ_0 is then given by

$$\tilde{\theta}_{0U} = \tilde{\theta}_0 \exp\left(\sum_{i=1}^{k} D_i\psi(r_i)\right)\prod_{i=1}^{k}\frac{\Gamma(r_i)}{\Gamma(r_i+D_i)}. \tag{4}$$

Furthermore, if $r_i + 2D_i > 0$ $(i = 1,2,\ldots,k)$, then the variance of $\tilde{\theta}_{0U}$ exists, and is given by

$$Var(\tilde{\theta}_{0U}) = \left(\prod_{i=1}^{k}\frac{\Gamma(r_i)\Gamma(r_i+2D_i)}{\Gamma^2(r_i+D_i)} - 1\right)\theta_0^2.$$

2) If $r_i+2D_i > 0, (i = 1,2,\ldots,k)$, then $\tilde{\theta}_{0U}$ has the smaller mean squared error than that of $\tilde{\theta}_0$.

In summary, contrary to the MLE in this case, whose estimators are asymptotic unbiased with asymptotic variances, we have obtained exact unbiased estimators of parameters $(\alpha_0,\alpha_1,\theta_0)$, and exact variances of these estimators.

B. The Unknown Shape Parameter Case

Now we consider the case with unknown shape parameter β. For each $i = 1,2,\ldots,k, j = 1,2,\ldots,r_i$, let

$$S_{i,j} = \sum_{l=1}^{j}(R_{i,l}+1)T_{i,l}^{\beta} + \left[n_i - \sum_{l=1}^{j}(R_{i,l}+1)\right]T_{i,j}^{\beta}.$$

From Wang *et al.* [41], we have

$$W_i(\beta) = 2\sum_{j=1}^{r_i-1}\log\left(\frac{S_{i,r_i}}{S_{i,j}}\right) \sim \chi^2(2r_i-2), \quad i = 1,2,\ldots,k,$$

and $W_i(\beta)$ is a strictly monotone function of β.

Notice that $W_1(\beta),\ldots,W_k(\beta)$ are s-independent; thus, we define

$$W(\beta) = \sum_{i=1}^{k} W_i(\beta) = 2\sum_{i=1}^{k}\sum_{j=1}^{r_i-1}\log\left(\frac{S_{i,r_i}}{S_{i,j}}\right), \tag{5}$$

and $W(\beta) \sim \chi^2(2\sum_{i=1}^{k} r_i - 2k)$.

Then, based on the inverse transformation method proposed by Wang, Yu, and Jones [41], the shape parameter β can be estimated from the solution of

$$\sum_{i=1}^{k}\sum_{j=1}^{r_i-1}\log\left(\frac{S_{i,r_i}}{S_{i,j}}\right) = \sum_{i=1}^{k} r_i - k - 1. \tag{6}$$

Due to the strictly increasing function of β, (6) has exactly one unique solution. Let $\hat{\beta}$ be a solution of (6). Then plugging $\hat{\beta}$ into (3) and (4), we obtain the estimators $(\hat{\alpha}_0,\hat{\alpha}_1,\hat{\theta}_0)$ of $(\alpha_0,\alpha_1,\theta_0)$:

$$\hat{\alpha}_0 = \frac{G\hat{H} - I\hat{M}}{\hat{\beta}(FG-I^2)}, \tag{7}$$

$$\hat{\alpha}_1 = \frac{F\hat{M} - I\hat{H}}{\hat{\beta}(FG-I^2)}, \tag{8}$$

$$\hat{\theta}_0 = \exp\left(\hat{\alpha}_0 + \hat{\alpha}_1 x_0 + \sum_{i=1}^{k}\hat{D}_i\psi(r_i)\right) \times \prod_{i=1}^{k}\frac{\Gamma(r_i)}{\Gamma(r_i+\hat{D}_i)}, \tag{9}$$

where

$$\hat{U}_i = \log\left(\sum_{j=1}^{r_i}(R_{i,j}+1)T_{i,j}^{\hat{\beta}}\right) - \psi(r_i),$$
$$\hat{H} = \sum_{i=1}^{k}[\psi'(r_i)]^{-1}\hat{U}_i,$$
$$\hat{M} = \sum_{i=1}^{k}[\psi'(r_i)]^{-1}x_i\hat{U}_i,$$
$$\hat{D}_i = \frac{[G-(x_0+x_i)I+x_0x_iF]}{\left[\hat{\beta}\psi'(r_i)(FG-I^2)\right]}.$$

The estimators $(\hat{\beta},\hat{\alpha}_0,\hat{\alpha}_1,\hat{\theta}_0)$ of $(\alpha_0,\alpha_1,\theta_0)$ given by (6) through (9) are new estimators of the parameters $(\beta,\alpha_0,\alpha_1,\theta_0)$. We shall study the finite sample properties of the proposed estimators in Section V.

IV. Interval Estimation of Unknown Parameters

In this section, we will first obtain an exact CI for the shape parameter, then derive the generalized CIs for other parameters, and some important quantities of the Weibull distribution at designed stress level x_0, such as its mean, quantiles, and the reliability function.

A. Exact CI for the Shape Parameter

Consider the pivotal quantity $W(\beta)$. Note that $W(\beta)$ is a function of β only, and *does not depend on other parameters*. Hence, we obtain an exact CI for the shape parameter β as follows.

Theorem 2: Suppose $(T_{i,1},T_{i,2},\ldots,T_{i,r_i}), i = 1,2,\ldots,k$ are progressively Type-II censored samples from the Weibull CSALT with the progressive censoring scheme $(R_{i,1},R_{i,2},\ldots,R_{i,r_i}), i = 1,2,\ldots,k$. Then, for any $0 < \gamma < 1$,

$$\left[W^{-1}\left\{\chi^2_{1-\gamma/2}\left(2\sum_{i=1}^{k}r_i - 2k\right)\right\},\right.$$
$$\left.W^{-1}\left\{\chi^2_{\frac{\gamma}{2}}\left(2\sum_{i=1}^{k}r_i - 2k\right)\right\}\right]$$

is a $1-\gamma$ CI for the shape parameter β, where $\chi^2_\gamma(v)$ is the upper γ percentile of the χ^2 distribution with v degrees of freedom, and for $w > 0$, $W^{-1}(w)$ is the solution in β of the equation $W(\beta) = w$.

B. Generalized CIs for Other Parameters

We now derive generalized CIs for other parameters, and some important quantities of the Weibull distribution at designed stress level x_0.

Let

$$V_1 = \frac{\sum_{i=1}^k [\psi'(r_i)]^{-1}(G - x_i I)\log(2S_i)}{FG - I^2} - \alpha_0\beta, \tag{10}$$

$$V_2 = \frac{\sum_{i=1}^k [\psi'(r_i)]^{-1}(x_i F - I)\log(2S_i)}{FG - I^2} - \alpha_1\beta. \tag{11}$$

Then,

$$V_1 = \frac{\sum_{i=1}^k [\psi'(r_i)]^{-1}(G - x_i I)\log(T_i)}{FG - I^2}, \tag{12}$$

$$V_2 = \frac{\sum_{i=1}^k [\psi'(r_i)]^{-1}(x_i F - I)\log(T_i)}{FG - I^2}, \tag{13}$$

where $T_i = 2S_i/\theta_i^\beta \sim \chi^2(2r_i)$. It is obvious from (12) and (13) that the distributions of V_1 and V_2 do not depend on any unknown parameters. Thus V_1, and V_2 are pivotal quantities.

Note that $W(\beta)$ is a strictly increasing function of β. Then $W(\beta) = W$ has the unique solution $g(W,T)$, where $W \sim \chi^2(2\sum_{i=1}^k r_i - 2k)$. In addition, from (10) and (11), we have

$$\alpha_0 = \frac{\sum_{i=1}^k [\psi'(r_i)]^{-1}(G - x_i I)\log(2S_i)}{\beta(FG - I^2)} - \frac{V_1}{\beta}, \tag{14}$$

$$\alpha_1 = \frac{\sum_{i=1}^k [\psi'(r_i)]^{-1}(x_i F - I)\log(2S_i)}{\beta(FG - I^2)} - \frac{V_2}{\beta}. \tag{15}$$

According to the substitution method given by Weerahandi [42], [43], we substitute $g(W,T)$ for β in the expression for α_0 and α_1 in (14) and (15); and we obtain the following generalized pivotal quantities for the parameters α_0 and α_1.

$$Y_0 = \frac{\sum_{i=1}^k [\psi'(r_i)]^{-1}(G - x_i I)\log(2s_i)}{g(W,t)(FG - I^2)} - \frac{V_1}{g(W,t)}, \tag{16}$$

$$Y_1 = \frac{\sum_{i=1}^k [\psi'(r_i)]^{-1}(x_i F - I)\log(2s_i)}{g(W,t)(FG - I^2)} - \frac{V_2}{g(W,t)}, \tag{17}$$

where $s_i = \sum_{j=1}^{r_i}(R_{i,j}+1)t_{i,j}^{g(W,t)}$.

Notice that Y_0, and Y_1 respectively reduce to α_0, and α_1 when $T = t$; and the distributions of Y_0, and Y_1 are free of any unknown parameters, thus Y_0, and Y_1 are indeed generalized pivotal quantities. If $Y_{0,\gamma}$, and $Y_{1,\gamma}$ denote the upper γ percentiles of Y_0, and Y_1, then $[Y_{0,1-\gamma/2}, Y_{0,\gamma/2}]$, and $[Y_{1,1-\gamma/2}, Y_{1,\gamma/2}]$ are the $1-\gamma$ generalized CIs for α_0, and α_1 respectively.

The percentiles of Y_0 and Y_1 can be obtained from (16) and (17) using the following Monte Carlo simulation algorithm.

Step 1) For a given data set (n, m, t, R), generate $W \sim \chi^2(2\sum_{i=1}^k r_i - 2k), T_1 \sim \chi^2(2r_1), \ldots, T_k \sim \chi^2(2r_k)$ separately and s-independently. Using these values, compute $g(W,t)$, V_1, and V_2 from $W(\beta) = W$, (12), and (13).

Step 2) In terms of (16) and (17), compute the values of Y_0 and Y_1.

Step 3) Repeat Steps 1 and 2 a large number of times, say, $m_1(\geq 10{,}000)$ times. The m_1 values of Y_0 and Y_1 can be obtained respectively.

Step 4) Arrange all Y_0 and Y_1 values in ascending order respectively: $Y_{0,1} < Y_{0,2} < \ldots < Y_{0,m_1}$ and $Y_{1,1} < Y_{1,2} < \ldots < Y_{1,m_1}$. Then the γ percentile of Y_0, and Y_1 are estimated by $Y_{0,\gamma m_1}$, and $Y_{1,\gamma m_1}$ respectively.

Now note that the mean, pth quantile $(0 < p < 1)$, and reliability function of the Weibull distribution at designed stress level x_0 are given by $\mu = \theta_0\Gamma(1 + 1/\beta)$, $t_p = \theta_0[-\log(1-p)]^{1/\beta}$, and $R(t_0) = \exp[-(t_0/\theta_0)^\beta]$ respectively. Along the same lines as the derivation of Y_0 and Y_1 for the parameters α_0 and α_1, we obtain the generalized pivotal quantities Y_2, Y_3, and Y_4 for μ, x_p, and $R(x_0)$ respectively:

$$Y_2 = e^{Y_0 + Y_1 x_0}\Gamma\left(1 + \frac{1}{g(W,t)}\right), \tag{18}$$

$$Y_3 = e^{Y_0 + Y_1 x_0}\left[-\log(1-p)\right]^{\frac{1}{g(W,t)}}, \tag{19}$$

$$Y_4 = \exp\left[-\left(t_0 e^{-Y_0 - Y_1 x_0}\right)^{g(W,t)}\right]. \tag{20}$$

Let $Y_{2,\gamma}, Y_{3,\gamma}, Y_{4,\gamma}$ denote the upper γ percentiles of Y_2, Y_3, Y_4 respectively. Then $Y_{2,\gamma}$, $Y_{3,\gamma}$, and $Y_{4,\gamma}$ are the $1-\gamma$ upper confidence limits for μ, t_p, and $R(t_0)$, respectively. Just as in the cases of Y_0 and Y_1, the percentiles of Y_2, Y_3, Y_4 can be obtained by Monte Carlo simulations.

We study the performance of coverage probabilities of these CIs by simulation. Such simulation results are reported in Section V.

V. Simulation Study

To evaluate and compare the performance of the MLE and proposed estimators with the RVT method, we perform simulation comparisons with data generated via various scenarios. Because the estimators are appropriately scale equivariant and invariant, without loss of generality we take $x_0 = 0$ in our simulation study. We consider different stress levels ($k = 2$, 3, 4 for simulation design scenarios 1 through 3, 4 through 6, and 7 through 9, respectively), combined with different censoring schemes (for example, progressive and conventional Type-II censoring). Details of the simulation design scenarios are summarized in Table I. For each scenario, 10,000 replicates of progressively Type-II censored samples were generated from the Weilbull distribution, as specified in (1), with three different parameter settings: 1) $(\beta, \alpha_0, \alpha_1) = (0.5, 5, -1)$, 2) $(\beta, \alpha_0, \alpha_1) = (1, 5, -1)$, 3) $(\beta, \alpha_0, \alpha_1) = (3, 5, -1)$, respectively.

Then Tables II to IV compare the relative-biases and relative-MSE (mean squared error) values of parameter estimators from the proposed RVT method with the MLEs of those parameters under different simulation scenarios, with respect to three different parameter settings. The relative-biases, and the relative-MSE are defined as follows.

$$\text{relative} - \text{biases} = \sum_{i=1}^n \frac{\hat{\xi}_i - \xi}{\xi},$$

$$\text{relative} - \text{MSE} = \sum_{i=1}^n \frac{(\hat{\xi}_i - \xi)^2}{\xi^2},$$

TABLE I
THE SIMULATION DESIGN SCENARIOS

S/N.	$x_1, \dots, x_k$	$n_1, \dots, n_k$	$r_1, \dots, r_k$	$R_1, \dots, R_k$
1	(0.5, 1)	(20, 10)	(12, 6)	$R_1 = (0, \dots, 0, 8)$ $R_2 = (0, \dots, 0, 4)$
2	(0.5, 1)	(20, 10)	(12, 6)	$R_1 = (8, 0, \dots, 0)$ $R_2 = (4, 0, \dots, 0)$
3	(0.5, 1)	(20, 10)	(12, 6)	$R_1 = (4, 0, \dots, 0, 4)$ $R_2 = (2, 0, \dots, 0, 2)$
4	(0.5, 0.75, 1)	(20, 15, 10)	(12, 9, 6)	$R_1 = (0, \dots, 0, 8)$ $R_2 = (0, \dots, 0, 6)$ $R_3 = (0, \dots, 0, 4)$
5	(0.5, 0.75, 1)	(20, 15, 10)	(12, 9, 6)	$R_1 = (8, 0, \dots, 0)$ $R_2 = (6, 0, \dots, 0)$ $R_3 = (4, 0, \dots, 0)$
6	(0.5, 0.75, 1)	(20, 15, 10)	(12, 9, 6)	$R_1 = (4, 0, \dots, 0, 4)$ $R_2 = (3, 0, \dots, 0, 3)$ $R_3 = (2, 0, \dots, 0, 2)$
7	(0.5, 0.75, 1, 1.25)	(30, 20, 15, 10)	(18, 12, 9, 6)	$R_1 = (0, \dots, 0, 12)$ $R_2 = (0, \dots, 0, 8)$ $R_3 = (0, \dots, 0, 6)$ $R_4 = (0, \dots, 0, 4)$
8	(0.5, 0.75, 1, 1.25)	(30, 20, 15, 10)	(18, 12, 9, 6)	$R_1 = (12, 0, \dots, 0)$ $R_2 = (8, 0, \dots, 0)$ $R_3 = (6, 0, \dots, 0)$ $R_4 = (4, 0, \dots, 0)$
9	(0.5, 0.75, 1, 1.25)	(30, 20, 15, 10)	(18, 12, 9, 6)	$R_1 = (6, 0, \dots, 0, 6)$ $R_2 = (4, 0, \dots, 0, 4)$ $R_3 = (3, 0, \dots, 0, 3)$ $R_4 = (2, 0, \dots, 0, 2)$

TABLE II
RELATIVE-BIAS AND RELATIVE-MSE OF MLE ESTIMATES AND THE NEW METHOD'S ESTIMATES. SAMPLES GENERATED WITH $(\beta, \alpha_0, \alpha_1) = (0.5, 5, -1)$. 10000 REPLICATES

	relative-bias							
S/N	MLE				RVT			
	β	α_0	α_1	θ_0	β	α_0	α_1	θ_0
1	0.165	-0.006	0.238	1.822	-0.004	0.013	0.046	-0.049
2	0.102	0.007	0.218	2.036	-0.011	-0.011	-0.013	-0.006
3	0.140	-0.005	0.191	1.693	-0.005	-0.003	-0.021	-0.109
4	0.104	-0.015	0.077	1.679	0.000	0.003	-0.009	-0.049
5	0.067	0.001	0.104	1.884	-0.004	-0.006	-0.005	0.007
6	0.085	-0.005	0.110	1.742	-0.006	0.001	0.011	-0.044
7	0.059	-0.005	0.056	0.492	0.000	0.004	0.008	-0.006
8	0.040	0.003	0.061	0.550	-0.002	-0.002	0.000	-0.012
9	0.050	-0.001	0.069	0.535	-0.002	0.002	0.017	0.005
	relative-MSE							
	MLE				RVT			
	β	α_0	α_1	θ_0	β	α_0	α_1	θ_0
1	0.107	0.086	4.407	55.653	0.059	0.087	4.358	5.728
2	0.053	0.087	4.430	84.562	0.035	0.087	4.406	12.960
3	0.083	0.083	4.306	44.638	0.048	0.083	4.280	4.361
4	0.053	0.085	4.065	50.759	0.038	0.085	4.076	6.066
5	0.029	0.085	4.108	70.623	0.023	0.086	4.136	11.685
6	0.041	0.083	4.039	64.457	0.031	0.083	4.051	10.532
7	0.025	0.035	1.321	3.425	0.021	0.035	1.346	1.363
8	0.015	0.035	1.302	3.532	0.014	0.035	1.318	1.408
9	0.021	0.035	1.338	3.628	0.018	0.035	1.359	1.477

where ξ denotes the true value, and $\hat{\xi}$ denotes its estimator.

Observe from Tables II, III, and IV that the relative-bias and relative-MSE of the RVT method for β is significantly smaller than those from the MLE method. The new β estimator is almost unbiased, and very accurate. The MLE-based β estimator shows slight over-estimation, as biases are all positive.

For α_0, both RVT and MLE methods have their estimators with small relative-bias and relative-MSE. The performances of both methods are very close. In both cases, the MSE of α_0 decrease, as the true value of β increases, namely, the right tail of the Weibull distribution becomes thinner. For example, when

TABLE III
SAMPLES GENERATED WITH $(\beta, \alpha_0, \alpha_1) = (1, 5, -1)$. 10000 REPLICATES

	relative-bias							
S/N	MLE				RVT			
	β	α_0	α_1	θ_0	β	α_0	α_1	θ_0
1	0.162	-0.007	0.079	0.255	-0.006	0.002	-0.017	-0.034
2	0.107	0.005	0.113	0.336	-0.007	-0.004	-0.001	-0.034
3	0.140	0.000	0.105	0.298	-0.005	0.001	0.000	-0.031
4	0.101	-0.008	0.026	0.249	-0.003	0.000	-0.017	-0.028
5	0.064	0.002	0.057	0.312	-0.007	-0.002	0.005	-0.018
6	0.085	-0.003	0.048	0.283	-0.005	0.000	-0.002	-0.020
7	0.060	-0.003	0.027	0.096	0.000	0.002	0.003	0.001
8	0.035	0.001	0.031	0.120	-0.006	-0.001	0.000	-0.009
9	0.049	-0.003	0.018	0.097	-0.003	-0.001	-0.009	-0.012
	relative-MSE							
	MLE				RVT			
	β	α_0	α_1	θ_0	β	α_0	α_1	θ_0
1	0.106	0.022	1.067	1.118	0.058	0.022	1.062	0.623
2	0.055	0.022	1.112	1.331	0.036	0.022	1.105	0.670
3	0.082	0.021	1.087	1.222	0.048	0.021	1.075	0.643
4	0.053	0.021	1.032	1.145	0.037	0.021	1.034	0.646
5	0.028	0.021	1.036	1.249	0.023	0.022	1.037	0.679
6	0.041	0.021	1.027	1.211	0.031	0.021	1.028	0.668
7	0.025	0.009	0.327	0.290	0.021	0.009	0.332	0.232
8	0.015	0.009	0.334	0.312	0.013	0.009	0.339	0.241
9	0.021	0.009	0.335	0.290	0.018	0.009	0.335	0.232

TABLE IV
SAMPLES GENERATED WITH $(\beta, \alpha_0, \alpha_1) = (2, 5, -1)$. 10000 REPLICATES

	relative-bias							
S/N	MLE				RVT			
	β	α_0	α_1	θ_0	β	α_0	α_1	θ_0
1	0.163	-0.003	0.045	0.053	-0.005	0.002	-0.003	-0.003
2	0.106	0.002	0.056	0.084	-0.009	-0.002	-0.002	-0.014
3	0.145	0.000	0.054	0.068	0.000	0.000	0.002	-0.007
4	0.099	-0.004	0.013	0.062	-0.004	0.000	-0.010	-0.009
5	0.065	0.000	0.023	0.065	-0.006	-0.002	-0.004	-0.015
6	0.087	-0.002	0.019	0.058	-0.003	-0.001	-0.005	-0.008
7	0.063	-0.002	0.010	0.016	0.003	0.000	-0.002	0.000
8	0.037	0.000	0.014	0.027	-0.004	-0.001	-0.001	-0.007
9	0.052	-0.002	0.008	0.018	-0.001	-0.001	-0.006	-0.007
	relative-MSE							
	MLE				RVT			
	β	α_0	α_1	θ_0	β	α_0	α_1	θ_0
1	0.104	0.006	0.290	0.163	0.057	0.005	0.270	0.138
2	0.056	0.005	0.280	0.180	0.036	0.006	0.278	0.146
3	0.086	0.005	0.275	0.169	0.050	0.005	0.273	0.142
4	0.051	0.005	0.263	3.017	0.036	0.005	0.254	0.135
5	0.029	0.005	0.253	0.158	0.023	0.005	0.254	0.133
6	0.042	0.005	0.265	0.166	0.031	0.005	0.263	0.144
7	0.026	0.002	0.080	0.056	0.021	0.002	0.081	0.054
8	0.015	0.002	0.081	0.058	0.013	0.002	0.083	0.055
9	0.021	0.002	0.082	0.056	0.018	0.002	0.083	0.054

$\beta = 0.5$, the relative-MSE of α_0 from MLE and RVT lie in the interval between $0.087 \sim 0.035$. When $\beta = 1$, the interval reduced to $0.022 \sim 0.009$; and when $\beta = 2$, the interval reduced to $0.006 \sim 0.002$.

For α_1, its RVT-based estimator has smaller relative-bias, whereas the MLE estimator tends to over-estimate. The relative-MSE for both methods are about the same, and significantly decrease as the true value of β increases. For θ_0, the MLE estimator still tends to over-estimate, while its RVT-based estimator seems to slightly under-estimate θ_0 for most cases. The MLE has much larger relative-MSE than the new method does, especially when the true value of β is small (heavy-tailed). For example, in Table II, the relative-MSE for θ_0 for simulation scheme 2 are 84.562 for the MLE, and 12.960 for the RVT estimator. Also, as the value of θ depends on values of α_0

TABLE V
AVERAGE CP AND INTERVAL LENGTH (IN PARENTHESES) OF 95% CI ESTIMATION. SAMPLES GENERATED WITH $(\beta, \alpha_0, \alpha_1) = (1, 5, -1)$. 1000 REPLICATES

S/N	MLE				RVT			
	β	α_0	α_1	$\log(\theta_0)$	β	α_0	α_1	$\log(\theta_0)$
1	0.966 (1.048)	0.887 (2.524)	0.884 (3.553)	0.887 (2.574)	0.944 (0.934)	0.945 (3.174)	0.95 (4.497)	0.945 (3.228)
3	0.970 (0.961)	0.906 (3.613)	0.913 (627.152)	0.909 (2.642)	0.955 (0.858)	0.941 (4.437)	0.953 (924.62)	0.954 (3.208)
4	0.969 (0.792)	0.925 (2.658)	0.928 (3.676)	0.925 (2.559)	0.952 (0.753)	0.952 (3.068)	0.943 (4.258)	0.952 (2.926)
6	0.957 (0.533)	0.922 (3.689)	0.932 (678.736)	0.932 (2.690)	0.943 (0.692)	0.953 (4.205)	0.951 (883.011)	0.953 (3.096)
7	0.964 (0.581)	0.928 (1.733)	0.933 (2.127)	0.928 (1.736)	0.965 (0.568)	0.944 (1.899)	0.953 (2.35)	0.944 (1.886)
9	0.967 (0.533)	0.919 (2.168)	0.916 (326.839)	0.916 (1.795)	0.953 (0.521)	0.939 (2.355)	0.942 (370.504)	0.944 (1.965)

and α_1, estimation bias and MSE for θ_0 significantly decrease as the true value of β increases. For example, when $\beta = 2$, the relative-MSE for θ_0 under simulation scheme 2 are 0.180 for the MLE, and 0.146 for the RVT estimator (in Table IV). Overall, as the number of stress levels increasing leads to larger sample sizes, estimation bias and MSE decrease as sample size increases.

To sum up, simulation for parameter estimation of the Weibull distribution shows that, in terms of estimation bias and MSE, the performance of the proposed RVT method is significantly better than that of the MLE method. The performance of both methods are somewhat sensitive to the value of the shape parameter β of the Weibull distribution. A smaller value of β leads to less accurate results, as the Weibull distribution becomes more heavily tailed.

We also compare the estimation of the CI from the MLE method and the RVT method. 1000 replicates of progressively Type-II censored samples were generated from a Weibull distribution with parameters $(\beta, \alpha_0, \alpha_1) = (1, 5, -1)$, under simulation design scenarios 1, 4, and 7 (conventional Type-II censoring), and scenarios 3, 6, and 9 (progressive censoring). We calculate the 95% CI based on the MLE method, and the RVT method, for different estimators. The average interval lengths and coverage probabilities of the two methods were reported in Table V. It is obvious that MLE-based 95% CIs have smaller interval lengths than the RVT-based CIs under small samples, except for the CI of β, but the coverage probabilities (CP) from MLE-based CIs are significantly poorer than those from RVT-based CIs. The CPs of MLE-based CIs are lower than the nominal confidence level for all tested statistics. On the other hand, the RVT-based generalised CIs have better CPs which are around the $\pm 1\%$ range of the nominal 95% confidence level. For example, for samples generated from simulation scheme 1, MLE-based CIs have smaller interval lengths, but poor average CPs that are all under 90%. RVT-based generalised CIs have CPs between 94.5% and 95.5%.

VI. A REAL EXAMPLE AND ITS ANALYSIS

Nelson [44] presented some data on the times to breakdown of a type of electrical insulting fluid subject to various constant voltage stresses. The purpose of the experiment was to estimate the distribution of time to breakdown at 20 kilovolt (kv). For the purpose of illustrating the methods presented in this paper, two Type-II progressively censored samples have been randomly generated from the $n_1 = 11$, and $n_2 = 15$ observations recorded at 30, and 36 kilovolts in Nelson [44] respectively. The observations and the progressive censored plans are reported in Table VI. The design stress level $x_0 = 20$ kv. Parameter estimation and CI estimation results are shown in Table VII.

TABLE VI
DATA OF THE TIMES TO BREAKDOWN OF A TYPE OF ELECTRICAL INSULTING FLUID SUBJECT TO VARIOUS CONSTANT VOLTAGE STRESSES AS IN NELSON [44]

Voltage level	n_i	R_i	Breakdown times
30 kv	11	(2, 0, ..., 0, 2)	7.74, 17.05, 21.02, 43.40, 47.30, 139.07, 144.12
36 kv	15	(4, 0, ..., 0, 1)	0.35, 0.96, 1.69, 1.97, 2.58, 2.71, 3.67, 3.99, 5.35, 13.77

TABLE VII
VALUES OF ESTIMATORS AND 95% CI (IN PARENTHESES) OF THE LIFETIME WEIBULL DISTRIBUTION OF THE ELECTRICAL INSULTING FLUID SUBJECTS

	MLE	RVT
β	1.02 (0.93, 1.11)	0.93 (0.64, 1.37)
α_0	19.54 (14.23, 24.86)	19.84 (13.98, 26.48)
α_1	-0.50 (-0.65, -0.34)	-0.50 (-0.7, -0.33)
$\log(\theta_0)$	9.61 (7.42, 11.79)	9.03 (7.44, 12.56)
μ	14740.47 (1656.63, 131158.94)	8613.56 (1786.85, 309930.2)

The estimates from RVT method for the parameter θ_0 and for the mean time to breakdown μ largely depart from the estimates of the MLE. For example, the mean time to breakdown estimated using the proposed RVT method is 8613.56, which is approximately 40% shorter than the value estimated by MLE, 14740.47. Note that, in the simulation tests, we found that the MLE tend to overestimate θ_0 by as much as nearly one third. Hence, in these data, the mean time to breakdown estimated by MLE is possibly also overestimated. See Fig. 1 for the difference.

VII. CONCLUSION

In this paper, we have considered a constant-stress ALT model with a Weibull distribution when the data are progressively censored. A new method, based on random variable transformation (RVT), and totally different from MLE-based inference, is proposed. We have derived the estimators of unknown parameters, the exact confidence interval of shape parameters, and the generalized CIs of other parameters. The numerical analysis and comparison show that the RVT method is promising, particularly for small samples, and different censoring rates or schemes.

APPENDIX

Proof of (3): Let

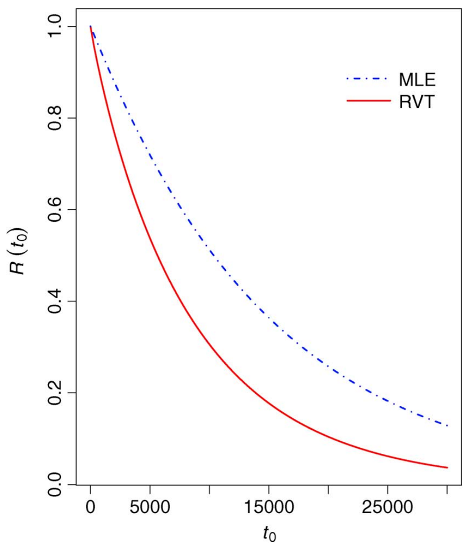

Fig. 1. The reliability plot $R(t_0)$ over a range of time.

$$V = \beta^{-2} diag\left(\psi'(r_1), \ldots, \psi'(r_k)\right), \; Y = \beta^{-1}(U_1, \ldots, U_k)^T$$

, and

$$Z = \begin{pmatrix} 1 & x_1 \\ \cdots & \cdots \\ 1 & x_k \end{pmatrix}.$$

Then the unbiased estimators of (α_0, α_1) are given by

$$\begin{pmatrix} \tilde{\alpha}_0 \\ \tilde{\alpha}_1 \end{pmatrix} = \left(Z^T V^{-1} Z\right)^{-1} Z^T V^{-1} Y = \frac{1}{\beta(FG - I^2)} \begin{pmatrix} GH - IM \\ FM - IH \end{pmatrix},$$

and the covariance matrix of the unbiased estimators $(\tilde{\alpha}_0, \tilde{\alpha}_1)$ is given by

$$Var\begin{pmatrix} \tilde{\alpha}_0 \\ \tilde{\alpha}_1 \end{pmatrix} = \left(Z^T V^{-1} Z\right)^{-1} = \frac{1}{\beta^2(FG - I^2)} \begin{pmatrix} G & -I \\ -I & F \end{pmatrix}.$$

The proof is completed.

Proof of Theorem 1: Let $Y_i = \log(S_i/\theta_i^\beta) - \psi(r_i)$. Notice that

$$H = \sum_{i=1}^k \left[\psi'(r_i)\right]^{-1} Y_i + \alpha_0 \beta F + \alpha_1 \beta I \hat{=} H_1 + \alpha_0 \beta F + \alpha_1 \beta I,$$

and

$$M = \sum_{i=1}^k \left[\psi'(r_i)\right]^{-1} x_i Y_i + \alpha_0 \beta I + \alpha_1 \beta G \hat{=} M_1 + \alpha_0 \beta I + \alpha_1 \beta G,$$

so we have

$$\tilde{\alpha}_0 = \frac{GH_1 - IM_1}{\beta(FG - I^2)} + \alpha_0, \tilde{\alpha}_1 = \frac{FM_1 - IH_1}{\beta(FG - I^2)} + \alpha_1.$$

Hence,

$$\log(\tilde{\theta}_0) - \log(\theta_0) = \tilde{\alpha}_0 + \tilde{\alpha}_1 x_0 - \alpha_0 - \alpha_1 x_0 = \sum_{i=1}^k D_i \log\left(\frac{S_i}{\theta_i^\beta}\right) - \sum_{i=1}^k D_i \psi(r_i).$$

Because $2S_i/\theta_i^\beta \sim \chi^2(2r_i)$, we have

$$E\left(\frac{\tilde{\theta}_0}{\theta_0}\right) = \exp\left(-\sum_{i=1}^k D_i \psi(r_i)\right) \prod_{i=1}^k E\left(\frac{T_i}{\theta_i}\right)^{D_i} = \exp\left(-\sum_{i=1}^k D_i \psi(r_i)\right) \prod_{i=1}^k \frac{\Gamma(r_i + D_i)}{\Gamma(r_i)}.$$

Thus, $\tilde{\theta}_{0U}$ is the unbiased estimator of θ_0. Similarly, we can derive the variance of $\tilde{\theta}_{0U}$.

Similar to the proof of Theorem 6 in Wang and Yu [36], we can prove that $\tilde{\theta}_{0U}$ has a smaller mean squared error than that of $\tilde{\theta}_0$. The proof is completed.

Proof of Theorem 2: Because $W(\beta)$ has the χ^2 distribution with v degrees of freedom, and $W(\beta)$ is strictly increasing on $(0, +\infty)$, we have

$$P\left(W^{-1}\left(\chi^2_{1-\gamma/2}(v)\right) \le \beta \le W^{-1}\left(\chi^2_{\frac{\gamma}{2}}(v)\right)\right) = P\left(\chi^2_{1-\gamma/2}(v) \le W(\beta) \le \chi^2_{\frac{\gamma}{2}}(v)\right) = 1 - \gamma,$$

where $v = 2\sum_{i=1}^k r_i - 2k$. The proof is completed.

Acknowledgment

The authors would like to thank the anonymous referees for their valuable comments and suggestions to improve the quality of this paper.

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