

# General Theory of Geometric Lévy Models for Dynamic Asset Pricing

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The geometric Lévy model (GLM) is a natural generalisation of the geometric Brownian motion model (GBM) used in the derivation of the Black-Scholes formula. The theory of such models simplifies considerably if one takes a pricing kernel approach. In one dimension, once the underlying Lévy process has been specified, the GLM has four parameters: the initial price, the interest rate, the volatility, and the risk aversion. The pricing kernel is the product of a discount factor and a risk aversion martingale. For GBM, the risk aversion parameter is the market price of risk. For a GLM, this interpretation is not valid: the excess rate of return is a nonlinear function of the volatility and the risk aversion. It is shown that for positive volatility and risk aversion the excess rate of return above the interest rate is positive, and is increasing with respect to these variables. In the case of foreign exchange, Siegel’s paradox implies that one can construct foreign exchange models for which the excess rate of return is positive both for the exchange rate and the inverse exchange rate. This condition is shown to hold for any geometric Lévy model for foreign exchange in which volatility exceeds risk aversion.

**Keywords:** Lévy processes, asset pricing, risk premium, risk aversion, Siegel’s paradox.

## I. INVESTMENT-GRADE ASSETS AND EXCESS RATE OF RETURN

The goal of this paper is to understand better the nature of the “risk premium” associated with jumps in asset prices. The idea is to work in a rather general setting, without being tied too much to any particular model. For mathematical simplicity, we shall assume that the dynamics of asset prices are driven by Lévy processes. This already encompasses a large class of models—it includes, above all, all the Brownian motion based models—so we need not worry that we are being too restrictive. We aim to clarify the relation between risk, risk aversion, and the excess rate of return (above the interest rate) offered by risky assets in such a context. With this end in mind, let us recall the setup in the geometric Brownian motion (GBM) model. The GBM model is very simple, but it captures a number of the main features of the relation between risk, risk aversion, and the excess rate of return. We shall adopt a pricing kernel approach, which turns out to be particularly useful in the Lévy case since it allows one to distinguish clearly between pricing issues and hedging issues. See, for example, Cochrane (2005) for an overview of the application of pricing kernel methods.

In the one-dimensional case, we have a Brownian motion  $\{W_t\}_{t \geq 0}$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and the associated augmented filtration  $\{\mathcal{F}_t\}_{t \geq 0}$ . Here  $\mathbb{P}$  represents the physical measure, and  $\{\mathcal{F}_t\}$  is taken to be the market filtration. The model consists of: (a) a pricing kernel; and (b) a collection of one or more “investment-grade” assets. For simplicity, we

assume for the time being that the assets under consideration pay no dividends over the time horizon considered. We relax this assumption in Section VI.

The idea of an investment-grade asset is that it should offer a positive excess rate of return above the interest rate. There are respectable assets that do not have this property—such assets are typically held alongside investment-grade assets for hedging. One can check, for example, in the context of the Black-Scholes model, that the price process of a put option has a drift that is less than the interest rate. The pricing kernel in the GBM model is

$$\pi_t = e^{-rt} e^{-\lambda W_t - \frac{1}{2}\lambda^2 t}, \quad (1)$$

where  $r > 0$  is the interest rate, and  $\lambda > 0$  is the risk aversion factor, both assumed to be constant. For a typical investment-grade asset we then have

$$S_t = S_0 e^{(r+\lambda\sigma)t} e^{\sigma W_t - \frac{1}{2}\sigma^2 t}, \quad (2)$$

where  $\sigma > 0$  is the volatility. The term  $\lambda\sigma$  is called the “risk premium” or “excess rate of return”, and is clearly positive under the assumptions made.

We observe that the risk premium is increasing with respect to both the volatility and the risk aversion. Since  $\lambda\sigma$  is linear in each factor, we call  $\lambda$  the “market price of risk” in the GBM model. It should be evident, however, that there is no *a priori* reason why the excess rate of return should be bilinear. Indeed, we shall demonstrate that in a general Lévy model the excess rate of return is a nonlinear function of  $\lambda$  and  $\sigma$ . The reason that the pricing kernel is such a useful concept in finance is that market equilibrium and the absence of arbitrage are both built into the idea that the product of the pricing kernel with the price of any asset paying no dividend is a martingale. In the GBM case, for example, we have

$$\pi_t S_t = S_0 e^{(\sigma-\lambda)W_t - \frac{1}{2}(\sigma-\lambda)^2 t}. \quad (3)$$

We shall use this property of the pricing kernel to establish the general form of an arbitrage-free Lévy-driven asset-pricing model. In Section II we look at one-dimensional geometric Lévy models with constant volatility and risk aversion, and in Section III we establish the positivity and monotonicity of the risk premium for such models. In Section IV we consider models for foreign exchange, and establish conditions sufficient to ensure that both the exchange rate and the inverse exchange rate have a positive risk premium. Examples of geometric Lévy models are studied in detail in Section V, where we note the fact that, unlike the GBM case, option prices in general depend on the risk aversion level; then in Section VI we extend the model to include dividends. In Section VII we consider models in which the market is driven by a vector of Lévy processes, and in which the volatility and risk aversion are predictable processes. In both situations we establish conditions sufficient to ensure the positivity of the excess rate of return. In conclusion we comment on the advantage of the use of pricing kernel methods in the theory of Lévy models, and how this allows us to unify earlier work on the subject, leading to a coherent framework for asset pricing.

## II. PRICING KERNEL APPROACH TO LÉVY MODELS FOR ASSET PRICES

Let us construct a family of Lévy models in the spirit of the GBM model. We shall call these geometric Lévy models (GLMs). Here we consider the one-dimensional case. In what follows we shall assume that the reader is familiar with basic aspects of the theory of Lévy

processes, as represented for example in Appelbaum (2004), Bertoin (2004), Cont & Tankov (2004), Kyprianou (2006), Protter (1990), Sato (1999), or Schoutens (2004). We recall that a Lévy process on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is a process  $\{X_t\}$  such that  $X_0 = 0$ ,  $X_t - X_s$  is independent of  $\mathcal{F}_s$  for  $t \geq s$  (independent increments), and

$$\mathbb{P}(X_t - X_s \leq y) = \mathbb{P}(X_{t+h} - X_{s+h} \leq y) \quad (4)$$

(stationary increments). Here  $\{\mathcal{F}_t\}$  denotes the augmented filtration generated by  $\{X_t\}$ . For  $\{X_t\}$  to give rise to a GLM, we require that

$$\mathbb{E}[e^{\alpha X_t}] < \infty \quad (5)$$

for all  $t \geq 0$ , for  $\alpha$  in some connected interval  $A$  containing the origin. Henceforth we consider Lévy processes satisfying such a moment condition. It follows by the stationary and independent increments property that there exists a function  $\psi(\alpha)$ , the so-called Lévy exponent, such that

$$\mathbb{E}[e^{\alpha X_t}] = e^{t\psi(\alpha)} \quad (6)$$

for  $\alpha \in A$ . The process  $\{M_t\}$  defined by

$$M_t = e^{\alpha X_t - t\psi(\alpha)} \quad (7)$$

is then called the geometric Lévy martingale (or Esscher martingale) associated with  $\{X_t\}$ , with volatility  $\alpha$ . Indeed, by the stationary and independent increments property we have:

$$\mathbb{E}_s[M_t] = \mathbb{E}_s[e^{\alpha X_t}] e^{-t\psi(\alpha)} = \mathbb{E}_s[e^{\alpha(X_t - X_s)}] e^{\alpha X_s - t\psi(\alpha)} = e^{(t-s)\psi(\alpha)} e^{\alpha X_s - t\psi(\alpha)} = M_s. \quad (8)$$

Our geometric Lévy model for asset prices will be put together as follows. First we construct the pricing kernel  $\{\pi_t\}_{t \geq 0}$ . Let  $\lambda > 0$  and assume that  $-\lambda \in A$ . Then set

$$\pi_t = e^{-rt} e^{-\lambda X_t - t\psi(-\lambda)}. \quad (9)$$

For a consistent pricing theory we require that the product of the pricing kernel and the asset price should be a martingale, which we shall assume is of the form

$$\pi_t S_t = S_0 e^{\beta X_t - t\psi(\beta)} \quad (10)$$

for some  $\beta \in A$ . From the formulae above we deduce that

$$S_t = S_0 e^{rt} e^{(\beta+\lambda)X_t + t\psi(-\lambda) - t\psi(\beta)} = S_0 e^{rt} e^{\sigma X_t + t\psi(-\lambda) - t\psi(\sigma - \lambda)}, \quad (11)$$

where  $\sigma = \beta + \lambda$ . We shall assume that  $\sigma > 0$  and that  $\sigma \in A$ . It follows that the asset price can be expressed in the form

$$S_t = S_0 e^{rt} e^{R(\lambda, \sigma)t} e^{\sigma X_t - t\psi(\sigma)}, \quad (12)$$

where

$$R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda). \quad (13)$$

### III. ON THE RISK PREMIUM ASSOCIATED WITH LÉVY MODELS

One sees that the function  $R(\sigma, \lambda)$  is the risk premium, that is to say, the excess rate of return above the interest rate. Indeed, we have:

$$\mathbb{E}[S_t] = S_0 e^{rt+R(\lambda, \sigma)t}. \quad (14)$$

The following result establishes a rather general property of geometric Lévy models, and is indicative of why such models are of interest.

**Proposition 1.** *The excess rate of return in a geometric Lévy model is positive, and is increasing with respect to the risk aversion and the volatility.*

*Proof.* We have  $\psi(\alpha) = t^{-1} \ln \mathbb{E} [e^{\alpha X_t}]$ , and thus

$$\psi''(\alpha) = \frac{1}{t} \frac{\mathbb{E} [(X_t - \bar{X}_t)^2 e^{\alpha X_t}]}{\mathbb{E} [e^{\alpha X_t}]}, \quad \text{where} \quad \bar{X}_t = \frac{\mathbb{E} [X_t e^{\alpha X_t}]}{\mathbb{E} [e^{\alpha X_t}]}. \quad (15)$$

Formula (15) shows that  $\psi''(\alpha) > 0$ , and thus that the Lévy exponent is *convex* as a function of  $\alpha$ . Indeed, for any random variable  $\xi$  satisfying  $m(\alpha) := \mathbb{E} [e^{\alpha \xi}] < \infty$  for  $\alpha$  in some interval containing the origin,  $\ln m(\alpha)$  is convex (see, e.g., Billingsley 1995). Now consider four values of  $\alpha$  in  $A$  such that  $\alpha_1 < \alpha_2 \leq \alpha_3 < \alpha_4$ , and for some  $h > 0$  suppose that  $\alpha_3 = \alpha_1 + h$  and  $\alpha_4 = \alpha_2 + h$ . Then we have:

$$\psi(\alpha_1) + \psi(\alpha_4) > \psi(\alpha_2) + \psi(\alpha_3). \quad (16)$$

To derive this inequality we note that the convexity of  $\psi(\alpha)$  implies that  $\psi'(x+h) - \psi'(x) > 0$  for  $x$  and  $x+h$  in  $A$ . Integrating  $\psi'(x+h) - \psi'(x)$  with respect to  $x$  between  $\alpha_1$  and  $\alpha_2$  one obtains (16). Then since  $\psi(0) = 0$ , and either  $-\lambda < 0 \leq \sigma - \lambda < \sigma$  or  $-\lambda < \sigma - \lambda \leq 0 < \sigma$ , it follows from (16) by letting either  $h = \sigma$  or  $h = \lambda$  that

$$\psi(-\lambda) + \psi(\sigma) > \psi(\sigma - \lambda). \quad (17)$$

Therefore,  $R(\lambda, \sigma) > 0$ . Furthermore, the convexity of the Lévy exponent implies that

$$\frac{\partial R(\lambda, \sigma)}{\partial \lambda} = \psi'(\sigma - \lambda) - \psi'(-\lambda) > 0 \quad \text{and} \quad \frac{\partial R(\lambda, \sigma)}{\partial \sigma} = \psi'(\sigma) - \psi'(\sigma - \lambda) > 0. \quad (18)$$

Thus,  $R(\lambda, \sigma)$  is increasing with respect to both  $\lambda$  and  $\sigma$ .  $\square$

We observe that the risk premium  $R(\lambda, \sigma)$  is in general a nonlinear function of the risk aversion (represented by  $\lambda$ ) and the risk (represented by  $\sigma$ ). This suggests that the notion of “market price of risk”, so common in the finance literature, is somehow linked specifically to models based on Brownian motion, and is not quite the right idea in the context of general Lévy models. Rather, risk premium is the more useful notion. In Section V we show that the only GLM leading to a bilinear risk premium is the GBM model.

Properties of the risk premium can be examined further by use of the Lévy-Khintchine representation for  $\psi(\alpha)$ , which in the case of a Lévy process admitting exponential moments takes the form

$$\psi(\alpha) = p\alpha + \frac{1}{2}q\alpha^2 + \int_{-\infty}^{\infty} (e^{\alpha x} - 1 - \alpha x \mathbb{1}\{|x| < 1\}) \nu(dx), \quad (19)$$

where  $p$  and  $q > 0$  are constants and  $\nu(dx)$  is the Lévy measure (see, e.g., Sato 1999, Theorem 25.17). For any measurable set  $B \in \mathbb{R}$ , the expected rate at which jumps occur for which the jump size lies in the range  $B$  is  $\nu(B)$ . It follows from (19) that the risk premium is given by

$$R(\lambda, \sigma) = \frac{1}{2}q\lambda\sigma + \int_{-\infty}^{\infty} (e^{\sigma x} - 1)(1 - e^{-\lambda x}) \nu(dx), \quad (20)$$

from which various of its properties can be deduced. In particular, the statement of Proposition 1 can be seen to follow from the fact that the argument of the integrand in (20) is positive for  $\sigma, \lambda > 0$ , and that its first derivatives are positive. One can also calculate the higher derivatives of the risk premium with respect to risk aversion and volatility, and one deduces, for example, that in the case of a spectrally negative Lévy process (downward jumps), we have  $\partial_{\sigma}^2 R < 0$  and  $\partial_{\lambda}^2 R > 0$ , and that these inequalities are reversed in the case of a spectrally positive process. One also observes that, providing the tails of the Lévy measure are not too fat, for small values of the risk aversion and volatility the risk premium is approximately bilinear.

#### IV. LÉVY MODELS FOR FOREIGN EXCHANGE

When the geometric Lévy model is extended to the case of foreign exchange, additional features arise that are of some interest. It is reasonable to require “numeraire symmetry” in the sense that if, for example, the dollar price of one pound sterling offers a positive excess rate of return above the interest rate differential, then the sterling price of one dollar should offer a positive excess rate of return above the reverse interest rate differential.

We examine the GBM case first, where the situation is transparent. Let the dollar be the domestic currency, and the pound the foreign currency. Let  $S_t$  denote the price of one pound in dollars, and  $\tilde{S}_t$  the price of one dollar in pounds. We write  $r$  for the domestic (dollar) interest rate, and  $f$  for the foreign (sterling) interest rate, both assumed constant. Let  $\lambda$  and  $\sigma$  be positive constants, and let the dollar pricing kernel be given by (1). Then the GBM model for the foreign exchange rate (the dollar price of one pound) is given by

$$S_t = S_0 e^{(r-f)t} e^{\lambda\sigma t} e^{\sigma W_t - \frac{1}{2}\sigma^2 t}. \quad (21)$$

We observe that the excess rate of return above the interest rate differential  $r - f$  is the product  $\lambda\sigma$ , which is positive. For the corresponding inverse exchange rate we obtain

$$\tilde{S}_t = \tilde{S}_0 e^{(f-r)t} e^{(\sigma-\lambda)\sigma t} e^{-\sigma W_t - \frac{1}{2}\sigma^2 t}. \quad (22)$$

In the case of  $\tilde{S}_t$ , we see that the excess rate of return above the reverse interest rate differential  $f - r$  is positive if and only if  $\sigma > \lambda$ . In equilibrium, we may presume that investors on both sides of the Atlantic wish to see the exchange rate promising a positive excess rate of return. The argument above shows that in a GBM model this possibility can be realised if  $\sigma > \lambda$ . This is the essence of the so-called Siegel (1972) paradox.

Now let us look at the analogous situation in the context of a geometric Lévy model. In the case of a GLM our elementary model for the exchange rate takes the form

$$S_t = S_0 e^{(r-f)t} e^{R(\lambda,\sigma)t} e^{\sigma X_t - t\psi(\sigma)}, \quad (23)$$

where  $\lambda$  and  $\sigma$  are positive constants, and  $R(\lambda, \sigma)$  is given by (13), as before. Thus for the inverse exchange rate we obtain

$$\tilde{S}_t = \tilde{S}_0 e^{(f-r)t} e^{\tilde{R}(\lambda, \sigma)t} e^{-\sigma X_t - t\psi(-\sigma)}, \quad (24)$$

where  $\tilde{R}(\lambda, \sigma) = -R(\lambda, \sigma) + \psi(\sigma) + \psi(-\sigma)$ . A short calculation shows that

$$\tilde{R}(\lambda, \sigma) = \psi(-\sigma) + \psi(\sigma - \lambda) - \psi(-\lambda). \quad (25)$$

Now suppose that  $\sigma > \lambda$ . Then  $-\sigma < -\lambda < 0 < \sigma - \lambda$ , which by taking  $h = \sigma$  in equation (16) implies that  $\tilde{R}(\lambda, \sigma) > 0$ . On the other hand, suppose that  $\sigma \leq \lambda$ . Then since either  $-\lambda \leq -\sigma \leq \sigma - \lambda \leq 0$  or  $-\lambda \leq \sigma - \lambda \leq -\sigma \leq 0$ , it follows from (16) by letting either  $h = \sigma$  or  $h = \lambda$  that  $\tilde{R}(\lambda, \sigma) \leq 0$ . One thus deduces the following:

**Proposition 2.** *If the volatility exceeds the risk aversion, then both (a) the excess rate of return on the FX rate, and (b) the excess rate of return on the inverse FX rate, are positive in a geometric Lévy model for foreign exchange.*

One observes that the volatility of the inverse exchange rate in (24), as matters stand, appears to be negative, which is not consistent with our original characterisation of an investment-grade asset. We can however put the inverse exchange rate into “canonical” form by regarding it as being driven by the mirror process  $\tilde{X}_t = -X_t$ . Then  $\sigma$  is the volatility, and we can regard the quantity  $\tilde{\lambda} = \sigma - \lambda$  as being the associated risk aversion parameter for foreign agents. Writing  $\tilde{\sigma} = \sigma$ , and regarding  $\tilde{\sigma}$  and  $\tilde{\lambda}$  as independent variables, one can check that the inverse excess rate of return is monotonically increasing in both the volatility  $\tilde{\sigma}$  and the foreign risk aversion factor  $\tilde{\lambda}$ . Indeed, if we write  $\tilde{\psi}(\alpha) = \psi(-\alpha)$  for the Lévy exponent associated with the mirror process, then it is an exercise to verify that

$$\tilde{R}(\lambda, \sigma) = \tilde{\psi}(\tilde{\sigma}) + \tilde{\psi}(-\tilde{\lambda}) - \tilde{\psi}(\tilde{\sigma} - \tilde{\lambda}), \quad (26)$$

and one sees that the form of the foreign excess rate of return, when expressed in terms of the relevant Lévy exponent, is identical to that of the domestic excess rate of return. The requirement  $\sigma > \lambda$  can be understood as an assertion that the foreign risk aversion is positive. Thus one might regard this constraint as a necessary feature of the model.

**Proposition 3.** *In a geometric Lévy model for foreign exchange with  $\sigma > \lambda > 0$ , the excess rate of return on the inverse FX rate is increasing with respect to the independent variables  $\tilde{\lambda} = \sigma - \lambda$  and  $\tilde{\sigma} = \sigma$ .*

## V. EXAMPLES OF GEOMETRIC LÉVY MODELS

It will be instructive to look at various explicit examples of geometric Lévy models for asset prices, noting in particular the structure of the excess rate of return function in each case.

**Example 1: Brownian motion.** In the case of a standard geometric Brownian motion model the Lévy exponent is given by  $\psi(\alpha) = \frac{1}{2}\alpha^2$ , and hence

$$R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda) = \frac{1}{2}\sigma^2 + \frac{1}{2}\lambda^2 - \frac{1}{2}(\sigma - \lambda)^2 = \sigma\lambda, \quad (27)$$

which is positive. Further, for the inverse excess rate of return we have

$$\tilde{R}(\lambda, \sigma) = \psi(-\sigma) + \psi(\sigma - \lambda) - \psi(-\lambda) = \frac{1}{2}\sigma^2 + \frac{1}{2}(\sigma - \lambda)^2 - \frac{1}{2}\lambda^2 = \sigma(\sigma - \lambda). \quad (28)$$

Thus  $\tilde{R}(\lambda, \sigma)$  is positive if and only if  $\sigma > \lambda$ . We observe that if one treats the quantities  $\tilde{\sigma} = \sigma$  and  $\tilde{\lambda} = \sigma - \lambda$  as independent variables, then the inverse excess rate of return function is increasing with respect to each.

One can ask to what extent the bilinear form of the excess rate of return determines the underlying Lévy process. Is it uniquely Brownian motion that has this property? If we consider expression (13) and set  $R(\lambda, \sigma) = \lambda\sigma$ , then by taking two derivatives we deduce that  $\psi''(\alpha) = 1$ . Integrating twice and imposing the condition  $\psi(0) = 0$ , we conclude that the general form of the Lévy exponent compatible with a bilinear excess rate of return is

$$\psi(\alpha) = p\alpha + \frac{1}{2}\alpha^2, \quad (29)$$

where  $p$  is a constant, and we obtain a standard Brownian motion with drift. But the addition of a drift to the driving Lévy process is irrelevant to the resulting pricing model in our scheme, since it cancels out in the formula for the geometric Lévy martingale. Thus:

**Proposition 4.** *The geometric Brownian motion model is the only geometric Lévy model with an excess rate of return that is bilinear in the risk aversion and the volatility.*

In fact, in any GLM the excess rate of return function is sufficient to determine the driving Lévy process, up to an irrelevant drift. To establish that this is the case, we observe that if we differentiate each side of (13) with respect to  $\lambda$  and  $\sigma$ , and then take the limit as  $\lambda$  approaches zero, we obtain

$$\psi''(\sigma) = \left. \frac{\partial^2 R(\lambda, \sigma)}{\partial \lambda \partial \sigma} \right|_{\lambda=0}. \quad (30)$$

Integrating twice, and fixing the constant, we obtain the Lévy exponent, modulo a drift.

**Example 2: Poisson process.** Let  $\{N_t\}$  be a standard Poisson process with jump rate  $m > 0$ . Then for any nonnegative integer  $n$  the distribution of  $N_t$  is given by

$$\mathbb{P}(N_t = n) = e^{-mt} \frac{(mt)^n}{n!}. \quad (31)$$

It follows that  $\mathbb{E}[X_t] = mt$ , and that the Lévy exponent is  $\psi(\alpha) = m(e^\alpha - 1)$ . The associated geometric Lévy martingale with volatility  $\alpha$  in this example is thus

$$M_t = \exp[\alpha N_t - mt(e^\alpha - 1)]. \quad (32)$$

A calculation then shows that the excess rate of return function is manifestly positive, and increasing with respect to its arguments:

$$R(\lambda, \sigma) = m(1 - e^{-\lambda})(e^\sigma - 1). \quad (33)$$

We remark that since the jumps in the geometric Poisson model are upward, the “risk” that an investor faces is that there may be fewer jumps than one hopes for. This is made evident



if we combine the expressions for the geometric martingale and the excess rate of return function to obtain the following formula for the price of a non-dividend-paying asset:

$$S_t = S_0 \exp [rt + \sigma N_t - mt e^{-\lambda}(e^\sigma - 1)]. \quad (34)$$

Thus, the effect of investor risk aversion is to reduce the downward drift rate in the compensator term by attaching the factor  $e^{-\lambda}$  to it. For the associated pricing kernel one has

$$\pi_t = \exp [-rt - \lambda N_t - mt(e^{-\lambda} - 1)], \quad (35)$$

and it is an exercise to check that the product of  $\pi_t$  and  $S_t$  gives a geometric Poisson martingale with volatility  $\sigma - \lambda$ . In the event that  $S_t$  represents the price of a unit of foreign currency, then we include the foreign interest rate by setting

$$S_t = S_0 \exp [(r - f)t + \sigma N_t - mt e^{-\lambda}(e^\sigma - 1)]. \quad (36)$$

For the corresponding inverse exchange rate we obtain

$$\tilde{S}_t = \tilde{S}_0 \exp [(f - r)t + \tilde{R}(\lambda, \sigma) - \sigma N_t + mt(1 - e^{-\sigma})], \quad (37)$$

where  $\tilde{R}(\lambda, \sigma) = m(e^{\sigma-\lambda} - 1)(1 - e^{-\sigma})$ , in agreement with equation (25). If  $\sigma > \lambda$  then the excess rate of return of the inverse exchange rate is evidently positive, and has the property of being increasing with respect to the independent variables  $\sigma$  and  $\sigma - \lambda$ .

**Example 3: Compound Poisson process.** Let  $\{N_t\}$  be a standard Poisson process with rate  $m$ , and let  $\{Y_k\}_{k \in \mathbb{N}}$  be a collection of identical independent copies of a random variable  $Y$  with the property that

$$\phi(\alpha) := \mathbb{E}[e^{\alpha Y}] < \infty \quad (38)$$

for  $\alpha$  in some connected interval  $A$  containing the origin. Writing  $\mathbf{1}\{-\}$  for the indicator function, one can check that

$$X_t = \sum_{k=1}^{\infty} \mathbf{1}\{k \leq N_t\} Y_k \quad (39)$$

defines a Lévy process, and that the associated Lévy exponent is given by the formula  $\psi(\alpha) = m(\phi(\alpha) - 1)$ . In this example the excess rate of return function is

$$R(\lambda, \sigma) = m(\phi(\sigma) + \phi(-\lambda) - \phi(\sigma - \lambda) - 1), \quad (40)$$

and the fact that it is positive and is bi-monotonic in its arguments is evident as a consequence of the convexity of  $\phi(\alpha)$ , which follows from (38). If  $S_t$  represents the price of a unit of foreign currency, then the resulting geometric Lévy model for the exchange rate is

$$S_t = S_0 \exp [(r - f)t + \sigma X_t + mt(\phi(-\lambda) - \phi(\sigma - \lambda))]. \quad (41)$$

**Example 4: Jump diffusion process.** This example is a generalisation of the Merton (1976) jump diffusion model. The driver is a vector Lévy process, one component being a standard Brownian motion, and the other a compound Poisson process with normally



distributed jumps. Write  $\lambda$  and  $\sigma$  for the risk aversion and volatility of the Brownian component, and  $\beta$  and  $\theta$  for the risk aversion and volatility of the jump component. Let  $Y$  be normally-distributed with mean zero and variance one. Then  $\phi(\alpha) = \exp(\frac{1}{2}\alpha^2)$ , and the excess rate of return function is of the form

$$R(\lambda, \sigma, \beta, \theta) = \lambda\sigma + m \left( e^{\frac{1}{2}\theta^2} + e^{\frac{1}{2}\beta^2} - e^{\frac{1}{2}(\theta-\beta)^2} - 1 \right), \quad (42)$$

which is positive and is monotonic in each of the variables. In Merton (1976) a key notion used to price options is the idea that there is no risk premium offered by the market in connection with idiosyncratic firm-specific risk. Merton assumes that jump risk is purely idiosyncratic and can be diversified away by holding a suitably broad portfolio. Merton argues that since the risk can be diversified away, the market awards no risk premium to investors who hold such assets. From a modern point of view, the assumption that jump risk is necessarily idiosyncratic is questionable: this is one of the lessons of the 2008 credit crisis. In our version of Merton's model, however, jump risk is being priced.

**Example 5: Gamma process.** By a standard gamma process with growth rate  $m$ , we mean a process  $\{\gamma_t\}$  that has gamma-distributed stationary and independent increments, and satisfies  $\mathbb{E}[\gamma_t] = mt$  and  $\text{Var}[\gamma_t] = mt$ . The density of  $\gamma_t$  is given by

$$\mathbb{P}(\gamma_t \in dx) = \mathbf{1}\{x > 0\} \frac{x^{mt-1}e^{-x}}{\Gamma[mt]} dx, \quad (43)$$

where  $\Gamma[a]$  denotes the gamma function. The identity  $\Gamma[a+1] = a\Gamma[a]$  implies that the mean of  $\gamma_t$  is  $mt$ , thus justifying the interpretation of  $m$  as the growth rate. The associated moment generating function is

$$\mathbb{E}[e^{\alpha\gamma_t}] = (1 - \alpha)^{-mt} = e^{-mt \ln(1-\alpha)}, \quad (44)$$

and hence the Lévy exponent is

$$\psi(\alpha) = -m \ln(1 - \alpha), \quad (45)$$

which is well-defined for  $\alpha < 1$ . For some applications it is useful to consider the two-parameter family of so-called scaled gamma processes. By a scaled gamma process with growth rate  $\mu$  and variance rate  $\nu^2$  we mean a process  $\{G_t\}_{0 \leq t < \infty}$  with stationary and independent increments such that  $G_0 = 0$  and such that  $G_t$  has a gamma distribution with mean  $\mu t$  and variance  $\nu^2 t$ , where  $\mu$  and  $\nu$  are parameters. Setting  $m = \mu^2/\nu^2$  and  $\kappa = \nu^2/\mu$ , one finds that  $\mu = \kappa m$  and  $\nu^2 = \kappa^2 m$ . One can thus think of  $m$  as a standardised growth rate, and  $\kappa$  as a scale parameter. The density of  $G_t$  is given by

$$\mathbb{P}(G_t \in dx) = \mathbf{1}\{x > 0\} \frac{\kappa^{-mt} x^{mt-1} e^{-x/\kappa}}{\Gamma[mt]} dx. \quad (46)$$

For fixed  $t$  the product  $mt$  is the so-called shape parameter of the distribution of the random variable  $G_t$ . If  $\{\gamma_t\}$  is a standard gamma process with growth rate  $m$ , then the process  $\{G_t\}$  defined by  $G_t = \kappa\gamma_t$  is evidently a scaled gamma process with standardised growth rate  $m$  and scale parameter  $\kappa$ , and for its moment generating function we have

$$\mathbb{E}[e^{\alpha G_t}] = (1 - \alpha\kappa)^{-mt}. \quad (47)$$

See Dufresne *et al.* (1991), Dickson & Waters (1993), Madan *et al.* (1998), Baxter (2007), Yor (2007), and Brody *et al.* (2008) for various aspects of the theory of the gamma process.

Now let  $\{\gamma_t\}$  be a standard gamma process with growth rate  $m$ , and let  $\sigma$  be a constant such that  $0 < \sigma < 1$ . The associated geometric Lévy martingale takes the form

$$M_t = (1 - \sigma)^{mt} e^{\sigma \gamma_t}. \quad (48)$$

The jumps are upward, and the compensator is a deterministic decreasing process. It follows from (13) and (45) that the excess rate of return function is of the form

$$R(\lambda, \sigma) = m \ln \frac{1 - \sigma + \lambda}{(1 - \sigma)(1 + \lambda)}. \quad (49)$$

If one takes the difference between the numerator and the denominator in the argument of the logarithm in (49), the result is  $(1 - \sigma + \lambda) - (1 - \sigma)(1 + \lambda) = \sigma \lambda$ , which is positive. It follows that  $R(\lambda, \sigma) > 0$ . By (23), the corresponding model for the foreign exchange rate is

$$S_t = S_0 e^{(r-f)t} \left(1 - \frac{\sigma}{1 + \lambda}\right)^{mt} e^{\sigma \gamma_t}. \quad (50)$$

Variants of the geometric gamma model appear in Heston (1993), Gerber & Shiu (1994), and Chan (1999). We observe that the effect of risk aversion is to reduce the rate at which the compensator decreases, thus encouraging investors who might otherwise be concerned over the possibility of an insufficient rise in the underlying gamma process. For small  $\lambda$  and  $\sigma$  the risk premium is given approximately by  $\lambda \sigma$ .

As for the excess rate of return associated with the inverse exchange rate, by use of (25) we obtain the following expression:

$$\tilde{R}(\lambda, \sigma) = m \ln \frac{1 + \lambda}{(1 + \sigma)(1 - \sigma + \lambda)}. \quad (51)$$

We observe that  $(1 + \lambda) - (1 + \sigma)(1 - \sigma + \lambda) = \sigma(\sigma - \lambda)$ , from which it follows that  $\tilde{R}(\lambda, \sigma) > 0$  if and only if  $\sigma > \lambda$ . Numeraire symmetry thus imposes a bound on the risk aversion factor and we have  $0 < \lambda < \sigma < 1$ . Inverting (50), and writing  $\tilde{\sigma} = \sigma$  and  $\tilde{\lambda} = \sigma - \lambda$ , we find that

$$\tilde{S}_t = \tilde{S}_0 e^{(f-r)t} \left(1 + \frac{\tilde{\sigma}}{1 - \tilde{\lambda}}\right)^{mt} e^{-\tilde{\sigma} \gamma_t}. \quad (52)$$

Thus  $\tilde{S}_t$  is driven by a negative gamma process, which jumps downward, and the compensator is a deterministic increasing process. Since  $0 < \tilde{\lambda} < 1$ , the effect of foreign risk aversion is to increase the rate at which the compensator increases. For small values of  $\lambda$  and  $\sigma$  the inverse excess rate of return is given approximately by  $\tilde{\lambda} \tilde{\sigma}$ .

**Example 6: Variance gamma process.** It will be convenient first to discuss the symmetric variance gamma (VG) process. This is the process considered by Madan & Seneta (1990) and Madan & Milne (1991). Then in the next example we discuss the more general asymmetric or “drifted” VG process of Madan *et al.* (1998). Both of these processes are of interest from a mathematical perspective and as a basis for financial modelling. There is a further extension of the model, due to Carr *et al.* (2002), which will not be discussed here.

The VG model relies on the use of a gamma process as a subordinator. Thus we begin with a standard gamma process  $\{\gamma_t\}$  with rate  $m$ , and give it the dimensionality of time by dividing it by  $m$ . In this way we define a scaled gamma process  $\{\Gamma_t\}$  by setting  $\Gamma_t = m^{-1}\gamma_t$ , and we observe that  $\mathbb{E}[\Gamma_t] = t$ . We call  $\{\Gamma_t\}$  a standard gamma subordinator. The symmetric VG process  $\{V_t\}$ , with parameter  $m$ , is defined by letting  $\{W_t\}$  be a standard Brownian motion and setting  $V_t = W_{\Gamma_t}$ . The associated moment generating function is thus

$$\mathbb{E}[\exp(\alpha V_t)] = \mathbb{E}[\exp(\alpha W_{\Gamma_t})] = \mathbb{E}\left[\exp\left(\frac{1}{2}\alpha^2 \Gamma_t\right)\right] = \left(1 - \frac{\alpha^2}{2m}\right)^{-mt}, \quad (53)$$

which is defined for  $\alpha^2 < 2m$ . Clearly  $\alpha$  must have units of inverse square-root time, since  $m$  has units of inverse time; but this is consistent with the fact that  $V_t$  has units of square-root time, like the Wiener process. The associated Lévy exponent is

$$\psi(\alpha) = -m \ln\left(1 - \frac{\alpha^2}{2m}\right), \quad (54)$$

and one can check the convexity of  $\psi(\alpha)$  in this example by observing that

$$\psi''(\alpha) = m \left(1 - \frac{\alpha^2}{2m}\right)^{-2}. \quad (55)$$

As a consequence the geometric Lévy martingale in the symmetric VG case takes the form

$$M_t = \left(1 - \frac{\alpha^2}{2m}\right)^{mt} \exp(\alpha W_{\Gamma_t}), \quad (56)$$

and the excess rate of return function, which is positive and monotonic, is

$$R(\lambda, \sigma) = m \ln \left[ \left(1 - \frac{(\sigma - \lambda)^2}{2m}\right) \left(1 - \frac{\lambda^2}{2m}\right)^{-1} \left(1 - \frac{\sigma^2}{2m}\right)^{-1} \right]. \quad (57)$$

The corresponding VG foreign exchange rate is thus given by

$$S_t = S_0 e^{(r-f)t} \left(1 - \frac{(\sigma - \lambda)^2}{2m}\right)^{mt} \left(1 - \frac{\lambda^2}{2m}\right)^{-mt} \exp(\sigma W_{\Gamma_t}). \quad (58)$$

We remark that in the case of the VG model one finds by use of (55) that the risk premium satisfies  $\partial_\sigma^2 R > 0$  iff  $\sigma > |\sigma - \lambda|$ , and  $\partial_\lambda^2 R > 0$  iff  $\lambda > |\sigma - \lambda|$ .

A well known alternative characterisation of the VG process is as follows. Let  $\{\gamma_t^1\}$  and  $\{\gamma_t^2\}$  be a pair of independent standard gamma processes, each with rate  $m$ . Then the process defined by the difference between these two processes has both upward and downward jumps, and is symmetrical about the origin in distribution, with mean zero. If we normalise the difference by setting

$$V_t = \frac{1}{\sqrt{2m}} (\gamma_t^1 - \gamma_t^2), \quad (59)$$

then it is easy to check that the variance of  $V_t$  is  $t$ , and so we get a pure jump process that has some properties in common with Brownian motion. Indeed if we consider the moment generating function we find by virtue of the independence of the two gamma processes that

$$\begin{aligned}\mathbb{E}[\exp(\alpha V_t)] &= \mathbb{E}\left[\exp\left(\alpha\frac{1}{\sqrt{2m}}(\gamma_t^1 - \gamma_t^2)\right)\right] \\ &= \left(1 - \frac{\alpha}{\sqrt{2m}}\right)^{-mt} \left(1 + \frac{\alpha}{\sqrt{2m}}\right)^{-mt} \\ &= \left(1 - \frac{\alpha^2}{2m}\right)^{-mt},\end{aligned}\tag{60}$$

and it is evident (Madan & Senata 1990) that (59) has the law of a VG process. For large values of  $m$ , the distribution of  $V_t$  is approximately Gaussian. In particular, we have:

$$\lim_{m \rightarrow \infty} \mathbb{E}[\exp(\alpha V_t)] = \exp\left(\frac{1}{2}\alpha^2 t\right).\tag{61}$$

**Example 7: Asymmetric VG process.** The representation of the VG process as the normalised difference between two independent gamma processes suggests two generalisations. One is that of Madan *et al.* (1998), where we consider an asymmetric difference between two independent standard gamma processes. Thus writing

$$U_t = \kappa_1 \gamma_t^1 - \kappa_2 \gamma_t^2,\tag{62}$$

where  $\kappa_1$  and  $\kappa_2$  are nonnegative constants, a calculation of the respective moment generating functions shows that  $U_t$  is identical in law to a “drifted” VG process of the form

$$U_t = \mu \Gamma_t + \sigma W_{\Gamma_t},\tag{63}$$

where  $\mu$  and  $\sigma$  are constants. The relationship between  $\mu$ ,  $\sigma$ ,  $\kappa_1$ ,  $\kappa_2$ , and  $m$  is given by  $\mu = m(\kappa_1 - \kappa_2)$  and  $\sigma^2 = 2m\kappa_1\kappa_2$ , together with

$$\kappa_1 = \frac{1}{2m} \left(\mu + \sqrt{\mu^2 + 2m\sigma^2}\right) \quad \text{and} \quad \kappa_2 = \frac{1}{2m} \left(-\mu + \sqrt{\mu^2 + 2m\sigma^2}\right).\tag{64}$$

The Lévy exponent  $\psi(\alpha) = -m \ln(1 - (\kappa_1 - \kappa_2)\alpha - \kappa_1\kappa_2\alpha^2)$ , which can be worked out by use of (47), can be equivalently written in the form

$$\psi(\alpha) = -m \ln \left(1 - \frac{\mu}{m}\alpha - \frac{\sigma^2}{2m}\alpha^2\right),\tag{65}$$

where the range of  $\alpha$  is  $-1/\kappa_2 < \alpha < 1/\kappa_1$ . It is straightforward to write down the associated excess rate of return function, and the corresponding expression for an asset price. In this example there is a single risk aversion factor.

On the other hand, one can also envisage the situation where the two gamma drivers are regarded as separate sources of risk, each being assessed independently by the market. This situation arises in instances where investors are for some reason more worried about downward jumps than upward ones. More specifically, let us suppose that investors are more concerned about excessive losses than about insufficient gains. It is said that studies

in behavioural finance suggest that this may actually be the case. One can model such a situation rigorously by introducing an asymmetric pricing kernel of the form

$$\pi_t = e^{-rt} (1 - \lambda_1)^{mt} (1 + \lambda_2)^{mt} e^{-\lambda_1 \gamma_t^1} e^{\lambda_2 \gamma_t^2}, \quad (66)$$

and an asset price process of the form

$$S_t = S_0 e^{rt} \left(1 - \frac{\sigma_1}{1 + \lambda_1}\right)^{mt} \left(1 + \frac{\sigma_2}{1 - \lambda_2}\right)^{mt} e^{\sigma_1 \gamma_t^1} e^{-\sigma_2 \gamma_t^2}. \quad (67)$$

Thus we have separate risk aversion factors for the upward jumps and the downward jumps. It is interesting to observe that in the case of “behavioural asymmetry” both the asset price and the pricing kernel are driven by extended VG processes—but there are two distinct such processes, one driving the pricing kernel, and the other driving the asset price. Indeed, the pricing kernel is driven by  $\lambda_1 \gamma_t^1 - \lambda_2 \gamma_t^2$ , whereas the asset price is driven by  $\sigma_1 \gamma_t^1 - \sigma_2 \gamma_t^2$ . These processes are synchronised in the sense that the times of their upward and downward jumps coincide and the magnitudes are proportional for a given jump type. Alternatively, we can model the two driving processes as different linear combinations of  $\Gamma_t$  and  $W_{\Gamma_t}$ .

**Example 8: Negative binomial process.** This process is a species of compound Poisson process, and can be viewed as a special case of Example 3. It is nevertheless of considerable interest in its own right. See, for example, Kozubowski & Krzysztof (2009) for a general overview. The negative binomial process has a Lévy exponent of the form

$$\psi(\alpha) = m \ln \left( \frac{1 - q}{1 - qe^\alpha} \right), \quad (68)$$

where  $0 < q < 1$ , and  $m > 0$  is a rate parameter. A short calculation shows that

$$R(\lambda, \sigma) = m \ln \left[ \frac{(1 - q)(1 - qe^{\sigma - \lambda})}{(1 - qe^\sigma)(1 - qe^{-\lambda})} \right]. \quad (69)$$

To see explicitly that this is positive we take the difference between the numerator and the denominator inside the logarithm to obtain  $q(e^\sigma - 1)(1 - e^{-\lambda})$ , which clearly is positive since  $\sigma, \lambda > 0$ . It follows that the argument of the logarithm is larger than one, and we have  $R(\lambda, \sigma) > 0$ . For the inverse excess rate of return we obtain

$$\tilde{R}(\lambda, \sigma) = m \ln \left[ \frac{(1 - q)(1 - qe^{-\lambda})}{(1 - qe^{-\sigma})(1 - qe^{\sigma - \lambda})} \right]. \quad (70)$$

To analyse the positivity of  $\tilde{R}(\lambda, \sigma)$  we again take the difference between the numerator and the denominator in the argument of the logarithm in (70). We get  $q(e^{\sigma - \lambda} - 1)(1 - e^{-\sigma})$ , which is positive only if  $\sigma > \lambda$ .

The jumps of the negative binomial process are positive integers. There are two distinct representations for the process. The first of these takes the form of a compound Poisson process with the following characteristics: (a) the jump sizes have a so-called logarithmic distribution, given in the notation of Example 3 by

$$\mathbb{P}(Y = n) = -\frac{1}{\ln(1 - q)} \frac{1}{n} q^n; \quad (71)$$

and (b) the intensity  $\mu$  of the underlying Poisson process is of the form  $\mu = -m \ln(1 - q)$ . A straightforward calculation of the moment generating function of  $Y$  gives

$$\phi(\alpha) = \frac{\ln(1 - qe^\alpha)}{\ln(1 - q)}. \quad (72)$$

By the general theory of the compound Poisson process we know that  $\psi(\alpha) = \mu(\phi(\alpha) - 1)$ , which immediately leads to the Lévy exponent (68). If we write  $X_t$  for the value of the negative binomial process at time  $t$ , we find that its probability mass function is given by

$$\mathbb{P}(X_t = k) = \frac{\Gamma(k + mt)}{\Gamma(mt)\Gamma(k + 1)} q^k (1 - q)^{mt}, \quad (73)$$

which is the negative binomial distribution; and for the geometric Lévy martingale we have

$$M_t = \left( \frac{1 - qe^\alpha}{1 - q} \right)^{mt} e^{\alpha X_t}. \quad (74)$$

The second representation of the negative binomial process is reminiscent of the theory of the VG process. We take a standard Poisson process, with intensity

$$\Lambda = m \frac{q}{(1 - q)}, \quad (75)$$

where  $0 < q < 1$  and  $m > 0$  as before, and subordinate it with a gamma subordinator  $\{\Gamma_t\}$  with standardised rate parameter  $m$ . Thus the expectation of  $\Gamma_t$  is  $t$ , and its variance is  $t/m$ , as in the theory of the VG process. The associated moment generating function is thus

$$\mathbb{E}[\exp(\alpha N_{\Gamma_t})] = \mathbb{E}[\exp(\Lambda(e^\alpha - 1)\Gamma_t)] = \left( 1 - \frac{\Lambda(e^\alpha - 1)}{m} \right)^{-mt} = \left( \frac{1 - q}{1 - qe^\alpha} \right)^{mt}, \quad (76)$$

by virtue of the chosen intensity (75), and we are lead directly to the Lévy exponent (68).

It is natural to ask in the context of these various examples what information can be extracted (or “implied”) about the values of model parameters when one is given option prices. In the case of the GBM model, for example, it is known that one can infer the value of volatility  $\sigma$ , but that the option price is independent of risk aversion  $\lambda$ . This can be checked directly by working out the price of a call option with strike  $K$  and expiry  $T$  by inserting (1) and (2) into the valuation formula  $C_0 = \mathbb{E}[\pi_T (S_T - K)^+]$ . In a general geometric Lévy model, this is no longer the case: option prices depend on both the risk aversion and the volatility. Indeed, a variety of different situations can arise, each with its own character. Thus in the Poisson model, there are two nontrivial model parameters—the risk aversion, and the jump rate  $m$  (the volatility is easily determined by observation of the price process); and a calculation shows that option prices depend on  $me^{-\lambda}$ , but not on  $m$  or  $\lambda$  separately. Thus if we can estimate the value of the actual jump rate  $m$  by observations of the asset price, then  $\lambda$  can be inferred from option prices. In the case of the gamma model, there are three nontrivial model parameters—the risk aversion, the volatility, and the jump rate. A calculation shows that option prices depend on  $m$  and on  $\sigma/(1 + \lambda)$ , but not on  $\sigma$  and  $\lambda$  separately, so neither  $\lambda$  nor  $\sigma$  can be determined exactly from option prices.

## VI. DIVIDEND PAYING ASSETS

Thus far we have considered the case of non-dividend paying assets. From a conceptual point of view it is better, however, to think of an asset price as being determined by the dividend stream or cash flow produced by the asset. Hence with the inclusion of dividends the pricing model is characterised by: (a) a pricing kernel  $\{\pi_t\}$ ; and (b) the dividend stream  $\{D_t\}$  generated by the asset. The value of the asset at time  $t$  is regarded as a *derived* quantity which can be worked out by means of the fundamental relation

$$S_t = \frac{1}{\pi_t} \mathbb{E}_t \left[ \int_t^\infty \pi_s D_s ds \right]. \quad (77)$$

With this in mind, let us consider how one extends the GBM model when dividends are included. The answer to this problem is well known, but rather than assuming the conclusion we shall derive it from first principles by modelling the dividend stream and the pricing kernel, and working out the resulting price process for the asset. For a typical investment asset in the GBM situation we model the dividend stream by setting

$$D_t = D_0 e^{\gamma t} e^{\sigma W_t - \frac{1}{2} \sigma^2 t}, \quad (78)$$

where  $D_0$  is the initial rate at which dividends are paid,  $\gamma$  is the growth rate of the dividend, and the constant  $\sigma > 0$  characterises the volatility of the dividend rate. We shall assume that the pricing kernel is of the form (1). Substituting (1) and (78) into equation (77) and performing a short calculation under the assumption that  $r + \lambda\sigma > \gamma$ , we deduce that

$$S_t = \frac{1}{r + \lambda\sigma - \gamma} D_0 e^{\gamma t + \sigma W_t - \frac{1}{2} \sigma^2 t}. \quad (79)$$

Thus we obtain a stochastic generalisation of the Gordon (1959) growth model, and at time zero we have the following valuation formula:

$$S_0 = \frac{D_0}{r + \lambda\sigma - \gamma}. \quad (80)$$

We observe that an increase in risk aversion has the effect of lowering the asset price, everything else being the same. Defining the proportional dividend rate by setting  $\delta = r + \lambda\sigma - \gamma$ , which by assumption is positive, we are able to deduce the relation  $D_t = \delta S_t$ , and we find that the asset price process is given by

$$S_t = S_0 e^{(r - \delta + \lambda\sigma)t} e^{\sigma W_t - \frac{1}{2} \sigma^2 t}. \quad (81)$$

As we stated earlier, the resulting expressions for  $\{S_t\}$  and  $\{D_t\}$  are of course familiar: the point is that we derive these formulae here rather than assume them.

In the situation of a geometric Lévy model it is remarkable that essentially the same line of argument carries through. Thus we assume a pricing kernel of the form (9), and a volatile dividend stream of the form

$$D_t = D_0 e^{\gamma t} e^{\sigma X_t - t\psi(\sigma)}. \quad (82)$$

It is then an exercise to check that the fundamental relation gives

$$S_t = S_0 e^{(r - \delta)t} e^{R(\lambda, \sigma)t} e^{\sigma X_t - t\psi(\sigma)}, \quad (83)$$



and a proportional dividend flow  $D_t = \delta S_t$ , where  $\delta = r + R(\lambda, \sigma) - \gamma$ . The resulting initial valuation formula is

$$S_0 = \frac{D_0}{r + R(\lambda, \sigma) - \gamma}. \quad (84)$$

This relation ties together in the context of a general geometric Lévy model the values of the initial asset price, the initial dividend rate, the interest rate, the risk aversion level, the dividend volatility, and the dividend growth rate.

## VII. MULTI-FACTOR MODELS WITH PREDICTABLE VOLATILITY

There are several reasons for extending the analysis to higher dimensions. First, we would like to consider models for a market consisting of a number of different assets. Second, even in the consideration of a single asset, it is natural to introduce the additional complexity of a higher-dimensional process to describe its dynamics. Both situations are familiar in the context of Brownian motion driven models. It seems to be advantageous to envisage the entire market as being driven by a *single* higher-dimensional Lévy process. We can use essentially the same notation as in the one-dimensional case. Now  $\{X_t\}$  is understood to be a vector Lévy process. For the Lévy exponent we still have (6), but now  $\alpha$  is understood to be a vector, and there is an implicit inner product between  $\alpha$  and  $X_t$  in the exponent on the left hand side of (6). The Lévy exponent is a function of the  $n$  components of  $\alpha$ .

The model is thus determined as in Section I, with the assumption that the market filtration is generated by a vector Lévy process, and with the specification of (i) a pricing kernel, and (ii) a collection of investment grade assets, driven collectively by  $\{X_t\}$ . We are accustomed, in the multi-dimensional Brownian case, to regard such a higher dimensional driver as being built from a set of independent drivers that can be isolated after a suitable linear transformation. No such simplification is readily at hand for a general vector Lévy process. Nevertheless for applications it is useful to consider the case where the components are assumed to be independent. This encompasses a large class of models, including the higher-dimensional Brownian motion models.

The pricing kernel is a process of the form (9) where  $\lambda$  is now understood to be a vector risk aversion factor. If the components of the vector Lévy process are independent, the Lévy exponent separates into a sum of terms, one for each component of its argument, each term being the marginal Lévy exponent associated with one of the risk factors. Next we introduce a set of investment-grade assets, each of the form (12) for some choice of the vector volatility  $\sigma$ . We require that  $\lambda$  and  $\sigma$  are “positive” vectors—that is, they belong to the cone of vectors with the property that all components are nonnegative and at least one component is positive. For a generic asset the excess rate of return takes the form (13), only now the arguments are understood to be vectors. When the Lévy process has independent components, the excess rate of return separates into a sum of terms, each being the excess rate of return associated with one of the components. In that case, we see by use of the arguments presented earlier that each term is nonnegative, and at least one is positive; as a consequence the total excess rate of return function is positive. Similarly one sees that in the case of independent components the excess rate of return is increasing with respect to the individual components of the risk aversion vector and the volatility vector. Thus we obtain the following result.

**Proposition 5.** *The excess rate of return function  $R(\lambda, \sigma)$  in a multi-dimensional geometric Lévy model with independent Lévy drivers is positive, and is increasing with respect to each of the components of the risk aversion vector  $\lambda$  and the volatility vector  $\sigma$ .*

In the discussion so far, we have assumed for simplicity that the interest rate, the risk aversion, and the asset price volatilities are constant. Indeed, as with many financial models, various characteristic features of the model are already present under the assumption of constant coefficients; but for practical applications, and to take the theory further, we need to relax this condition. Thus in the case of geometric Brownian motion models we consider the situation where the risk aversion  $\{\lambda_t\}$  and volatility  $\{\sigma_t\}$  are adapted vector-valued processes, and are chosen in such a way as to ensure that the process defined by the expression

$$M_t = \exp \left( \int_0^t \alpha_s dW_s - \frac{1}{2} \int_0^t \alpha_s^2 ds \right) \quad (85)$$

is an  $\{\mathcal{F}_t\}$ -martingale for  $\alpha_t = -\lambda_t$ ,  $\alpha_t = \sigma_t$ , and  $\alpha_t = \sigma_t - \lambda_t$ . It suffices that  $\{\alpha_t\}$  should be bounded. More generally, we consider the situation where  $\{X_t\}$  is a Lévy process with exponential moments,  $\{\mathcal{F}_t\}$  is the associated filtration, and  $\{\alpha_t\}$  is a *predictable* process, adapted to  $\{\mathcal{F}_t\}$ , chosen in such a way that  $\alpha_t \in A$  for  $t \geq 0$  and that the local martingale defined by

$$M_t = \exp \left( \int_0^t \alpha_s dX_s - \int_0^t \psi(\alpha_s) ds \right) \quad (86)$$

is a martingale. If a predictable vector process  $\{\alpha_t\}$  satisfies these conditions then we say it is admissible. Thus we consider a market model of the following form. Let the exogenously specified short rate process  $\{r_t\}$  be adapted to  $\{\mathcal{F}_t\}$ , and be such that the unit-initialised money market account

$$B_t = \exp \left( \int_0^t r_s ds \right) \quad (87)$$

is finite almost surely for  $t > 0$ . Let the  $\{\mathcal{F}_t\}$ -adapted vector risk aversion and volatility processes  $\{\lambda_t\}$  and  $\{\sigma_t\}$  be positive, and be such that the processes  $\{-\lambda_t\}$ ,  $\{\sigma_t\}$ , and  $\{\sigma_t - \lambda_t\}$  are admissible in the sense described above. The pricing kernel is taken to be of the form

$$\pi_t = \exp \left( - \int_0^t r_s ds - \int_0^t \lambda_s dX_s - \int_0^t \psi(-\lambda_s) ds \right). \quad (88)$$

The corresponding expression for the price of a typical non-dividend-paying asset is then

$$S_t = S_0 \exp \left( \int_0^t r_s ds + \int_0^t R(\lambda_s, \sigma_s) ds + \int_0^t \sigma_s dX_s - \int_0^t \psi(\sigma_s) ds \right), \quad (89)$$

where  $R(\lambda, \sigma)$  is the excess rate of return function associated with the given Lévy exponent. Clearly both  $\{\pi_t B_t\}$  and  $\{\pi_t S_t\}$  are martingales. It should also be evident that the following statement holds, which is to be understood as an expression of the fact that the asset offers a rate of return greater than the interest rate.

**Proposition 6.** *Let the asset price in a model with predictable volatility and risk aversion, driven by a vector of independent Lévy processes, be given by (89), and define the money market account by (87). Then the asset price, expressed in units of the money market account, is a submartingale.*

In the literature, stochastic volatility models are often introduced by the method of a time change. In the context of simple parametric volatility this amounts to the observation that if  $\{W_t\}$  is a standard Brownian motion, then the processes defined by  $Z_t = \sigma W_t$  and  $Z'_t = W_{\sigma^2 t}$  have the same law. Thus instead of introducing the volatility as a coefficient measuring the sensitivity of the asset price to the underlying Brownian motion, it is introduced by “speeding up” (or “slowing down”) the Brownian motion: the effect is equivalent. In the case of jump processes, the two transformations are clearly inequivalent; thus, for example, in the case of the Poisson process the effect of scaling the process (magnifying the jumps) is quite different from that of scaling the time (speeding up the arrival of jumps). It seems that in the general situation one wishes to consider both effects. Thus if the market is driven by a vector Lévy process  $\{X_t\}$ , then we introduce a vector of sensitivity parameters  $\sigma$ , as well as a time dilation factor  $c$ , and let a typical asset be driven by the process  $\{\sigma X_{ct}\}$ . Again, we think of the entire market as being driven by a single vector Lévy process so it is consistent that there is but a single overall time change for that process. On the other hand if the time change is random then we introduce a subordinator  $\{c_t\}$  and the time-changed process is given by  $\{\sigma X_{c_t}\}$ . If the subordinator is itself a Lévy process then one stays within the category of models already under consideration—thus the VG process can be obtained by subordinating Brownian motion with a gamma process, and the negative binomial process can be obtained by subordinating a Poisson process with a gamma process, as we have seen.

## VIII. CONCLUDING REMARKS

Geometric Lévy models have a surprisingly wide range of desirable properties. As we have seen, once suitable inequalities are imposed on the volatility and risk aversion parameters, the convexity of the Lévy exponent ensures that the excess rate of return function is positive and is monotonic. In foreign exchange models, numeraire symmetry can be ensured by imposing a further inequality on the relation between the volatility and the risk aversion. In the extended version of the model, the market is driven by a vector of independent Lévy processes, and the risk aversion and volatility coefficients are taken to be predictable processes satisfying suitable integrability conditions, these conclusions remain valid. Our approach is based on use of the physical measure  $\mathbb{P}$ . We emphasise the importance of the pricing kernel method, because this leads to a unified view of the role of Lévy models in finance, allowing one to separate pricing issues from hedging issues. In particular, we make no use of the idea of “trying to find an equivalent martingale measure” by some recipe when one is given a set of price processes. Rather, the pricing kernel is to be regarded as an essential component of the theory from the beginning. One needs the pricing kernel before one can speak of prices, because the value of a security is determined by the random cash flows that it produces, and these need to be valued by use of the pricing kernel. In this respect our point of view diverges in spirit from the earlier literature on Lévy models in finance, as represented by Gerber & Shiu (1994), Eberlein & Keller (1995), Eberlein & Jacod (1997), Chan (1999), Raible (2000), Kallsen & Shiryaev (2002), Fujiwara & Miyahara (2003), Esche & Schweizer (2005), Eberlein *et al.* (2005), Hubalek & Sgarra (2006), and others. Exceptions

include: Madan & Milne (1991) who are able to identify the rate of return in their study of the VG model; Heston (1993) who with the introduction of a “pricing operator” in his study of the gamma model offers a point of view similar in some respects to ours; and Madan (2006) where risks are priced by such kernels in a particular equilibrium.

The approach we have outlined for asset pricing in a Lévy setting with predictable risk aversion and volatility is also useful in the theory of interest rates, since it allows one to generalise the HJM (Heath *et al.* 1992) framework in a natural way to the Lévy category, without the need of introducing instantaneous forward rates, but in a way that guarantees positive excess rates of returns on bonds, and is formulated in the  $\mathbb{P}$ -measure, making it suitable as a practical basis for risk management, forecasting, and scenario analysis.

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