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IDENTIFYING LONG-RUN BEHAVIOUR WITH NON-STATIONARY DATA

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Abstract

Results for the identification of non-linear models are used to support the traditional form of the order condition by sufficient conditions. The sufficient conditions reveal a two step procedure for firstly checking generic identification and then testing identifiability. This approach can be extended to sub-blocks of the system and it generalizes to non-linear restrictions. The procedure is applied to an empirical model of the exchange rate, which is identified by diagonalising the system.

Keywords: Cointegration, Identification, Identifiability, Order Condition, Sufficient Conditions.

JEL classification: C10, C22.

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1 Introduction

In this article, identification and identifiability are assumed to be nonlinear in nature, which differentiates this work from Johansen (1995) and Boswijk (1996). Generic identification follows from the existence of at least one solution relating the long-run parameters matrix (Π) to the cointegrating vectors (β) and the loadings matrix (α). Firstly, the restrictions are checked using a systems wide test, which has degrees of freedom based on the degree of over-identification of the system. Secondly, the existence of a solution to the system, depends on two rank conditions. Should such conditions fail, then the part of Π selected to identify is invalid. The latter test provides confirmation that the generic solution is empirically valid.

The approach can be extended to handle any number of sub-systems. As a result, the existence of cointegrating or weakly exogenous variables can be used to identify the system to a sub-block (section 3). The approach is applied to a model of the UK effective exchange rate (section 4).

2 Identifying Long-run Behaviour

This section addresses the question of long-run identification and identifiability, firstly in terms of satisfying the generic conditions and then via an empirical test of the restrictions and the conditions required for identifiability. Prior to any testing a solution is found to link the well defined parts of Π to the unrestricted elements of α and β, this is equivalent to one of the approaches adopted by Sargan (1983) to identify non-linear parameters. Failure of the test of over-identifying restrictions is sufficient to reject the solution to the generic problem, whereas acceptance of the test is only necessary, but not sufficient for identification. Should the test of over-identifying restrictions be satisfied, then the sufficient conditions for generic identification are tested via a pair of rank conditions. Existence of a solution to the generic problem is sufficient for identification, while the invertability of two $r \times r$ sub-matrices (A and B) is necessary and sufficient for the existence of a solution.

Let Π represent the matrix of long-run parameters for an n equation system (omitting deterministic terms and lags)

$$\Delta x_t = \Pi x_{t-1} + \epsilon_t, \tag{1}$$

where the usual conditions for cointegration hold: $rank(\Pi) = r$, $\Pi{=}\alpha\beta'$, $rank(\alpha) = rank(\beta) = r$, and α and β are $n \times r$ dimensioned matrices (see e.g. Johansen, 1991). It follows from the definition of cointegration, that there is a set of r linearly independent row vectors (Π_i) and column vectors ($\Pi_{.j}$), which are $r \times n$ and $n \times r$ dimensioned sub-matrices of Π. If α and β are stacked into a vector $\theta = vec[\alpha : \beta]$, then this vector has $2nr - r$ elements, assuming β is normalized by setting an element in each column to unity. Since Π has $2nr - r^2$ free parameters, relative to $2nr - r$ in θ, the order condition for the identification of θ is:

$$j \geq r^2 - r, \tag{2}$$

where j is the number of systems wide restrictions on α and β.

Subject to the order condition that precludes possible over-parameterisation, the following theorem provides sufficient conditions for the existence of a unique solution to a vector function relating the identifiable elements of Π, that is $\xi = vec(\Pi_r)$, where $\Pi_r = [\pi_{ij} \in \Pi_i \cup \Pi_{.j}]$, to the unknown parameters in α and β, encapsulated in θ.

Theorem 1 *In the cointegration case a sufficient condition for a solution to the vector function $\theta = g(\xi)$ is the existence of two $r \times r$ dimensioned non-singular sub-matrices A and B, in α and β respectively.*

Proof: $Rank(\alpha) = r$ is equivalent to the existence of a sub-matrix A such that $rank(A) = r$. There are $\frac{n!}{(n-r)!r!}$ possible alternative combinations of rows of α from which A might be formed. It follows that each A has a related sub-matrix Π_i of Π such that $rank(A) = r \Leftrightarrow rank(\Pi_i) = r$ and $\Pi_i = A\beta'$. Vectorising Π_i implies that $vec(\Pi_i) = vec(A\beta') = (I_n \otimes A)vec(\beta')$. Following the argument in Sargan (1983; p282-283), β is identifiable when A has full rank as firstly a unique solution results:

$$vec(\beta') = (I_n \otimes A)^{-1} vec(\Pi_i), \tag{3}$$

and secondly $rank(\frac{\partial vec(\beta')}{\partial vec(\Pi i)'}) = nr$, if the normalisation is ignored. By similar argument, α is identifiable when there exists two matrices $\Pi_{.j}$ and B for which $\Pi_{.j} = \alpha B'$ and B is non-singular. As a result, a unique solution for α exists of the form:

$$vec(\alpha) = (B \otimes I_n)^{-1} vec(\Pi_{.j}). \tag{4}$$

$\diamond$

The existence of one or more solutions to (**??**) and (**??**) is sufficient for identification given (**??**). Finding such solutions negates the need to undertake the test in Johansen (1995).

Johansen (1995) considers a stochastic system of linear equations $\beta' x_t = \eta_t$ subject to a number of linear restrictions $R_i'\beta_i = 0$ or $\beta_i = H_i\phi_i$ for $i = 1, ..r$,where R_i is a known $n \times r_i$ matrix or rank r_i, β_i is column i of β, H_i is a $n \times s_i$ (with $s_i = n - r_i$) selection matrix such that $H_i'R_i = 0$, and ϕ_i a $s_i \times 1$ vector of unrestricted parameters. Such restrictions are identifying for each cointegrating vector when Theorem 3 in Johansen (1995) is satisfied. Each block of restrictions R_i, for $i = 1, ..r$, identifies a cointegrating vector if and only if

$$rank(R_i'H_{i_1}\ R_i'H_{i_2} \ldots R_i'H_{i_k}) \geq k \tag{5}$$

for each $k = 1, ..., r-1$ and any set of indices $1 \leq i_1 < i_2 < \ldots < i_k \leq r$ not containing i. In Johansen's approach, the rank condition must be checked to confirm the 'linear independence' of the restrictions applied to each cointegrating vector in turn. The test is linear in nature as it relates to homogenous restrictions applied to β alone. If joint restrictions on α and β are considered, then the linear result breaks down.

Linearity or the need to consider α and β does not present a problem for the condition in Theorem 1 that may be applied sequentially to α and β to yield a sufficient set of solutions. Empirical verification of the generic result follows from a direct test of the over-identifying restrictions:

$$\begin{array}{llcl} (I) & H_\beta & : & \phi_\beta + R_\beta vec(\beta) = 0 \\ & H_\alpha & : & R_\alpha vec(\alpha) = 0 \end{array}$$

Now ϕ_β is a $j_\beta \times 1$ vector of known constants (normalisations), R_β and R_α are $j_\beta \times nr$ and $j_\alpha \times nr$ matrices, which select all the j_β and j_α restrictions on β and α respectively, and $j = j_\beta + j_\alpha$.[1] The degrees of freedom of the test are calculated from the number of solutions to (**??**) and (**??**). If (I) is rejected, then this is sufficient for non-identification, which requires a different set of restrictions. However, acceptance is only necessary for identification as their may be a sequence of models, that accept either the over-identifying restrictions or Johansen's test (Johansen, 1995).

To solve this problem, Boswijk (1996) provides two further conditions for what he terms identifiability. According to Boswijk β is non-identifiable when the normalisation fails or some of the remaining parameters are not significant:

$$H_{02} : \beta \in B_3 \cup B_4 = \{\beta : rank(R_1^{*\prime}\beta) \leq r - 1\},$$

where $R_1^{*\prime}$ is the restriction matrix including the normalisation and $B_3 \cup B_4$ defines the null associated with non-identifiability. Consider the following example, developed from Boswijk (1996), $n = 3$, $r = 2$ and (2) is satisfied when $j = 2 = r^2 - r$ restrictions identify β:

$$\beta' = \begin{bmatrix} a & 0 & b \\ c & d & 0 \end{bmatrix} \text{ and } H_2 = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}.$$

Subject to a normalisation, $a = 1$ and $d = 1$, then following Boswijk (1996), the first vector in β' is identifiable when a matrix H_2 has full rank.[2] However, given acceptance of the Johansen condition ($rank(\beta') = r$), β is identifiable as r linearly independent cointegrating vectors exist and given acceptance of the over-identifying restrictions (I), then the first vector is identified when $I(0)$ variables are precluded. Here, it is suggested that the choice of normalisation must hang on the exogeneity conditions of variables in the system; exogeneity is discussed in more detail in the next section. Following Boswijk, should the first vector be identifiable, then further rank conditions are tested for each vector.

In this article an alternative approach follows from the sufficient conditions for a solution to (**??**) and (**??**) given in Theorem 1:

$$(II) \text{ Test identifiability} : rank(B) = r \text{ and } rank(A) = r.$$

[1]In the case where more complex restrictions apply, then the general restriction condition and procedure in Doornik and Hendry (1997) apply.

[2]To discriminate between failure of normalisation and other types of failure, a further rank test is applied to an $r - 1$ dimensioned sub-matrix:

$$H_{03} : \beta \in B_4 = \{\beta : rank(R_1'\beta) \leq r - 2\}$$

For the example rank failure implies $a = 0$ (normalisation) or $d = 0$. Here, H_{03}: $d = 0$.

The existence of a solution to (??) and (??) implies the system is generically identified. As Boswijk, suggests on empirical grounds identification may fail due to insignificance of certain parameters. Here, identifiability follows from the existence of sufficient information in certain rows and columns of Π to identify α and β (Sargan, 1983). Clearly, many such orientations related to particular over-identifying restrictions may exist. However, it is sufficient to find one such orientation of the system to empirically accept the generic solution. Consider the above example, where for comparison with Boswijk $B = H_2$. When $rank(H_2) = r$,[3] then the necessary and sufficient condition for the existence of a solution to (??) and (??) is satisfied. From Theorem 1, the rank condition identifies α based on the restrictions in (I). Therefore, discovery of one such matrix (B) is sufficient for identification.

If the variable chosen for normalisation is invalid ($a = 0$ and $rank(H_2) < r$), then failure of the rank condition yields an additional restriction on the set of cointegrating vectors (β'). Therefore α is identified based on a new orientation:

$$\beta' = \begin{bmatrix} 0 & 0 & b \\ c & d & o \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & b \\ d & 0 \end{bmatrix}.$$

The system is overidentified as $j = 3 > r^2 - r$. Given acceptance of the Johansen rank condition $|B| = 0$ only occurs when $d = 0$ and $x_1$and x_3 are both stationary variables.[4]

Boswijk and Johansen emphasize a limited information approach associated with linear restrictions, that can only be applied to α and β in turn. In this article restrictions can be applied to both α and β, they can be non-linear and they apply to the system as a whole.

In the next section, the results are extended further to take account of exogeneity.

[3] β is identifiable for the restrictions in (I) when the selected columns of Π yield a matrix A of rank r, but for expositional reasons it is assumed at this point.

[4] For $n = 4$, the approach discussed above can be shown to identify. Let:

$$\beta' = \begin{bmatrix} a & 0 & b & c \\ d & e & f & 0 \end{bmatrix} \text{ and } B = H_2 = \begin{bmatrix} a & 0 \\ d & e \end{bmatrix}.$$

Following Boswijk (1996), identifiability is lost when a normalisation is invalid (i.e., $a = 0 \Rightarrow rank(H_2) < r$), but with this new restriction $[\alpha : \beta]$ is overidentified as $j = 3 > r^2 - r$. Selecting a new orientation, ensuring the generic result associated with Theorem 1 holds, then:

$$\beta'_{(1)} = \begin{bmatrix} 0 & 0 & b & c \\ d & e & f & 0 \end{bmatrix} \text{ and } B_{(1)} = \begin{bmatrix} b & c \\ f & 0 \end{bmatrix}.$$

This orientation is rejected when $x_t \sim I(1)$ and $f = 0$. Now α is not identifiable, but for the following orientation:

$$\beta'_{(2)} = \begin{bmatrix} 0 & 0 & b & c \\ d & e & 0 & 0 \end{bmatrix}, \; B_{(2)} = \begin{bmatrix} 0 & b \\ e & 0 \end{bmatrix}$$

and $rank(B) = r$. Hence, $[\alpha : \beta_{(2)}]$ is always empirically identified and identifiable.

3 Exogeneity and Identification

Traditional econometric methodology assumes the existence of a set of exogenous variables, where as the notion of cointegration and Vector Auto-Regressive (VAR) modelling negates this. Cointegration is multi-causal and the VAR treats all variables as endogenous, but within such a system, it is feasible to test a number of notions of long-run exogeneity. The interested reader is directed to Ericsson and Irons (1994) and Ericsson et al (1998). To help motivate the example in the next section, some further discussion of long-run exogeneity and identification is required.

Let us partition the system (**??**) into two sub-models, corresponding to a partition of x_t into y_t and z_t, of dimensions n_1 and n_2, respectively, and conformable partitioning of α and β:[5]

$$\begin{aligned}\Delta y_t &= (\alpha_{11}\beta'_{11} + \alpha_{12}\beta'_{12})y_{t-1} + (\alpha_{11}\beta'_{21} + \alpha_{12}\beta'_{22})z_{t-1} + \epsilon_{1t} &(6)\\ \Delta z_t &= (\alpha_{21}\beta'_{11} + \alpha_{22}\beta'_{12})y_{t-1} + (\alpha_{21}\beta'_{21} + \alpha_{22}\beta'_{22})z_{t-1} + \epsilon_{2t}, &(7)\end{aligned}$$

where $(\epsilon'_{1t}\,\epsilon'_{2t})' \sim N(0,\Omega)$, independently over $t = 1, ..., T$. It is well known that when $[\alpha_{21} : \alpha_{22}] = [0 : 0]$, then z_t is weakly exogenous for β (Johansen, 1992).

However, such restrictions do not directly assist in the identification of the long-run parameters as they apply to a part of α which is non-informative. In terms of the requirement to find a solution to (**??**) and (**??**), weak exogeneity is of direct use when there are $n - r$ weakly exogenous variables as the only basis for a choice of A is the matrix $[\alpha_{11} : \alpha_{12}]$, which is then by definition of rank r.

Otherwise, one might consider weak exogeneity associated with a sub-block of cointegrating vectors. To discuss issues of exogeneity it is useful to look at the conditional model for y_t given z_t (Johansen, 1992):

$$\begin{aligned}\Delta y_t &= [(\alpha_{11}\beta'_{11} + \alpha_{12}\beta'_{12}) - \omega(\alpha_{21}\beta'_{11} + \alpha_{22}\beta'_{12})]y_{t-1} + \omega\Delta z_t\\ &+ [(\alpha_{11}\beta'_{21} + \alpha_{12}\beta'_{22}) - \omega(\alpha_{21}\beta'_{21} + \alpha_{22}\beta'_{22})]z_{t-1} + \epsilon_{1t} - \omega\epsilon_{2t}\end{aligned} \quad (8)$$

where $\omega = \Omega_{12}\Omega_{22}^{-1}$. One set of sufficient conditions for weak exogeneity of z_t for $\beta'_{.1} = [\beta'_{11} : \beta'_{21}]$ is $\alpha_{12} - \omega\alpha_{22} = 0$ and $\alpha_{21} = 0$, see Lemma 2 in Ericsson et al (1998). Combining, (**??**) with (**??**) yields a system, which to a non-singular transformation matrix is equivalent to the original VAR. If ($\alpha_{12} = 0$, $\alpha_{21} = 0$) is applied to (**??**) and (**??**), then the VAR has a quasi-diagonal long-run structure (Hunter, 1992). For weak exogeneity additional restrictions may apply as $\alpha_{12} - \omega\alpha_{22} = 0$. Should $\alpha_{12} = 0$, then $\omega\alpha_{22} = 0$ is sufficient for weak exogeneity. This result can be associated with three possible conditions: i) $\omega = 0$, ii) $\alpha_{22} = 0$ or iii) ω is a left hand side annihilation matrix of α_{22}. Under cointegration (ii) does not apply as $rank(\alpha_{22}) = r_2$. Case (i) is consistent with Lemma 2 in Ericsson et al (1998). For case (iii), the quasi-diagonality restriction ($\alpha_{12} = 0$, $\alpha_{21} = 0$) combined with $\omega\alpha_{22} = 0$ is sufficient for weak exogeneity of z_t for $\beta_{.1}$.

[5] The matrices α_{ij} and β_{ij} have the dimensions $n_i \times r_j$, for $i = 1, 2$ and $j = 1, 2$. For example, the matrix β is partitioned into two blocks of columns, $\beta_{.1}$ of dimensions $n \times r_1$, and $\beta_{.2}$ of dimensions $n \times r_2$, then each block is itself cut into two blocks of rows.

Weak exogeneity for a sub-block implies that analysis may be undertaken at the level of the sub-system. More specifically, identification conditions now apply at the level of the sub-system, as previously at the level of the full system. Let Π_1 denote an $n_1 \times n$ sub-matrix of Π for which $rank(\Pi_1) = r_1$ and $n_1 > r_1 \geq 1$. If $\Pi_{1(r_1)}$ defines an $r_1 \times n$ sub-matrix of Π_1 for which the maximum rank is given by its smallest dimension, then an equivalent column matrix exists which is $n_1 \times r_1$ and has full column rank. Given the quasi-diagonality restriction, it follows that:

$$\Pi_1 = \alpha_{11}\beta'_{.1} \text{ and } \Pi_{1(r_1)} = A_1\beta'_{.1}, \tag{9}$$

where A_1 is a square matrix of full rank r_1 suitably extracted from α_{11} (by selecting r_1 rows). To identify α_{11} and $\beta_{.1}$ subject to a standard normalisation (i.e. r_1 restrictions) the following sub-system order condition is now applicable:

$$r_1 n + r_1 n_1 - r_1 \leq r_1 n + r_1 n_1 - r_1^2 \Leftrightarrow r_1^2 - r_1 \leq j_1,$$

where j_1 is the number of restrictions associated with the sub-system. Now, $r_1 - 1$ restrictions apply to each equation in the first sub-block as compared with $r - 1$ when the full system condition is used. Hence, r_2 variables are viewed as exogenous to the sub-system.

Theorem 2 *A sufficient condition for the existence of a solution to the vector sub-system:* $vec(\beta'_{.1}) = (I_n \otimes A_1)^{-1} vec(\Pi_{1(r_1)})$ *is the existence of a matrix* A_1 *of full rank* r_1 *constructed by selection of* r_1 *rows of* α_{11}.

Proof: By analogy with the proof of Theorem 1, $vec(\beta_{.1})$, which follows from vectorising (**??**), is identifiable when A_1 has full rank. $\diamond$

A special case arises when $r_1 = 1$ and excepting the choice of normalisation no further restrictions are required to identify $\beta_{.1}$.

Corollary 1 *If* $r_1 = 1$*, then subject to a normalisation, weak exogeneity is sufficient for identification of the long-run parameters* $\beta_{.1}$ *associated with the first sub-block.*

If in addition, $r_2 = 1$, then weak exogeneity is sufficient for the identification of β when $r_1 + r_2 = r$. It follows from weak exogeneity that identification is a natural consequence of the partition. In more general sub-systems, the type of conditions derived in the previous section are relevant.

It can readily be shown that a similar result to Theorem 2 applies to any subsequent sub-system. Hence, $vec(\beta_{.2})$ is identified when a sub-matrix A_2 of α_2 has full rank. There are now at least two sub-systems which can be separately estimated and identified based on the above conditions. However, the quasi-diagonal form of weak exogeneity implies that while y is dependent in the long-run on z in the first sub-block, then z is also dependent on y in the second block. The latter statement does not appear to be consistent with the idea that in the long-run the notions of exogeneity and causality are coherent.

To address the above concern, attention is focused on cointegrating exogeneity, the restrictions $\beta_{12} = 0$ combined with $\alpha_{21} = 0$ imply that z is not long-run caused by y and as a result $\Pi_{21} = 0$ (Hunter, 1992). Restrictions associated with cointegrating exogeneity direct attention towards the identification of the long-run parameters in a sub-block. However, such restrictions only identify β to the sub-block as ($\beta_{12} = 0$) implies that the same restrictions are applied to all the rows of $\beta_{.2}$. However, the order condition per sub-block is now less onerous ($r_2 - 1$ restrictions). And when $r_2 = 1$, then β_{22} is identified via a normalized coefficient. When compared with the impact of quasi-diagonalising the system, cointegrating exogeneity applies only to the set of identified sub-system relationships. In terms of identifying that sub-block, the following relationship is of interest:

$$\Pi_{22} = \alpha_{22}\beta'_{22}.$$

If $rank(\Pi_{22}) = r_2$, then there is a sub-matrix $\Pi_{2(r_2)}$ of dimension $r_2 \times n_2$, and a matrix of column vectors dimensioned $n_2 \times r_2$, both of rank r_2. Now the order condition for this sub-system is:

$$r_2 n + r_2 n_2 - r_2 \leq r_2 n + r_2 n_2 - r_2^2 \Leftrightarrow r_2^2 - r_2 \leq j_2.$$

Even with all of the zero restrictions in the second block of cointegrating vectors, the number of relevant restrictions in the order condition for the sub-block remains unchanged at the level of the sub-block. Subject to an appropriate number of identifying restrictions, then a sufficient condition for the existence of a solution to the system associated with β_{22} is the existence of A_2, an $r_2 \times r_2$ sub-matrix of α_{22}. By analogy with the result in Theorem 2, the following relationship exists for β_{22}:

$$vec(\beta'_{22}) = (I_{n_2} \otimes A_2)^{-1} vec(\Pi_{1(r_2)}).$$

Further, when z_t is also cointegrating exogenous, then the long-run behaviour of the sub-system for z_t does not depend on the endogenous variables. If z_t is both weakly exogenous for $\beta_{.1}$ and z_t is not long-run caused by y_t, then z_t is termed long-run strongly exogenous for $\beta_{.1}$. Therefore, strong exogeneity combines the restrictions associated with weak exogeneity and the restrictions appropriate for cointegrating exogeneity.

In the next section, identification and identifiability of a model involving weak, cointegrating and strongly exogenous variables is addressed.

4 An empirical example

To motivate the analytic solution and empirical results discussed in this article, the method is applied to the data set analyzed by Johansen and Juselius (1992) and Hunter (1992).[6] The system of equations associated with Theorem 1 is observed to have a number of solutions, which directly relate to the correct degrees of freedom

[6] A VAR(2) was estimated using the same method and period as Johansen and Juselius (1992). The model involved the following variables: (e_{12}) the UK effective exchange rate, (p_o) a real oil price, (p_1) UK prices, (p_2) foreign prices, (i_1) a home interest rate, (i_2) a foreign interest rate (all data series are in logarithms). See Hunter (1992) for more detail.

for the test of over-identifying restrictions. Emphasis is placed on a model, that is identified via restrictions on α discussed in section 3.

For generic identification of a system with $r = 2$ cointegrating vectors $r^2 - r = 2$ restrictions are required with normalisation and r^2 without. The following α imposes the quasi-diagonal structure, discussed above:

$$\alpha' = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{42} & \alpha_{52} & \alpha_{62} \end{bmatrix}. \tag{10}$$

The only restrictions applied to β are those associated with the normalisation ($\beta_{41} = -1$, $\beta_{52} = 1$).

$$\begin{array}{cc} & \begin{array}{cccccc} p_o & p_1 & p_2 & e_{12} & i_1 & i_2 \end{array} \\ \beta' = & \begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} & -1 & \beta_{51} & \beta_{61} \\ \beta_{12} & \beta_{22} & \beta_{32} & \beta_{42} & 1 & \beta_{62} \end{bmatrix}. \end{array}$$

Now consider the orientation of the system or the selection of the appropriate r-dimensioned square matrices A and B. A valid choice for A is based on the 3rd and 6th rows from α. A solution is required for:

$$vec(\beta') = (I_6 \otimes A)^{-1} vec(\Pi_3), \tag{11}$$

where:

$$A = \begin{bmatrix} \alpha_{31} & 0 \\ 0 & \alpha_{62} \end{bmatrix} \text{ and } \Pi_3 = \begin{bmatrix} \pi_{31} & \pi_{32} & \pi_{33} & \pi_{34} & \pi_{35} & \pi_{36} \\ \pi_{61} & \pi_{62} & \pi_{63} & \pi_{64} & \pi_{65} & \pi_{66} \end{bmatrix}.$$

A possible choice of B is based on the fourth and fifth columns of Π, so that:

$$vec(\alpha) = (B \otimes I_6)^{-1} \begin{bmatrix} vec(\pi'_{.4}) \\ vec(\pi'_{.5}) \end{bmatrix} \text{ and } B' = \begin{bmatrix} -1 & \beta_{42} \\ \beta_{51} & 1 \end{bmatrix} \tag{12}$$

where $\pi'_{.j} = [\pi_{1j}\pi_{2j}...\pi_{6j}]$ for $j = 4, 5$. The following solution is derived from (**??**) and (**??**).[7]

$$\begin{aligned} \theta &= [\ \alpha_{11} \quad \alpha_{21} \quad \alpha_{31} \quad \alpha_{42} \quad \alpha_{52} \quad \alpha_{62} \quad \beta_{11} \quad \beta_{21} \quad \beta_{31} \quad \beta_{51} \quad \beta_{61} \quad \beta_{12} \\ &\qquad \beta_{22} \quad \beta_{32} \quad \beta_{42} \quad \beta_{62}\ ] \\ &= [\ \tfrac{1}{\delta}\pi_{14} - \tfrac{\beta_{51}}{\delta}\pi_{15} \quad \tfrac{1}{\delta}\pi_{24} - \tfrac{\beta_{51}}{\delta}\pi_{25} \quad \tfrac{1}{\delta}\pi_{34} - \tfrac{\beta_{51}}{\delta}\pi_{35} \quad -\tfrac{\beta_{51}}{\delta}\pi_{44} - \tfrac{1}{\delta}\pi_{45} \\ &\qquad -\tfrac{\beta_{51}}{\delta}\pi_{54} - \tfrac{1}{\delta}\pi_{55} \quad -\tfrac{\beta_{51}}{\delta}\pi_{64} - \tfrac{1}{\delta}\pi_{65} \quad \alpha_{31}^{-1}\pi_{31} \quad \alpha_{31}^{-1}\pi_{32} \quad \alpha_{31}^{-1}\pi_{33} \\ &\qquad \alpha_{31}^{-1}\pi_{35} \quad \alpha_{31}^{-1}\pi_{36} \quad \alpha_{62}^{-1}\pi_{61} \quad \alpha_{62}^{-1}\pi_{62} \quad \alpha_{62}^{-1}\pi_{63} \quad \alpha_{62}^{-1}\pi_{64} \quad \alpha_{62}^{-1}\pi_{66}\ ] \\ &= g^{-1}(\xi), \text{ where } \delta = -1 - \beta_{42}\beta_{51}. \end{aligned}$$

[7] See the Appendix.

The discovery of 4 valid solutions implies that the model has 4 over-identifying restrictions.

To test the over-identifying restrictions and identifiability, the likelihood ratio test discussed in Johansen and Juselius (1992) and implemented in Doornik and Hendry (1997) are used.

Using the results in the section 3, α and β can be identified via a normalisation and the restrictions associated with quasi-diagonal α. The necessary conditions are met as: i) six restrictions are imposed ($j \geq r^2 - r = 2$) and ii) the test of over-identifying restrictions is accepted at the 5% level (see the p-value in Table 1, I). Furthermore, the restrictions associated with weak exogeneity ($\alpha_{12} = \omega\alpha_{22}$ and $\alpha_{21} = 0$) are also accepted (Table 1, Ia). If $\alpha_{12} = \omega\alpha_{22}$ and $\alpha_{12} = 0$, then the quasi-diagonal system and that satisfying the weak exogeneity conditions are observationaly equivalent. Hence the condition discussed in the previous section ($\omega\alpha_{22} = 0$) is satisfied and in this case quasi-diagonality is sufficient for the interest rates and exchange rate to be weakly exogenous for $\beta_{.1}$.

Theorem 2 implies that a sufficient condition for the existence of a solution to the vector system associated with the first r_1 cointegrating vectors is the existence of a matrix A_1 such that:

$$vec(\beta'_{.1}) = (I_6 \otimes A_1)^{-1} vec(\Pi_{1(r_1)}).$$

From Corollary 3, when $r_1 = 1$, then the existence of a block of weakly exogenous variables is a sufficient condition for identifiability of the cointegrating vectors in the first block. By analogy the second block is also identified, when $r_2 = 1$. The system is sequentially identifiable from the restrictions on α alone and the selection of the normalisation. In this case, the long-run is partitioned into two sub-systems for which $r_i = 1$ and consequently each vector is identified by the normalisation alone.

The sufficient conditions for a solution are accepted when A and B have an inverse. These tests are formulated as non-identifiability tests and they imply non-singularity of A and B. To test the orientation, identifiability is tested prior to any restriction (see Table 1, II) and after the imposition of weak exogeneity (Table 1, IIa). Under the null the determinant of B is set to zero, the test is $\chi^2(1)$ and from the critical value non-identifiability can be rejected at the 5% level, whether or not weak exogeneity is imposed. Hence, no further testing is required to identify α.

Now consider β. As discussed in section 2, identifiability of β depends on the rejection of the condition $|A| = 0$ that occurs when α_{31}=0 or α_{62}=0. Both of these tests are applied under a null of non-identifiability of β (Table 1, IIb and IIc), the tests are $\chi^2(1)$ and the null is rejected at the 5% level.

If $\alpha_{12} = \omega\alpha_{22} = 0$ and cointegrating exogeneity is combined with either weak exogeneity or quasi-diagonality, then the interest rates and exchange rate are strongly exogenous for $\beta_{.1}$ (Table 1, Ib).

5 Conclusion

The procedure outlined can be applied using standard packages and identifiability is a product of the conditions required for generic identification. The procedure requires identification to be checked on an a priori basis. The test of the existence of the sufficient conditions associated with Sargan (1983) stems from the application of restrictions to both α and β, and the whole approach can be made operational with a plethora of non-linear restrictions.

The method was applied to data well known in the cointegration literature.[8] The discovery of a solution to the vector conditions associated with Theorem 1, verifies the restrictions as over-identifying and determines the degree of over-identification. Identifiability of α is accepted on the basis of a test similar to the H_{02} in Boswijk (1996). However, this test confirms that it is appropriate to solve the system using the selected rows and columns of Π. Hence, the orientation of the system and the solution uncovered are empirically identified. Identifiability of β follows from restrictions on α that relate to the exogeneity of the variables selected. The question of which variables are exogenous would appear to be of importance when normalisation is at issue.

Based on the results in section 4, the system was identified by imposing a quasi-diagonality restriction on α and by normalising with respect to r coefficients in β. It is shown that quasi-diagonality, subject to additional covariance restrictions, implies weak exogeneity for a sub-block of β. Finally the joint acceptance of weak exogeneity and cointegrating exogeneity tests for the interest rates implies that they are long-run strongly exogenous for the first cointegrating vector.

Appendix

From (**??**), which can be written as:

$$\begin{bmatrix} vec(\beta_{11}\beta_{12}) \\ vec(\beta_{21}\beta_{22}) \\ vec(\beta_{31}\beta_{32}) \\ vec(\beta_{41}\beta_{42}) \\ vec(\beta_{51}\beta_{52}) \\ vec(\beta_{61}\beta_{62}) \end{bmatrix} = \begin{bmatrix} A^{-1}vec([\pi_{31}\pi_{61}]) \\ A^{-1}vec([\pi_{32}\pi_{62}]) \\ A^{-1}vec([\pi_{33}\pi_{63}]) \\ A^{-1}vec([\pi_{34}\pi_{64}]) \\ A^{-1}vec([\pi_{35}\pi_{65}]) \\ A^{-1}vec([\pi_{36}\pi_{66}]) \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} \frac{1}{\alpha_{31}} & 0 \\ 0 & \frac{1}{\alpha_{62}} \end{bmatrix},$$

and upon using the restrictions embodied in (**??**), we obtain:

$$\begin{aligned} \beta_{i1} &= \alpha_{31}^{-1}\pi_{3i}, \text{ for } i = 1,2,3,5,6, \ \beta_{i2} = \alpha_{62}^{-1}\pi_{6i}, \text{ for } i = 1,2,3,4,6, \\ 1 &= \alpha_{31}^{-1}\pi_{34}, \ 1 = \alpha_{62}^{-1}\pi_{65}. \end{aligned}$$

[8] The original source of the data is the National Institute of Economic Research, that has been kindly passed on to us by Paul Fisher and Ken Wallis.

Similarly for (**??**):

$$\begin{bmatrix} vec(\alpha_{11}\alpha_{12}) \\ vec(\alpha_{21}\alpha_{22}) \\ vec(\alpha_{31}\alpha_{32}) \\ vec(\alpha_{41}\alpha_{42}) \\ vec(\alpha_{51}\alpha_{52}) \\ vec(\alpha_{61}\alpha_{62}) \end{bmatrix} = (B^{-1} \otimes I_6) \begin{bmatrix} vec(\pi_{14}\pi_{24}...\pi_{64}) \\ vec(\pi_{15}\pi_{25}...\pi_{65}) \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} \frac{1}{\delta} & \frac{-\beta_{51}}{\delta} \\ -\frac{\beta_{42}}{\delta} & \frac{-1}{\delta} \end{bmatrix}$$

where $\delta = -1 - \beta_{42}\beta_{51}$. Solving the former equation, subject to the restrictions on α:

$$\begin{aligned}
\alpha_{11} &= \frac{1}{\delta}\pi_{14} - \frac{\beta_{51}}{\delta}\pi_{15}, \alpha_{21} = \frac{1}{\delta}\pi_{24} - \frac{\beta_{51}}{\delta}\pi_{25}, \alpha_{12} = -\frac{\beta_{42}}{\delta}\pi_{14} - \frac{1}{\delta}\pi_{15} = 0, \\
\alpha_{22} &= -\frac{\beta_{42}}{\delta}\pi_{24} - \frac{1}{\delta}\pi_{25} = 0, \alpha_{31} = \frac{1}{\delta}\pi_{34} - \frac{\beta_{51}}{\delta}\pi_{35}, \\
\alpha_{41} &= \frac{1}{\delta}\pi_{44} - \frac{\beta_{42}}{\delta}\pi_{45} = 0, \alpha_{51} = \frac{1}{\delta}\pi_{54} - \frac{\beta_{42}}{\delta}\pi_{55} = 0, \\
\alpha_{61} &= \frac{1}{\delta}\pi_{64} - \frac{\beta_{42}}{\delta}\pi_{65} = 0, \alpha_{32} = -\frac{\beta_{42}}{\delta}\pi_{34} - \frac{1}{\delta}\pi_{35} = 0, \\
\alpha_{42} &= -\frac{\beta_{51}}{\delta}\pi_{44} - \frac{1}{\delta}\pi_{45}, \alpha_{52} = -\frac{\beta_{51}}{\delta}\pi_{54} - \frac{1}{\delta}\pi_{55}, \alpha_{62} = -\frac{\beta_{51}}{\delta}\pi_{64} - \frac{1}{\delta}\pi_{65}.
\end{aligned}$$

As the parameters are over-identified one only needs to consider the following results: $\alpha_{11}, \alpha_{21}, \alpha_{31}, \alpha_{42}, \alpha_{52}$ and α_{62}. This yields the solution given in section 4 that exists if A and B are non-singular.

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Table 1 Tests of Identification and Identifiability

Test	Null	*Statistic* [*p* − *value*]
(I)Quasi-Diagonality$\|r=2$	$\alpha_{i1}=0$ for $i=4,5,6;\beta_{41}=-1$ $\alpha_{i2}=0$ for $i=1,2,3;\beta_{52}=1.$	$\chi^2(4)=3.9595$ [0.4115]
(Ia)Weak Exogeneity$\|r=2$	$\alpha_{i1}=0$ for $i=4,5,6;\beta_{41}=-1$ $\alpha_{i2}=\omega_{i1}\alpha_{42}+\omega_{i2}\alpha_{52}+\omega_{i3}\alpha_{62}$ for $i=1,2,3;\beta_{52}=1.$	$\chi^2(4)=2.5132$ [0.6423]
(Ib)Strong Exogeneity$\|r=2$ (Weak+Cointegrating Exogeneity)	$\alpha_{i1}=0$ for $i=4,5,6$ $\alpha_{i2}=0$ for $i=1,2,3$ $\beta_{i2}=0$ for $i=1,...,4.$	$\chi^2(8)=12.708$ [0.1223]
(II)Non-Identifiability$\|r=2$	$\beta_{41}\beta_{52}-\beta_{42}\beta_{52}=0$	$\chi^2(1)=3.9087$ [0.0481]
(IIa)Non-identifiability$\|(I)$	$\beta_{41}\beta_{52}-\beta_{42}\beta_{52}=0$ $\alpha_{i1}=0$ for $i=4,5,6$ $\alpha_{i2}=0$ for $i=1,2,3$	$\chi^2(5)=12.078$ [0.0337]
(IIb)Non-identifiability$\|(I)$	$\alpha_{31}=0$ $\alpha_{i1}=0$ for $i=4,5,6$ $\alpha_{i2}=0$ for $i=1,2,3$	$\chi^2(5)=24.399$ [0.0002]
(IIc)Non-identifiability$\|(I)$	$\alpha_{62}=0,$ $\alpha_{i1}=0$ for $i=4,5,6$ $\alpha_{i2}=0$ for $i=1,2,3$	$\chi^2(5)=13.262$ [0.0210]