

Some inequalities for the Tutte polynomial

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Abstract

We prove that the Tutte polynomial of a coloopless paving matroid is convex along the portions of the line segments $x + y = p$ lying in the positive quadrant. Every coloopless paving matroid is in the class of matroids which contain two disjoint bases or whose ground set is the union of two bases. For this latter class we give a proof that $T_M(a, a) \leq \max\{T_M(2a, 0), T_M(0, 2a)\}$ for $a \geq 2$. We conjecture that $T_M(1, 1) \leq \max\{T_M(2, 0), T_M(0, 2)\}$ for the same class of matroids. We also prove this conjecture for some families of graphs and matroids.

1 Introduction

The Tutte polynomial is a two variable polynomial which can be defined for a graph G or more generally a matroid M . The Tutte polynomial has many interesting combinatorial interpretations when evaluated at different points (x, y) and along several algebraic curves. For example, for a graph G , the Tutte polynomial along the line $y = 0$ is the chromatic polynomial, after a suitable change of variable and multiplication by an easy term. In similar ways, we can get the flow polynomial of a graph, the all terminal reliability of a network and the partition function of the Q -state Potts model. When considering a $\text{GF}(q)$ -representable matroid, the Tutte polynomial gives us the weight enumerator of linear codes over $\text{GF}(q)$ associated to M . All the necessary background on the Tutte polynomial is contained in Section 2.

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For a convex set S , a function $f : S \rightarrow \mathbb{R}$ is a *convex function* if for all $x_1, x_2 \in S$ and $t \in (0, 1)$, $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$. In this work we mainly concentrate on proving the convexity of one-variable polynomials but in the conclusion we also consider the convexity of two-variable polynomials over the convex set positive quadrant.

It is well-known [3] that the Tutte polynomial of a matroid M has an expansion

$$T_M(x, y) = \sum_{i,j} t_{ij} x^i y^j,$$

in which each coefficient t_{ij} is non-negative. Consequently, for $m \geq 0$ and for any b , $T_M(x, y)$ increases along the portion of the line $y = mx + b$ lying in the positive quadrant, as x increases. The simplicity of the behaviour of T along lines with positive gradient suggests the study of the behaviour of T_M along lines with negative gradient in the positive quadrant. Merino and Welsh [17] were the first to consider this and were particularly interested in resolving the question of whether the Tutte polynomial is convex along the portion of the line $x + y = 2$ lying in the positive quadrant. They made the following intriguing conjecture.

Conjecture 1.1. *Let G be a 2-connected graph with no loops. Then*

$$\max\{T_{M(G)}(2, 0), T_{M(G)}(0, 2)\} \geq T_{M(G)}(1, 1). \quad (1)$$

Notice that this is a necessary condition for T to be a convex function along the portion of the line mentioned above. Any graph with at least one loop and at least one coloop fails to satisfy (1), so (1) cannot hold for all graphs. The main reason for the particular interest in the points $(2, 0)$, $(0, 2)$ and $(1, 1)$ is that in a connected graph G , $T_{M(G)}(2, 0)$, $T_{M(G)}(0, 2)$ and $T_{M(G)}(1, 1)$ give the number of acyclic orientations, totally cyclic orientations and spanning trees in G , respectively. Definitions of acyclic and totally cyclic orientations are contained in Section 2.

A related question is to determine whether any loopless and bridgeless graph G satisfies the apparently stronger requirement

$$T_{M(G)}(2, 0)T_{M(G)}(0, 2) \geq (T_{M(G)}(1, 1))^2.$$

Relatively little progress has been made to resolve these questions. However, Jackson in [14] has shown, with a clever argument, that for any loopless and bridgeless graph G , and a and b real positive numbers with $b \geq a(a + 2)$,

$$T_{M(G)}(b, 0)T_{M(G)}(0, b) \geq T_{M(G)}(a, a).$$

In this paper we make three contributions. First, in Section 4, we show in Theorem 4.9 that the Tutte polynomial T of a coloopless paving matroid satisfies the inequality

$$tT(x_1, y_1) + (1-t)T(x_2, y_2) \geq T(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2), \quad (2)$$

where $0 \leq t \leq 1$ and x_1, x_2, y_1, y_2 are non-negative and satisfy $x_1 + y_1 = x_2 + y_2$. That is, T is convex along the portions of the line segments $x + y = p$ lying in the positive quadrant. A paving matroid is one in which all circuits have size at least $r(M)$. Interest in them stems from a conjecture in [15] which says that asymptotically every matroid is paving. The special case of (2), obtained by setting $x_1 = y_2 = 2$, $x_2 = y_1 = 0$ and $t = 1/2$, establishes (1) for the class of paving matroids. Therefore if the above conjecture is true then we have established (1) for, asymptotically all coloopless matroids.

Second, in Section 5, we prove that (1) holds for some smaller classes of matroids and graphs that are not paving matroids. Finally, in Section 3, we prove that if the ground set of M contains two disjoint bases then $T_M(0, 2a) \geq T_M(a, a)$, whenever $a \geq 2$, and dually if the ground set of M is the union of two bases then $T_M(2a, 0) \geq T_M(a, a)$. These results cannot be obtained with the methods used by Jackson in [14].

We conclude with a brief discussion of the natural question of for which matroids is T_M a convex function in the positive quadrant?

2 Preliminaries

We assume that the reader has some familiarity with matroid and graph theory. For matroid theory we follow Oxley's book [21] and for graph theory we follow Diestel's book [9].

The Tutte polynomial is a matroid invariant over the ring $\mathbb{Z}[x, y]$. Further details of many of the concepts treated here can be found in Welsh [26] and Oxley and Brylawski [6].

Some of the richness of the Tutte polynomial is due to its numerous equivalent definitions. One of the simplest definitions, which is often the easiest way to prove properties of the Tutte polynomial, uses the notion of rank.

If $M = (E, r)$ is a matroid, where r is the rank-function of M , and $A \subseteq E$, we denote $r(E) - r(A)$ by $z(A)$ and $|A| - r(A)$ by $n(A)$.

Definition 2.1. *The Tutte polynomial of M , $T_M(x, y)$, is defined as follows:*

$$T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{z(A)} (y - 1)^{n(A)}. \quad (3)$$

Almost immediately we see that $T_M(1, 1)$ equals the number of bases of M and $T_M(2, 2)$ equals $2^{|E|}$. Recall that if $M = (E, r)$ is a matroid, then $M^* = (E, r^*)$ is its dual matroid, where $r^*(A) = |A| - r(E) + r(E \setminus A)$. Because $z_{M^*}(A) = n_M(E \setminus A)$ and $n_{M^*}(A) = z_M(E \setminus A)$ it follows that $T_M(x, y) = T_{M^*}(y, x)$.

For a graphic matroid $M(G)$, the evaluations of the Tutte polynomial at $(2, 0)$ and $(0, 2)$ equal the number of acyclic orientations and the number of totally cyclic orientations of G , respectively. An acyclic orientation of a graph G is an orientation where there are no directed cycles. A totally cyclic orientation is an orientation where every edge is in a directed cycle. See [6] for a proof of this

result. We let $\alpha(G)$ and $\alpha^*(G)$ denote $T_{M(G)}(2, 0)$ and $T_{M(G)}(0, 2)$ respectively. If G is connected, the number of spanning trees of G is the evaluation of the Tutte polynomial at $(1, 1)$ and this quantity is denoted by $\tau(G)$.

The Tutte polynomial may be also defined by a linear recursion relation given by deleting and contracting elements that are neither loops nor coloops.

Definition 2.2. *If M is a matroid, and e is an element that is neither a coloop nor a loop, then*

$$T_M(x, y) = T_{M \setminus e}(x, y) + T_{M/e}(x, y). \quad (4)$$

If there is no such element e , then $T_M(x, y) = x^i y^j$ where i and j are the number of coloops and loops of M respectively.

The proof that Definition 2.1 and 2.2 are equivalent can be found in [6]. We still require another (equivalent) definition of the Tutte polynomial but first we introduce the relevant notions.

Let us fix an ordering \prec on the elements of M , say $E = \{e_1, \dots, e_m\}$, where $e_i \prec e_j$ if $i < j$. Given a fixed basis S , an element e is called *internally active* if $e \in S$ and it is the smallest edge with respect to \prec in the only cocircuit disjoint from $S \setminus \{e\}$. Dually, an element f is *externally active* if $f \notin S$ and it is the smallest element in the only circuit contained in $S \cup \{f\}$. We define t_{ij} to be the number of bases with i internally active elements and j externally active elements. In [24] Tutte defined T_M using these concepts. A proof of the equivalence with Definition 2.1 can be found in [3].

Definition 2.3. *If $M = (E, r)$ is a matroid with a total order on its ground set, then*

$$T_M(x, y) = \sum_{i,j} t_{ij} x^i y^j. \quad (5)$$

In particular, the coefficients t_{ij} are independent of the total order used on the ground set.

By an inductive argument using Equation (4), it can be proved that $t_{10} = t_{01}$ when $E(M) \geq 2$. This is one of a number of identities known to hold for the coefficients t_{ij} . For a complete characterization of all the affine linear relations that hold among the coefficients t_{ij} see Theorem 6.2.13 in [6]. From there we extract the relations that we need.

Theorem 2.4. *If a rank- r matroid M with m elements has neither loops nor coloops, then*

- (a) $t_{ij} = 0$, whenever $i > r$ or $j > m - r$;
- (b) $t_{r0} = 1$ and $t_{0, m-r} = 1$;
- (c) $t_{rj} = 0$ for all $j > 0$ and $t_{i, m-r} = 0$ for all $i > 0$.

The previous result follows easily from Definition 2.3. In [6] the statement is for simple matroids (geometries) but it is easy to extend it to matroids with parallel elements.

3 Some inequalities for the Tutte polynomial

From the results in the previous section it is easy to prove the following result stated in [16].

Theorem 3.1. *If a matroid M has neither loops nor coloops, then*

$$\max\{T_M(4, 0), T_M(0, 4)\} \geq T_M(2, 2).$$

Proof. Let r be the rank and m the number of elements of M .

$$\begin{aligned} \max\{T_M(4, 0), T_M(0, 4)\} &\geq \max\{4^r, 4^{m-r}\} \\ &= \max\{2^{2r}, 2^{2(m-r)}\} \\ &\geq 2^m = T_M(2, 2), \end{aligned}$$

where the first inequality follows from Equation (5) combined with Theorem 2.4 (b). \square

Note that, for a matroid $M = (E, r)$ with dual $M^* = (E, r^*)$, the following inequalities are equivalent for any $A \subseteq E$.

$$|A| \leq |E| - 2(r(E) - r(A)), \quad (6)$$

$$|E \setminus A| \leq 2r^*(E \setminus A) \quad (7)$$

and

$$z(A) + n(A) \leq |E| - r. \quad (8)$$

We now restrict attention to matroids M in which all subsets A of the ground set E satisfy the (equivalent) inequalities above. By a classical result of J. Edmonds [10], these are the matroids that contain two disjoint bases; by duality, these are the matroids M whose ground set is the union of two bases of M^* .

The monomial of maximum degree of any term $(x-1)^{z(A)}(y-1)^{n(A)}$ in T_M is $x^{z(A)}y^{n(A)}$. Hence, the following theorem follows directly from the set of inequalities above.

Theorem 3.2. *If a matroid M contains two disjoint bases, then $t_{ij} = 0$, for all i and j such that $i + j > m - r$. Dually, if its ground set is the union of two bases, then $t_{ij} = 0$, for all i and j such that $i + j > r$.*

Now, it is easy to prove an infinite set of inequalities for the Tutte polynomial of a matroid that contains two disjoint bases or whose ground set is the union of two bases. This theorem was stated in [16].

Theorem 3.3. *If a matroid M contains two disjoint bases, then*

$$T_M(0, 2a) \geq T_M(a, a), \quad (9)$$

for all $a \geq 2$. Dually, if its ground set is the union of two bases, then

$$T_M(2a, 0) \geq T_M(a, a), \quad (10)$$

for all $a \geq 2$.

Proof. Let us consider just the case when M has two disjoint bases: the other case follows from duality. In this situation $m - r \geq r$. From the proof of Theorem 3.1 and Equation (5) we have $4^{m-r} \geq T_M(2, 2) = \sum_{i,j} t_{ij} 2^{i+j}$. Multiplying this inequality by $(a/2)^{m-r}$ we get

$$(2a)^{m-r} \geq \sum_{i,j} t_{ij} \left(\frac{a}{2}\right)^{m-r} 2^{i+j} \geq \sum_{i,j} t_{ij} \left(\frac{a}{2}\right)^{i+j} 2^{i+j} = \sum_{i,j} t_{ij} a^{i+j}.$$

The second inequality follows from Theorem 3.2. Thus

$$T_M(0, 2a) \geq (2a)^{m-r} \geq \sum_{i,j} t_{ij} a^{i+j} = T_M(a, a).$$

□

We can sum up the previous result by saying that if M contains two disjoint bases or its ground set is the union of two bases then

$$\max\{T_M(2a, 0), T_M(0, 2a)\} \geq T_M(a, a), \quad (11)$$

for $a \geq 2$. That is, along the portion of the line $x + y = 2a$ lying in the positive quadrant, the value of T at one of the endpoints of the line segment is greater than the value of T at its midpoint. This is a necessary condition for T to be a convex function along these line segments. Some classes of matroids which contain two disjoint bases or whose ground set is the union of two bases are mentioned in the following

Corollary 3.4. *For a matroid M , T_M satisfies (11), for all $a \geq 2$ whenever M is one of the following:*

- an identically self-dual matroid M ,
- a rank- r projective geometry over $GF(q)$ or its dual, for $r \geq 2$.

Proof. A matroid $M = (E, r)$ is identically self-dual if $M = M^*$, so, B is a basis of M if and only if $E - B$ is a basis of M .

For $r \geq 3$, the graphic matroid W_{r+1} , the $r + 1$ -wheel (with $r + 2$ vertices), is a submatroid of the matroid $PG(r, q)$, see [21] and contains two disjoint bases. Thus, $PG(r, q)$ contains two disjoint bases. The matroid $U_{2,4} \oplus_2 U_{2,4}$ is a submatroid of a projective plane of order $m \geq 4$ and again contains two disjoint bases. Thus, such a projective plane contains two disjoint bases. The only projective plane of order 3 is the Fano matroid which clearly contains two disjoint bases. □

There are more classes of matroids that can be added to the previous list, for instance, coloopless paving matroids. However, in the next section we will prove a much stronger result for them. The graphic matroids corresponding to the families of graphs in our next result may also be added to the list.

Corollary 3.5. *For a simple graph G , $T_{M(G)}$ satisfies (11), for all $a \geq 2$ whenever G is one of the following:*

- a 4-edge-connected graph,
- a 2-connected threshold graph,
- a complete bipartite graph,
- a series-parallel graph,
- a 3-regular graph,
- a bipartite planar graph,
- a Laman graph,
- a triangulation,
- the wheel graph W_n , for $n \geq 2$,
- the square lattice L_n , for $n \geq 2$,
- the n -cycle $n \geq 2$,
- a tree with n edges, for $n \geq 1$.

Proof. By the classical result in [25] every 4-edge-connected graph has two edge-disjoint spanning trees. It is easy to see that 2-connected threshold and wheel graphs have two edge-disjoint spanning trees. Using the expression for computing the arboricity of a graph given in [20] we see that simple series-parallel, 3-regular, bipartite planar, and Laman graphs all have arboricity two, which is equivalent to having two spanning trees that cover all the edges of the graph. Triangulations are geometric duals of 3-regular planar graphs, so they have two edge-disjoint spanning trees.

It is easy to see that each of $K_{2,m}$ for $m \geq 2$, $K_{3,3}$, the square lattice L_n for $n \geq 2$, the n -cycle for $n \geq 2$, and a tree have two spanning trees which cover all the edges in the graph. With the exception of the case $n = m = 3$, if both n and m are at least 3, then $K_{n,m}$ always has two edge-disjoint spanning trees. \square

4 Paving matroids

A paving matroid $M = (E, r)$ is a matroid whose circuits all have size at least r . Paving matroids are closed under minors and the set of excluded minors for the class consists of the matroid $U_{2,2} \oplus U_{0,1}$, see for example [21] (page 132, exercise 8). The interest in paving matroids goes back to 1976 when Dominic Welsh asked if most matroids are paving, see [21]. More recently, the authors in [15] pose as a conjecture that asymptotically every matroid is paving.

First, we prove that most paving matroids either contain two disjoint bases or their ground set is the union of two bases. Consequently coloopless paving matroids fall within the class of matroids considered in the previous section.

Theorem 4.1. *Let $M = (E, r)$ be a rank- r paving matroid with n elements,*

- *if $2r > n$, then E is the union of two bases,*
- *if $2r \leq n$ and M is coloopless, then M contains two disjoint bases.*

Proof. In the first case, take B_1 to be a basis of M , then $I_2 = E \setminus B_1$ has size $n - r < r$, so it is independent and we can extend it to a basis B_2 . Thus $E = B_1 \cup B_2$.

In the second case, if M has a circuit C of size $r + 1$, then $C' = E \setminus C$ has size $n - r - 1 \geq r - 1$. Let I be a set of size $r - 1$ contained in C' . As I is independent and C is spanning, there exists $a \in C \setminus I$ such that $I \cup \{a\}$ is a basis. But $C \setminus \{a\}$ is also a basis. Thus, we have two disjoint bases.

Let M be a coloopless paving matroid with no circuits of size $r + 1$ and suppose that $2r \leq n$. Let B be a basis of M . Then either $E \setminus B$ contains a basis, in which case we have finished the proof, or $r(E \setminus B) = r - 1$. In the latter case, let H be the hyperplane defined as the closure of $E \setminus B$, and $I = E \setminus H \subseteq B$. The set I has size $p + 1$ with $p \geq 1$ as M is coloopless.

We show that in this case M also has two disjoint bases. Let $I' = I \setminus \{a\}$, for some $a \in I$. Then, I' is a non-empty independent set of size p with the property that for any circuit C of size r contained in H , $I' \cup C$ contains a basis of M . Thus, there is a basis B_1 of M of the form $I' \cup A_1$ for some subset A_1 of H of size $r - p$. Now, let $B_2 = \{a\} \cup A_2$ for some $A_2 \subseteq H \setminus A_1$ of size $r - 1$. This is possible as $|H \setminus A_1| = (n - p - 1) - (r - p) = (n - r) - 1 \geq r - 1$. Thus, B_1 and B_2 are disjoint bases of M . \square

The main goal of this section is to prove that for any coloopless paving matroid

$$tT(x_1, y_1) + (1 - t)T(x_2, y_2) \geq T(tx_1 + (1 - t)x_2, ty_1 + (1 - t)y_2), \quad (12)$$

whenever $0 \leq t \leq 1$ and x_1, x_2, y_1, y_2 are non-negative and satisfy $x_1 + y_1 = x_2 + y_2$. Notice that this inequality is a much stronger statement than (10), as it says that T is a convex function along the portions of the line $x + y = p$ lying in the positive quadrant, rather than merely saying that the value of T at one of the endpoints of the line segment is greater than the value of T at its midpoint.

Our main tools for establishing the convexity of T are the following results.

Lemma 4.2. *Let M be a matroid. Either, both $T_M(x, y)$ and $T_{M^*}(x, y)$ are convex along the portion of the line $x + y = p$ lying in the positive quadrant or neither is.*

Proof. This follows directly from the equality $T_M(x, y) = T_{M^*}(y, x)$. \square

Lemma 4.3. *Let M be a matroid and e in M be neither a loop nor a coloop. If $T_{M \setminus e}$ and $T_{M/e}$ are both convex along the portion of the line $x + y = p$ lying in the positive quadrant, then T_M is also convex on the same domain.*

Proof. This follows directly from the deletion-contraction formula (4) and the fact that the sum of convex functions is also a convex function. \square

The following three results deal with the convexity of T for some coloopless paving matroids. We use these cases as bases for an inductive argument later on.

Lemma 4.4. *If M is isomorphic to the paving matroid $U_{1,k+1} \oplus U_{0,l}$, where $l \geq 0$ and $k \geq 1$, then T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant.*

Proof. We have

$$T_M(x, y) = y^l(y^k + \cdots + y + x) = py^l + \sum_{m=l+2}^{l+k} y^m.$$

Since y^m is convex for all $m \geq 0$ in the given region and the sum of convex functions is convex, the result follows. \square

Lemma 4.5. *The Tutte polynomial T_M is a convex function in the positive quadrant when M is a uniform matroid. In particular, T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant.*

Proof. The Tutte polynomial of a uniform matroid can be computed easily using (3).

$$T_{U_{r,n}}(x, y) = \sum_{i=0}^{r-1} \binom{n}{i} (x-1)^{r-i} + \binom{n}{r} + \sum_{i=r+1}^n \binom{n}{i} (y-1)^{i-r}.$$

This can be expanded into the following expression, which may also be established directly using (5).

$$T_{U_{r,n}}(x, y) = \sum_{j=1}^{n-r} \binom{n-j-1}{r-1} y^j + \sum_{i=1}^r \binom{n-i-1}{n-r-1} x^i,$$

when $0 < r < n$, while $T_{U_{n,n}}(x, y) = x^n$ and $T_{U_{0,n}}(x, y) = y^n$.

As each term is a convex function, the result follows. \square

Theorem 4.6. *If M is a rank-2 loopless and coloopless matroid, then T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant.*

Proof. If M is isomorphic to the uniform matroid $U_{2,n}$, the result follows from applying the previous lemma. Otherwise, M is isomorphic to a matroid with parallel elements whose simplification is isomorphic to $U_{2,n}$.

If $n \geq 3$ or there is a parallel class of size at least 3, we can choose an element e in a non-trivial parallel class of M such that $M \setminus e$ does not have a coloop. In this case M/e is isomorphic to $U_{1,k+1} \oplus U_{0,l}$, where $l \geq 1$ and $k \geq 1$ and $M \setminus e$ is a rank-2 loopless and coloopless matroid. The result follows from Lemma 4.4, induction and Lemma 4.3.

In the last case, the simplification of M is isomorphic to $U_{2,2}$ and every element is in a parallel class of size 2. Then M is isomorphic to $U_{1,2} \oplus U_{1,2}$. Then, $T_M = (x + y)^2$ which is convex (in fact is constant) along $x + y = p$ for $p > 0$ and $0 \leq y \leq p$. \square

In order to establish our main result, we need the following structural result about coloopless paving matroids. The 2-thickening of a matroid M is obtained from M by replacing each non-loop element by 2 parallel elements and replacing each loop by 2 loops. The 2-stretching of a matroid M is the dual matroid of the 2-thickening of M^* , that is, performing a 2-stretch on M amounts to replacing each of its elements by 2 elements in series.

Lemma 4.7. *Let M be a rank- r coloopless paving matroid. If for every element e of M , $M \setminus e$ has a coloop, then one of the following three cases happens.*

- (a) M is isomorphic to $U_{r,r+1}$, $r \geq 1$.
- (b) M is the 2-stretching of a uniform matroid $U_{s,s+2}$, for some $s \geq 1$.
- (c) M is isomorphic to $U_{1,2} \oplus U_{1,2}$.

Proof. If e is such that $M \setminus e$ has a coloop f , then either $\{e, f\}$ are in series or form a parallel class. If there is a parallel class in a paving matroid, its rank is either 1 or 2. Thus, if $\{e, f\}$ are in a parallel class, M is isomorphic to $U_{1,2} \oplus U_{1,2}$ or $U_{1,2}$.

Therefore, we can assume that M contains no non-trivial parallel classes. Hence every element belongs to a series class of size at least two. Suppose that there is a series class containing at least three elements e, f, g . In this case, $M \setminus e$ will have at least 2 coloops. But as M is paving all its minors are also paving. Thus, $M \setminus e$, being a paving matroid with at least 2 coloops, cannot have circuits and $M \setminus e$ is isomorphic to $U_{r,r}$. In this case, since M is coloopless, we conclude that M is isomorphic to $U_{r,r+1}$.

To finish, we suppose that every element in M is in a series class of size 2. In this case, M is the 2-stretching of a rank- s matroid N with m elements and $s \geq 1$. N is paving because it is a minor of M and it must have circuits as M is coloopless.

If the minimum size of a circuit in N is s , then M has a circuit of size $2s$. But the rank of M is $s + m$ as it is the 2-stretching of N . Thus $2s \geq s + m$

and so $s = m$. In this case, N would be isomorphic to $U_{s,s}$ and we arrive at a contradiction. Thus, N does not have circuits of size s .

Hence all the circuits of N have size $s + 1$ and N is uniform. Then, there is a circuit in M of size $2s + 2 \geq s + m$, and $s + 2 \geq m \geq s + 1$. Thus, N is isomorphic to $U_{s,s+1}$ or $U_{s,s+2}$. But when M is the 2-stretching of $U_{s,s+1}$, M is isomorphic to $U_{2s+1,2s+2}$ and is covered in case (a). \square

Lemma 4.8. *Let M be a rank- r coloopless paving matroid. If for every element e of M , $M \setminus e$ has a coloop, then T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant.*

Proof. We analyse the cases for M given in the previous lemma. If M is isomorphic to $U_{r,r+1}$, the result follows from Lemma 4.5. If M is isomorphic to $U_{1,2} \oplus U_{1,2}$ or $U_{1,2}$, the corresponding Tutte polynomials are $(x + y)^2$ and $x + y$, which are both convex.

If M is the 2-stretching of $U_{s,s+2}$, then M^* is the 2-thickening of $U_{2,n}$ which is a rank-2 matroid and the result follows from Theorem 4.6 and Lemma 4.2. \square

Finally, we arrive at the main result of this section.

Theorem 4.9. *If M is a coloopless paving matroid, then T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant.*

Proof. If M has a loop, then M has rank 1, it is isomorphic to $U_{1,k+1} \oplus U_{0,l}$ with $l, k \geq 1$ and the result follows from Lemma 4.4.

Otherwise, every element of M is neither a loop nor a coloop. If there is an element e such that $M \setminus e$ has no coloop, then both M/e and $M \setminus e$ are coloopless paving matroids and the result follows from Lemma 4.3.

So, we can assume that for all e , $M \setminus e$ has a coloop. Then the result follows from Lemma 4.8. \square

Paving matroids are not closed under duality but using Lemma 4.2 we obtain the convexity of the Tutte polynomial for a bigger class of matroids.

Corollary 4.10. *If M or M^* is a coloopless paving matroid, then T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant.*

By Theorem 4.1, the class of matroids M such that either M or M^* is a coloopless paving matroid is contained in the class of matroids that contains two disjoint bases or whose ground set is the union of two bases. Thus, we have a strengthening of Theorem 3.3.

Corollary 4.11. *If M or M^* is a coloopless paving matroid, then T_M satisfies inequality (11) for all $a \geq 0$.*

5 The Merino-Welsh conjecture

In this section we return to the original Merino–Welsh conjecture (Conjecture 1.1) and establish that inequality (1) of the Merino-Welsh conjecture holds for some fairly specific classes of graphs and matroids. Recall that the conclusion of the conjecture is certainly not true for all graphs. Taking any graph and adding a loop and a bridge results in a graph that does not satisfy (1). However, the condition on the connectivity may not be the most natural because if G consists of 2 cycles of length 2 sharing a common vertex, then the graphic matroid $M(G)$ satisfies (9) for all $a \geq 0$. So (1) is satisfied by some graphs that are not 2-connected.

5.1 Wheels and whirls

In this subsection we consider wheels, a well-known class of self-dual planar graphs, and whirls, a related class of matroids that are also self-dual. The wheel graph W_n has $n + 1$ vertices and $2n$ edges. The whirl W^n is the matroid with ground set $E(W^n) = E(W_n)$, while the set of bases of W^n consists of the edge set in the n -cycle of W_n together with all edge sets of spanning trees of W_n , see [21].

It is well-known that $\tau(W_n) = L_{2n} - 2$, for $n \geq 1$, where L_k is the k th-Lucas number which is defined recursively by $L_1 = 1$, $L_2 = 3$ and $L_k = L_{k-1} + L_{k-2}$ for $k \geq 3$. This result was proved by Sedláček [22] and also by Myers [19]. Using the analogy of Binet’s Fibonacci formula for Lucas numbers we get

$$\tau(W_n) = \left(\frac{3 + \sqrt{5}}{2}\right)^n + \left(\frac{3 - \sqrt{5}}{2}\right)^n - 2.$$

The same formula can be obtained directly by using Equation (4) for $T_{W_n}(1, 1)$ and then solving the corresponding recurrence relation.

The chromatic polynomial of W_n is known, see [2], and is equal to $\chi_{W_n}(x) = x(x - 2)^n + (-1)^n x(x - 2)$. Now, applying the famous result of R. Stanley [23] that relates the number of acyclic orientations and the chromatic polynomial, namely $\alpha(G) = |\chi_G(-1)|$, we get $\alpha(W_n) = 3^n - 3$. These results together yield the following.

Theorem 5.1. *For all $n \geq 2$, $\alpha(W_n) \geq \tau(W_n)$ and $M(W_n)$ satisfies Conjecture 1.1.*

The Tutte polynomials of W^n and $M(W_n)$ are related by the equality, $T_{W^n}(x, y) = T_{W_n}(x, y) - xy + x + y$. Thus, we obtain the following result.

Theorem 5.2. *For all $n \geq 2$, $T_{W^n}(2, 0) \geq T_{W^n}(1, 1)$ and W^n satisfies Conjecture 1.1.*

5.2 3-regular graphs with girth at least 5

The following lower bound on the number of acyclic orientations of 3-regular graphs with girth at least 5 comes from [13].

$$\alpha(G) \geq (2^{3/8} 3^{3/8} 4^{1/8})^n,$$

where n is the number of vertices of G . On the other hand, the following upper bound for the number of spanning trees in a 3-regular graph G is given in [7].

$$\tau(G) \leq \frac{2\beta}{3n} e^{\frac{12}{\sqrt{\pi}} \left(\frac{1}{\beta}\right)^{\frac{5}{2}}} \left(\frac{4}{\sqrt{3}}\right)^n,$$

where $\beta = \lceil \ln(n)/\ln(9/8) \rceil$. From the formulae we obtain the following.

Theorem 5.3. *If G is a 3-regular graph of girth at least 5, we have $\tau(G) < \alpha(G)$ and $M(G)$ satisfies Conjecture 1.1.*

5.3 Complete graphs

It is natural to check if Conjecture 1.1 is true for complete graphs and complete bipartite graphs.

A classical result of Cayley [2] states that $\tau(K_n) = n^{n-2}$. For K_3 we have $\alpha(K_3) = 6 > 3 = \tau(K_3)$, thus K_3 satisfies Conjecture 1.1.

We use the following lemma which has an easy proof, see [8].

Lemma 5.4. *If G is a 2-connected graph with a vertex v of degree d , then $(2^d - 2)\alpha^*(G - v) \leq \alpha^*(G)$.*

We will prove that $\alpha^*(K_n) \geq n^{n-2}$, for $n \geq 4$. When $n = 4$, we have $\alpha^*(K_4) = 24 > 16 = \tau(K_4)$. We proceed by induction on n .

$$\begin{aligned} \tau(K_{n+1}) &= (n+1)^{n-1} = \left(\frac{n+1}{n}\right)^n \left(\frac{n}{n+1}\right)^2 (n+1)\tau(K_n) \\ &\leq e(n+1)\tau(K_n) \leq (2^n - 2)\tau(K_n) \\ &\leq (2^n - 2)\alpha^*(K_n). \end{aligned}$$

The last quantity is less than or equal $\alpha^*(K_{n+1})$ by the previous lemma.

Theorem 5.5. *For all $n \geq 3$, $M(K_n)$ satisfies Conjecture 1.1.*

The technique used for complete graphs can be used to prove Conjecture 1.1 in the case of threshold graphs, a type of chordal graphs, see [8]. Also in [8] complete bipartite graphs are considered and the authors prove the following.

Theorem 5.6. *For all $m \geq n \geq 2$, $M(K_{n,m})$ satisfies Conjecture 1.1.*

5.4 Catalan matroids

A *Dyck path* of length $2n$ is a path in the plane from $(0,0)$ to $(2n,0)$, with steps $(1,1)$, called *up-steps*, and $(1,-1)$, called *down-steps*. It is well-known that the number of Dyck paths of length $2n$ is the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. Each Dyck path P defines an *up-step set*, consisting of the integers i , $1 \leq i \leq 2n$, for which the i th-step of P is an up-step. The collection of up-step sets of all Dyck paths of length $2n$ forms the bases of a matroid M_n over $\{1, 2, \dots, 2n\}$. These matroids are called Catalan matroids and have recently been studied extensively, see [4] or [1].

We consider the matroids N_n , $n \geq 2$, obtained from M_n by deleting the elements 1 and $2n$. This corresponds to deleting the loop and coloop of M_n . From the results in [4] it follows that the matroid N_n is self-dual, but not identically self-dual. An expression for the Tutte polynomial of N_n follows from Corollary 5.8 of [4].

$$T_{N_n}(x, y) = \sum_{i, j > 0} \frac{i + j - 2}{n - 1} \binom{2n - i - j - 1}{n - i - j + 1} x^{i-1} y^{j-1}.$$

After some algebraic manipulations we get a formula for the evaluations at $(2,0)$ and $(0,2)$.

$$T_{N_n}(2, 0) = T_{N_n}(0, 2) = \sum_{k=0}^m \frac{k}{m} \binom{2m - k - 1}{m - k} 2^k,$$

where $m = n - 1$. This quantity equals $\binom{2m}{m}$ as follows.

$$\begin{aligned} \sum_{k=0}^m \frac{k}{m} \binom{2m - k - 1}{m - k} 2^k &= \sum_{k=0}^m \left(\binom{2m - k - 1}{m - 1} - \binom{2m - k - 1}{m} \right) 2^k \\ &= \sum_{k=0}^m \sum_{j=0}^k \left(\binom{2m - k - 1}{m - 1} - \binom{2m - k - 1}{m} \right) \binom{k}{j} \\ &= \sum_{j=0}^m \sum_{k=j}^m \left(\binom{2m - k - 1}{m - 1} - \binom{2m - k - 1}{m} \right) \binom{k}{j} \\ &= \sum_{j=0}^m \left(\binom{2m}{m + j} - \binom{2m}{m + j + 1} \right) \\ &= \binom{2m}{m}. \end{aligned}$$

The key step in the middle uses the convolution identity $\sum_{k=0}^{2m-1} \binom{2m-k-1}{q} \binom{k}{j} = \binom{2m}{q+j+1}$ which is the basic identity (5.6) in [11]. The value of $T_{N_n}(1, 1)$ is clearly $C_n = \frac{1}{n+1} \binom{2n}{n}$.

Theorem 5.7. *For all $n \geq 2$, N_n satisfies Conjecture 1.1.*

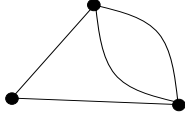


Figure 1:

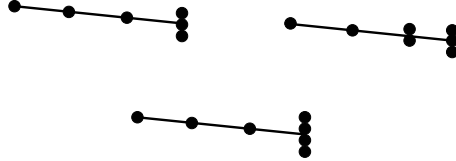


Figure 2:

Notice that in all of the classes that we have considered, either the ground set contains two disjoint bases or is the union of two bases. We therefore propose the following conjecture which is a weaker form of Conjecture 1.1 and may turn out to be more tractable.

Conjecture 5.8. *If M contains two disjoint bases or its ground set is the union of two bases then $\max\{T_M(2, 0), T_M(0, 2)\} \geq T_M(1, 1)$.*

6 Conclusion and Discussion

We have proved that T_M is convex along the portion of the line $x + y = p$ lying in the positive quadrant, whenever M is a coloopless paving matroid. By Definition 2.3, T_M is convex along the semilines $y = mx + b$ for $m \geq 0$ and $b \in \mathbb{R}$ in the positive quadrant. It is natural to ask for which matroids is T_M convex in the positive quadrant?

There is no clear link between convexity of the Tutte polynomial in the positive quadrant and the classes of matroids that we have considered. Coloopless paving matroids may or may not have Tutte polynomials that are convex in the positive quadrant. For example, the Tutte polynomials of uniform matroids and the graphic matroid $M(K_4)$ are convex in the positive quadrant; on the other hand the Tutte polynomial $y^l(y^k + \dots + y + x)$ of the paving matroid $U_{1,k+1} \oplus U_{0,l}$, where $l \geq 1$ and $k \geq 1$ is not a convex or concave function. There are also non-paving matroids whose Tutte polynomial is convex, for example $U_{n,n}^2$, for $n \geq 3$, the 2-thickening of $U_{n,n}$. The Tutte polynomial of this matroid is $(x + y)^n$ which is clearly convex. Note however that this latter class of matroids has two disjoint bases.

Establishing the convexity of the Tutte polynomials of matroids within a given large class seems to be a difficult problem. The Tutte polynomials of the graphs at the top of Fig. 1 are convex functions while the Tutte polynomial of the graph at the bottom is neither convex nor concave. A similar situation holds for the matroids in Fig. 2: the Tutte polynomials of the two matroids at the top of the figure are convex functions while the polynomial for the matroid at the bottom is neither convex nor concave.

We proved Conjecture 1.1 for some families of graphs and matroids. There are some more families for which the conjecture holds: for example it is not difficult to prove that $\tau(G) \leq \alpha(G)$ when G is a simple outerplanar graph.

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References

- [1] F. Ardila, The Catalan matroid. *J. Combin. Theory Ser. A*, **104** (2003), 49-62.
- [2] N. Biggs, Algebraic Graph Theory. Cambridge University Press, Cambridge, 1996.
- [3] A. Björner, Homology and shellability of matroids and geometric lattices. In: White, N. (ed) Matroid Applications, Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1992.
- [4] J. Bonin, A. de Mier and M. Noy, Lattice path matroids: enumerative aspects and Tutte polynomials. *J. Combin. Theory Ser. A*, **104** (2003), 63-94.
- [5] T. Brylawski, A decomposition for combinatorial geometries. *Trans. Amer. Math. Soc.*, **171** (1972), 235-282.
- [6] T. Brylawski and J. Oxley, The Tutte polynomial and its applications. In N. White, editor, Matroid Applications, Encyclopedia of Mathematics and its Applications, 123-225. Cambridge University Press, Cambridge, 1992.
- [7] F. Chung and S. T. Yau, Coverings, heat kernels and spanning trees. *Electron. J. Combin.*, **6** R12, (1999), 21pp.
- [8] R. Conde and C. Merino, Comparing the number of acyclic and totally cyclic orientations with the number of spanning trees of a graph. *International Journal of Mathematical Combinatorics*, **2** (2009), 78-89.
- [9] R. Diestel, Graph Theory, Graduate Texts in Mathematics. Springer, New York, 2000.
- [10] J. Edmonds, Lehman's Switching Game and a Theorem of Tutte and Nash-Williams. *J. Res. Natl. Bur. Stand.*, **69B** (1965), 73-77.
- [11] R. L. Graham, D. E. Knuth, and O. Patashnik, Concrete Mathematics: A Foundation for Computer Science. Addison-Wesley, New York, 1990.
- [12] R. Grone and R. Merris, A bound for the complexity of a simple graph. *Discrete Math.*, **69** (1988), 97-99.

- [13] N. E. Kahale and L. J. Schulman, Bounds on the chromatic polynomial and on the number of acyclic orientations of a graph. *Combinatorica*, **16** (1996), 383–397.
- [14] B. Jackson, An Inequality for Tutte Polynomials. *Combinatorica*, **30** (2010), 68–81.
- [15] D. Mayhew, M. Newman, D. Welsh and G. Whittle, The asymptotic proportion of connected matroids. To appear in *European J. Combin.*
- [16] C. Merino, M. Ibañez and M. G. Rodríguez, A note on some inequalities for the Tutte polynomial of a matroid. *Electron. Notes Discrete Math.* **34** (2009), 603–607.
- [17] C. Merino and D.J.A. Welsh, Forests, colourings and acyclic orientations of the square lattice. *Ann. Combin.*, **3** (1999), 417–429.
- [18] R. Merris, Degree maximal graphs are Laplacian integral. *Linear Algebra Appl.*, **199** (1994), 381–389.
- [19] B. R. Myers, Number of spanning trees in a wheel. *IEEE Trans. Circuit Theory*, **CT-18** (1971), 387–391.
- [20] C. St. J. A. Nash-Williams, Decomposition of finite graphs into forests. *J. Lond. Math. Soc.*, **39** (1964), 12.
- [21] J. G. Oxley, *Matroid Theory*. Oxford University Press, New York, 1992.
- [22] J. Sedláček, Lucas numbers in graph theory (Czech. English summary). *Mathematics (Geometry and Graph Theory)* (Czech) 111–115, Univ. Karlova, Prague, 1970.
- [23] R.P. Stanley, Acyclic orientations of graphs. *Discrete Math.*, **5** (1973), 172–178,
- [24] W. T. Tutte, A contribution to the theory of chromatic polynomials. *Canad. J. Math.*, **6**, (1954), 80–91.
- [25] W. T. Tutte, On the problem of decomposing a graph into n connected factors. *J. Lond. Math. Soc.*, **36** (1961), 221–230.
- [26] D. J. A. Welsh, *Complexity: Knots, Colourings and Counting*. Cambridge University Press, Cambridge, 1993.