FINITE-DIFFERENCE SOLUTIONS OF TENTH-ORDER BOUNDARY-VALUE PROBLEMS

A THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

BY

Aijaz Ahmad Siddiqui

DEPARTMENT OF MATHEMATICS AND STATISTICS,

BRUNEL UNIVERSITY,

UXBRIDGE, MIDDLESEX, ENGLAND. UB8 3PH

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Dedication

TO MY

Late MOTHER

This Thesis is dedicated to my dear Mother who passed away during the second year of my research. She was always there when I needed Her and my only regret is that She is not here with us when I have completed this book.

May Allah(swt) rest Her soul in eternal peace and grant Her a place in Jannatul-Fidaus.

Abstract

In this thesis finite difference methods are used to obtain numerical solutions for a class of high-order ordinary differential equations with applications to eigenvalue problems.

Two families of numerical methods are developed for tenth-order boundary-value problems and global extrapolations on two and three grids are considered for the special problem.

Special nonlinear tenth-order boundary-value problems are solved using a family of direct finite difference methods which are adapted to solve a general linear and nonlinear boundary-value problem. These methods convert the ordinary differential equation into a set of algebraic equations. If the original ordinary differential equations are linear, the finite difference equations will give linear algebraic equations. If the ordinary differential equation are nonlinear, the resulting finite difference equations will be nonlinear algebraic equations. These nonlinear equations are first linearized by Newton's method. The methods developed are of orders two, four, six, eight, ten and twelve. The error analyses are discussed. A generalized form is given to solve a class of high-order boundary-value problems by converting the differential equation to a system of first-order equations. The method based on using a Padé rational approximant to the exponential function for general boundary-value problems is applied to a tenth-order eigenvalue problem associated with instability in a Bénard layer and numerical results are compared with asymtotic estimates appearing in the literature. This method may be implemented on a parallel computer. The method is extended to a twelfth-order eigenvalue problem in an appendix. The algorithms developed are tested on a variety of problems from the literature. The REDUCE package is used to obtain the parameters in the numerical methods and all computations are carried out on a Sun Workstation at Brunel University using Fortran 77 with double precision arithmetic.

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Contents

Chapter 1

INTRODUCTION

1.1 SUMMARY

Finite-difference methods are developed and analysed for the solution of linear and non-linear tenth-order boundary-value problems (BVPs).

1.2 INTRODUCTION

Boundary-value problems are manifest in many branches of science. These differential equations arise frequently in a wide variety of applications and can also be found in many engineering studies. Descriptions of some high-order boundary-value problems now follow.

1.2.1 Fourth-order problems

In the theory of vibrations, for instance, a fourth-order boundary-value problem called " The Euler-Bernoulli Beam Equation " arises. With some given end conditions on both sides of the beam, this equation governs the vibration of a non-uniform beam. In real life this beam can be a bridge, a ship hull, an aeroplane wing, a structure of a building, etc.

The Euler-Bernoulli beam equation is defined as follows:
the free undamped infinitesimal tranverse vibrations, of frequency ω , of a thin straight beam of length l, are governed by the Euler-Bernoulli ordinary differential equation (ODE)

$$\frac{d^2}{dt^2}(EI(t)\frac{d^2y(t)}{dt^2}) = \rho\, \omega^2 A(t)\, y(t), \quad 0 \le t \le l. \tag{1.1}$$

Here E is the Young's modulus and ρ is the density, both assumed constant. A(t) is the cross-sectional area at section t, and I(t) is the second moment of this area about the axis through the centriod at right angles to the plane of vibration (the neutral axis).

$$Let\ t = lx, y(x) = y(t), p(x) = \frac{I(x)}{I(t_0)}, s(x) = \frac{A(t)}{A(t_0)}, \lambda = \frac{[A(t_0)\rho l^4\omega^2]}{(EI(t_0))},$$

where t_0 is a chosen point in [0, l], then equation (1.1) becomes,

$$[p(x)\ y''(x)]'' - \lambda\ s(x)\ y(x) = 0, \quad 0 \le x \le 1.$$

Both p(x) and s(x) should be twice continuously differentiable and positive functions.

For a beam the most common end conditions are

$$\textit{clamped} \qquad y = 0 = y'; \tag{1.2}$$

$$\textit{pinned} \qquad y = 0 = y''; \tag{1.3}$$

$$\textit{sliding} \qquad y' = 0 = y'''; \tag{1.4}$$

$$\textit{free} \qquad y'' = 0 = y'''. \tag{1.5}$$

The boundary conditions are a combination of the conditions (1.2)—(1.5) on the boundaries x = 0, x = 1. There are boundary conditions that have been widely utilized in the literature and involve only the shape of the beam deflection curve at its boundaries. These are

i) free-free beams.

$$y''(0) = y'''(0) = 0 \ \text{ and } \ y''(1) = y'''(1) = 0.$$

ii) clamped-free beams.

$$y(0) = y'(0) = 0 \quad \text{and} \quad y''(1) = y'''(1) = 0.$$

iii) clamped-clamped beams.

$$y(0) = y'(0) = 0 \quad \text{and} \quad y(1) = y'(1) = 0.$$

iv) simple-simple beams.

$$y(0) = y''(0) = 0 \quad \text{and} \quad y(1) = y''(1) = 0.$$

v) clamped-simple beams.

$$y(0) = y'(0) = 0 \quad \text{and} \quad y(1) = y''(1) = 0.$$

vi) simple-free beams.

$$y(0) = y''(0) = 0 \quad \text{and} \quad y''(1) = y'''(1) = 0.$$

See Gorman (1975) and Thomson (1981) for more details.

The beam will have movements as a rigid-body under some boundary conditions. These possible movements are called rigid body modes. They are eigenmodes of the equation (1.1).

The Euler-Bernoulli Beam Equation (1.1) with end conditions is self-adjoint. This ensures that the eigenvalues are real; in particular , the eigenvalues of (1.1) are non-negative, and are positive if and only if the system is positive. See Theorem 10.1.2 in Gladwell (1985).

A ship's hull is regarded as a free-free beam the section of which does not distort when it bends or twists (see Bishop and Price (1979)). If the thickness of the slice is Δx and the shearing force and bending moment applied to it are V and M respectively then the upward force Z(x,t) per unit length applied to the slice includes contributions from weight, buoyancy and all other forces.

Motion of the slice of the beam in the vertical direction is governed by the equation

$$V_1 - V_2 + Z(x,t)\Delta x = \mu(x)\Delta x \frac{\partial^2 \omega(x,t)}{\partial t^2},$$

where $\mu(x)$ is the mass per unit length of the beam and $\omega(x,t)$ is the upward deflection. Hence

$$\frac{\partial V}{\partial x} + Z(x,t) = \mu(x)\frac{\partial^2 \omega(x,t)}{\partial t^2}. \tag{1.6}$$

If rotatory inertia of the beam is neglected, $M_1 - M_2 + V\Delta x = 0$,
so that $V = -\frac{\partial M}{\partial x}$.

According to elementary beam theory

$$M = EI(x)\frac{\partial^2 \omega(x,t)}{\partial x^2} + \beta(x)\frac{\partial^3 \omega(x,t)}{\partial x^2 \partial t},$$

where EI(X) is the flexural rigidity and $\beta(x)$ represents viscous structural damping. It follows that

$$V = -\frac{\partial}{\partial x}[EI(x)\frac{\partial^2 \omega(x,t)}{\partial x^2}]\frac{\partial}{\partial x}[\beta(x)\frac{\partial^3 \omega(x,t)}{\partial x^2 \partial t}].$$

Denoting the partial differentiation with respect to x by a prime ($\prime$) and partial differentiation with respect to t by a dot ($\cdot$), the equation of flexural motion may be written as

$$\mu(x)\ \ddot{\omega}(x,t) + [EI(x)\ \omega''(x,t)]'' + [\beta(x)\ \dot{\omega}''(x,t)]'' = Z(x,t).$$

This is the equation of vertical symmetric bending of the dry hull.

In free vibration of the undamped dry beam, $Z(x,t) = 0 = \beta(x)$ for all positions x on the beam and at all times t so that the trial solution

$$\omega(x,t) = f(x)\ \sin\omega t$$

requires that

$$-\mu(x)\ \omega^2 f(x) + [EI(x) f''(x)]'' = 0,$$

where the prime now represents a total derivative with respect to x. The function f(x) has also to satisfy the boundary conditions $f''(0) = f''(1) = 0$. The values of ω, say $\omega_1, \omega_2, \omega_3, \omega_4, \ldots$ are the natural frequencies of the beam. This sequence of principal modes for the free-free non-uniform beam will have different shapes like cargo ship and a small warship etc.

1.2.2 Sixth-order problems

Sixth-order boundary-value problems are found to have applications in astrophysics, as A-type stars. Chandrasekhar (1961) and Baldwin (1987) noted that if the level of the temperature gradient at which the instability occurs is not at a boundary, then the motion may be modelled by the eigenvalue problem

$$(D^2 - a^2)^3\, y(x) + Ra^2(1 - x^2)\, y(x) = 0, \tag{1.7}$$

with

$$y(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm\, \infty. \tag{1.8}$$

In this problem $D = \frac{d}{dx}$, x is a dimensionless boundary layer coordinate, $y = y(x)$ is a dimensionless vertical velocity, a is horizontal wave number and R is a Rayleigh number.

1.2.3 Eighth- and tenth-order problems

Chandrasekhar (1961) also noted that the effect of both rotation and a magnetic field impart to the fluid a certain rigidity while at the same time they impart to it certain properties of elasticity that enable it to transmit disturbances by new modes of wave propagation. These effects can be represented by eighth-order and tenth-order equations. The eighth-order ODE has the form

$$\begin{aligned}&(D^2 - a^2 - \rho\sigma)[(D^2 - a^2 - \sigma)^2(D^2 - a^2) + TD^2]y(x)\\ &= -R(D^2 - a^2 - \sigma)y(x), \quad 0 < x < 1,\end{aligned} \tag{1.9}$$

with free-free boundary conditions

$$\begin{aligned} &y(0) = D^2y(0) = D^4y(0) = D^6y(0) = 0, \\ &y(1) = D^2y(1) = D^4y(1) = D^6y(1) = 0. \end{aligned} \tag{1.10}$$

In (1.9), y(x) is the vertical flow of a fluid heated below and under the effect of rotation.

The tenth-order equation is given by

$$\begin{aligned} &(D^2 - a^2)[(D^2 - a^2)^2 - QD^2]^2 + TD^2(D^2 - a^2)y(x) \\ &= -Ra^2[(D^2 - a^2) - QD^2]y(x); \quad 0 < x < 1, \end{aligned} \tag{1.11}$$

with free-free boundary conditions

$$\begin{aligned} &y(0) = D^2y(0) = D^4y(0) = D^6y(0) = D^8y(0) = 0, \\ &y(1) = D^2y(1) = D^4y(1) = D^6y(1) = D^8y(0) = 0. \end{aligned} \tag{1.12}$$

1.2.4 Literature survey

Whereas the qualitative theory of differential equtions, in former years, mostly occupied itself with systems of the second-order and their solution trajectories, attention of late is more and more focussed on systems of a higher order. In this thesis a brief survey will be given of some of the results which have been achieved in the meantime. In addition, numerous papers on the behaviour of the solutions of more or less special systems of non-linear differential equations or the properties of general dynamical systems have appeared in various periodicals or Academy publications.

In short, several methods are currently used for the numerical solution of boundary-value problems and the literature associated with each mehod is abundant. Some references related to these methods are now listed.

Finite difference methods :

Boutayeb (1990), Boutayeb and Twizell (1991,1992,1993), Chawla and Katti (1979), Collatz (1966, 1986), Keller (1968), Djidjeli et al. (1993), Fox (1962), Collatz (1966, 1986), Keller (1968), Twizell and Boutayeb (1990), Twizell

(1988a,b), Twizell et al. (1994), Usmani (1978, 1981).

Finite element methods :

Davies (1980) and Wait and Mitchell (1986).

Shooting methods :

Keller (1968) and Twizell (1988a).

Collocation methods :

Russel and Shampine (1977).

Quasilinearization methods :

Bellman and Kalaba (1966), Lee (1968) and Agarwal (1986).

Orthonormalization methods :

Godunove (1961) and Scott and Watts (1977).

Variational methods :

Bailey et al. (1968).

Repeated integration methods :

Fröberg (1985).

Other methods :

Keller (1975) has written a survey paper, covering a general outline of these techniques. Aktas and Stetter (1977) gave a classification and survey of numerical methods for BVPs. Also, Daniel, in Childs et al. (1978), wrote a "road map" of methods for approximating solutions of two-point BVPs.

In fact, most of the authors cited above deal with more than one method and most of them give detailed bibliographies on boundary-value problems.

However, when they treat high-order equations, the majority of the authors concentrate on the fourth-order. The numerical analysis literature on higher-order boundary-value problems remains sparse, although such problems are contained implicity in some papers and, as noted by Keller (1968), high-order differential equations can always be converted to a system of first-order equations for which well known numerical methods may be applied.

A second-order convergent method is outlined in Twizell (1988b) for sixth-order problems. Scott and Watts (1977) treated a linear eighth-order problem.

One method of solving a general-order boundary-value problem is to convert the differential equation $y^{(n)} = f(x,y)$; $a < x < b$, with boundary conditions specified, to a system of first-order equations and then to use appropriate methods for this kind of problem (see, for example, Keller (1968), Matar (1990)). This technique will be followed in Chapters 4, 5 and 6.

Twizell and Tirmizi (1986) developed a sixth-order multiderivative method for the numerical solution of fourth-order boundary-value problems. The method is derived from a five-point recurrence relation involving exponential terms, the multiderivatives being obtained by replacing the exponentials by Padé approximants. The method is adopted from the numerical solution of the problem of bending a simply-supported beam.

1.3 PADÉ APPROXIMANTS

Padé approximants are defined as follows:

Let $f(z)$, $z \in \mathbf{C}$, be an analytic function in a region of the complex plane containing the origion z=0. A Padé approximant $R_{\mu,\kappa}(z)$ to the function $f(z)$ is then defined by

$$f(z) = \frac{P^*_\kappa(z)}{Q^*_\mu(z)},$$

where $P^*_\kappa(z)$ and $Q^*_\mu(z)$ are polynomials of degree κ and μ, respectively with leading coefficient unity.

For the function f(z)=exp(z), Varga (1962), the polynomials $P^*_{\kappa(z)}$ and $Q^*_\mu(z)$ are given explicitly as

$$P^*_\kappa(z) = \Sigma_{j=0}^{\kappa} \frac{(\mu+\kappa-j)!\kappa!}{(\mu+\kappa)!j!(\kappa-j)!}(z)^j,$$

$$Q^*_\mu(z) = \Sigma_{j=0}^{\mu} \frac{(\mu+\kappa-j)!\mu!}{(\mu+\kappa)!j!(\mu-j)!}(-z)^j,$$

and if

$$\exp(z) = \frac{P^*{}_\kappa(z)}{Q^*{}_\mu(z)} + T_{\mu,\kappa}(z),$$

then the remainder $T_{\mu,\kappa}(z)$ is given by

$$T_{\mu,\kappa}(z) = \frac{(-1)^{\kappa+1} z^{(\mu+\kappa+1)}}{(\mu+\kappa)! Q^*{}_\mu(z)} \int_0^1 \exp(z)(1-u)u^\kappa (1-u)^\mu du.$$

For $\mu = 0,1,2,3,4$ and $\kappa = 0,1,2,3,4$ the first fifteen entries of the Padé approximants for f(z)=exp(z) are given in Tirmizi (1984).

Twizell (1978 and 1980) has used Padé approximants and has developed various finite difference methods which cover wide areas in ordinary and partial differential equations for both initial- and boundary-value problems. More about the use of Padé approximants is contained in Khaliq (1983) and Tirmizi (1984).

1.4 GENERAL PADÉ-BASED NUMERICAL SCHEMES FOR LINEAR BOUNDARY-VALUE PROBLEMS

The general n-th order linear boundary-value problem consists of the ODE

$$y^{(n)}(x) + \Sigma_{i=0}^{n-1} P_i y^{(i)}(x) = r(x), \quad a < x < b, \tag{1.13}$$

with the boundary conditions

$$B_a \, \mathbf{Y}(a) + B_b \, \mathbf{Y}(b) = \mathbf{c}, \tag{1.14}$$

where $\mathbf{Y}(a) = [y^{(n-1)}(a), y^{(n-2)}(a), y^{(n-3)}(a), \ldots, y''(a), y'(a), y(a)]^T$, $\mathbf{Y}(b) = [y^{(n-1)}(b), y^{(n-2)}(b), y^{(n-3)}(b), \ldots, y''(b), y'(b), y(b)]^T$, $\mathbf{c}$ is a constant n-dimensional vector and B_a, B_b are constant matrices of order $n \times n$.

This differential equation can be written as a system of n first-order differential equations. Introducing the variables $y_0 = y_0(x), \quad y_1 = y_1(x), y_2 = y_2(x), \ldots$

and $y_{n-1} = y_{n-1}(x)$, these are defined by
$y_0 = y_0(x),\ y_1 = y'(x),\ y_2 = y''(x),\ y_3 = y'''(x), \ldots\ldots$ and $y_{n-1} = y^{(n-1)}(x)$, it follows that $y_0' = y_1,\ y_1' = y_2,\ y_2' = y_3,\ y_3' = y_4, \ldots\ldots,$
$y_{n-1}' = -(P_0(x)y_0 + P_1(x)y_1 + P_2(x)y_2 + \ldots + P_{n-1}(x)y_{n-1}) + r(x).$
The n first-order linear boundary-value problems can be written as

$$\mathbf{D}\mathbf{Y}(x) = Q(x)\mathbf{Y}(x) + \mathbf{P}(x),\ a < x < b, \tag{1.15}$$

with boundary conditions (1.14) in which $D \equiv \text{diag}\{\frac{d}{dx}\}$ is a digonal matrix of order $n \times n$, Q is an $n \times n$ matrix with entries q_{ij} given by $q_{i+1,i} = 1$ $[i = 1, 2, 3, 4, 5, \ldots, (n-1)]$, $q_{1,j} = -P_{n-j}$ $(j = 1, 2, 3, \ldots, n)$ and the other entries are zero, that is

$$Q = \begin{bmatrix} -P_{n-1} & -P_{n-2} & -P_{n-3} & . & . & . & -P_2 & -P_1 & -P_0 \\ 1 & 0 & & & & & & & \\ & 1 & 0 & & & & & & \\ & & 1 & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & \ddots & & & & \\ & & & & \cdots & & & & 0 \\ & & & & & & & 1 & 0 \end{bmatrix},$$

$\mathbf{Y}$ and $\mathbf{P}$ are defined respectively as $\mathbf{Y}(x) = [y_{n-1}, y_{n-2}, y_{n-3}, \ldots, \ldots, y_2, y_1, y_0]^T$, $\mathbf{P}(x) = [r(x), 0, 0, 0, \ldots, 0, 0, 0]^T$.

Consider the grid

$$\Pi_1 :\quad a = x_0 < x_1 < x_2 < x_3 < \ldots < x_N < x_{N+1} = b,$$

obtained by discretizing the interval [a,b] into N+1 subintervals each of width $h = \frac{b-a}{(N+1)}$ where N is a positive integer.

The solution $y(x)$ will be computed at the points x_k where $x_k = a + kh$ $(k = 0, 1, 2, 3, \ldots, N+1)$ of Π_1. In particular the values of the $n \times 1$ vectors $\mathbf{y}_k$ $(k = 1, 2, 3, \ldots, N)$ will be computed, and also the vector $\mathbf{y}_{N+1}$ will be computed where this last vector consists of all unknown values of y and its derivatives y', y'', y''', $y^{(iv)}, \ldots, y^{(n-1)}$ at both boundaries $x = a$ and $x = b$. Define the unknown vector

$$\mathbf{y}_{N+1} = [y_1, y_2, y_3, y_4, \ldots, \ldots, y_N]^T,$$

where its set of elements $y_1, y_2, y_3, y_4, \ldots, \ldots, y_N$ will be a subset of

$$[y(a), y(b), y'(a), y'(b), y''(a), y''(b), \ldots, \ldots, y^{(n-1)}(a), y^{(n-1)}(b)]^T.$$

Then, the total number of unknowns that will be approximated is n(N+1), where n is the order of the differential equation.

Applying the (μ, κ) Padé approximant to the exponential term in $\mathbf{y}(x_k + h) = [\exp(hD)]\mathbf{y}(x_k)$, the result will be

$$Q^*_\mu(hD)\mathbf{y}_{k+1} = P^*_\kappa(hD)\mathbf{y}_k + O(h^{\mu+\kappa+1}), \qquad (1.16)$$

where the operator functions Q^*_μ and P^*_κ are defined as

$$Q^*_\mu(hD) = \Sigma^{\mu}_{j=0}\frac{(\mu+\kappa-j)!\mu!}{(\mu+\kappa)!j!(\mu-j)!}(-hD)^j,$$

and

$$P^*_\kappa(hD) = \Sigma^{\kappa}_{j=0}\frac{(\mu+\kappa-j)!\kappa!}{(\mu+\kappa)!j!(\kappa-j)!}(-hD)^j.$$

Equation (1.16) can be rewritten in a more explicit form by

$$\begin{aligned} & [I - \alpha_1 hD + \alpha_2(hD)^2 - \alpha_3(hD)^3 + \ldots + (-1)^\mu \alpha_\mu (hD)^\mu]\mathbf{y}_{k+1} \\ = \; & [I + \beta_1 hD + \beta_2(hD)^2 + \beta_3(hD)^3 + \ldots + \beta_\kappa(hD)^\kappa]\mathbf{y}_k + O(h^{\mu+\kappa+1}), \end{aligned} \qquad (1.17)$$

where

$$\alpha_j = \frac{(\mu+\kappa-j)!\mu!}{(\mu+\kappa)!j!(\mu-j)!}\;; \quad \beta_j = \frac{(\mu+\kappa-j)!\kappa!}{(\mu+\kappa)!j!(\kappa-j)!}.$$

To find the jth derivative of the vector $\mathbf{Y}, D^j\mathbf{Y}(x)$, equation (1.15) will be used first of all. Rename Q^* and $\mathbf{P}$ by Q_1 and $\mathbf{P}_1$, respectively. Then

$$D^2\mathbf{Y} = Q_2\mathbf{Y} + \mathbf{P}_2,$$

where $Q_2 = [DQ_1 + Q_1^2]$ and $\mathbf{P}_2 = [Q_1\mathbf{P}_1 + D\mathbf{P}_1]$.
Therefore,

$$D^i\mathbf{Y} = Q_j\mathbf{Y} + \mathbf{P}_j, \qquad (1.18)$$

where $Q_j = DQ_{j-1} + Q_{j-1}\ Q_1$ and $\mathbf{P}_j = Q_{j-1}\mathbf{P}_1 + D\mathbf{P}_{j-1},\ j = 2, 3, 4, \ldots$.
Secondly, substitute equation (1.18) into (1.17) to give

$$A_{k+1}\mathbf{Y}_{k+1} + B_k\mathbf{Y}_k = \mathbf{E}_{k+1} + \mathbf{F}_k, \qquad (1.19)$$

where

$$A_{k+1} = I + \Sigma_{j=1}^{\mu}(-1)^j\alpha_j h^j Q_j, \qquad (1.20)$$

and

$$B_k = -I - \Sigma_{j=1}^{\kappa}\beta_j h^j Q_j, \qquad (1.21)$$

and also the right side is

$$\mathbf{E}_{k+1} = -\Sigma_{j=1}^{\mu}(-1)^j\alpha_j h^j\mathbf{P}_j \text{ and } \mathbf{F}_k = \Sigma_{j=1}^{\kappa}\beta_j h^j\mathbf{P}_j.$$

Denote the right side of (1.19) by $\mathbf{g}_{k+1}$, then

$$\mathbf{g}_{k+1} = \mathbf{E}_{k+1} + \mathbf{F}_k,$$

and the final form of (1.19) is

$$A_{k+1}\mathbf{y}_{k+1} + B_k\mathbf{y}_k = \mathbf{g}_{k+1}.$$

This vector-matrix equation is to be applied to the discrete points $x_0, x_1, x_2, \ldots, x_N$. The result will be a of the form system of linear equations with $n(N+n)$ equations with $n(N+n)$ unknowns

$$A\ \mathbf{Y} = \mathbf{G},$$

which can be written in a block vector-matrix form as

$$\begin{bmatrix} A_1 & & & & & & & & B_0 \\ B_1 & A_2 & & & & & & & \\ & B_2 & A_3 & & & & & & \\ & & B_3 & A_4 & & & & & \\ & & & \cdot & \cdot & & & & \\ & & & & \cdot & \cdot & & & \\ & & & & & \cdot & \cdot & & \\ & & & & & & \cdot & \cdot & \\ & & & & & & & B_N & A_{N+1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdot \\ \cdot \\ \cdot \\ y_N \\ y_{N+1} \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ \cdot \\ \cdot \\ \cdot \\ g_N \\ g_{N+1} \end{bmatrix} . \quad (1.22)$$

The vector $\mathbf{Y}_{N+1}$ will contain all the unknown elements on the two boundaries $x = a$ and $x = b$. (See Matar (1990).) This approach of transforming a high-order problem into a system of first-order problems will be utilized in Chapter 4 for linear tenth-order problems, where a parallel algorithm will be developed.

1.5 GENERAL PADÉ-BASED NUMERICAL SCHEMES FOR NON-LINEAR BOUNDARY-VALUE PROBLEMS

The general n-th order non linear boundary-value problem has the form

$$y^{(n)}(x) = f(x, y(x), y', y''(x), \ldots, y^{(n-1)}(x)), \quad a < x < b, \qquad (1.23)$$

with the boundary conditions

$$B_a \, \mathbf{Y}(a) + B_b \, \mathbf{Y}(b) = \mathbf{c}, \qquad (1.24)$$

where $\mathbf{Y}(a) = [y^{(n-1)}(a), y^{(n-2)}(a), y^{(n-3)}(a), \ldots, y''(a), y'(a), y(a)]^T$, $\mathbf{Y}(b) = [y^{(n-1)}(b), y^{(n-2)}(b), y^{(n-3)}(b), \ldots, y''(b), y'(b), y(b)]^T$, $\mathbf{c}$ is a constant

n-dimensional vector and B_a, B_b are constant matrices of order $n \times n$.
Let $y_0 = y(x),\ y_1 = y'(x),\ y_2 = y''(x),\ y_3 = y'''(x), \ldots\ldots$ and $y_{(n-1)} = y^{(n-1)}(x)$, then the n-th order boundary-value problem can be written as a system of first-order differential equations

$$DY(x) = Q(x)Y(x) + P(x, Y),\ a < x < b \tag{1.25}$$

with boundary conditions (1.24). Again, $D \equiv \mathrm{diag}\{\frac{d}{dx}\}$ is a matrix of order $n \times n$, Q is an $n \times n$ matrix with entries q_{ij} given by $q_{i+1,i} = 1$ $[i = 1, 2, 3, \ldots, (n-1)]$ and the other entries are zero, i.e.

$$Q = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 1 & 0 & & & & & & & \\ & 1 & 0 & & & & & & \\ & & 1 & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & \ddots & & & & \\ & & & & \cdots & & & & 0 \\ & & & & & & & 1 & 0 \end{bmatrix},$$

$\mathbf{Y}$ and $\mathbf{P}$ are defined, respectively, as $\mathbf{Y} = [y_{n-1}, y_{n-2}, y_{n-3}, \ldots, \ldots, y_2, y_1, y_0]^T$ and $\mathbf{P} = [f(x, y_0, y_1, y_2, y_3, \ldots, \ldots, y_{n-1}), 0, \ldots, \ldots, 0, 0, 0]^T$.

Applying the (μ, κ) Padé approximant in the same steps as in the linear case an equation similar to (1.17) will be produced, but with the jth derivative of the vector $\mathbf{Y}$ written in a different manner; in this nonlinear case $DQ = 0$ and

$$Df = \frac{\partial f}{\partial x} + y_1\frac{\partial f}{\partial y_0} + y_2\frac{\partial f}{\partial y_1} + y_3\frac{\partial f}{\partial y_2} + \ldots + y_{n-1}\frac{\partial f}{\partial y_{n-2}} + f\frac{\partial f}{\partial y_{n-1}}.$$

Therefore, using (1.26), $D\mathbf{Y} = Q\mathbf{Y} + \mathbf{P}$, then $D^2\mathbf{Y} = Q^2 + Q\mathbf{P} + D\mathbf{P}$ and finally

$$D^j\mathbf{Y} = Q^j\mathbf{Y} + \Sigma_{i=0}^{j-1} Q^{j-i-1} D^i\mathbf{P} \quad j = 2, 3, 4, \ldots.$$

Substituting (1.26) into equation (1.17) the result will be

$$A_{k+1}\mathbf{Y}_{k+1} + B_k\mathbf{Y}_k = \mathbf{E}_{k+1}(x, \mathbf{Y}_{k+1}) + \mathbf{F}_k(x, \mathbf{Y}_k), \tag{1.26}$$

where

$$A_{k+1} = I + \Sigma_{j=1}^{\mu}(-1)^j \alpha_j h^j Q^j, \tag{1.27}$$

and

$$B_k = -I - \Sigma_{j=1}^{\kappa}\beta_j h^j Q^j. \tag{1.28}$$

The nonlinear part of the right side of (1.26) is

$$\mathbf{E}_{k+1} = -\Sigma_{j=1}^{\mu}(-1)^j \alpha_j h^j Q^j (\Sigma_{i=0}^{j-1} Q^{j-i-1} D^i \mathbf{P})$$

and

$$\mathbf{F}_k = \Sigma_{j=1}^{\kappa}\beta_j h^j (\Sigma_{i=0}^{j-1} Q^{j-i-1} D^i \mathbf{P}).$$

Define the non-linear right side of (1.26) by Φ_{k+1}, then $\Phi_{k+1} = \mathbf{E}_{k+1} + \mathbf{F}_k$ and the final form of (1.26) is

$$A_{k+1}\ \mathbf{Y}_{k+1} + B_k\ \mathbf{Y}_k = \Phi_{k+1}.$$

The elements of the matrices A_{k+1} and B_k are given by

$$a_{ij} = \begin{pmatrix} 0 & : 1 < j \\ 1 & : 1 = j \\ (-1)^{i-j}\alpha_{i-j}h^{i-j} & : i > j \end{pmatrix} \text{ and } b_{ij} = \begin{pmatrix} 0 & : 1 < j \\ -1 & : 1 = j \\ -\beta_{i-j}h^{i-j} & : i > j \end{pmatrix}$$

respectively.

The vector-matrix equation (1.26) is to be applied to the discrete points $x_0, x_1, x_2, x_3, x_4, \ldots, x_N$. The result will be a system of n(N+1) non-linear equations with n(N+1) unknowns, which can be written as

$$A\mathbf{Y} + \Phi(\mathbf{Y}) = \mathbf{0}, \tag{1.29}$$

where A is a block bi-diagonal matrix, except for the first row, similar in structure to (1.22), $\Phi(\mathbf{Y})$ is an $n(N+1)$-dimension non-linear vector defined by

$$\Phi(\mathbf{Y}) = -\begin{bmatrix} E_1 + F_0 \\ E_2 + F_1 \\ E_3 + F_2 \\ E_4 + F_3 \\ E_5 + F_4 \\ \cdot \\ \cdot \\ \cdot \\ E_{N-1} + F_N \\ E_N + F_{N-1} \\ E_{N+1} + F_N \end{bmatrix}$$

This approach will be used in Chapter 5 where a parallel algorithm will be developed for the solution of a nonlinear tenth-order boundary-value problem.

1.6 NEWTON'S METHOD

The best-known and the most popular method for solving non-linear algebraic equations is Newton's method. Let

$$\mathbf{F}(\mathbf{Y}) \equiv A\mathbf{Y} + \Phi(\mathbf{Y}) = \mathbf{0},$$

$\mathbf{F} = [F_1, F_2, F_3, \ldots, F_{n(N+1)}]^T$. To solve this system of $n(N+1)$ non-linear equations, the Jacobian matrix of $\mathbf{F}(\mathbf{Y})$ should be defined. The Jacobian matrix $J(\mathbf{Y})$ which can alternatively be denoted by $\frac{\partial \mathbf{F}}{\partial \mathbf{Y}}$, will be a block matrix similar to the block matrix A in equation (1.22), i.e.

$$J(\mathbf{Y}) = \begin{bmatrix} A_1 & & & & & & & & B_0 \\ B_1 & A_2 & & & & & & & \\ & B_2 & A_3 & & & & & & \\ & & B_3 & A_4 & & & & & \\ & & & \cdot & \cdot & & & & \\ & & & & \cdot & \cdot & & & \\ & & & & & \cdot & \cdot & & \\ & & & & & & \cdot & \cdot & \\ & & & & & & & B_N & A_{N+1} \end{bmatrix}. \qquad (1.30)$$

The above Jacobian (1.30) consists of non-zero $n \times n$ matrices A_k $(k = 1, 2, 3, \ldots, N + 1)$ and B_k $(k = 0, 1, 2, \ldots, N)$. The (i,j)th elements of the matrices A_k $(k = 1, 2, 3, \ldots)$, B_k $(k = 1, 2, \ldots, N)$ are

$$a_{k,i,j} = \frac{\partial F_{(k-1)n+i}}{\partial y_{nj_k}}, \quad b_{k,i,j} = \frac{\partial F_{(k-1)n+i}}{\partial y_{nj_k}},$$

respectively. Correspondingly, the elements of the two matrices A_{N+1} and B_0 are

$$a_{N+1,i,j} = \frac{\partial F_{nN+i}}{\partial y_j}, \quad b_{0,i,j} = \frac{\partial F_i}{\partial y_j},$$

y_j $(j = 1, 2, 3, \ldots, n)$ are elements of the vector $\mathbf{y}_{N+1}$.

Let $\mathbf{Y}^{(0)}$ be some initial value of the vector. It can be determined by giving starting values to the unknown y and its derivatives $y', y'', \ldots, y^{(n-1)}$, at the discrete points $x_k = a + \frac{k(b-a)}{N+1}$, $(k = 0, 1, 2, \ldots, N + 1)$. Assume ξ is the iteration number. The solution $\mathbf{Z}^{(\xi)}$ of the linear system

$$J(\mathbf{Y}^{(\xi)})\mathbf{Z}^{(\xi)} = -\mathbf{F}(\mathbf{Y}^{(\xi)}), \qquad (1.31)$$

can be solved in the same manner as in the linear case. After $\mathbf{Z}^{(\xi)}$ is computed, $\mathbf{Y}^{(\xi+1)}$ is easily computed from

$$\mathbf{Y}^{(\xi+1)} = \mathbf{Y}^{(\xi)} + \mathbf{Z}^{(\xi)}, \; (\xi = 0, 1, 2, \ldots). \qquad (1.32)$$

The process (1.31) and (1.32) will be repeated for a few iterations until convergence occurs. In general, the iteratons of Newton's method cannot be guaranteed to converge, but it is usually successful if the system has a solution, the system is not seriously unstable for step-by-step solution, and a good initial estimate can be found for the unknown values of vector $\mathbf{Y}^{(0)}$. It may be necessary to simplify the problem and perform some preliminary calculations in order to get suitable starting values.

The Newton-Raphson method will be adapted for use in Chapter 5.

Chapter 2

SPECIAL NONLINEAR TENTH-ORDER BOUNDARY-VALUE PROBLEMS

2.1 A FAMILY OF NUMERICAL METHODS

Consider the problem

$$y^{(x)}(x) = f(x, y), \quad a < x < b; a, b, x \in \Re, \tag{2.1}$$

$$y^{(2i)}(a) = A_{2i}, \quad y^{(2i)} = B_{2i} \ (i = 0, 1, 2, 3, 4). \tag{2.2}$$

It is assumed that f(x,y) is as many times differentiable as required, is real and that A_{2i}, B_{2i} $(i = 0, 1, 2, 3, 4)$ are real finite constants.

Consider first the mesh G_1, obtained by discretizing the interval $a \leq x \leq b$ into N+1 subintervals each of width $h = \frac{(b-a)}{N+1}$ where $N \geq 9$ is an integer. The solution y(x) will be computed at the mesh points $x_n = x_n^{(1)} = a + nh$ $(n = 1, 2, 3, 4, \ \ldots\ldots, N)$ of mesh G_1 and the notation $y_n = y_n^{(1)}$ will be adopted to denote the solution of an approximating difference scheme at the grid point $x_n^{(1)}$. It is clear that, according to (2.2),

$$y_0^{(1)} = A_0 \quad \text{and} \quad y_{N+1}^{(1)} = B_0.$$

A general family of symmetric numerical methods is given by

$$\begin{aligned} & y_{n-5} - 10y_{n-4} + 45y_{n-3} - 120y_{n-2} + 210y_{n-1} - 252y_n \\ & +210y_{n+1} - 120y_{n+2} + 45y_{n+3} - 10y_{n+4} + y_{n+5} \\ = \; & h^{10}[\alpha f_{n-5} + \beta f_{n-4} + \gamma f_{n-3} + \delta f_{n-2} + \epsilon f_{n-1} + \sum f_n \\ & +\epsilon f_{n+1} + \delta f_{n+2} + \gamma f_{n+3} + \beta f_{n+4} + \alpha f_{n+5}], \end{aligned} \tag{2.3}$$

$\alpha, \beta, \gamma, \delta, \epsilon$ are parameters chosen to ensure consistency as a minimum requirement and $\sum = 1 - 2(\alpha + \beta + \gamma + \delta + \epsilon)$.

2.2 A SECOND-ORDER METHOD

We note that a well-known second-order central-difference approximation to the tenth derivative $y^{(x)}(x_n)$ is given by

$$\begin{aligned} y^{(x)}(x_n) = \; & h^{-10}[y_{n-5} - 10y_{n-4} + 45y_{n-3} - 120y_{n-2} \\ & +210y_{n-1} - 252y_n + 210y_{n+1} - 120y_{n+2} \\ & +45y_{n+3} - 10y_{n+4} + y_{n+5}] + O(h^2). \end{aligned} \tag{2.4}$$

Given the ordinary differential equation $y^{(x)} = f(x, y)$, at point n of the discretization $x_1, x_2, x_3, \quad \ldots, x_n,$ we have

$$\begin{aligned} & y_{n-5} - 10y_{n-4} + 45y_{n-3} - 120y_{n-2} + 210y_{n-1} - 252y_n \\ & +210y_{n+1} - 120y_{n+2} + 45y_{n+3} - 10y_{n+4} + y_{n+5} \\ & = h^{10}f_n, \end{aligned} \tag{2.5}$$

which is the simplest example of (2.3), having $\alpha = \beta = \gamma = \delta = \epsilon = 0$ and $\sum = 1$.

This is written as

$$\begin{aligned}
&-y_{n-5}+10y_{n-4}-45y_{n-3}+120y_{n-2}-210y_{n-1}+252y_{n}\\
&-210y_{n+1}+120y_{n+2}-45y_{n+3}-10y_{n+4}-y_{n+5}+h^{10}f_{n} \qquad (2.6)\\
&=0, \quad \text{for} \quad n=5,6,7,\dots,N-5,N-4.
\end{aligned}$$

Note: This is equivalent to writing the ODE as $-y^{(x)}+f(x,y)=0$.

The local truncation error (l.t.e.) of this numerical method at any point is given by

$$\begin{aligned}
L[y(x);h]= \; &-y(x-5h)+10y(x-4h)-45y(n-3)+120y(x-2h)\\
&-210y(x-h)+252y(x)-210y(x+h)+120y(x+2h) \qquad (2.7)\\
&-45y(x+3h)-10y(x+4h)-y(x+5h)+h^{10}y^{(x)}(x).
\end{aligned}$$

Writing (2.7) as a Taylor series about y(x) gives

$$\begin{aligned}
L[y(x);h]= \; &-[y-5hy'+\frac{5^2h^2}{2!}y''-\frac{5^3h^3}{3!}y'''+\frac{5^4h^4}{4!}y^{(iv)}-\frac{5^5h^5}{5!}y^{(v)}\\
&+\frac{5^6h^6}{6!}y^{(vi)}-\frac{5^7h^7}{7!}y^{(vii)}+\frac{5^8h^8}{8!}y^{(viii)}-\frac{5^9h^9}{9!}y^{(ix)}+\frac{5^{10}h^{10}}{10!}y^{(x)}\\
&-\frac{5^{11}h^{11}}{11!}y^{(xi)}+\frac{5^{12}h^{12}}{12!}y^{(xii)}-\frac{5^{13}h^{13}}{13!}y^{(xiii)}+\frac{5^{14}h^{14}}{14!}y^{(xiv)}\\
&-\frac{5^{15}h^{15}}{15!}y^{(xv)}+\frac{5^{16}h^{16}}{16!}y^{(xvi)}-\frac{5^{17}h^{17}}{17!}y^{(xvii)}+\frac{5^{18}h^{18}}{18!}y^{(xviii)}\\
&-\frac{5^{19}h^{19}}{19!}y^{(xix)}+\frac{5^{20}h^{20}}{20!}y^{(xx)}-\dots]\\
&+10[y-4hy'+\frac{4^2h^2}{2!}y''-\frac{4^3h^3}{3!}+\frac{4^4h^4}{4!}y^{(iv)}-\frac{4^5h^5}{5!}y^{(v)}\\
&+\frac{4^6h^6}{6!}y^{(vi)}-\frac{4^7h^7}{7!}y^{(vii)}+\frac{4^8h^8}{8!}y^{(viii)}-\frac{4^9h^9}{9!}y^{(ix)}+\frac{4^{10}h^{10}}{10!}y^{(x)}\\
&-\frac{4^{11}h^{11}}{11!}y^{(xi)}+\frac{4^{12}h^{12}}{12!}y^{(xii)}-\frac{4^{13}h^{13}}{13!}y^{(xiii)}+\frac{4^{14}h^{14}}{14!}y^{(xiv)}\\
&-\frac{4^{15}h^{15}}{15!}y^{(xv)}+\frac{4^{16}h^{16}}{16!}y^{(xvi)}-\frac{4^{17}h^{17}}{17!}y^{(xvii)}+\frac{4^{18}h^{18}}{18!}y^{(xviii)}\\
&-\frac{4^{19}h^{19}}{19!}y^{(xix)}+\frac{4^{20}h^{20}}{20!}y^{(xx)}-\dots]\\
&-45[y-3hy'+\frac{3^2h^2}{2!}y''-\frac{3^3h^3}{3!}y'''+\frac{3^4h^4}{4!}y^{(iv)}-\frac{3^5h^5}{5!}y^{(v)}\\
&+\frac{3^6h^6}{6!}y^{(vi)}-\frac{3^7h^7}{7!}y^{(vii)}+\frac{3^8h^8}{8!}y^{(viii)}-\frac{3^9h^9}{9!}y^{(ix)}+\frac{3^{10}h^{10}}{10!}y^{(x)}\\
&-\frac{3^{11}h^{11}}{11!}y^{(xi)}+\frac{3^{12}h^{12}}{12!}y^{(xii)}-\frac{3^{13}h^{13}}{13!}y^{(xiii)}+\frac{3^{14}h^{14}}{14!}y^{(xiv)}\\
&-\frac{3^{15}h^{15}}{15!}y^{(xv)}+\frac{3^{16}h^{1}6}{16!}y^{(xvi)}-\frac{3^{17}h17}{17!}y^{(xvii)}+\frac{3^{18}h^{18}}{18!}y^{(xviii)}\\
&-\frac{3^{19}h^{19}}{19!}y^{(xix)}+\frac{3^{20}h^{20}}{20!}y^{(xx)}-\dots]
\end{aligned}$$

$$
\begin{aligned}
&+120[y-2hy'+\tfrac{2^2h^2}{2!}y''-\tfrac{2^3h^3}{3!}y'''+\tfrac{2^4h^6}{4!}y^{(iv)}-\tfrac{2^5h^5}{5!}y^{(v)}+\tfrac{2^6}{6!}y^{(vi)}-\tfrac{2^7h^7}{7!}y^{(vii)}\\
&+\tfrac{2^8}{8!}y^{(viii)}-\tfrac{2^9h^9}{9!}y^{(ix)}+\tfrac{2^{10ah^{10}}}{10!}y^{(x)}-\tfrac{2^{11}h^{11}}{11!}y^{(xi)}+\tfrac{2^{12}h^{12}}{12!}y^{(xii)}-\tfrac{2^{13}h^{13}}{13!}y^{(xiii)}\\
&+\tfrac{2^{14}h^{14}}{14!}y^{(xiv)}-\tfrac{2^{15}h^{15}}{15!}y^{(xv)}+\tfrac{2^{16}h^{16}}{16!}y^{(xvi)}-\tfrac{2^{17}h^{17}}{17!}y^{(xvii)}+\tfrac{2^{18}h^{18}}{18!}y^{(xviii)}\\
&-\tfrac{2^{19}h^{19}}{19!}y^{(xix)}+\tfrac{2^{20}h^{20}}{20!}y^{(xx)}-\ldots]\\
&-210[y-hy'+\tfrac{h^2}{2!}y''-\tfrac{h^3}{3!}y'''+\tfrac{h^4}{4!}y^{(iv)}-\tfrac{h^5}{5!}y^{(v)}+\tfrac{h^6}{6!}y^{(vi)}-\tfrac{h^7}{7!}y^{(vii)}+\tfrac{h^8}{8!}y^{(viii)}\\
&-\tfrac{h^9}{9!}y^{(ix)}+\tfrac{h^{10}}{10!}y^{(x)}-\tfrac{h^{11}}{11!}y^{(xi)}+\tfrac{h^{12}}{12!}y^{(xii)}-\tfrac{h^{13}}{13!}y^{(xiii)}+\tfrac{h^{14}}{14!}y^{(xiv)}-\tfrac{h^{15}}{15!}y^{(xv)}\\
&+\tfrac{h^{16}}{16!}y^{(xvi)}-\tfrac{h^{17}}{17!}y^{(xvii)}+\tfrac{h^{18}}{18!}y^{(xviii)}-\tfrac{h^{19}}{19!}y^{(xix)}+\tfrac{h^{20}}{20!}y^{(xx)}-\ldots]\\
&+252y(x)\\
&-210[y+hy'+\tfrac{h^2}{2!}y''+\tfrac{h^3}{3!}y'''+\tfrac{h^4}{4!}y^{(iv)}+\tfrac{h^5}{5!}y^{(v)}+\tfrac{h^6}{6!}y^{(vi)}+\tfrac{h^7}{7!}y^{(vii)}\\
&+\tfrac{h^8}{8!}y^{(viii)}+\tfrac{h^9}{9!}y^{(ix)}+\tfrac{h^{10}}{10!}y^{(x)}+\tfrac{h^{11}}{11!}y^{(xi)}+\tfrac{h^{12}}{12!}y^{(xii)}+\tfrac{h^{13}}{13!}y^{(xiii)}+\tfrac{h^{14}}{14!}y^{(xiv)}\\
&+\tfrac{h^{15}}{15!}y^{(xv)}+\tfrac{h^{16}}{16!}y^{(xvi)}+\tfrac{h^{17}}{17!}y^{(xvii)}+\tfrac{h^{18}}{18!}y^{(xviii)}+\tfrac{h^{19}}{19!}y^{(xix)}+\tfrac{h^{20}}{20!}y^{(xx)}-\ldots]\\
&+120[y+2hy'+\tfrac{2^2h^2}{2!}y''+\tfrac{2^3h^3}{3!}y'''+\tfrac{2^4h^6}{4!}y^{(iv)}+\tfrac{2^5h^5}{5!}y^{(v)}+\tfrac{2^6}{6!}y^{(vi)}\\
&+\tfrac{2^7h^7}{7!}y^{(vii)}+\tfrac{2^8}{8!}y^{(viii)}+\tfrac{2^9h^9}{9!}y^{(ix)}+\tfrac{2^{10}h^{10}}{10!}y^{(x)}+\tfrac{2^{11}h^{11}}{11!}y^{(xi)}\\
&+\tfrac{2^{12}h^{12}}{12!}y^{(xii)}+\tfrac{2^{13}h^{13}}{13!}y^{(xiii)}+\tfrac{2^{14}h^{14}}{14!}y^{(xiv)}+\tfrac{2^{15}h^{15}}{15!}y^{(xv)}+\tfrac{2^{16}h^{16}}{16!}y^{(xvi)}\\
&+\tfrac{2^{17}h^{17}}{17!}y^{(xvii)}+\tfrac{2^{18}h^{18}}{18!}y^{(xviii)}+\tfrac{2^{19}h^{19}}{19!}y^{(xix)}+\tfrac{2^{20}h^{20}}{20!}y^{(xx)}-\ldots]\\
&-45[y-3hy'+\tfrac{3^2h^2}{2!}y''-\tfrac{3^3h^3}{3!}y'''+\tfrac{3^4h^4}{4!}y^{(iv)}-\tfrac{3^5h^5}{5!}y^{(v)}+\tfrac{3^6h^6}{6!}y^{(vi)}\\
&-\tfrac{3^7h^7}{7!}y^{(vii)}+\tfrac{3^8h^8}{8!}y^{(viii)}-\tfrac{3^9h^9}{9!}y^{(ix)}+\tfrac{3^{10}h^{10}}{10!}y^{(x)}-\tfrac{3^{11}h^{11}}{11!}y^{(xi)}+\tfrac{3^{12}h^{12}}{12!}y^{(xii)}\\
&-\tfrac{3^{13}h^{13}}{13!}y^{(xiii)}+\tfrac{3^{14}h^{14}}{14!}y^{(xiv)}-\tfrac{3^{15}h^{15}}{15!}y^{(xv)}+\tfrac{3^{16}h^{16}}{16!}y^{(xvi)}-\tfrac{3^{17}h^{17}}{17!}y^{(xvii)}\\
&+\tfrac{3^{18}h^{18}}{18!}y^{(xviii)}-\tfrac{3^{19}h^{19}}{19!}y^{(xix)}+\tfrac{3^{20}h^{20}}{20!}y^{(xx)}-\ldots]\\
&+10[y+4hy'+\tfrac{4^2h^2}{2!}y''+\tfrac{4^3h^3}{3!}y'''+\tfrac{4^4h^4}{4!}y^{(iv)}+\tfrac{4^5h^5}{5!}y^{(v)}+\tfrac{4^6h^6}{6!}y^{(vi)}\\
&+\tfrac{4^7h^7}{7!}y^{(vii)}+\tfrac{4^8h^8}{8!}y^{(viii)}+\tfrac{4^9h^9}{9!}y^{(ix)}+\tfrac{4^{10}h^{10}}{10!}y^{(x)}+\tfrac{4^{11}h^{11}}{11!}y^{(xi)}+\tfrac{4^{12}h^{12}}{12!}y^{(xii)}\\
&+\tfrac{4^{13}h^{13}}{13!}y^{(xiii)}+\tfrac{4^{14}h^{14}}{14!}y^{(xiv)}+\tfrac{4^{15}h^{15}}{15!}y^{(xv)}+\tfrac{4^{16}h^{16}}{16!}y^{(xvi)}-\tfrac{4^{17}h^{17}}{17!}y^{(xvii)}\\
&+\tfrac{4^{18}h^{18}}{18!}y^{(xviii)}+\tfrac{4^{19}h^{19}}{19!}y^{(xix)}+\tfrac{4^{20}h^{20}}{20!}y^{(xx)}-\ldots]\\
&-[y+5hy'+\tfrac{5^2h^2}{2!}y''+\tfrac{5^3h^3}{3!}y'''+\tfrac{5^4h^4}{4!}y^{(iv)}+\tfrac{5^5h^5}{5!}y^{(v)}+\tfrac{5^6h^6}{6!}y^{(vi)}\\
&+\tfrac{5^7h^7}{7!}y^{(vii)}+\tfrac{5^8h^8}{8!}y^{(viii)}+\tfrac{5^9h^9}{9!}y^{(ix)}+\tfrac{5^{10}h^{10}}{10!}y^{(x)}+\tfrac{5^{11}h^{11}}{11!}y^{(xi)}+\tfrac{5^{12}h^{12}}{12!}y^{(xii)}\\
&+\tfrac{5^{13}h^{13}}{13!}y^{(xiii)}+\tfrac{5^{14}h^{14}}{14!}y^{(xiv)}+\tfrac{5^{15}h^{15}}{15!}y^{(xv)}+\tfrac{5^{16}h^{16}}{16!}y^{(xvi)}+\tfrac{5^{17}h^{17}}{17!}y^{(xvii)}\\
&+\tfrac{5^{18}h^{18}}{18!}y^{(xviii)}+\tfrac{5^{19}h^{19}}{19!}y^{(xix)}+\tfrac{5^{20}h^{20}}{20!}y^{(xx)}-\ldots]\ +h^{10}y^{(x)}
\end{aligned}
$$

$$
=-\frac{5}{12}h^{12}y^{(xii)}(x)-\frac{1}{12}h^{14}y^{(xiv)}(x)-\frac{43}{4032}h^{16}y^{(xvi)}(x)-\ \ldots\ldots \qquad (2.8)
$$

The local truncation error $t_n^{(1)}$ at the point $x_n^{(1)}$ is then given by

$$\begin{aligned} t_n^{(1)} = \; & c_{11}h^{11}y^{(xi)}(x_n^{(1)}) + c_{12}h^{12}y^{(xii)}(x_n^{(1)}) + c_{13}h^{13}y^{(xiii)}(x_n^{(1)}) \\ & + c_{14}h^{14}y^{(xiv)}(x_n^{(1)}) + \; \ldots; \end{aligned} \tag{2.9}$$

in (2.9) the $c_{11}, c_{12}, c_{13}, c_{14}, \ldots$ are constants with $c_{11} = c_{13} = c_{15} = c_{17} = c_{19}$ $= c_{21} = \; \ldots \ldots \ldots = 0$ because of symmetry.

Equation (2.3) is applicable only to the N-8 mesh points $x_n^{(1)}$ $(n = 5, 6, 7, 8, 9, 10, \ldots, \; N-6, \; N-5, \; N-4)$ of G_1. In order to be able to implement global extrapolation procedures special formulae are needed for the other mesh points n=1,2,3,4 and $n = N-3, N-2, N-1, N$ which must also have local truncation error with principal part $\frac{-5}{12}h^{12}y^{(xii)}(x)$ in (2.8). These formulae will be assumed to be consistent.

It will be covenient in the convergence analysis on grid G_1 to introduce the matrix J_1 of order N given by i.e.

$$J_1 = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix} \tag{2.10}$$

for which it is known that

$$||J_1^{-1}||_\infty = \frac{(N+1)^2}{8}. \tag{2.11}$$

In order to use the powers of the matrix J_1 , these special end-point formulae will be assumed to be of the forms (2.12)—(2.19), as follows

$$
\begin{aligned}
&132y_1 - 165y_2 + 110y_3 - 44y_4 + 10y_5 - y_6 + a_0y_0 + a_2h^2y_0''\\
&+a_4h^4y_0^{(iv)} + a_6h^6y_0^{(vi)} + a_8h^8y_0^{(viii)}\\
&+h^{10}[\alpha_0f_0 + \alpha_1f_1 + \alpha_2f_2 + \alpha_3f_3 + \alpha_4f_4 + \quad \ldots + \alpha_{12}f_{12}]\\
&= 0,
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
&-165y_1 + 242y_2 - 209y_3 + 120y_4 - 45y_5 + 10y_6 - y_7 + b_0y_0\\
&+b_2h^2y_0'' + b_4h^4y_0^{(iv)} + b_6h^6y_0^{(vi)} + b_8h^8y_0^{(viii)}\\
&+h^{10}[\beta_0f_0 + \beta_1f_1 + \beta_2f_2 + \beta_3f_3 + \beta_4f_4 + \quad \ldots + \beta_{12}f_{12}]\\
&= 0,
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
&110y_1 - 209y_2 + 252y_3 - 210y_4 + 120y_5 - 45y_6 + 10y_7 - y_8\\
&+c_0y_0 + c_2h^2y_0'' + c_4h^4y_0^{(iv)} + c_6h^6y_0^{(vi)} + c_8h^8y_0^{(viii)}\\
&+h^{10}[\gamma_0f_0 + \gamma_1f_1 + \gamma_2f_2 + \gamma_3f_3 + \gamma_4f_4 + \quad \ldots + \gamma_{12}f_{12}]\\
&= 0,
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
&-44y_1 + 120y_2 - 210y_3 + 252y_4 - 210y_5 + 120y_6 - 45y_7 + 10y_8\\
&-y_9 + d_0y_0 + d_2h^2y_0'' + d_4h^4y_0^{(iv)} + d_6h^6y_0^{(vi)} + d_8h^8y_0^{(viii)}\\
&+h^{10}[\delta_0f_0 + \delta_1f_1 + \delta_2f_2 + \delta_3f_3 + \delta_4f_4 + \quad \ldots + \delta_{12}f_{12}]\\
&= 0.
\end{aligned} \tag{2.15}
$$

At the other end of the array, the special end-point formulae are as follows

$$
\begin{aligned}
&-y_{N-8} + 10y_{N-7} - 45y_{N-6} + 120y_{N-5} - 210y_{N-4} + 252y_{N-3}\\
&-210y_{N-2} + 120y_{N-1} - 44y_N + d_0y_{N+1} + d_2h^2y_{N+1}^{(ii)} + d_4h^4y_{N+1}^{(iv)}\\
&+d_6h^6y_{N+1}^{(vi)} + d_8h^8y_{N+1}^{(viii)}\\
&+h^{10}[\delta_0f_{N+1} + \delta_1f_N + \delta_2f_{N-1} + \delta_3f_{N-2} + \delta_4f_{N-3} + \quad \ldots + \delta_{12}f_{N-11}]\\
&= 0,
\end{aligned} \tag{2.16}
$$

$$\begin{aligned}
&-y_{N-7}+10y_{N-6}-45y_{N-5}+120y_{N-4}-210y_{N-3}+252y_{N-2}\\
&-209y_{N-1}+110y_N+c_0y_{N+1}+c_2h^2y^{(ii)}_{N+1}+c_4h^4y^{(iv)}_{N+1}\\
&+c_6h^6y^{(vi)}_{N+1}+c_8h^8y^{(viii)}_{N+1}\\
&+h^{10}[\gamma_0f_{N+1}+\gamma_1f_N+\gamma_2f_{N-1}+\gamma_3f_{N-2}+\gamma_4f_{N-3}+\ \ldots+\gamma_{12}f_{N-11}]\\
&=0,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
&-y_{N-6}+10y_{N-5}-45y_{N-4}+120y_{N-3}-209y_{N-2}+242y_{N-1}\\
&-165y_N+b_0y_{N+1}+b_2h^2y^{(ii)}_{N+1}+b_4h^4y^{(iv)}_{N+1}+b_6h^6y^{(vi)}_{N+1}\\
&+b_8h^8y^{(viii)}_{N+1}\\
&+h^{10}[\beta_0f_{N+1}+\beta_1f_N+\beta_2f_{N-1}+\beta_3f_{N-2}+\beta_4f_{N-3}+\ \ldots+\beta_{12}f_{N-11}]\\
&=0,
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
&-y_{N-5}+10y_{N-4}-44y_{N-3}+110y_{N-2}-165y_{N-1}+132y_N\\
&+a_0y_{N+1}+a_2h^2y^{(ii)}_{N+1}+a_4h^4y^{(iv)}_{N+1}+a_6h^6y^{(vi)}_{N+1}+a_8h^8y^{(viii)}_{N+1}\\
&+h^{10}[\alpha_0f_{N+1}+\alpha_1f_N+\alpha_2f_{N-1}+\alpha_3f_{N-2}+\alpha_4f_{N-3}+\ \ldots+\alpha_{12}f_{N-11}]\\
&=0.
\end{aligned} \tag{2.19}$$

The $a_i,\ b_i,\ c_i,\ d_i\ (i=0,2,4,6,8)$ and $\alpha_i,\ \beta_i, \gamma_i,\ \delta_i\ (i=0,1,2,3,\ldots,12)$ are parameters which must be chosen so that the local truncation errors of (2.12)—(2.19) are identical with the (2.9) to the order required in sections 2.4, 2.5. Note: n=5 and n=N − 4 do not need special formulae, though these do use boundary values.

Clearly, the family of numerical methods is described by the set of equations $\{(2.12),(2.13),(2.14),(2.15),(2.16),(2.17),(2.18),(2.19)\}$ and the solution vector $\mathbf{Y}^{(1)}=[y^{(1)}_1,y^{(2)}_2,y^{(3)}_3,y^{(4)}_4,\ldots,\ y^{(1)}_N]^T$, T denoting transpose, is obtained by solving a non-linear algebraic system of order N which has the form

$$J^5_1\mathbf{Y}^{(1)}+h^{10}M_1\mathbf{f}^{(1)}(x,\mathbf{Y}^{(1)})-\mathbf{b}^{(1)}=\mathbf{0}^{(1)}, \tag{2.20}$$

the vector $\mathbf{f}^{(1)}$ of order N has the form

$$\mathbf{f}^{(1)} = [f_1^{(1)}, f_2^{(2)}, f_3^{(3)}, f_4^{(4)}, f_5^{(5)}, \ldots, f_N^{(1)}]^T,$$

the constant vector $\mathbf{b}^{(1)}$ is of order N and is given by

$$\mathbf{b}^{(1)} = \begin{bmatrix} a_0A_0 + a_2h^2A_2 + a_4h^4A_4 + a_6h^6A_6 + a_8h^8A_8 + a_{10}h^{10}y_0^{(x)} \\ b_0A_0 + b_2h^2A_2 + b_4h^4A_4 + b_6h^6A_6 + b_8h^8A_8 + b_{10}h^{10}y_0^{(x)} \\ c_0A_0 + c_2h^2A_2 + c_4h^4A_4 + c_6h^6A_6 + c_8h^8A_8 + c_{10}h^{10}y_0^{(x)} \\ d_0A_0 + d_2h^2A_2 + d_4h^4A_4 + d_6h^6A_6 + d_8h^8A_8 + d_{10}h^{10}y_0^{(x)} \\ -A_0 + h^{10}\alpha_0 y_0^{(x)} \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ 0 \\ -B_0 + h^{10}\alpha_0 y_{N+1}^{(x)} \\ d_0B_0 + d_2h^2B_2 + d_4h^4B_4 + d_6h^6B_6 + d_8h^8B_8 + d_{10}h^{10}y_{N+1}^{(x)} \\ c_0B_0 + c_2h^2B_2 + c_4h^4B_4 + c_6h^6B_6 + c_8h^8B_8 + c_{10}h^{10}y_{N+1}^{(x)} \\ b_0B_0 + b_2h^2B_2 + b_4h^4B_4 + b_6h^6B_6 + b_8h^8B_8 + b_{10}h^{10}y_{N+1}^{(x)} \\ a_0B_0 + a_2h^2B_2 + a_4h^4B_4 + a_6h^6B_6 + a_8h^8B_8 + a_{10}h^{10}y_{N+1}^{(x)} \end{bmatrix} \quad (2.21)$$

and $\mathbf{0}^{(1)}$ is the zerocolumn vector of order N. The matrix M_1 in (2.20), of order

N, is given by

$$
M_1 = \begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \alpha_9 & \alpha_{10} & \alpha_{11} & \alpha_{12} & & & \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 & \beta_{10} & \beta_{11} & \beta_{12} & & & \\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & & & \\
\delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 & \delta_7 & \delta_8 & \delta_9 & \delta_{10} & \delta_{11} & \delta_{12} & & & \\
\beta & \gamma & \delta & \epsilon & \sum & \epsilon & \delta & \gamma & \beta & \alpha & & & & & \\
\alpha & \beta & \gamma & \delta & \epsilon & \sum & \epsilon & \delta & \gamma & \beta & \alpha & & & & \\
 & \alpha & \beta & \gamma & \delta & \epsilon & \sum & \epsilon & \delta & \gamma & \beta & \alpha & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & \alpha & \beta & \gamma & \delta & \epsilon & \sum & \epsilon & \delta & \gamma & \beta & \alpha & \\
 & & & & \alpha & \beta & \gamma & \delta & \epsilon & \sum & \epsilon & \delta & \gamma & \beta & \alpha \\
 & & & & & \alpha & \beta & \gamma & \delta & \epsilon & \sum & \epsilon & \delta & \gamma & \beta \\
 & & & \delta_{12} & \delta_{11} & \delta_{10} & \delta_9 & \delta_8 & \delta_7 & \delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\
 & & & \gamma_{12} & \gamma_{11} & \gamma_{10} & \gamma_9 & \gamma_8 & \gamma_7 & \gamma_6 & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\
 & & & \beta_{12} & \beta_{11} & \beta_{10} & \beta_9 & \beta_8 & \beta_7 & \beta_6 & \beta_5 & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\
 & & & \alpha_{12} & \alpha_{11} & \alpha_{10} & \alpha_9 & \alpha_8 & \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1
\end{bmatrix} \tag{2.22}
$$

The exact solution vector $\mathbf{y}^{(1)} = [y(x_1^{(1)}), y(x_2^{(1)}), y(x_3^{(1)}), \ldots, y(x_N^{(1)})]^T$ satisfies

$$J_1^5 \mathbf{y}^{(1)} + h^{10} M_1 \, \mathbf{f}^{(1)}(x, \mathbf{y}^{(1)}) - \mathbf{b}^{(1)} - \mathbf{t}^{(1)} = \mathbf{0}^{(1)}, \tag{2.23}$$

where, $\mathbf{t}^{(1)} = [t_1^{(1)}, t_2^{(1)}, t_3^{(1)}, \ldots, t_N^{(1)}]^T$ is the vector of local truncation errors .

2.3 CONVERGENCE ANALYSIS OF THE SECOND-ORDER METHOD

For the convergence analysis we must obtain a bound on $||\mathbf{z}^{(1)}||_\infty$, where $\mathbf{z}^{(1)} = \mathbf{y}^{(1)} - \mathbf{Y}^{(1)}$. For this purpose, the following lemma is used. (It will be assumed throughout the thesis that the norm used is the infinity norm.)

Lemma 2.1 If A is a square matrix of order N and $||A|| < 1$, then $(I - A)^{-1}$ exists , where I is the identity matrix of order N and

$$||(I - A)^{-1}|| < \frac{1}{(1 - ||A||)}. \tag{2.24}$$

Equations (2.20) and (2.23) give

$$[J_1^5 - h^{10} M_1 \; F_1] \; \mathbf{z}^{(1)} - \mathbf{t}^{(1)} = \mathbf{0}^{(1)} \tag{2.25}$$

where, $F_1 = \text{diag}\left(\frac{\partial f_n^{(1)}}{\partial y_n^{(1)}}\right)$ and for which Lemma 2.1 can be applied to obtain

$$||\mathbf{z}^{(1)}|| \leq \frac{(b-a)^{10}}{32768-(b-a)^{10}M^*F^*}[|c_{12}|h^2 V_{12} + |c_{14}|h^4 V_{14} + |c_{16}|h^6 V_{16} + \ldots], \tag{2.26}$$

where

$$V_i = \max_{a \leq x \leq b} |\frac{d^i y(x)}{dx^i}|$$

and

$$F^* = \max_{a \leq x \leq b} |\frac{\partial f}{\partial y(x)}|,$$

provided

$$F^* < \frac{32768}{(b-a)^{10} M^*}$$

and the parameters in (2.12)—(2.19) are chosen to ensure that $c_{11} = c_{13} = 0$. The order of convergence of the numerical method is p if c_{p+10} is the first non-vanishing constant on the right-hand side of (2.9).

2.4 GLOBAL EXTRAPOLATION ON TWO GRIDS

Suppose, now, that the interval $a \leq x \leq b$ is subdivided into 2N+2 subintervals each of width $\frac{1}{2}h$ giving a finer grid G_2 of interior points called $x_1^{(2)}, x_2^{(2)}$, $x_3^{(2)}, x_4^{(2)}, \ldots, x_{2N+1}^{(2)}$. Clearly, the points $x_{2i}^{(1)}$ of the fine grid G_2 coincide with the points $x_i^{(1)}$ of the coarse grid G_1 $(i = 1, 2, 3, 4 \ldots, N)$.

The finite difference formulae (2.12), (2.13), (2.14), (2.15), (2.3), (2.16), (2.17), (2.18), (2.19) are modified for use on G_2 by replacing h with $\frac{h}{2}$. They may be written in matrix-vector form as

$$J_2^5 \mathbf{Y}^{(2)} - \left(\frac{h}{2}\right)^{10} M_2\, \mathbf{f}^{(2)}(x, \mathbf{Y}^{(2)}) - \mathbf{b}^{(2)} = \mathbf{0}^{(2)}, \tag{2.27}$$

in which J_2 and M_2 are matrices of order 2N+1 which may be written down immadiately from (2.10) and (2.22). All vectors in (2.27) have 2N+1 elements; $\mathbf{b}^{(2)}$ is obtained from $\mathbf{b}^{(1)}$ by replacing h with $\frac{h}{2}$, $\mathbf{Y}^{(2)}$ and $\mathbf{f}^{(2)}$ follow in an obvious way from $\mathbf{Y}^{(1)}$ and $\mathbf{f}^{(1)}$, as do $\mathbf{y}^{(2)}$ from $\mathbf{y}^{(1)}$ and $\mathbf{z}^{(2)}$ from $\mathbf{z}^{(1)}$.

In the convergence analysis on G_2, $||\mathbf{z}^{(2)}||$ satisfies

$$\begin{aligned} ||\mathbf{z}^{(2)}|| &\leq \tfrac{(b-a)^{10}}{32768-(b-a)^{10}M^*F^*} \\ &\left[|c_{12}|(\tfrac{h}{2})^2 V_{12} + |c_{14}|(\tfrac{h}{2})^4 V_{14} + |c_{16}|(\tfrac{h}{2})^6 V_{16} + \cdots\right] \end{aligned} \tag{2.28}$$

(it should be noted that $M^* = ||M_2|| = ||M_1||$) .

Introduce, now, an extrapolation vector $\mathbf{z}^{(E)}$ of order N defined by

$$\mathbf{z}^{(E)} = q I^h_{\frac{1}{2}h} \mathbf{z}^{(2)} + (1-q)\mathbf{z}^{(1)}$$

where $I^h_{\frac{1}{2}h}$ is a fine-to-coarse grid restriction operator with

$$I^h_{\frac{1}{2}h} \mathbf{z}^{(2)} = \left[z_2^{(2)}, z_4^{(2)}, z_6^{(2)}, z_8^{(2)}, \ldots, z_{2N}^{(2)}\right]^T$$

and

$$I^h_{\frac{1}{2}h} \mathbf{Y}^{(2)} = \left[y_2^{(2)}, y_4^{(2)}, y_6^{(2)}, y_8^{(2)}, \ldots, y_{2N}^{(2)}\right]^T.$$

It is easy to see from (2.26) and (2.28) that

$$q||\mathbf{z}^{(2)}|| + (1-q)||\mathbf{z}^{(1)}|| = O(h^4)$$

provided

$$q = \frac{2^p}{(2^p - 1)} \quad , \tag{2.29}$$

where p is the order of convergence of the numerical method (clearly p=2 for the method given in (2.6)).

Now from the inequalities

$$|\ ||\mathbf{a}|| - ||\mathbf{b}||\ | \ \leq ||\mathbf{a} - \mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||$$

and using

$$||I^h_{\frac{1}{2}h}|| = 1,$$

it follows that

$$q||\mathbf{z}^{(2)}|| + (1-q)||\mathbf{z}^{(1)}|| \leq ||\mathbf{z}^{(E)}|| \leq |q|||\mathbf{z}^{(2)}|| + |1-q|||\mathbf{z}^{(1)}||$$

(Boutayeb,1990) and that one of the two possibilities must hold

$$||\mathbf{z}^{(E)}|| \leq c_1 h^{p+2} V_{p+4} + O(h^{p+4})$$

or

$$c_1 h^{p+2} V_{p+4} + O(h^{p+4}) \leq ||\mathbf{z}^{(E)}|| \leq c_2 h^p V_{p+2} + O(h^{p+2})$$

provided q takes the value given by (2.29). The order of convergence of the global extrapolation vector

$$\mathbf{Y}^{(E)} = q I^h_{\frac{1}{2}h} \mathbf{Y}^{(2)} + (1-q)\mathbf{Y}^{(1)} \tag{2.30}$$

can be either four or between two and four in the case of the method in (2.6).

2.5 GLOBAL EXTRAPOLATION ON THREE GRIDS

Consider, next, a third grid G_3 of step size $\frac{1}{3}h$. The interval $a \leq x \leq b$ is thus divided into 3N+3 subintervals and the interior points of G_3 are named

$x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, x_4^{(3)}, \dots, x_{3N+2}^{(3)}$. Clearly, the points $x_{3i}^{(1)}$ of the third grid G_3 coincide with the points $x_i^{(1)}$ of the original grid $G_1 (i = 1, 2, 3, 4 \dots, N)$. The solution vector

$$\mathbf{Y}^{(3)} = [y_1^{(3)}, y_2^{(3)}, y_3^{(3)}, \dots, y_{3N+2}^{(3)}]^T,$$

on G_3 is obtained from the non-linear algebraic system

$$J_3^5 \mathbf{Y}^{(3)} - \left(\frac{h}{3}\right)^{10} M_3 \, \mathbf{f}^{(3)}(x, \mathbf{Y}^{(3)}) - \mathbf{b}^{(3)} = \mathbf{0}^{(3)}, \tag{2.31}$$

in which J_3, M_3, $\mathbf{f}^{(3)}$ and $\mathbf{b}^{(3)}$ are obtained in an obvious way as section 2.4.

In the convergence analysis on G_3, $\mathbf{z}^{(3)}$ satisfies

$$\begin{aligned} &||\mathbf{z}^{(3)}|| \\ &\leq \frac{(b-a)^{10}}{32768-(b-a)^{10}M^*F^*} \left[|c_{12}|(\tfrac{h}{3})^2 V_{12} + |c_{14}|(\tfrac{h}{3})^4 V_{14} + |c_{16}|(\tfrac{h}{3})^6 V_{16} + \dots\right] \end{aligned} \tag{2.32}$$

(it should be noted that $M^* = ||M_3||$).

The extrapolation formula

$$\mathbf{z}^{(E)} = r I^h_{\frac{1}{3}h} \mathbf{z}^{(3)} + s I^h_{\frac{1}{2}h} \mathbf{z}^{(2)} + (1 - r - s) \mathbf{z}^{(1)},$$

in which the fine-to-coarse grid restriction operator $I^h_{\frac{1}{3}h}$ is such that

$$I^h_{\frac{1}{3}h} \mathbf{z}^{(3)} = \left[\mathbf{z}_3^{(3)}, \mathbf{z}_6^{(3)}, \mathbf{z}_9^{(3)}, \mathbf{z}_{12}^{(3)}, \dots, \mathbf{z}_{3N}^{(3)}\right]^T$$

and

$$I^h_{\frac{1}{3}h} \mathbf{Y}^{(3)} = \left[y_3^{(3)}, y_6^{(3)}, y_9^{(3)}, y_{12}^{(3)}, \dots, y_{3N}^{(3)}\right]^T,$$

satisfies

$$||\mathbf{z}^{(E)}|| \leq |r| ||\mathbf{z}^{(3)}|| + |s| ||\mathbf{z}^{(2)}|| + |1 - r - s| ||\mathbf{z}^{(1)}||.$$

From (2.26), (2.28) and (2.32) it can be shown that

$$r ||\mathbf{z}^{(3)}|| + s ||\mathbf{z}^{(2)}|| + (1 - r - s) ||\mathbf{z}^{(1)}|| = O(h^{p+4}).$$

provided

$$r = \frac{3^{p+3}}{(5 + 3^{p+3} - 2^{p+5})} \quad \text{and} \quad s = \frac{-2^{p+5}}{(5 + 3^{p+3} - 2^{p+5})} \tag{2.33}$$

and, thus,

$$1 - \mathrm{r} - \mathrm{s} = \frac{5}{(5 + 3^{\mathrm{p}+3} - 2^{\mathrm{p}+5})};$$

clearly p=2 for the method in (2.6). However, in contrast to the global extrapolation on two grids, we are unable to prove that $||\mathbf{z}^{(\mathrm{E})}||$ is at most $O(\mathrm{h}^{\mathrm{p}+4})$ although the numerical results reported by Boutayeb (1990) for sixth- and eight-order boundary-value problems show that the global extrapolation algorithm

$$\mathbf{Y}^{(\mathrm{E})} = \mathrm{r}\mathrm{I}^{\mathrm{h}}_{\frac{1}{3}\mathrm{h}}\mathbf{Y}^{(3)} + \mathrm{s}\mathrm{I}^{\mathrm{h}}_{\frac{1}{2}\mathrm{h}}\mathbf{Y}^{(2)} + (1 - \mathrm{r} - \mathrm{s})\mathbf{Y}^{(1)} \tag{2.34}$$

is likely to be of order p+4, where p is the order of convergence of the numerical method, provided r and s take the values indicated by (2.33).

2.6 CONSTRUCTION OF A SECOND-ORDER METHOD

Writing $\alpha = \beta = \gamma = \delta = \epsilon = 0$ in (2.3) gives as has already been seen,

$$c_{12} = \frac{-5}{12}, c_{14} = \frac{-1}{12} \tag{2.35}$$

in (2.9), thus verifying that (2.3) is a second-order method. To be able to implement global extrapolation on two and three grids the parameters in the special end-point formulae (2.12)—(2.19) must be chosen so that $\mathrm{c}_{11} = \mathrm{c}_{13} = 0$ in (2.9) and so that c_{12} and c_{14} in (2.9), with $\mathrm{n} = 1, 2, 3, 4, \mathrm{N}-3, \mathrm{N}-2, \mathrm{N}-1$, or N agree with (2.35).

Using the method of undetermined coefficients reveals that, for the point $x = x_1$ this is achieved provided

$$\mathrm{a}_0 = -42, \mathrm{a}_2 = 14, \mathrm{a}_4 = \frac{-23}{6}, \mathrm{a}_6 = \frac{217}{180}, \mathrm{a}_8 = \frac{-809}{1440}, \tag{2.36}$$

together with parameters $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{12}$ calculated from the local truncation error of (2.12) which is

$$\left.\begin{aligned} L[y(x_1);h] = \; & 132y(x) - 165y(x+h) + 110y(x+2h) - 44y(x+3h) \\ & +10y(x+4h) - y(x+5h) - 42y(x-h) + 14h^2y''(x-h) \\ & -\tfrac{23}{6}h^4y^{(iv)}(x-h) + \tfrac{217}{180}h^6y^{(vi)}(x-h) \\ & -\tfrac{809}{1440}h^8y^{(viii)}(x-h) + h^{10}[\alpha_0 y^{(x)}(x-h) \\ & +\alpha_1 y^{(x)}(x) + \alpha_2 y^{(x)}(x+h) + \alpha_3 y^{(x)}(x+2h) \\ & +\alpha_4 y^{(x)}(x+3h) + \alpha_5 y^{(x)}(x+4h) + \alpha_6 y^{(x)}(x+5h) \\ & +\alpha_7 y^{(x)}(x+6h) + \alpha_8 y^{(x)}(x+7h) + \alpha_9 y^{(x)}(x+8h) \\ & +\alpha_{10} y^{(x)}(x+9h) + \alpha_{11} y^{(x)}(x+10h) + \alpha_{12} y^{(x)}(x+11h) + \ldots]. \end{aligned}\right\} \quad (2.37)$$

Expanding the function terms and their derivatives in (2.37) by the Taylor expansion gives, at the point $x = x_1$,

$$\begin{aligned} L[y(x_1);h] = \; & 132y(x) \\ & -165[y + hy' + \tfrac{h^2}{2!}y'' + \tfrac{h^3}{3!}y''' + \tfrac{h^4}{4!}y^{(iv)} + \tfrac{h^5}{5!}y^{(v)} + \tfrac{h^6}{6!}y^{(vi)} \\ & +\tfrac{h^7}{7!}y^{(vii)} + \tfrac{h^8}{8!}y^{(viii)} + \tfrac{h^9}{9!}y^{(ix)} + \tfrac{h^{10}}{10!}y^{(x)} + \tfrac{h^{11}}{11!}y^{(xi)} \\ & +\tfrac{h^{12}}{12!}y^{(xii)} + \tfrac{h^{13}}{13!}y^{(xiii)} + \tfrac{h^{14}}{14!}y^{(xiv)} + \tfrac{h^{15}}{15!}y^{(xv)} + \tfrac{h^{16}}{16!}y^{(xvi)} \\ & +\tfrac{h^{17}}{17!}y^{(xvii)} + \tfrac{h^{18}}{18!}y^{(xviii)} + \tfrac{h^{19}}{19!}y^{(xix)} + \tfrac{h^{20}}{20!}y^{(xx)} + \ldots] \\ & +110[y + 2hy' + \tfrac{2^2h^2}{2!}y'' + \tfrac{2^3h^3}{3!}y''' + \tfrac{2^4h^6}{4!}y^{(iv)} + \tfrac{2^5h^5}{5!}y^{(v)} \\ & +\tfrac{2^6}{6!}y^{(vi)} + \tfrac{2^7h^7}{7!}y^{(vii)} + \tfrac{2^8}{8!}y^{(viii)} + \tfrac{2^9h^9}{9!}y^{(ix)} + \tfrac{2^{10ah^{10}}}{10!}y^{(x)} \\ & +\tfrac{2^{11}h^{11}}{11!}y^{(xi)} + \tfrac{2^{12}h^{12}}{12!}y^{(xii)} + \tfrac{2^{13}h^{13}}{13!}y^{(xiii)} + \tfrac{2^{14}h^{14}}{14!}y^{(xiv)} \\ & +\tfrac{2^{15}h^{15}}{15!}y^{(xv)} + \tfrac{2^{16}h^{16}}{16!}y^{(xvi)} + \tfrac{2^{17}h^{17}}{17!}y^{(xvii)} + \tfrac{2^{18}h^{18}}{18!}y^{(xviii)} \\ & +\tfrac{2^{19}h^{19}}{19!}y^{(xix)} + \tfrac{2^{20}h^{20}}{20!}y^{(xx)} + \ldots] \\ & -44[y + 3hy' + \tfrac{3^2h^2}{2!}y'' + \tfrac{3^3h^3}{3!}y''' + \tfrac{3^4h^4}{4!}y^{(iv)} + \tfrac{3^5h^5}{5!}y^{(v)} \\ & +\tfrac{3^6h^6}{6!}y^{(vi)} + \tfrac{3^7h^7}{7!}y^{(vii)} + \tfrac{3^8h^8}{8!}y^{(viii)} + \tfrac{3^9h^9}{9!}y^{(ix)} + \tfrac{3^{10}h^{10}}{10!}y^{(x)} \\ & +\tfrac{3^{11}h^{11}}{11!}y^{(xi)} + \tfrac{3^{12}h^{12}}{12!}y^{(xii)} + \tfrac{3^{13}h^{13}}{13!}y^{(xiii)} + \tfrac{3^{14}h^{14}}{14!}y^{(xiv)} \\ & +\tfrac{3^{15}h^{15}}{15!}y^{(xv)} + \tfrac{3^{16}h^16}{16!}y^{(xvi)} + \tfrac{3^{17}h17}{17!}y^{(xvii)} + \tfrac{3^{18}h^{18}}{18!}y^{(xviii)} \\ & +\tfrac{3^{19}h^{19}}{19!}y^{(xix)} + \tfrac{3^{20}h^{20}}{20!}y^{(xx)} + \ldots] \end{aligned}$$

$$
\begin{aligned}
&+10[y+4hy'+\tfrac{4^2h^2}{2!}y''+\tfrac{4^3h^3}{3!}y'''+\tfrac{4^4h^4}{4!}y^{(iv)}+\tfrac{4^5h^5}{5!}y^{(v)}+\tfrac{4^6h^6}{6!}y^{(vi)}\\
&+\tfrac{4^7h^7}{7!}y^{(vii)}+\tfrac{4^8h^8}{8!}y^{(viii)}+\tfrac{4^9h^9}{9!}y^{(ix)}+\tfrac{4^{10}h^{10}}{10!}y^{(x)}+\tfrac{4^{11}h^{11}}{11!}y^{(xi)}\\
&+\tfrac{4^{12}h^{12}}{12!}y^{(xii)}+\tfrac{4^{13}h^{13}}{13!}y^{(xiii)}+\tfrac{4^{14}h^{14}}{14!}y^{(xiv)}+\tfrac{4^{15}h^{15}}{15!}y^{(xv)}+\tfrac{4^{16}h^{16}}{16!}y^{(xvi)}\\
&-\tfrac{4^{17}h^{17}}{17!}y^{(xvii)}]+\tfrac{4^{18}h^{18}}{18!}y^{(xviii)}+\tfrac{4^{19}h^{19}}{19!}y^{(xix)}+\tfrac{4^{20}h^{20}}{20!}y^{(xx)}+\ldots]\\
&-[y+5hy'+\tfrac{5^2h^2}{2!}y''+\tfrac{5^3h^3}{3!}y'''+\tfrac{5^4h^4}{4!}y^{(iv)}+\tfrac{5^5h^5}{5!}y^{(v)}+\tfrac{5^6h^6}{6!}y^{(vi)}\\
&+\tfrac{5^7h^7}{7!}y^{(vii)}+\tfrac{5^8h^8}{8!}y^{(viii)}+\tfrac{5^9h^9}{9!}y^{(ix)}+\tfrac{5^{10}h^{10}}{10!}y^{(x)}+\tfrac{5^{11}h^{11}}{11!}y^{(xi)}\\
&+\tfrac{5^{12}h^{12}}{12!}y^{(xii)}+\tfrac{5^{13}h^{13}}{13!}y^{(xiii)}+\tfrac{5^{14}h^{14}}{14!}y^{(xiv)}+\tfrac{5^{15}h^{15}}{15!}y^{(xv)}+\tfrac{5^{16}h^{16}}{16!}y^{(xvi)}\\
&+\tfrac{5^{17}h^{17}}{17!}y^{(xvii)}+\tfrac{5^{18}h^{18}}{18!}y^{(xviii)}+\tfrac{5^{19}h^{19}}{19!}y^{(xix)}+\tfrac{5^{20}h^{20}}{20!}y^{(xx)}+\ldots]\\
&-42[y-hy'+\tfrac{h^2}{2!}y''-\tfrac{h^3}{3!}y'''+\tfrac{h^4}{4!}y^{(iv)}-\tfrac{h^5}{5!}y^{(v)}+\tfrac{h^6}{6!}y^{(vi)}\\
&-\tfrac{h^7}{7!}y^{(vii)}+\tfrac{h^8}{8!}y^{(viii)}-\tfrac{h^9}{9!}y^{(ix)}+\tfrac{h^{10}}{10!}y^{(x)}-\tfrac{h^{11}}{11!}y^{(xi)}+\tfrac{h^{12}}{12!}y^{(xii)}\\
&-\tfrac{h^{13}}{13!}y^{(xiii)}+\tfrac{h^{14}}{14!}y^{(xiv)}-\tfrac{h^{15}}{15!}y^{(xv)}+\tfrac{h^{16}}{16!}y^{(xvi)}-\tfrac{h^{17}}{17!}y^{(xvii)}+\tfrac{h^{18}}{18!}y^{(xviii)}\\
&-\tfrac{h^{19}}{19!}y^{(xvii)}+\tfrac{h^{20}}{20!}y^{(xx)}+\ldots]\\
&+14h^2[y''-hy'''+\tfrac{h^2}{2!}y^{iv}-\tfrac{h^3}{3!}y^{(v)}+\tfrac{h^4}{4!}y^{(vi)}-\tfrac{h^5}{5!}y^{(vii)}+\tfrac{h^6}{6!}y^{(viii)}\\
&-\tfrac{h^7}{7!}y^{(ix)}+\tfrac{h^8}{8!}y^{(x)}-\tfrac{h^9}{9!}y^{(xi)}+\tfrac{h^{10}}{10!}y^{(xii)}-\tfrac{h^{11}}{11!}y^{(xiii)}+\tfrac{h^{12}}{12!}y^{(xiv)}-\tfrac{h^{13}}{13!}y^{(xv)}\\
&+\tfrac{h^{14}}{14!}y^{(xvi)}-\tfrac{h^{15}}{15!}y^{(xvii)}+\tfrac{h^{16}}{16!}y^{(xviii)}-\tfrac{h^{17}}{17!}y^{(xix)}+\tfrac{h^{18}}{18!}y^{(xx)}+\ldots]\\
&-\tfrac{23}{6}h^4[y^{(iv)}-hy^{(v)}+\tfrac{h^2}{2!}y^{(vi)}-\tfrac{h^3}{3!}y^{(vii)}+\tfrac{h^4}{4!}y^{(viii)}-\tfrac{h^5}{5!}y^{(ix)}+\tfrac{h^6}{6!}y^{(x)}\\
&-\tfrac{h^7}{7!}y^{(xi)}+\tfrac{h^8}{8!}y^{(xii)}-\tfrac{h^9}{9!}y^{(xiii)}+\tfrac{h^{10}}{10!}y^{(xiv)}-\tfrac{h^{11}}{11!}y^{(xv)}+\tfrac{h^{12}}{12!}y^{(xvi)}\\
&-\tfrac{h^{13}}{13!}y^{(xvii)}+\tfrac{h^{14}}{14!}y^{(xviii)}-\tfrac{h^{15}}{15!}y^{(xix)}+\tfrac{h^{16}}{16!}y^{(xx)}+\ldots]\\
&+\tfrac{217}{180}h^6[y^{(vi)}-hy^{(vii)}+\tfrac{h^2}{2!}y^{(viii)}-\tfrac{h^3}{3!}y^{(ix)}+\tfrac{h^4}{4!}y^{(x)}-\tfrac{h^5}{5!}y^{(xi)}+\tfrac{h^6}{6!}y^{(xii)}\\
&-\tfrac{h^7}{7!}y^{(xiii)}+\tfrac{h^8}{8!}y^{(xiv)}-\tfrac{h^9}{9!}y^{(xv)}+\tfrac{h^{10}}{10!}y^{(xvi)}-\tfrac{h^{11}}{11!}y^{(xvii)}+\tfrac{h^{12}}{12!}y^{(xviii)}\\
&-\tfrac{h^{13}}{13!}y^{(xix)}+\tfrac{h^{14}}{14!}y^{(xx)}+\ldots]\\
&-\tfrac{809}{1440}h^8[y^{(viii)}-hy^{(ix)}+\tfrac{h^2}{2!}y^{(x)}-\tfrac{h^3}{3!}y^{(xi)}+\tfrac{h^4}{4!}y^{(xii)}-\tfrac{h^5}{5!}y^{(xiii)}+\tfrac{h^6}{6!}y^{(xiv)}\\
&-\tfrac{h^7}{7!}y^{(xv)}+\tfrac{h^8}{8!}y^{(xvi)}-\tfrac{h^9}{9!}y^{(xvii)}+\tfrac{h^{10}}{10!}y^{(xviii)}-\tfrac{h^{11}}{11!}y^{(xix)}+\tfrac{h^{12}}{12!}y^{(xx)}+\ldots]\\
&+h^{10}\alpha_0[y-hy'+\tfrac{h^2}{2!}y''-\tfrac{h^3}{3!}y'''+\tfrac{h^4}{4!}y^{(iv)}-\tfrac{h^5}{5!}y^{(v)}+\tfrac{h^6}{6!}y^{(vi)}-\tfrac{h^7}{7!}y^{(vii)}\\
&+\tfrac{h^8}{8!}y^{(viii)}-\tfrac{h^9}{9!}y^{(ix)}+\tfrac{h^{10}}{10!}y^{(x)}-\tfrac{h^{11}}{11!}y^{(xi)}+\tfrac{h^{12}}{12!}y^{(xii)}-\tfrac{h^{13}}{13!}y^{(xiii)}+\tfrac{h^{14}}{14!}y^{(xiv)}\\
&-\tfrac{h^{15}}{15!}y^{(xv)}+\tfrac{h^{16}}{16!}y^{(xvi)}-\tfrac{h^{17}}{17!}y^{(xvii)}+\tfrac{h^{18}}{18!}y^{(xviii)}-\tfrac{h^{19}}{19!}y^{(xix)}+\tfrac{h^{20}}{20!}y^{(xx)}+\ldots]\\
&+h^{10}\alpha_1y^{(x)}\\
&+h^{10}\alpha_2[y+hy'+\tfrac{h^2}{2!}y''+\tfrac{h^3}{3!}y'''+\tfrac{h^4}{4!}y^{(iv)}+\tfrac{h^5}{5!}y^{(v)}+\tfrac{h^6}{6!}y^{(vi)}+\tfrac{h^7}{7!}y^{(vii)}
\end{aligned}
$$

$$+\frac{h^8}{8!}y^{(viii)}+\frac{h^9}{9!}y^{(ix)}+\frac{h^{10}}{10!}y^{(x)}+\frac{h^{11}}{11!}y^{(xi)}+\frac{h^{12}}{12!}y^{(xii)}+\frac{h^{13}}{13!}y^{(xiii)}+\frac{h^{14}}{14!}y^{(xiv)}$$
$$+\frac{h^{15}}{15!}y^{(xv)}+\frac{h^{16}}{16!}y^{(xvi)}+\frac{h^{17}}{17!}y^{(xvii)}+\frac{h^{18}}{18!}y^{(xviii)}+\frac{h^{19}}{19!}y^{(xix)}+\frac{h^{20}}{20!}y^{(xx)}]+\ldots]$$
$$+h^{10}\alpha_3[y+2hy'+\frac{2^2h^2}{2!}y''+\frac{2^3h^3}{3!}y'''+\frac{2^4h^6}{4!}y^{(iv)}+\frac{2^5h^5}{5!}y^{(v)}+\frac{2^6h^6}{6!}y^{(vi)}$$
$$+\frac{2^7h^7}{7!}y^{(vii)}+\frac{2^8h^8}{8!}y^{(viii)}+\frac{2^9h^9}{9!}y^{(ix)}+\frac{2^{10}h^{10}}{10!}y^{(x)}+\frac{2^{11}h^{11}}{11!}y^{(xi)}+\frac{2^{12}h^{12}}{12!}y^{(xii)}]$$
$$+\frac{2^{13}h^{13}}{13!}y^{(xiii)}+\frac{2^{14}h^{14}}{14!}y^{(xiv)}+\frac{2^{15}h^{15}}{15!}y^{(xv)}+\frac{2^{16}h^{16}}{16!}y^{(xvi)}+\frac{2^{17}h^{17}}{17!}y^{(xvii)}]$$
$$+\frac{2^{18}h^{18}}{18!}y^{(xviii)}+\frac{2^{19}h^{19}}{19!}y^{(xix)}+\frac{2^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\alpha_4[y+3hy'+\frac{3^2h^2}{2!}y''+\frac{3^3h^3}{3!}y'''+\frac{3^4h^4}{4!}y^{(iv)}+\frac{3^5h^5}{5!}y^{(v)}+\frac{3^6h^6}{6!}y^{(vi)}$$
$$+\frac{3^7h^7}{7!}y^{(vii)}+\frac{3^8h^8}{8!}y^{(viii)}+\frac{3^9h^9}{9!}y^{(ix)}+\frac{3^{10}h^{10}}{10!}y^{(x)}+\frac{3^{11}h^{11}}{11!}y^{(xi)}$$
$$+\frac{3^{12}h^{12}}{12!}y^{(xii)}+\frac{3^{13}h^{13}}{13!}y^{(xiii)}+\frac{3^{14}h^{14}}{14!}y^{(xiv)}+\frac{3^{15}h^{15}}{15!}y^{(xv)}+\frac{3^{16}h^16}{16!}y^{(xvi)}$$
$$+\frac{3^{17}h17}{17!}y^{(xvii)}+\frac{3^{18}h^{18}}{18!}y^{(xviii)}+\frac{3^{19}h^{19}}{19!}y^{(xix)}+\frac{3^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\alpha_5[y+4hy'+\frac{4^2h^2}{2!}y''+\frac{4^3h^3}{3!}y'''+\frac{4^4h^4}{4!}y^{(iv)}+\frac{4^5h^5}{5!}y^{(v)}+\frac{4^6h^6}{6!}y^{(vi)}$$
$$+\frac{4^7h^7}{7!}y^{(vii)}+\frac{4^8h^8}{8!}y^{(viii)}+\frac{4^9h^9}{9!}y^{(ix)}+\frac{4^{10}h^{10}}{10!}y^{(x)}+\frac{4^{11}h^{11}}{11!}y^{(xi)}+\frac{4^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{4^{13}h^{13}}{13!}y^{(xiii)}+\frac{4^{14}h^{14}}{14!}y^{(xiv)}+\frac{4^{15}h^{15}}{15!}y^{(xv)}+\frac{4^{16}h^{16}}{16!}y^{(xvi)}+\frac{4^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{4^{18}h^{18}}{18!}y^{(xviii)}+\frac{4^{19}h^{19}}{19!}y^{(xix)}+\frac{4^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\alpha_6[y+5hy'+\frac{5^2h^2}{2!}y''+\frac{5^3h^3}{3!}y'''+\frac{5^4h^4}{4!}y^{(iv)}+\frac{5^5h^5}{5!}y^{(v)}+\frac{5^6h^6}{6!}y^{(vi)}$$
$$+\frac{5^7h^7}{7!}y^{(vii)}+\frac{5^8h^8}{8!}y^{(viii)}+\frac{5^9h^9}{9!}y^{(ix)}+\frac{5^{10}h^{10}}{10!}y^{(x)}+\frac{5^{11}h^{11}}{11!}y^{(xi)}+\frac{5^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{5^{13}h^{13}}{13!}y^{(xiii)}+\frac{5^{14}h^{14}}{14!}y^{(xiv)}+\frac{5^{15}h^{15}}{15!}y^{(xv)}+\frac{5^{16}h^{16}}{16!}y^{(xvi)}+\frac{5^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{5^{18}h^{18}}{18!}y^{(xviii)}+\frac{5^{19}h^{19}}{19!}y^{(xix)}+\frac{5^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\alpha_7[y+6hy'+\frac{6^2h^2}{2!}y''+\frac{6^3h^3}{3!}y'''+\frac{6^4h^4}{4!}y^{(iv)}+\frac{6^5h^5}{5!}y^{(v)}+\frac{6^6h^6}{6!}y^{(vi)}$$
$$+\frac{6^7h^7}{7!}y^{(vii)}+\frac{6^8h^8}{8!}y^{(viii)}+\frac{6^9h^9}{9!}y^{(ix)}+\frac{6^{10}h^{10}}{10!}y^{(x)}+\frac{6^{11}h^{11}}{11!}y^{(xi)}+\frac{6^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{6^{13}h^{13}}{13!}y^{(xiii)}+\frac{6^{14}h^{14}}{14!}y^{(xiv)}+\frac{6^{15}h^{15}}{15!}y^{(xv)}+\frac{6^{16}h^{16}}{16!}y^{(xvi)}+\frac{6^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{6^{18}h^{18}}{18!}y^{(xviii)}+\frac{6^{19}h^{19}}{19!}y^{(xix)}+\frac{6^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\alpha_8[y+7hy'+\frac{7^2h^2}{2!}y''+\frac{7^3h^3}{3!}y'''+\frac{7^4h^4}{4!}y^{(iv)}+\frac{7^5h^5}{5!}y^{(v)}+\frac{7^6h^6}{6!}y^{(vi)}$$
$$+\frac{7^7h^7}{7!}y^{(vii)}+\frac{7^8h^8}{8!}y^{(viii)}+\frac{7^9h^9}{9!}y^{(ix)}+\frac{7^{10}h^{10}}{10!}y^{(x)}+\frac{7^{11}h^{11}}{11!}y^{(xi)}+\frac{7^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{7^{13}h^{13}}{13!}y^{(xiii)}+\frac{7^{14}h^{14}}{14!}y^{(xiv)}+\frac{7^{15}h^{15}}{15!}y^{(xv)}+\frac{7^{16}h^{16}}{16!}y^{(xvi)}+\frac{7^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{7^{18}h^{18}}{18!}y^{(xviii)}+\frac{7^{19}h^{19}}{19!}y^{(xix)}+\frac{7^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\alpha_9[y+8hy'+\frac{8^2h^2}{2!}y''+\frac{8^3h^3}{3!}y'''+\frac{8^4h^4}{4!}y^{(iv)}+\frac{8^5h^5}{5!}y^{(v)}+\frac{8^6h^6}{6!}y^{(vi)}$$
$$+\frac{8^7h^7}{7!}y^{(vii)}+\frac{8^8h^8}{8!}y^{(viii)}+\frac{8^9h^9}{9!}y^{(ix)}+\frac{8^{10}h^{10}}{10!}y^{(x)}+\frac{8^{11}h^{11}}{11!}y^{(xi)}+\frac{8^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{8^{13}h^{13}}{13!}y^{(xiii)}+\frac{8^{14}h^{14}}{14!}y^{(xiv)}+\frac{8^{15}h^{15}}{15!}y^{(xv)}+\frac{8^{16}h^{16}}{16!}y^{(xvi)}+\frac{8^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{8^{18}h^{18}}{18!}y^{(xviii)}+\frac{8^{19}h^{19}}{19!}y^{(xix)}+\frac{8^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$

$$
\begin{aligned}
&+h^{10}\alpha_{10}[y + 9hy' + \tfrac{9^2h^2}{2!}y'' + \tfrac{9^3h^3}{3!}y''' + \tfrac{9^4h^4}{4!}y^{(iv)} + \tfrac{9^5h^5}{5!}y^{(v)} + \tfrac{9^6h^6}{6!}y^{(vi)} \\
&+\tfrac{9^7h^7}{7!}y^{(vii)} + \tfrac{9^8h^8}{8!}y^{(viii)} + \tfrac{9^9h^9}{9!}y^{(ix)} + \tfrac{9^{10}h^{10}}{10!}y^{(x)} + \tfrac{9^{11}h^{11}}{11!}y^{(xi)} + \tfrac{9^{12}h^{12}}{12!}y^{(xii)}] \\
&+\tfrac{9^{13}h^{13}}{13!}y^{(xiii)} + \tfrac{9^{14}h^{14}}{14!}y^{(xiv)} + \tfrac{9^{15}h^{15}}{15!}y^{(xv)} + \tfrac{9^{16}h^{16}}{16!}y^{(xvi)} + \tfrac{9^{17}h^{17}}{17!}y^{(xvii)} \\
&+\tfrac{9^{18}h^{18}}{18!}y^{(xviii)} + \tfrac{9^{19}h^{19}}{19!}y^{(xix)} + \tfrac{9^{20}h^{20}}{20!}y^{(xx)} + \ldots] \\
&+h^{10}\alpha_{11}[y + 10hy' + \tfrac{10^2h^2}{2!}y'' + \tfrac{10^3h^3}{3!}y''' + \tfrac{10^4h^4}{4!}y^{(iv)} + \tfrac{10^5h^5}{5!}y^{(v)} \\
&+\tfrac{10^6h^6}{6!}y^{(vi)} + \tfrac{10^7h^7}{7!}y^{(vii)} + \tfrac{10^8h^8}{8!}y^{(viii)} + \tfrac{10^9h^9}{9!}y^{(ix)} + \tfrac{10^{10}h^{10}}{10!}y^{(x)} + \tfrac{10^{11}h^{11}}{11!}y^{(xi)} \\
&+\tfrac{10^{12}h^{12}}{12!}y^{(xii)} + \tfrac{10^{13}h^{13}}{13!}y^{(xiii)} + \tfrac{10^{14}h^{14}}{14!}y^{(xiv)} + \tfrac{10^{15}h^{15}}{15!}y^{(xv)} + \tfrac{10^{16}h^{16}}{16!}y^{(xvi)} \\
&+\tfrac{10^{17}h^{17}}{17!}y^{(xvii)} + \tfrac{10^{18}h^{18}}{18!}y^{(xviii)} + \tfrac{10^{19}h^{19}}{19!}y^{(xix)} + \tfrac{10^{20}h^{20}}{20!}y^{(xx)} + \ldots] \\
&+h^{10}\alpha_{12}[y + 11hy' + \tfrac{11^2h^2}{2!}y'' + \tfrac{11^3h^3}{3!}y''' + \tfrac{11^4h^4}{4!}y^{(iv)} + \tfrac{11^5h^5}{5!}y^{(v)} \\
&+\tfrac{11^6h^6}{6!}y^{(vi)} + \tfrac{11^7h^7}{7!}y^{(vii)} + \tfrac{11^8h^8}{8!}y^{(viii)} + \tfrac{11^9h^9}{9!}y^{(ix)} + \tfrac{11^{10}h^{10}}{10!}y^{(x)} \\
&+\tfrac{11^{11}h^{11}}{11!}y^{(xi)} + \tfrac{11^{12}h^{12}}{12!}y^{(xii)} + \tfrac{11^{13}h^{13}}{13!}y^{(xiii)} + \tfrac{11^{14}h^{14}}{14!}y^{(xiv)} + \tfrac{11^{15}h^{15}}{15!}y^{(xv)} \\
&+\tfrac{11^{16}h^{16}}{16!}y^{(xvi)} + \tfrac{11^{17}h^{17}}{17!}y^{(xvii)} + \tfrac{11^{18}h^{18}}{18!}y^{(xviii)} + \tfrac{11^{19}h^{19}}{19!}y^{(xix)} \\
&+\tfrac{11^{20}h^{20}}{20!}y^{(xx)} + \ldots]
\end{aligned}
\tag{2.38}
$$

Considering (2.38) and on equating the coefficients of the derivatives $y^{(x)}, y^{(xi)}$, $y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}$ to those in (2.8) gives, respectively,

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \frac{655177}{907200}, \tag{2.39}$$

$$\alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 = \frac{252023}{907200}, \tag{2.40}$$

$$\frac{\alpha_0}{2!} + \alpha_2 + 2^2\frac{\alpha_3}{2!} + 3^2\frac{\alpha_4}{2!} = \frac{27438979}{119750400} - \frac{5}{12}, \tag{2.41}$$

$$-\frac{\alpha_0}{3!} + \alpha_2 + 2^3\frac{\alpha_3}{3!} + 3^3\frac{\alpha_4}{3!} = \frac{11368009}{119750400}, \tag{2.42}$$

$$\frac{\alpha_0}{4!} + \frac{\alpha_2}{4!} + 2^4\frac{\alpha_3}{4!} + 3^4\frac{\alpha_4}{4!} = \frac{131904163}{3113510400} - \frac{1}{12}. \tag{2.43}$$

Solving this system we get the parameters of the first end-point formula (i.e. $x = x_1$) for the second-order method. Thus

$$\left.\begin{aligned}\alpha_0 &= -\tfrac{1586842547}{3736212480},\\ \alpha_1 &= \tfrac{1683367717}{1167566400},\\ \alpha_2 &= -\tfrac{306653299}{622702080},\\ \alpha_3 &= \tfrac{2886847}{11675664},\\ \alpha_4 &= -\tfrac{927622183}{18681062400},\end{aligned}\right\}. \tag{2.44}$$

and it is noted that the parameters α_i $(i = 5, 6, 7, \ldots, 12)$ may then be arbitrarily given the value zero.

Using the method of undetermined coefficients reveals that for the point $x = x_2$ the first two non-vanishing terms in the local truncation error have the values given in (2.8) provided

$$b_0 = 48, b_2 = -14, b_4 = \frac{17}{6}, b_6 = \frac{-67}{180}, b_8 = \frac{-809}{1440}, \tag{2.45}$$

together with parameters β_i $(i = 1, 2, \ldots, 12)$ calculated from the expression

$$\left.\begin{aligned}L[y(x_2); h] = \; & -165y(x-h) + 242y(x) - 209y(x+h) + 120y(x+2h)\\ & -45y(x+3h) + 10y(x+4h) - y(x+5h)\\ & +48y(x-2h) - 14h^2y''(x-2h) + \tfrac{17}{6}h^4y^{(iv)}(x-2h)\\ & -\tfrac{67}{180}h^6y^{(vi)}(x-2h) - \tfrac{289}{1440}h^8y^{(viii)}(x-2h)\\ & +h^{10}[\beta_0y^{(x)}(x-2h) + \beta_1y^{(x)}(x-h) + \beta_2y^{(x)}(x)\\ & +\beta_3y^{(x)}(x+h) + \beta_4y^{(x)}(x+2h) + \beta_5y^{(x)}(x+3h)\\ & +\beta_6y^{(x)}(x+4h) + \beta_7y^{(x)}(x+5h) + \beta_8y^{(x)}(x+6h)\\ & +\beta_9y^{(x)}(x+7h) + \beta_{10}y^{(x)}(x+8h) + \beta_{11}y^{(x)}(x+9h)\\ & +\beta_{12}y^{(x)}(x+10h) + \ldots].\end{aligned}\right\}. \tag{2.46}$$

Expanding (2.46) as Taylor series gives, at the point $x = x_2$

$$\begin{aligned}L[y(x_2); h] = \; & [y - hy' + \tfrac{h^2}{2!}y'' - \tfrac{h^3}{3!}y''' + \tfrac{h^4}{4!}y^{(iv)} - \tfrac{h^5}{5!}y^{(v)} + \tfrac{h^6}{6!}y^{(vi)}\\ & -\tfrac{h^7}{7!}y^{(vii)} + \tfrac{h^8}{8!}y^{(viii)} - \tfrac{h^9}{9!}y^{(ix)} + \tfrac{h^{10}}{10!}y^{(x)} - \tfrac{h^{11}}{11!}y^{(xi)}\end{aligned}$$

$$+\frac{h^{12}}{12!}y^{(xii)} - \frac{h^{13}}{13!}y^{(xiii)} + \frac{h^{14}}{14!}y^{(xiv)} - \frac{h^{15}}{15!}y^{(xv)} + \frac{h^{16}}{16!}y^{(xvi)} - \frac{h^{17}}{17!}y^{(xvii)} + \frac{h^{18}}{18!}y^{(xviii)}$$

$$-\frac{h^{19}}{19!}y^{(xix)} + \frac{h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+242y(x)$$

$$-209[y + hy' + \frac{h^2}{2!}y'' + \frac{h^3}{3!}y''' + \frac{h^4}{4!}y^{(iv)} + \frac{h^5}{5!}y^{(v)} + \frac{h^6}{6!}y^{(vi)} + \frac{h^7}{7!}y^{(vii)} + \frac{h^8}{8!}y^{(viii)}$$

$$+\frac{h^9}{9!}y^{(ix)} + \frac{h^{10}}{10!}y^{(x)} + \frac{h^{11}}{11!}y^{(xi)} + \frac{h^{12}}{12!}y^{(xii)} + \frac{h^{13}}{13!}y^{(xiii)} + \frac{h^{14}}{14!}y^{(xiv)} + \frac{h^{15}}{15!}y^{(xv)} + \frac{h^{16}}{16!}y^{(xvi)}$$

$$+\frac{h^{17}}{17!}y^{(xvii)} + \frac{h^{18}}{18!}y^{(xviii)} + \frac{h^{19}}{19!}y^{(xix)} + \frac{h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+120[y + 2hy' + \frac{2^2h^2}{2!}y'' + \frac{2^3h^3}{3!}y''' + \frac{2^4h^6}{4!}y^{(iv)} + \frac{2^5h^5}{5!}y^{(v)} + \frac{2^6h^6}{6!}y^{(vi)} + \frac{2^7h^7}{7!}y^{(vii)}$$

$$+\frac{2^8h^8}{8!}y^{(viii)} + \frac{2^9h^9}{9!}y^{(ix)} + \frac{2^{10}h^{10}}{10!}y^{(x)} + \frac{2^{11}h^{11}}{11!}y^{(xi)} + \frac{2^{12}h^{12}}{12!}y^{(xii)} + \frac{2^{13}h^{13}}{13!}y^{(xiii)}$$

$$+\frac{2^{14}h^{14}}{14!}y^{(xiv)} + \frac{2^{15}h^{15}}{15!}y^{(xv)} + \frac{2^{16}h^{16}}{16!}y^{(xvi)} + \frac{2^{17}h^{17}}{17!}y^{(xvii)} + \frac{2^{18}h^{18}}{18!}y^{(xviii)}$$

$$+\frac{2^{19}h^{19}}{19!}y^{(xix)} + \frac{2^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$

$$-45[y + 3hy' + \frac{3^2h^2}{2!}y'' + \frac{3^3h^3}{3!}y''' + \frac{3^4h^4}{4!}y^{(iv)} + \frac{3^5h^5}{5!}y^{(v)} + \frac{3^6h^6}{6!}y^{(vi)} + \frac{3^7h^7}{7!}y^{(vii)}$$

$$+\frac{3^8h^8}{8!}y^{(viii)} + \frac{3^9h^9}{9!}y^{(ix)} + \frac{3^{10}h^{10}}{10!}y^{(x)} + \frac{3^{11}h^{11}}{11!}y^{(xi)} + \frac{3^{12}h^{12}}{12!}y^{(xii)} + \frac{3^{13}h^{13}}{13!}y^{(xiii)}$$

$$+\frac{3^{14}h^{14}}{14!}y^{(xiv)} + \frac{3^{15}h^{15}}{15!}y^{(xv)} + \frac{3^{16}h^{16}}{16!}y^{(xvi)} + \frac{3^{17}h^{17}}{17!}y^{(xvii)} + \frac{3^{18}h^{18}}{18!}y^{(xviii)} + \frac{3^{19}h^{19}}{19!}y^{(xix)}$$

$$+\frac{3^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+10[y + 4hy' + \frac{4^2h^2}{2!}y'' + \frac{4^3h^3}{3!}y''' + \frac{4^4h^4}{4!}y^{(iv)} + \frac{4^5h^5}{5!}y^{(v)} + \frac{4^6h^6}{6!}y^{(vi)} + \frac{4^7h^7}{7!}y^{(vii)}$$

$$+\frac{4^8h^8}{8!}y^{(viii)} + \frac{4^9h^9}{9!}y^{(ix)} + \frac{4^{10}h^{10}}{10!}y^{(x)} + \frac{4^{11}h^{11}}{11!}y^{(xi)} + \frac{4^{12}h^{12}}{12!}y^{(xii)} + \frac{4^{13}h^{13}}{13!}y^{(xiii)}$$

$$+\frac{4^{14}h^{14}}{14!}y^{(xiv)} + \frac{4^{15}h^{15}}{15!}y^{(xv)} + \frac{4^{16}h^{16}}{16!}y^{(xvi)} - \frac{4^{17}h^{17}}{17!}y^{(xvii)} + \frac{4^{18}h^{18}}{18!}y^{(xviii)} + \frac{4^{19}h^{19}}{19!}y^{(xix)}$$

$$+\frac{4^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$

$$-[y + 5hy' + \frac{5^2h^2}{2!}y'' + \frac{5^3h^3}{3!}y''' + \frac{5^4h^4}{4!}y^{(iv)} + \frac{5^5h^5}{5!}y^{(v)} + \frac{5^6h^6}{6!}y^{(vi)} + \frac{5^7h^7}{7!}y^{(vii)}$$

$$+\frac{5^8h^8}{8!}y^{(viii)} + \frac{5^9h^9}{9!}y^{(ix)} + \frac{5^{10}h^{10}}{10!}y^{(x)} + \frac{5^{11}h^{11}}{11!}y^{(xi)} + \frac{5^{12}h^{12}}{12!}y^{(xii)} + \frac{5^{13}h^{13}}{13!}y^{(xiii)}$$

$$+\frac{5^{14}h^{14}}{14!}y^{(xiv)} + \frac{5^{15}h^{15}}{15!}y^{(xv)} + \frac{5^{16}h^{16}}{16!}y^{(xvi)} + \frac{5^{17}h^{17}}{17!}y^{(xvii)} + \frac{5^{18}h^{18}}{18!}y^{(xviii)}$$

$$+\frac{5^{19}h^{19}}{19!}y^{(xix)} + \frac{5^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+48[y - 2hy' + \frac{2^2h^2}{2!}y'' - \frac{2^3h^3}{3!}y''' + \frac{2^4h^6}{4!}y^{(iv)} - \frac{2^5h^5}{5!}y^{(v)} + \frac{2^6}{6!}y^{(vi)} - \frac{2^7h^7}{7!}y^{(vii)}$$

$$+\frac{2^8}{8!}y^{(viii)} - \frac{2^9h^9}{9!}y^{(ix)} + \frac{2^{10}h^{10}}{10!}y^{(x)} - \frac{2^{11}h^{11}}{11!}y^{(xi)} + \frac{2^{12}h^{12}}{12!}y^{(xii)} - \frac{2^{13}h^{13}}{13!}y^{(xiii)}$$

$$+\frac{2^{14}h^{14}}{14!}y^{(xiv)} - \frac{2^{15}h^{15}}{15!}y^{(xv)} + \frac{2^{16}h^{16}}{16!}y^{(xvi)} - \frac{2^{17}h^{17}}{17!}y^{(xvii)} + \frac{2^{18}h^{18}}{18!}y^{(xviii)}$$

$$-\frac{2^{19}h^{19}}{19!}y^{(xix)} + \frac{2^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$

$$-14h^2[y'' - 2hy''' + \frac{2^2h^2}{2!}y^{iv} - \frac{2^3h^3}{3!}y^{(v)} + \frac{2^4h^4}{4!}y^{(vi)} - \frac{2^5h^5}{5!}y^{(vii)} + \frac{2^6h^6}{6!}y^{(viii)}$$

$$-\frac{2^7h^7}{7!}y^{(ix)} + \frac{2^8h^8}{8!}y^{(x)} - \frac{2^9h^9}{9!}y^{(xi)} + \frac{2^{10}h^{10}}{10!}y^{(xii)} - \frac{2^{11}h^{11}}{11!}y^{(xiii)} + \frac{2^{12}h^{12}}{12!}y^{(xiv)}$$

$$-\frac{2^{13}h^{13}}{13!}y^{(xv)} + \frac{2^{14}h^{14}}{14!}y^{(xvi)} - \frac{2^{15}h^{15}}{15!}y^{(xvii)} + \frac{2^{16}h^{16}}{16!}y^{(xviii)} - \frac{2^{17}h^{17}}{17!}y^{(xix)}$$

$$+\frac{2^{18}h^{18}}{18!}y^{(xx)} + \ldots]$$

$$+\frac{17}{6}h^4[y^{iv} - 2hy^{(v)} + \frac{2^2h^2}{2!}y^{(vi)} - \frac{2^3h^3}{3!}y^{(vii)} + \frac{2^4h^4}{4!}y^{(viii)} - \frac{2^5h^5}{5!}y^{(ix)}$$

$$+\frac{2^6h^6}{6!}y^{(x)} - \frac{2^7h^7}{7!}y^{(xi)} + \frac{2^8h^8}{8!}y^{(xii)} - \frac{2^9h^9}{9!}y^{(xiii)} + \frac{2^{10}h^{10}}{10!}y^{(xiv)} - \frac{2^{11}h^{11}}{11!}y^{(xv)}$$

$$+\frac{2^{12}h^{12}}{12!}y^{(xvi)} - \frac{2^{13}h^{13}}{13!}y^{(xvii)} + \frac{2^{14}h^{14}}{14!}y^{(xviii)} - \frac{2^{15}h^{15}}{15!}y^{(xix)} + \frac{2^{16}h^{16}}{16!}y^{(xx)} + \ldots]$$

$$-\frac{67}{180}h^6[y^{vi} - 2hy^{(vii)} + \frac{2^2h^2}{2!}y^{(viii)} - \frac{2^3h^3}{3!}y^{(ix)} + \frac{2^4h^4}{4!}y^{(x)} - \frac{2^5h^5}{5!}y^{(xi)}$$

$$+\frac{2^6h^6}{6!}y^{(xii)} - \frac{2^7h^7}{7!}y^{(xiii)} + \frac{2^8h^8}{8!}y^{(xiv)} - \frac{2^9h^9}{9!}y^{(xv)} + \frac{2^{10}h^{10}}{10!}y^{(xvi)} - \frac{2^{11}h^{11}}{11!}y^{(xvii)}$$

$$+\frac{2^{12}h^{12}}{12!}y^{(xviii)} - \frac{2^{13}h^{13}}{13!}y^{(xix)} + \frac{2^{14}h^{14}}{14!}y^{(xx)} + \ldots]$$

$$-\frac{289}{1440}h^8[y^{(viii)} - 2hy^{(ix)} + \frac{2^2h^2}{2!}y^{(x)} - \frac{2^3h^3}{3!}y^{(xi)} + \frac{2^4h^4}{4!}y^{(xii)} - \frac{2^5h^5}{5!}y^{(xiii)}$$

$$+\frac{2^6h^6}{6!}y^{(xiv)} - \frac{2^7h^7}{7!}y^{(xv)} + \frac{2^8h^8}{8!}y^{(xvi)} - \frac{2^9h^9}{9!}y^{(xvii)} + \frac{2^{10}h^{10}}{10!}y^{(xviii)}$$

$$-\frac{2^{11}h^{11}}{11!}y^{(xix)} + \frac{2^{12}h^{12}}{12!}y^{(xx)} + \ldots]$$

$$-\frac{289}{1440}h^8[y^{(viii)} - 2hy^{(ix)} + \frac{2^2h^2}{2!}y^{(x)} - \frac{2^3h^3}{3!}y^{(xi)} + \frac{2^4h^4}{4!}y^{(xii)} - \frac{2^5h^5}{5!}y^{(xiii)}$$

$$+\frac{2^6h^6}{6!}y^{(xiv)} - \frac{2^7h^7}{7!}y^{(xv)} + \frac{2^8h^8}{8!}y^{(xvi)} - \frac{2^9h^9}{9!}y^{(xvii)} + \frac{2^{10}h^{10}}{10!}y^{(xviii)}$$

$$-\frac{2^{11}h^{11}}{11!}y^{(xix)} + \frac{2^{12}h^{12}}{12!}y^{(xx)} + \ldots]$$

$$+h^{10}\beta_0[y - 2hy' + \frac{2^2h^2}{2!}y'' - \frac{2^3h^3}{3!}y''' + \frac{2^4h^6}{4!}y^{(iv)} - \frac{2^5h^5}{5!}y^{(v)} + \frac{2^6h^6}{6!}y^{(vi)}$$

$$-\frac{2^7h^7}{7!}y^{(vii)} + \frac{2^8h^8}{8!}y^{(viii)} - \frac{2^9h^9}{9!}y^{(ix)} + \frac{2^{10}h^{10}}{10!}y^{(x)} - \frac{2^{11}h^{11}}{11!}y^{(xi)}$$

$$+\frac{2^{12}h^{12}}{12!}y^{(xii)} - \frac{2^{13}h^{13}}{13!}y^{(xiii)} + \frac{2^{14}h^{14}}{14!}y^{(xiv)} - \frac{2^{15}h^{15}}{15!}y^{(xv)} + \frac{2^{16}h^{16}}{16!}y^{(xvi)}$$

$$-\frac{2^{17}h^{17}}{17!}y^{(xvii)} + \frac{2^{18}h^{18}}{18!}y^{(xviii)} - \frac{2^{19}h^{19}}{19!}y^{(xix)} + \frac{2^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+h^{10}\beta_1[y - hy' + \frac{h^2}{2!}y'' - \frac{h^3}{3!}y''' + \frac{h^4}{4!}y^{(iv)} - \frac{h^5}{5!}y^{(v)} + \frac{h^6}{6!}y^{(vi)} - \frac{h^7}{7!}y^{(vii)}$$

$$+\frac{h^8}{8!}y^{(viii)} - \frac{h^9}{9!}y^{(ix)} + \frac{h^{10}}{10!}y^{(x)} - \frac{h^{11}}{11!}y^{(xi)} + \frac{h^{12}}{12!}y^{(xii)} - \frac{h^{13}}{13!}y^{(xiii)} + \frac{h^{14}}{14!}y^{(xiv)}$$

$$-\frac{h^{15}}{15!}y^{(xv)} + \frac{h^{16}}{16!}y^{(xvi)} - \frac{h^{17}}{17!}y^{(xvii)} + \frac{h^{18}}{18!}y^{(xviii)} - \frac{h^{19}}{19!}y^{(xix)} + \frac{h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+h^{10}\beta_2y^{(x)}$$

$$+h^{10}\beta_3[y + hy' + \frac{h^2}{2!}y'' + \frac{h^3}{3!}y''' + \frac{h^4}{4!}y^{(iv)} + \frac{h^5}{5!}y^{(v)} + \frac{h^6}{6!}y^{(vi)} + \frac{h^7}{7!}y^{(vii)}$$

$$+\frac{h^8}{8!}y^{(viii)} + \frac{h^9}{9!}y^{(ix)} + \frac{h^{10}}{10!}y^{(x)} + \frac{h^{11}}{11!}y^{(xi)} + \frac{h^{12}}{12!}y^{(xii)} + \frac{h^{13}}{13!}y^{(xiii)} + \frac{h^{14}}{14!}y^{(xiv)}$$

$$+\frac{h^{15}}{15!}y^{(xv)} + \frac{h^{16}}{16!}y^{(xvi)} + \frac{h^{17}}{17!}y^{(xvii)} + \frac{h^{18}}{18!}y^{(xviii)} + \frac{h^{19}}{19!}y^{(xix)} + \frac{h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+h^{10}\beta_4[y + 2hy' + \frac{2^2h^2}{2!}y'' + \frac{2^3h^3}{3!}y''' + \frac{2^4h^6}{4!}y^{(iv)} + \frac{2^5h^5}{5!}y^{(v)} + \frac{2^6h^6}{6!}y^{(vi)}$$

$$+\frac{2^7h^7}{7!}y^{(vii)} + \frac{2^8h^8}{8!}y^{(viii)} + \frac{2^9h^9}{9!}y^{(ix)} + \frac{2^{10}h^{10}}{10!}y^{(x)} + \frac{2^{11}h^{11}}{11!}y^{(xi)}$$

$$+\frac{2^{12}h^{12}}{12!}y^{(xii)}+\frac{2^{13}h^{13}}{13!}y^{(xiii)}+\frac{2^{14}h^{14}}{14!}y^{(xiv)}+\frac{2^{15}h^{15}}{15!}y^{(xv)}+\frac{2^{16}h^{16}}{16!}y^{(xvi)}$$
$$+\frac{2^{17}h^{17}}{17!}y^{(xvii)}+\frac{2^{18}h^{18}}{18!}y^{(xviii)}+\frac{2^{19}h^{19}}{19!}y^{(xix)}+\frac{2^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\beta_5[y+3hy'+\frac{3^2h^2}{2!}y''+\frac{3^3h^3}{3!}y'''+\frac{3^4h^4}{4!}y^{(iv)}+\frac{3^5h^5}{5!}y^{(v)}+\frac{3^6h^6}{6!}y^{(vi)}$$
$$+\frac{3^7h^7}{7!}y^{(vii)}+\frac{3^8h^8}{8!}y^{(viii)}+\frac{3^9h^9}{9!}y^{(ix)}+\frac{3^{10}h^{10}}{10!}y^{(x)}+\frac{3^{11}h^{11}}{11!}y^{(xi)}]$$
$$+\frac{3^{12}h^{12}}{12!}y^{(xii)}+\frac{3^{13}h^{13}}{13!}y^{(xiii)}+\frac{3^{14}h^{14}}{14!}y^{(xiv)}+\frac{3^{15}h^{15}}{15!}y^{(xv)}+\frac{3^{16}h^{1}6}{16!}y^{(xvi)}$$
$$+\frac{3^{17}h^{17}}{17!}y^{(xvii)}+\frac{3^{18}h^{18}}{18!}y^{(xviii)}+\frac{3^{19}h^{19}}{19!}y^{(xix)}+\frac{3^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\beta_6[y+4hy'+\frac{4^2h^2}{2!}y''+\frac{4^3h^3}{3!}y^{(''')}+\frac{4^4h^4}{4!}y^{(iv)}+\frac{4^5h^5}{5!}y^{(v)}+\frac{4^6h^6}{6!}y^{(vi)}$$
$$+\frac{4^7h^7}{7!}y^{(vii)}+\frac{4^8h^8}{8!}y^{(viii)}+\frac{4^9h^9}{9!}y^{(ix)}+\frac{4^{10}h^{10}}{10!}y^{(x)}+\frac{4^{11}h^{11}}{11!}y^{(xi)}$$
$$+\frac{4^{12}h^{12}}{12!}y^{(xii)}+\frac{4^{13}h^{13}}{13!}y^{(xiii)}+\frac{4^{14}h^{14}}{14!}y^{(xiv)}+\frac{4^{15}h^{15}}{15!}y^{(xv)}+\frac{4^{16}h^{16}}{16!}y^{(xvi)}]$$
$$+\frac{4^{17}h^{17}}{17!}y^{(xvii)}+\frac{4^{18}h^{18}}{18!}y^{(xviii)}+\frac{4^{19}h^{19}}{19!}y^{(xix)}+\frac{4^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\beta_7[y+5hy'+\frac{5^2h^2}{2!}y''+\frac{5^3h^3}{3!}y'''+\frac{5^4h^4}{4!}y^{(iv)}+\frac{5^5h^5}{5!}y^{(v)}+\frac{5^6h^6}{6!}y^{(vi)}$$
$$+\frac{5^7h^7}{7!}y^{(vii)}+\frac{5^8h^8}{8!}y^{(viii)}+\frac{5^9h^9}{9!}y^{(ix)}+\frac{5^{10}h^{10}}{10!}y^{(x)}+\frac{5^{11}h^{11}}{11!}y^{(xi)}+\frac{5^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{5^{13}h^{13}}{13!}y^{(xiii)}+\frac{5^{14}h^{14}}{14!}y^{(xiv)}+\frac{5^{15}h^{15}}{15!}y^{(xv)}+\frac{5^{16}h^{16}}{16!}y^{(xvi)}+\frac{5^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{5^{18}h^{18}}{18!}y^{(xviii)}+\frac{5^{19}h^{19}}{19!}y^{(xix)}+\frac{5^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\beta_8[y+6hy'+\frac{6^2h^2}{2!}y''+\frac{6^3h^3}{3!}y'''+\frac{6^4h^4}{4!}y^{(iv)}+\frac{6^5h^5}{5!}y^{(v)}+\frac{6^6h^6}{6!}y^{(vi)}$$
$$+\frac{6^7h^7}{7!}y^{(vii)}+\frac{6^8h^8}{8!}y^{(viii)}+\frac{6^9h^9}{9!}y^{(ix)}+\frac{6^{10}h^{10}}{10!}y^{(x)}+\frac{6^{11}h^{11}}{11!}y^{(xi)}+\frac{6^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{6^{13}h^{13}}{13!}y^{(xiii)}+\frac{6^{14}h^{14}}{14!}y^{(xiv)}+\frac{6^{15}h^{15}}{15!}y^{(xv)}+\frac{6^{16}h^{16}}{16!}y^{(xvi)}+\frac{6^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{6^{18}h^{18}}{18!}y^{(xviii)}+\frac{6^{19}h^{19}}{19!}y^{(xix)}+\frac{6^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\beta_9[y+7hy'+\frac{7^2h^2}{2!}y''+\frac{7^3h^3}{3!}y'''+\frac{7^4h^4}{4!}y^{(iv)}+\frac{7^5h^5}{5!}y^{(v)}+\frac{7^6h^6}{6!}y^{(vi)}$$
$$+\frac{7^7h^7}{7!}y^{(vii)}+\frac{7^8h^8}{8!}y^{(viii)}+\frac{7^9h^9}{9!}y^{(ix)}+\frac{7^{10}h^{10}}{10!}y^{(x)}+\frac{7^{11}h^{11}}{11!}y^{(xi)}+\frac{7^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{7^{13}h^{13}}{13!}y^{(xiii)}+\frac{7^{14}h^{14}}{14!}y^{(xiv)}+\frac{7^{15}h^{15}}{15!}y^{(xv)}+\frac{7^{16}h^{16}}{16!}y^{(xvi)}+\frac{7^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{7^{18}h^{18}}{18!}y^{(xviii)}+\frac{7^{19}h^{19}}{19!}y^{(xix)}+\frac{7^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\beta_{10}[y+8hy'+\frac{8^2h^2}{2!}y''+\frac{8^3h^3}{3!}y'''+\frac{8^4h^4}{4!}y^{(iv)}+\frac{8^5h^5}{5!}y^{(v)}+\frac{8^6h^6}{6!}y^{(vi)}$$
$$+\frac{8^7h^7}{7!}y^{(vii)}+\frac{8^8h^8}{8!}y^{(viii)}+\frac{8^9h^9}{9!}y^{(ix)}+\frac{8^{10}h^{10}}{10!}y^{(x)}+\frac{8^{11}h^{11}}{11!}y^{(xi)}+\frac{8^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{8^{13}h^{13}}{13!}y^{(xiii)}+\frac{8^{14}h^{14}}{14!}y^{(xiv)}+\frac{8^{15}h^{15}}{15!}y^{(xv)}+\frac{8^{16}h^{16}}{16!}y^{(xvi)}+\frac{8^{17}h^{17}}{17!}y^{(xvii)}]$$
$$+\frac{8^{18}h^{18}}{18!}y^{(xviii)}+\frac{8^{19}h^{19}}{19!}y^{(xix)}+\frac{8^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\beta_{11}[y+9hy'+\frac{9^2h^2}{2!}y''+\frac{9^3h^3}{3!}y'''+\frac{9^4h^4}{4!}y^{(iv)}+\frac{9^5h^5}{5!}y^{(v)}+\frac{9^6h^6}{6!}y^{(vi)}$$
$$+\frac{9^7h^7}{7!}y^{(vii)}+\frac{9^8h^8}{8!}y^{(viii)}+\frac{9^9h^9}{9!}y^{(ix)}+\frac{9^{10}h^{10}}{10!}y^{(x)}+\frac{9^{11}h^{11}}{11!}y^{(xi)}+\frac{9^{12}h^{12}}{12!}y^{(xii)}]$$
$$+\frac{9^{13}h^{13}}{13!}y^{(xiii)}+\frac{9^{14}h^{14}}{14!}y^{(xiv)}+\frac{9^{15}h^{15}}{15!}y^{(xv)}+\frac{9^{16}h^{16}}{16!}y^{(xvi)}+\frac{9^{17}h^{17}}{17!}y^{(xvii)}$$

$$+\frac{9^{18}h^{18}}{18!}y^{(xviii)}+\frac{9^{19}h^{19}}{19!}y^{(xix)}+\frac{9^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\beta_{12}[y+10hy'+\frac{10^2h^2}{2!}y''+\frac{10^3h^3}{3!}y'''+\frac{10^4h^4}{4!}y^{(iv)}+\frac{10^5h^5}{5!}y^{(v)}+\frac{10^6h^6}{6!}y^{(vi)}$$
$$+\frac{10^7h^7}{7!}y^{(vii)}+\frac{10^8h^8}{8!}y^{(viii)}+\frac{10^9h^9}{9!}y^{(ix)}+\frac{10^{10}h^{10}}{10!}y^{(x)}+\frac{10^{11}h^{11}}{11!}y^{(xi)}$$
$$+\frac{10^{12}h^{12}}{12!}y^{(xii)}+\frac{10^{13}h^{13}}{13!}y^{(xiii)}+\frac{10^{14}h^{14}}{14!}y^{(xiv)}+\frac{10^{15}h^{15}}{15!}y^{(xv)}+\frac{10^{16}h^{16}}{16!}y^{(xvi)}$$
$$+\frac{10^{17}h^{17}}{17!}y^{(xvii)}+\frac{10^{18}h^{18}}{18!}y^{(xviii)}+\frac{10^{19}h^{19}}{19!}y^{(xix)}+\frac{10^{20}h^{20}}{20!}y^{(xx)}+\ldots] \tag{2.47}$$

Considering (2.47) and on equating the coefficients of the derivatives $y^{(x)}, y^{(xi)}$, $y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}$ to those in (2.8) gives

$$\beta_0+\beta_1+\beta_2+\beta_3+\beta_4=\frac{882773}{907200}, \tag{2.48}$$

$$-2\beta_0-\beta_1+\beta_3+2\beta_4=\frac{24427}{453600}, \tag{2.49}$$

$$2^2\frac{\beta_0}{2!}+\frac{\beta_1}{2!}+\frac{\beta_3}{2!}+2^2\frac{\beta_4}{2!}=\frac{43202009}{119750400}-\frac{5}{12}, \tag{2.50}$$

$$-2^3\frac{\beta_0}{3!}-\frac{\beta_1}{3!}+\frac{\beta_3}{3!}+2^3\frac{\beta_4}{3!}=\frac{2394839}{59875200}, \tag{2.51}$$

$$2^4\frac{\beta_0}{4!}+\frac{\beta_1}{4!}+\frac{\beta_3}{4!}+2^4\frac{\beta_4}{4!}=\frac{190486607}{3113510400}-\frac{1}{12}. \tag{2.52}$$

Solving this system, we get the parameters of the second end-point formula (i.e. $x = x_2$) for the second-order method. It is noted that the parameters β_i (i = 5,6,7, ... , 12) may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}\beta_0&=-\frac{123315019}{3736212480},\\ \beta_1&=\frac{4243927}{233513280},\\ \beta_2&=\frac{277359163}{283046400},\\ \beta_3&=\frac{5827189}{583783200},\\ \beta_4&=-\frac{7410451}{3736212480}.\end{aligned}\right\}. \tag{2.53}$$

Using the method of undetermined coefficients reveals that for the point $x = x_3$ the first two non-vanishing terms in the local truncation error have the values given in (2.8) provided

$$c_0=-27, c_2=6, c_4=\frac{-1}{2}, c_6=\frac{-3}{20}, c_8=\frac{-41}{3360}, \tag{2.54}$$

together with parameters γ_i $(i = 0, 1, 2, \ldots, 12)$ calculated from the expression

$$\left.\begin{aligned}L[y(x_3); h] = \;& 110y(x-2h) - 209y(x-h) + 252y(x) - 210y(x+h)\\ &+120y(x+2h) - 45y(x+3h) + 10y(x+4h) - y(x+5h)\\ &-27y(x-3h) + 6h^2y''(x-3h) - \tfrac{1}{2}h^4y^{(iv)}(x-3h)\\ &-\tfrac{3}{20}h^6y^{(vi)}(x-3h) - \tfrac{41}{3360}h^8y^{(viii)}(x-3h)\\ &+h^{10}[\gamma_0y^{(x)}(x-3h) + \gamma_1y^{(x)}(x-2h) + \gamma_2y^{(x)}(x-h)\\ &+\gamma_3y^{(x)}(x) + \gamma_4y^{(x)}(x+h) + \gamma_5y^{(x)}(x+2h)\\ &+\gamma_6y^{(x)}(x+3h) + \gamma_7y^{(x)}(x+4h) + \gamma_8y^{(x)}(x+5h)\\ &+\gamma_9y^{(x)}(x+6h) + \gamma_{10}y^{(x)}(x+7h) + \gamma_{11}y^{(x)}(x+8h)\\ &+\gamma_{12}y^{(x)}(x+9h) + \ldots].\end{aligned}\right\} . \tag{2.55}$$

Expanding the terms in (2.55) about $y(x)$ and its derivatives gives, at the point $x = x_3$,

$$\begin{aligned}L[y(x_3); h] = \;& 110[y - 2hy' + \tfrac{2^2h^2}{2!}y'' - \tfrac{2^3h^3}{3!}y''' + \tfrac{2^4h^4}{4!}y^{(iv)} - \tfrac{2^5h^5}{5!}y^{(v)}\\ &+\tfrac{2^6h^6}{6!}y^{(vi)} - \tfrac{2^7h^7}{7!}y^{(vii)} + \tfrac{2^8h^8}{8!}y^{(viii)} - \tfrac{2^9h^9}{9!}y^{(ix)} + \tfrac{2^{10}h^{10}}{10!}y^{(x)}\\ &-\tfrac{2^{11}h^{11}}{11!}y^{(xi)} + \tfrac{2^{12}h^{12}}{12!}y^{(xii)} - \tfrac{2^{13}h^{13}}{13!}y^{(xiii)} + \tfrac{2^{14}h^{14}}{14!}y^{(xiv)}\\ &-\tfrac{2^{15}h^{15}}{15!}y^{(xv)} + \tfrac{2^{16}h^{16}}{16!}y^{(xvi)} - \tfrac{2^{17}h^{17}}{17!}y^{(xvii)} + \tfrac{2^{18}h^{18}}{18!}y^{(xviii)}\\ &-\tfrac{2^{19}h^{19}}{19!}y^{(xix)} + \tfrac{2^{20}h^{20}}{20!}y^{(xx)} + \ldots]\\ &-209[y - hy' + \tfrac{h^2}{2!}y'' - \tfrac{h^3}{3!}y''' + \tfrac{h^4}{4!}y^{(iv)} - \tfrac{h^5}{5!}y^{(v)} + \tfrac{h^6}{6!}y^{(vi)}\\ &-\tfrac{h^7}{7!}y^{(vii)} + \tfrac{h^8}{8!}y^{(viii)} - \tfrac{h^9}{9!}y^{(ix)} + \tfrac{h^{10}}{10!}y^{(x)} - \tfrac{h^{11}}{11!}y^{(xi)} + \tfrac{h^{12}}{12!}y^{(xii)}\\ &-\tfrac{h^{13}}{13!}y^{(xiii)} + \tfrac{h^{14}}{14!}y^{(xiv)} - \tfrac{h^{15}}{15!}y^{(xv)} + \tfrac{h^{16}}{16!}y^{(xvi)} - \tfrac{h^{17}}{17!}y^{(xvii)}\\ &+\tfrac{h^{18}}{18!}y^{(xviii)} - \tfrac{h^{19}}{19!}y^{(xix)} + \tfrac{h^{20}}{20!}y^{(xx)} + \ldots] + 252y(x)\\ &-210[y + hy' + \tfrac{h^2}{2!}y'' + \tfrac{h^3}{3!}y''' + \tfrac{h^4}{4!}y^{(iv)} + \tfrac{h^5}{5!}y^{(v)} + \tfrac{h^6}{6!}y^{(vi)}\\ &+\tfrac{h^7}{7!}y^{(vii)} + \tfrac{h^8}{8!}y^{(viii)} + \tfrac{h^9}{9!}y^{(ix)} + \tfrac{h^{10}}{10!}y^{(x)} + \tfrac{h^{11}}{11!}y^{(xi)}\\ &+\tfrac{h^{12}}{12!}y^{(xii)} + \tfrac{h^{13}}{13!}y^{(xiii)} + \tfrac{h^{14}}{14!}y^{(xiv)} + \tfrac{h^{15}}{15!}y^{(xv)} + \tfrac{h^{16}}{16!}y^{(xvi)}]\\ &+\tfrac{h^{17}}{17!}y^{(xvii)} + \tfrac{h^{18}}{18!}y^{(xviii)} + \tfrac{h^{19}}{19!}y^{(xix)} + \tfrac{h^{20}}{20!}y^{(xx)} + \ldots]\\ &+120[y + 2hy' + \tfrac{2^2h^2}{2!}y'' + \tfrac{2^3h^3}{3!}y''' + \tfrac{2^4h^6}{4!}y^{(iv)} + \tfrac{2^5h^5}{5!}y^{(v)}\\ &+\tfrac{2^6}{6!}y^{(vi)} + \tfrac{2^7h^7}{7!}y^{(vii)} + \tfrac{2^8}{8!}y^{(viii)} + \tfrac{2^9h^9}{9!}y^{(ix)} + \tfrac{2^{10}h^{10}}{10!}y^{(x)}\end{aligned}$$

$$+\frac{2^{11}h^{11}}{11!}y^{(xi)}+\frac{2^{12}h^{12}}{12!}y^{(xii)}+\frac{2^{13}h^{13}}{13!}y^{(xiii)}+\frac{2^{14}h^{14}}{14!}y^{(xiv)}+\frac{2^{15}h^{15}}{15!}y^{(xv)}+\frac{2^{16}h^{16}}{16!}y^{(xvi)}$$
$$+\frac{2^{17}h^{17}}{17!}y^{(xvii)}+\frac{2^{18}h^{18}}{18!}y^{(xviii)}+\frac{2^{19}h^{19}}{19!}y^{(xix)}+\frac{2^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$-45[y+3hy'+\frac{3^2h^2}{2!}y''+\frac{3^3h^3}{3!}y'''+\frac{3^4h^4}{4!}y^{(iv)}+\frac{3^5h^5}{5!}y^{(v)}+\frac{3^6h^6}{6!}y^{(vi)}+\frac{3^7h^7}{7!}y^{(vii)}$$
$$+\frac{3^8h^8}{8!}y^{(viii)}+\frac{3^9h^9}{9!}y^{(ix)}+\frac{3^{10}h^{10}}{10!}y^{(x)}+\frac{3^{11}h^{11}}{11!}y^{(xi)}+\frac{3^{12}h^{12}}{12!}y^{(xii)}+\frac{3^{13}h^{13}}{13!}y^{(xiii)}$$
$$+\frac{3^{14}h^{14}}{14!}y^{(xiv)}+\frac{3^{15}h^{15}}{15!}y^{(xv)}+\frac{3^{16}h^{1}6}{16!}y^{(xvi)}+\frac{3^{17}h^{17}}{17!}y^{(xvii)}+\frac{3^{18}h^{18}}{18!}y^{(xviii)}+\frac{3^{19}h^{19}}{19!}y^{(xix)}$$
$$+\frac{3^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+10[y+4hy'+\frac{4^2h^2}{2!}y''+\frac{4^3h^3}{3!}y'''+\frac{4^4h^4}{4!}y^{(iv)}+\frac{4^5h^5}{5!}y^{(v)}+\frac{4^6h^6}{6!}y^{(vi)}+\frac{4^7h^7}{7!}y^{(vii)}$$
$$+\frac{4^8h^8}{8!}y^{(viii)}+\frac{4^9h^9}{9!}y^{(ix)}+\frac{4^{10}h^{10}}{10!}y^{(x)}+\frac{4^{11}h^{11}}{11!}y^{(xi)}+\frac{4^{12}h^{12}}{12!}y^{(xii)}+\frac{4^{13}h^{13}}{13!}y^{(xiii)}$$
$$+\frac{4^{14}h^{14}}{14!}y^{(xiv)}+\frac{4^{15}h^{15}}{15!}y^{(xv)}+\frac{4^{16}h^{16}}{16!}y^{(xvi)}-\frac{4^{17}h^{17}}{17!}y^{(xvii)}+\frac{4^{18}h^{18}}{18!}y^{(xviii)}+\frac{4^{19}h^{19}}{18!}y^{(xix)}$$
$$+\frac{4^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$-[y+5hy'+\frac{5^2h^2}{2!}y''+\frac{5^3h^3}{3!}y'''+\frac{5^4h^4}{4!}y^{(iv)}+\frac{5^5h^5}{5!}y^{(v)}+\frac{5^6h^6}{6!}y^{(vi)}+\frac{5^7h^7}{7!}y^{(vii)}$$
$$+\frac{5^8h^8}{8!}y^{(viii)}+\frac{5^9h^9}{9!}y^{(ix)}+\frac{5^{10}h^{10}}{10!}y^{(x)}+\frac{5^{11}h^{11}}{11!}y^{(xi)}+\frac{5^{12}h^{12}}{12!}y^{(xii)}+\frac{5^{13}h^{13}}{13!}y^{(xiii)}$$
$$+\frac{5^{14}h^{14}}{14!}y^{(xiv)}+\frac{5^{15}h^{15}}{15!}y^{(xv)}+\frac{5^{16}h^{16}}{16!}y^{(xvi)}+\frac{5^{17}h^{17}}{17!}y^{(xvii)}+\frac{5^{18}h^{18}}{18!}y^{(xviii)}$$
$$+\frac{5^{19}h^{19}}{19!}y^{(xix)}+\frac{5^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$-27[y-3hy'+\frac{2^2h^2}{2!}y''-\frac{3^3h^3}{3!}y'''+\frac{3^4h^6}{4!}y^{(iv)}-\frac{3^5h^5}{5!}y^{(v)}+\frac{3^6}{6!}y^{(vi)}$$
$$-\frac{3^7h^7}{7!}y^{(vii)}+\frac{3^8}{8!}y^{(viii)}-\frac{3^9h^9}{9!}y^{(ix)}+\frac{2^{10h^{10}}}{10!}y^{(x)}-\frac{3^{11}h^{11}}{11!}y^{(xi)}+\frac{3^{12}h^{12}}{12!}y^{(xii)}$$
$$-\frac{3^{13}h^{13}}{13!}y^{(xiii)}+\frac{3^{14}h^{14}}{14!}y^{(xiv)}-\frac{3^{15}h^{15}}{15!}y^{(xv)}+\frac{3^{16}h^{16}}{16!}y^{(xvi)}-\frac{3^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{3^{18}h^{18}}{18!}y^{(xviii)}-\frac{3^{19}h^{19}}{19!}y^{(xix)}+\frac{3^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+6h^2[y''-3hy'''+\frac{3^2h^2}{2!}y^{iv}-\frac{3^3h^3}{3!}y^{(v)}+\frac{3^4h^4}{4!}y^{(vi)}-\frac{3^5h^5}{5!}y^{(vii)}+\frac{3^6h^6}{6!}y^{(viii)}$$
$$-\frac{3^7h^7}{7!}y^{(ix)}+\frac{3^8h^8}{8!}y^{(x)}-\frac{3^9h^9}{9!}y^{(xi)}+\frac{3^{10}h^{10}}{10!}y^{(xii)}-\frac{3^{11}h^{11}}{11!}y^{(xiii)}+\frac{3^{12}h^{12}}{12!}y^{(xiv)}$$
$$-\frac{3^{13}h^{13}}{13!}y^{(xv)}+\frac{3^{14}h^{14}}{14!}y^{(xvi)}-\frac{3^{15}h^{15}}{15!}y^{(xvii)}+\frac{3^{16}h^{16}}{16!}y^{(xviii)}-\frac{3^{17}h^{17}}{17!}y^{(xix)}$$
$$+\frac{3^{18}h^{18}}{18!}y^{(xx)}+\ldots]$$
$$-\frac{1}{2}h^4[y^{iv}-3hy^{(v)}+\frac{3^2h^2}{2!}y^{(vi)}-\frac{3^3h^3}{3!}y^{(vii)}+\frac{3^4h^4}{4!}y^{(viii)}-\frac{3^5h^5}{5!}y^{(ix)}$$
$$+\frac{3^6h^6}{6!}y^{(x)}-\frac{3^7h^7}{7!}y^{(xi)}+\frac{3^8h^8}{8!}y^{(xii)}-\frac{3^9h^9}{9!}y^{(xiii)}+\frac{3^{10}h^{10}}{10!}y^{(xiv)}-\frac{3^{11}h^{11}}{11!}y^{(xv)}$$
$$+\frac{3^{12}h^{12}}{12!}y^{(xvi)}-\frac{3^{13}h^{13}}{13!}y^{(xvii)}+\frac{3^{14}h^{14}}{14!}y^{(xviii)}-\frac{3^{15}h^{15}}{15!}y^{(xix)}+\frac{3^{16}h^{16}}{16!}y^{(xx)}+\ldots]$$
$$-\frac{3}{20}h^6[y^{vi}-2hy^{(vii)}+\frac{2^2h^2}{2!}y^{(viii)}-\frac{2^3h^3}{3!}y^{(ix)}+\frac{2^4h^4}{4!}y^{(x)}-\frac{2^5h^5}{5!}y^{(xi)}$$
$$+\frac{2^6h^6}{6!}y^{(xii)}-\frac{2^7h^7}{7!}y^{(xiii)}+\frac{2^8h^8}{8!}y^{(xiv)}-\frac{2^9h^9}{9!}y^{(xv)}+\frac{2^{10}h^{10}}{10!}y^{(xvi)}$$
$$-\frac{2^{11}h^{11}}{11!}y^{(xvii)}+\frac{2^{12}h^{12}}{12!}y^{(xviii)}-\frac{2^{13}h^{13}}{13!}y^{(xix)}+\frac{2^{14}h^{14}}{14!}y^{(xx)}+\ldots]$$

$$-\frac{41}{3360}h^8[y^{(viii)} - 3hy^{(ix)} + \frac{3^2h^2}{2!}y^{(x)} - \frac{3^3h^3}{3!}y^{(xi)} + \frac{3^4h^4}{4!}y^{(xii)} - \frac{3^5h^5}{5!}y^{(xiii)}$$

$$+\frac{3^6h^6}{6!}y^{(xiv)} - \frac{3^7h^7}{7!}y^{(xv)} + \frac{3^8h^8}{8!}y^{(xvi)} - \frac{3^9h^9}{9!}y^{(xvii)} + \frac{3^{10}h^{10}}{10!}y^{(xviii)}$$

$$-\frac{3^{11}h^{11}}{11!}y^{(xix)} + \frac{3^{12}h^{12}}{12!}y^{(xx)} + \ldots]$$

$$+h^{10}\gamma_0[y - 3hy' + \frac{3^2h^2}{2!}y'' - \frac{3^3h^3}{3!}y''' + \frac{3^4h^6}{4!}y^{(iv)} - \frac{3^5h^5}{5!}y^{(v)} + \frac{3^6h^6}{6!}y^{(vi)}$$

$$-\frac{3^7h^7}{7!}y^{(vii)} + \frac{3^8h^8}{8!}y^{(viii)} - \frac{3^9h^9}{9!}y^{(ix)} + \frac{3^{10}h^{10}}{10!}y^{(x)} - \frac{3^{11}h^{11}}{11!}y^{(xi)} + \frac{3^{12}h^{12}}{12!}y^{(xii)}$$

$$-\frac{3^{13}h^{13}}{13!}y^{(xiii)} + \frac{3^{14}h^{14}}{14!}y^{(xiv)} - \frac{3^{15}h^{15}}{15!}y^{(xv)} + \frac{3^{16}h^{16}}{16!}y^{(xvi)} - \frac{3^{17}h^{17}}{17!}y^{(xvii)}$$

$$+\frac{3^{18}h^{18}}{18!}y^{(xviii)} - \frac{3^{19}h^{19}}{19!}y^{(xix)} + \frac{3^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+h^{10}\gamma_1[y - 2hy' + \frac{2^2h^2}{2!}y'' - \frac{2^3h^3}{3!}y''' + \frac{2^4h^6}{4!}y^{(iv)} - \frac{2^5h^5}{5!}y^{(v)} + \frac{2^6h^6}{6!}y^{(vi)}$$

$$-\frac{2^7h^7}{7!}y^{(vii)} + \frac{2^8h^8}{8!}y^{(viii)} - \frac{2^9h^9}{9!}y^{(ix)} + \frac{2^{10}h^{10}}{10!}y^{(x)} - \frac{2^{11}h^{11}}{11!}y^{(xi)} + \frac{2^{12}h^{12}}{12!}y^{(xii)}$$

$$-\frac{2^{13}h^{13}}{13!}y^{(xiii)} + \frac{2^{14}h^{14}}{14!}y^{(xiv)} - \frac{2^{15}h^{15}}{15!}y^{(xv)} + \frac{2^{16}h^{16}}{16!}y^{(xvi)} - \frac{2^{17}h^{17}}{17!}y^{(xvii)} + \frac{2^{18}h^{18}}{18!}y^{(xviii)}$$

$$-\frac{2^{19}h^{19}}{19!}y^{(xix)} + \frac{2^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+h^{10}\gamma_2[y - hy' + \frac{h^2}{2!}y'' - \frac{h^3}{3!}y''' + \frac{h^4}{4!}y^{(iv)} - \frac{h^5}{5!}y^{(v)} + \frac{h^6}{6!}y^{(vi)} - \frac{h^7}{7!}y^{(vii)}$$

$$+\frac{h^8}{8!}y^{(viii)} - \frac{h^9}{9!}y^{(ix)} + \frac{h^{10}}{10!}y^{(x)} - \frac{h^{11}}{11!}y^{(xi)} + \frac{h^{12}}{12!}y^{(xii)} - \frac{h^{13}}{13!}y^{(xiii)} + \frac{h^{14}}{14!}y^{(xiv)}$$

$$-\frac{h^{15}}{15!}y^{(xv)} + \frac{h^{16}}{16!}y^{(xvi)} - \frac{h^{17}}{17!}y^{(xvii)} + \frac{h^{18}}{18!}y^{(xviii)} - \frac{h^{19}}{19!}y^{(xix)} + \frac{h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+h^{10}\gamma_3 y^{(x)}$$

$$+h^{10}\gamma_4[y + hy' + \frac{h^2}{2!}y'' + \frac{h^3}{3!}y''' + \frac{h^4}{4!}y^{(iv)} + \frac{h^5}{5!}y^{(v)} + \frac{h^6}{6!}y^{(vi)} + \frac{h^7}{7!}y^{(vii)}$$

$$+\frac{h^8}{8!}y^{(viii)} + \frac{h^9}{9!}y^{(ix)} + \frac{h^{10}}{10!}y^{(x)} + \frac{h^{11}}{11!}y^{(xi)} + \frac{h^{12}}{12!}y^{(xii)} + \frac{h^{13}}{13!}y^{(xiii)} + \frac{h^{14}}{14!}y^{(xiv)}$$

$$+\frac{h^{15}}{15!}y^{(xv)} + \frac{h^{16}}{16!}y^{(xvi)} + \frac{h^{17}}{17!}y^{(xvii)} + \frac{h^{18}}{18!}y^{(xviii)} + \frac{h^{19}}{19!}y^{(xix)} + \frac{h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+h^{10}\gamma_5[y + 2hy' + \frac{2^2h^2}{2!}y'' + \frac{2^3h^3}{3!}y''' + \frac{2^4h^6}{4!}y^{(iv)} + \frac{2^5h^5}{5!}y^{(v)} + \frac{2^6h^6}{6!}y^{(vi)}$$

$$+\frac{2^7h^7}{7!}y^{(vii)} + \frac{2^8h^8}{8!}y^{(viii)} + \frac{2^9h^9}{9!}y^{(ix)} + \frac{2^{10}h^{10}}{10!}y^{(x)} + \frac{2^{11}h^{11}}{11!}y^{(xi)} + \frac{2^{12}h^{12}}{12!}y^{(xii)}$$

$$+\frac{2^{13}h^{13}}{13!}y^{(xiii)} + \frac{2^{14}h^{14}}{14!}y^{(xiv)} + \frac{2^{15}h^{15}}{15!}y^{(xv)} + \frac{2^{16}h^{16}}{16!}y^{(xvi)} + \frac{2^{17}h^{17}}{17!}y^{(xvii)}$$

$$+\frac{2^{18}h^{18}}{18!}y^{(xviii)} + \frac{2^{19}h^{19}}{19!}y^{(xix)} + \frac{2^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+h^{10}\gamma_6[y + 3hy' + \frac{3^2h^2}{2!}y'' + \frac{3^3h^3}{3!}y''' + \frac{3^4h^4}{4!}y^{(iv)} + \frac{3^5h^5}{5!}y^{(v)} + \frac{3^6h^6}{6!}y^{(vi)}$$

$$+\frac{3^7h^7}{7!}y^{(vii)} + \frac{3^8h^8}{8!}y^{(viii)} + \frac{3^9h^9}{9!}y^{(ix)} + \frac{3^{10}h^{10}}{10!}y^{(x)} + \frac{3^{11}h^{11}}{11!}y^{(xi)} + \frac{3^{12}h^{12}}{12!}y^{(xii)}$$

$$+\frac{3^{13}h^{13}}{13!}y^{(xiii)} + \frac{3^{14}h^{14}}{14!}y^{(xiv)} + \frac{3^{15}h^{15}}{15!}y^{(xv)} + \frac{3^{16}h^{16}}{16!}y^{(xvi)} + \frac{3^{17}h^{17}}{17!}y^{(xvii)}$$

$$+\frac{3^{18}h^{18}}{18!}y^{(xviii)} + \frac{3^{19}h^{19}}{19!}y^{(xix)} + \frac{3^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$

$$+h^{10}\gamma_7[y + 4hy' + \frac{4^2h^2}{2!}y'' + \frac{4^3h^3}{3!}y''' + \frac{4^4h^4}{4!}y^{(iv)} + \frac{4^5h^5}{5!}y^{(v)} + \frac{4^6h^6}{6!}y^{(vi)}$$

$$+\frac{4^7h^7}{7!}y^{(vii)} + \frac{4^8h^8}{8!}y^{(viii)} + \frac{4^9h^9}{9!}y^{(ix)} + \frac{4^{10}h^{10}}{10!}y^{(x)} + \frac{4^{11}h^{11}}{11!}y^{(xi)}$$
$$+\frac{4^{12}h^{12}}{12!}y^{(xii)} + \frac{4^{13}h^{13}}{13!}y^{(xiii)} + \frac{4^{14}h^{14}}{14!}y^{(xiv)} + \frac{4^{15}h^{15}}{15!}y^{(xv)} + \frac{4^{16}h^{16}}{16!}y^{(xvi)}$$
$$+\frac{4^{17}h^{17}}{17!}y^{(xvii)} + \frac{4^{18}h^{18}}{18!}y^{(xviii)} + \frac{4^{19}h^{19}}{19!}y^{(xix)}] + \frac{4^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$
$$+h^{10}\gamma_8[y + 5hy' + \frac{5^2h^2}{2!}y'' + \frac{5^3h^3}{3!}y''' + \frac{5^4h^4}{4!}y^{(iv)} + \frac{5^5h^5}{5!}y^{(v)} + \frac{5^6h^6}{6!}y^{(vi)}$$
$$+\frac{5^7h^7}{7!}y^{(vii)} + \frac{5^8h^8}{8!}y^{(viii)} + \frac{5^9h^9}{9!}y^{(ix)} + \frac{5^{10}h^{10}}{10!}y^{(x)} + \frac{5^{11}h^{11}}{11!}y^{(xi)}$$
$$+\frac{5^{12}h^{12}}{12!}y^{(xii)} + \frac{5^{13}h^{13}}{13!}y^{(xiii)} + \frac{5^{14}h^{14}}{14!}y^{(xiv)} + \frac{5^{15}h^{15}}{15!}y^{(xv)} + \frac{5^{16}h^{16}}{16!}y^{(xvi)}$$
$$+\frac{5^{17}h^{17}}{17!}y^{(xvii)} + \frac{5^{18}h^{18}}{18!}y^{(xviii)} + \frac{5^{19}h^{19}}{19!}y^{(xix)} + \frac{5^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$
$$+h^{10}\gamma_9[y + 6hy' + \frac{6^2h^2}{2!}y'' + \frac{6^3h^3}{3!}y''' + \frac{6^4h^4}{4!}y^{(iv)} + \frac{6^5h^5}{5!}y^{(v)} + \frac{6^6h^6}{6!}y^{(vi)}$$
$$+\frac{6^7h^7}{7!}y^{(vii)} + \frac{6^8h^8}{8!}y^{(viii)} + \frac{6^9h^9}{9!}y^{(ix)} + \frac{6^{10}h^{10}}{10!}y^{(x)} + \frac{6^{11}h^{11}}{11!}y^{(xi)} + \frac{6^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{6^{13}h^{13}}{13!}y^{(xiii)} + \frac{6^{14}h^{14}}{14!}y^{(xiv)} + \frac{6^{15}h^{15}}{15!}y^{(xv)} + \frac{6^{16}h^{16}}{16!}y^{(xvi)} + \frac{6^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{6^{18}h^{18}}{18!}y^{(xviii)} + \frac{6^{19}h^{19}}{19!}y^{(xix)} + \frac{6^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$
$$+h^{10}\gamma_{10}[y + 7hy' + \frac{7^2h^2}{2!}y'' + \frac{7^3h^3}{3!}y''' + \frac{7^4h^4}{4!}y^{(iv)} + \frac{7^5h^5}{5!}y^{(v)} + \frac{7^6h^6}{6!}y^{(vi)}$$
$$+\frac{7^7h^7}{7!}y^{(vii)} + \frac{7^8h^8}{8!}y^{(viii)} + \frac{7^9h^9}{9!}y^{(ix)} + \frac{7^{10}h^{10}}{10!}y^{(x)} + \frac{7^{11}h^{11}}{11!}y^{(xi)} + \frac{7^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{7^{13}h^{13}}{13!}y^{(xiii)} + \frac{7^{14}h^{14}}{14!}y^{(xiv)} + \frac{7^{15}h^{15}}{15!}y^{(xv)} + \frac{7^{16}h^{16}}{16!}y^{(xvi)} + \frac{7^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{7^{18}h^{18}}{18!}y^{(xviii)} + \frac{7^{19}h^{19}}{19!}y^{(xix)} + \frac{7^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$
$$+h^{10}\gamma_{11}[y + 8hy' + \frac{8^2h^2}{2!}y'' + \frac{8^3h^3}{3!}y''' + \frac{8^4h^4}{4!}y^{(iv)} + \frac{8^5h^5}{5!}y^{(v)} + \frac{8^6h^6}{6!}y^{(vi)}$$
$$+\frac{8^7h^7}{7!}y^{(vii)} + \frac{8^8h^8}{8!}y^{(viii)} + \frac{8^9h^9}{9!}y^{(ix)} + \frac{8^{10}h^{10}}{10!}y^{(x)} + \frac{8^{11}h^{11}}{11!}y^{(xi)} + \frac{8^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{8^{13}h^{13}}{13!}y^{(xiii)} + \frac{8^{14}h^{14}}{14!}y^{(xiv)} + \frac{8^{15}h^{15}}{15!}y^{(xv)} + \frac{8^{16}h^{16}}{16!}y^{(xvi)} + \frac{8^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{8^{18}h^{18}}{18!}y^{(xviii)} + \frac{8^{19}h^{19}}{19!}y^{(xix)} + \frac{8^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$
$$+h^{10}\gamma_{12}[y + 9hy' + \frac{9^2h^2}{2!}y'' + \frac{9^3h^3}{3!}y''' + \frac{9^4h^4}{4!}y^{(iv)} + \frac{9^5h^5}{5!}y^{(v)} + \frac{9^6h^6}{6!}y^{(vi)}$$
$$+\frac{9^7h^7}{7!}y^{(vii)} + \frac{9^8h^8}{8!}y^{(viii)} + \frac{9^9h^9}{9!}y^{(ix)} + \frac{9^{10}h^{10}}{10!}y^{(x)} + \frac{9^{11}h^{11}}{11!}y^{(xi)} + \frac{9^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{9^{13}h^{13}}{13!}y^{(xiii)} + \frac{9^{14}h^{14}}{14!}y^{(xiv)} + \frac{9^{15}h^{15}}{15!}y^{(xv)} + \frac{9^{16}h^{16}}{16!}y^{(xvi)} + \frac{9^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{9^{18}h^{18}}{18!}y^{(xviii)} + \frac{9^{19}h^{19}}{19!}y^{(xix)} + \frac{9^{20}h^{20}}{20!}y^{(xx)} + \ldots]. \tag{2.56}$$

Consider (2.56) and equate the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}$, $y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}$ to those in (2.8). This gives

$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = \frac{302231}{302400}, \tag{2.57}$$

$$-3\gamma_0 - 2\gamma_1 - \gamma_2 + 2\gamma_4 = \frac{169}{100800}, \tag{2.58}$$

$$3^2\frac{\gamma_0}{2!}+2^2\frac{\gamma_1}{2!}+\frac{\gamma_2}{2!}+\frac{\gamma_4}{2!}=\frac{5510311}{13305600}-\frac{5}{12}, \tag{2.59}$$

$$-3^3\frac{\gamma_0}{3!}-3^2\frac{\gamma_1}{3!}-\frac{\gamma_2}{3!}+\frac{\gamma_4}{3!}=\frac{11381}{4435200}, \tag{2.60}$$

$$3^4\frac{\gamma_0}{4!}+2^4\frac{\gamma_1}{4!}+\frac{\gamma_2}{4!}+\frac{\gamma_4}{4!}=\frac{591141643}{7264857600}-\frac{1}{12}. \tag{2.61}$$

Solving this system, we get the parameters of the third end-point formula (i.e.$x = x_3$) for the second-order method. It is noted that the parameters $\gamma_i (i = 5, 6, 7, \ldots, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned} \gamma_0 &= -\tfrac{160883}{264176640}, \\ \gamma_1 &= \tfrac{27127}{181621440}, \\ \gamma_2 &= \tfrac{132647}{807206400}, \\ \gamma_3 &= \tfrac{14190326}{14189175}, \\ \gamma_4 &= -\tfrac{46537}{2905943040}. \end{aligned}\right\} . \tag{2.62}$$

Using the method of undetermined coefficients reveals that for the point $x = x_4$ the first two nonvanishing terms in the local truncation error have the values given in (2.8) provided

$$d_0 = 8, d_2 = -1, d_4 = \frac{-1}{12}, d_6 = \frac{-1}{360}, d_8 = \frac{-1}{20160}, \tag{2.63}$$

together with parameters δ_i $(i = 0, 1, 2 \ldots, 12)$ calculated from the expression

$$\left.\begin{aligned} L[y(x_4); h] = \; & -44y(x-3h)+120y(x-2h)-210y(x-h) \\ & +252y(x)-210y(x+h)+120y(x+2h) \\ & -45y(x+3h)+10y(x+4h)-y(x+5h) \\ & +8y(x-4h)-h^2y''(x-4h)-\tfrac{1}{12}h^4y^{(iv)}(x-4h) \\ & -\tfrac{1}{360}h^6y^{(vi)}(x-4h)-\tfrac{1}{20160}h^8y^{(viii)}(x-4h) \\ & +h^{10}[\delta_0y^{(x)}(x-4h)+\delta_1y^{(x)}(x-3h) \\ & +\delta_2y^{(x)}(x-2h)+\delta_3y^{(x)}(x-h)+\delta_4y^{(x)}(x) \\ & +\delta_5y^{(x)}(x+h)+\delta_6y^{(x)}(x+2h)+\delta_7y^{(x)}(x+3h) \\ & +\delta_8y^{(x)}(x+4h)+\delta_9y^{(x)}(x+5h)+\delta_{10}y^{(x)}(x+6h) \\ & +\delta_{11}y^{(x)}(x+7h)+\delta_{12}y^{(x)}(x+8h)+\ldots]. \end{aligned}\right\} . \tag{2.64}$$

Expanding the terms in (2.64) about y(x) and its derivatives gives, at the point $x = x_4$,

$$
\begin{aligned}
L[y(x_4); h] = \; & -44[y - 3hy' + \tfrac{3^2h^2}{2!}y'' - \tfrac{3^3h^3}{3!}y''' + \tfrac{3^4h^4}{4!}y^{(iv)} - \tfrac{3^5h^5}{5!}y^{(v)} \\
& + \tfrac{3^6h^6}{6!}y^{(vi)} - \tfrac{3^7h^7}{7!}y^{(vii)} + \tfrac{3^8h^8}{8!}y^{(viii)} - \tfrac{3^9h^9}{9!}y^{(ix)} + \tfrac{3^{10}h^{10}}{10!}y^{(x)} \\
& - \tfrac{3^{11}h^{11}}{11!}y^{(xi)} + \tfrac{3^{12}h^{12}}{12!}y^{(xii)} - \tfrac{3^{13}h^{13}}{13!}y^{(xiii)} + \tfrac{3^{14}h^{14}}{14!}y^{(xiv)} \\
& - \tfrac{3^{15}h^{15}}{15!}y^{(xv)} + \tfrac{3^{16}h^{16}}{16!}y^{(xvi)} - \tfrac{3^{17}h^{17}}{17!}y^{(xvii)} + \tfrac{3^{18}h^{18}}{18!}y^{(xviii)} \\
& - \tfrac{3^{19}h^{19}}{19!}y^{(xix)} + \tfrac{3^{20}h^{20}}{20!}y^{(xx)} + \ldots] \\
& +120[y - 2hy' + \tfrac{2^2h^2}{2!}y'' - \tfrac{2^3h^3}{3!}y''' + \tfrac{2^4h^4}{4!}y^{(iv)} - \tfrac{2^5h^5}{5!}y^{(v)} + \tfrac{2^6h^6}{6!}y^{(vi)} \\
& - \tfrac{2^7h^7}{7!}y^{(vii)} + \tfrac{2^8h^8}{8!}y^{(viii)} - \tfrac{2^9h^9}{9!}y^{(ix)} + \tfrac{2^{10}h^{10}}{10!}y^{(x)} - \tfrac{2^{11}h^{11}}{11!}y^{(xi)} \\
& + \tfrac{2^{12}h^{12}}{12!}y^{(xii)} - \tfrac{2^{13}h^{13}}{13!}y^{(xiii)} + \tfrac{2^{14}h^{14}}{14!}y^{(xiv)} - \tfrac{2^{15}h^{15}}{15!}y^{(xv)} + \tfrac{2^{16}h^{16}}{16!}y^{(xvi)} \\
& - \tfrac{2^{17}h^{17}}{17!}y^{(xvii)} + \tfrac{2^{18}h^{18}}{18!}y^{(xviii)} - \tfrac{2^{19}h^{19}}{19!}y^{(xix)} + \tfrac{2^{20}h^{20}}{20!}y^{(xx)} + \ldots] \\
& -210[y - hy' + \tfrac{h^2}{2!}y'' - \tfrac{h^3}{3!}y''' + \tfrac{h^4}{4!}y^{(iv)} - \tfrac{h^5}{5!}y^{(v)} + \tfrac{h^6}{6!}y^{(vi)} - \tfrac{h^7}{7!}y^{(vii)} \\
& + \tfrac{h^8}{8!}y^{(viii)} - \tfrac{h^9}{9!}y^{(ix)} + \tfrac{h^{10}}{10!}y^{(x)} - \tfrac{h^{11}}{11!}y^{(xi)} + \tfrac{h^{12}}{12!}y^{(xii)} - \tfrac{h^{13}}{13!}y^{(xiii)} \\
& + \tfrac{h^{14}}{14!}y^{(xiv)} - \tfrac{h^{15}}{15!}y^{(xv)} + \tfrac{h^{16}}{16!}y^{(xvi)} - \tfrac{h^{17}}{17!}y^{(xvii)} + \tfrac{h^{18}}{18!}y^{(xviii)} - \tfrac{h^{19}}{19!}y^{(xix)} \\
& + \tfrac{h^{20}}{20!}y^{(xx)} + \ldots] + 252y(x) \\
& -210[y + hy' + \tfrac{h^2}{2!}y'' + \tfrac{h^3}{3!}y''' + \tfrac{h^4}{4!}y^{(iv)} + \tfrac{h^5}{5!}y^{(v)} + \tfrac{h^6}{6!}y^{(vi)} + \tfrac{h^7}{7!}y^{(vii)} \\
& + \tfrac{h^8}{8!}y^{(viii)} + \tfrac{h^9}{9!}y^{(ix)} + \tfrac{h^{10}}{10!}y^{(x)} + \tfrac{h^{11}}{11!}y^{(xi)} + \tfrac{h^{12}}{12!}y^{(xii)} + \tfrac{h^{13}}{13!}y^{(xiii)} \\
& + \tfrac{h^{14}}{14!}y^{(xiv)} + \tfrac{h^{15}}{15!}y^{(xv)} + \tfrac{h^{16}}{16!}y^{(xvi)} + \tfrac{h^{17}}{17!}y^{(xvii)} + \tfrac{h^{18}}{18!}y^{(xviii)} + \tfrac{h^{19}}{19!}y^{(xix)} \\
& + \tfrac{h^{20}}{20!}y^{(xx)} + \ldots] \\
& +120[y + 2hy' + \tfrac{2^2h^2}{2!}y'' + \tfrac{2^3h^3}{3!}y''' + \tfrac{2^4h^6}{4!}y^{(iv)} + \tfrac{2^5h^5}{5!}y^{(v)} + \tfrac{2^6h^6}{6!}y^{(vi)} \\
& + \tfrac{2^7h^7}{7!}y^{(vii)} + \tfrac{2^8h^8}{8!}y^{(viii)} + \tfrac{2^9h^9}{9!}y^{(ix)} + \tfrac{2^{10}h^{10}}{10!}y^{(x)} + \tfrac{2^{11}h^{11}}{11!}y^{(xi)} \\
& + \tfrac{2^{12}h^{12}}{12!}y^{(xii)} + \tfrac{2^{13}h^{13}}{13!}y^{(xiii)} + \tfrac{2^{14}h^{14}}{14!}y^{(xiv)} + \tfrac{2^{15}h^{15}}{15!}y^{(xv)} + \tfrac{2^{16}h^{16}}{16!}y^{(xvi)} \\
& + \tfrac{2^{17}h^{17}}{17!}y^{(xvii)} + \tfrac{2^{18}h^{18}}{18!}y^{(xviii)} + \tfrac{2^{19}h^{19}}{19!}y^{(xix)} + \tfrac{2^{20}h^{20}}{20!}y^{(xx)} + \ldots] \\
& -45[y + 3hy' + \tfrac{3^2h^2}{2!}y'' + \tfrac{3^3h^3}{3!}y''' + \tfrac{3^4h^4}{4!}y^{(iv)} + \tfrac{3^5h^5}{5!}y^{(v)} + \tfrac{3^6h^6}{6!}y^{(vi)} \\
& + \tfrac{3^7h^7}{7!}y^{(vii)} + \tfrac{3^8h^8}{8!}y^{(viii)} + \tfrac{3^9h^9}{9!}y^{(ix)} + \tfrac{3^{10}h^{10}}{10!}y^{(x)} + \tfrac{3^{11}h^{11}}{11!}y^{(xi)} \\
& + \tfrac{3^{12}h^{12}}{12!}y^{(xii)} + \tfrac{3^{13}h^{13}}{13!}y^{(xiii)} + \tfrac{3^{14}h^{14}}{14!}y^{(xiv)} + \tfrac{3^{15}h^{15}}{15!}y^{(xv)} + \tfrac{3^{16}h^{16}}{16!}y^{(xvi)} \\
& + \tfrac{3^{17}h17}{17!}y^{(xvii)} + \tfrac{3^{18}h^{18}}{18!}y^{(xviii)} + \tfrac{3^{19}h^{19}}{19!}y^{(xix)} + \tfrac{3^{20}h^{20}}{20!}y^{(xx)} + \ldots] \\
& +10[y + 4hy' + \tfrac{4^2h^2}{2!}y'' + \tfrac{4^3h^3}{3!}y''' + \tfrac{4^4h^4}{4!}y^{(iv)} + \tfrac{4^5h^5}{5!}y^{(v)} + \tfrac{4^6h^6}{6!}y^{(vi)} \\
& + \tfrac{4^7h^7}{7!}y^{(vii)} + \tfrac{4^8h^8}{8!}y^{(viii)} + \tfrac{4^9h^9}{9!}y^{(ix)} + \tfrac{4^{10}h^{10}}{10!}y^{(x)} + \tfrac{4^{11}h^{11}}{11!}y^{(xi)} + \tfrac{4^{12}h^{12}}{12!}y^{(xii)}
\end{aligned}
$$

$$+\frac{4^{13}h^{13}}{13!}y^{(xiii)}+\frac{4^{14}h^{14}}{14!}y^{(xiv)}+\frac{4^{15}h^{15}}{15!}y^{(xv)}+\frac{4^{16}h^{16}}{16!}y^{(xvi)}-\frac{4^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{4^{18}h^{18}}{18!}y^{(xviii)}+\frac{4^{19}h^{19}}{19!}y^{(xix)}+\frac{4^{24}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$-[y+5hy'+\frac{5^2h^2}{2!}y''+\frac{5^3h^3}{3!}y'''+\frac{5^4h^4}{4!}y^{(iv)}+\frac{5^5h^5}{5!}y^{(v)}+\frac{5^6h^6}{6!}y^{(vi)}+\frac{5^7h^7}{7!}y^{(vii)}$$
$$+\frac{5^8h^8}{8!}y^{(viii)}+\frac{5^9h^9}{9!}y^{(ix)}+\frac{5^{10}h^{10}}{10!}y^{(x)}+\frac{5^{11}h^{11}}{11!}y^{(xi)}+\frac{5^{12}h^{12}}{12!}y^{(xii)}+\frac{5^{13}h^{13}}{13!}y^{(xiii)}$$
$$+\frac{5^{14}h^{14}}{14!}y^{(xiv)}+\frac{5^{15}h^{15}}{15!}y^{(xv)}+\frac{5^{16}h^{16}}{16!}y^{(xvi)}+\frac{5^{17}h^{17}}{17!}y^{(xvii)}+\frac{5^{18}h^{18}}{18!}y^{(xviii)}$$
$$+\frac{5^{19}h^{19}}{19!}y^{(xix)}+\frac{5^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+8[y-4hy'+\frac{4^2h^2}{2!}y''-\frac{4^3h^3}{3!}y'''+\frac{4^4h^6}{4!}y^{(iv)}-\frac{4^5h^5}{5!}y^{(v)}+\frac{4^6h^6}{6!}y^{(vi)}-\frac{4^7h^7}{7!}y^{(vii)}$$
$$+\frac{4^8h^8}{8!}y^{(viii)}-\frac{4^9h^9}{9!}y^{(ix)}+\frac{4^{10}h^{10}}{10!}y^{(x)}-\frac{4^{11}h^{11}}{11!}y^{(xi)}+\frac{4^{12}h^{12}}{12!}y^{(xii)}-\frac{4^{13}h^{13}}{13!}y^{(xiii)}$$
$$+\frac{4^{14}h^{14}}{14!}y^{(xiv)}-\frac{4^{15}h^{15}}{15!}y^{(xv)}+\frac{4^{16}h^{16}}{16!}y^{(xvi)}-\frac{4^{17}h^{17}}{17!}y^{(xvii)}+\frac{4^{18}h^{18}}{18!}y^{(xviii)}$$
$$-\frac{4^{19}h^{19}}{19!}y^{(xix)}+\frac{4^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$-h^2[y''-3hy'''+\frac{4^2h^2}{2!}y^{iv}-\frac{4^3h^3}{3!}y^{(v)}+\frac{4^4h^4}{4!}y^{(vi)}-\frac{4^5h^5}{5!}y^{(vii)}+\frac{4^6h^6}{6!}y^{(viii)}$$
$$-\frac{4^7h^7}{7!}y^{(ix)}+\frac{4^8h^8}{8!}y^{(x)}-\frac{4^9h^9}{9!}y^{(xi)}+\frac{4^{10}h^{10}}{10!}y^{(xii)}-\frac{4^{11}h^{11}}{11!}y^{(xiii)}+\frac{4^{12}h^{12}}{12!}y^{(xiv)}$$
$$-\frac{4^{13}h^{13}}{13!}y^{(xv)}+\frac{4^{14}h^{14}}{14!}y^{(xvi)}-\frac{4^{15}h^{15}}{15!}y^{(xvii)}+\frac{4^{16}h^{16}}{16!}y^{(xviii)}-\frac{4^{17}h^{17}}{17!}y^{(xix)}$$
$$+\frac{4^{18}h^{18}}{18!}y^{(xx)}+\ldots]$$
$$-\frac{1}{12}h^4[y^{(iv)}-4hy^{(v)}+\frac{4^2h^2}{2!}y^{(vi)}-\frac{4^3h^3}{3!}y^{(vii)}+\frac{4^4h^4}{4!}y^{(viii)}-\frac{4^5h^5}{5!}y^{(ix)}$$
$$+\frac{4^6h^6}{6!}y^{(x)}-\frac{4^7h^7}{7!}y^{(xi)}+\frac{4^8h^8}{8!}y^{(xii)}-\frac{4^9h^9}{9!}y^{(xiii)}+\frac{4^{10}h^{10}}{10!}y^{(xiv)}-\frac{4^{11}h^{11}}{11!}y^{(xv)}$$
$$+\frac{4^{12}h^{12}}{12!}y^{(xvi)}-\frac{4^{13}h^{13}}{13!}y^{(xvii)}+\frac{4^{14}h^{14}}{14!}y^{(xviii)}-\frac{4^{15}h^{15}}{15!}y^{(xix)}+\frac{4^{16}h^{16}}{16!}y^{(xx)}+\ldots]$$
$$-\frac{1}{360}h^6[y^{(vi)}-4hy^{(vii)}+\frac{4^2h^2}{2!}y^{(viii)}-\frac{4^3h^3}{3!}y^{(ix)}+\frac{4^4h^4}{4!}y^{(x)}-\frac{4^5h^5}{5!}y^{(xi)}$$
$$+\frac{4^6h^6}{6!}y^{(xii)}-\frac{4^7h^7}{7!}y^{(xiii)}+\frac{4^8h^8}{8!}y^{(xiv)}-\frac{4^9h^9}{9!}y^{(xv)}+\frac{4^{10}h^{10}}{10!}y^{(xvi)}-\frac{4^{11}h^{11}}{11!}y^{(xvii)}$$
$$+\frac{4^{12}h^{12}}{12!}y^{(xviii)}-\frac{4^{13}h^{13}}{13!}y^{(xix)}+\frac{4^{14}h^{14}}{14!}y^{(xx)}+\ldots]$$
$$-\frac{1}{20160}h^8[y^{(viii)}-4hy^{(ix)}+\frac{4^2h^2}{2!}y^{(x)}-\frac{4^3h^3}{3!}y^{(xi)}+\frac{4^4h^4}{4!}y^{(xii)}-\frac{4^5h^5}{5!}y^{(xiii)}$$
$$+\frac{4^6h^6}{6!}y^{(xiv)}-\frac{4^7h^7}{7!}y^{(xv)}+\frac{4^8h^8}{8!}y^{(xvi)}-\frac{4^9h^9}{9!}y^{(xvii)}+\frac{4^{10}h^{10}}{10!}y^{(xviii)}$$
$$-\frac{4^{11}h^{11}}{11!}y^{(xix)}+\frac{4^{12}h^{12}}{12!}y^{(xx)}+\ldots]$$
$$+h^{10}\delta_0[y-4hy'+\frac{4^2h^2}{2!}y''-\frac{4^3h^3}{3!}y'''+\frac{4^4h^6}{4!}y^{(iv)}-\frac{4^5h^5}{5!}y^{(v)}+\frac{4^6h^6}{6!}y^{(vi)}$$
$$-\frac{4^7h^7}{7!}y^{(vii)}+\frac{4^8h^8}{8!}y^{(viii)}-\frac{4^9h^9}{9!}y^{(ix)}+\frac{4^{10}h^{10}}{10!}y^{(x)}-\frac{4^{11}h^{11}}{11!}y^{(xi)}+\frac{4^{12}h^{12}}{12!}y^{(xii)}$$
$$-\frac{4^{13}h^{13}}{13!}y^{(xiii)}+\frac{4^{14}h^{14}}{14!}y^{(xiv)}-\frac{4^{15}h^{15}}{15!}y^{(xv)}+\frac{4^{16}h^{16}}{16!}y^{(xvi)}-\frac{4^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{4^{18}h^{18}}{18!}y^{(xviii)}-\frac{4^{19}h^{19}}{19!}y^{(xix)}+\frac{4^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\delta_1[y-3hy'+\frac{3^2h^2}{2!}y''-\frac{3^3h^3}{3!}y'''+\frac{3^4h^6}{4!}y^{(iv)}-\frac{3^5h^5}{5!}y^{(v)}+\frac{3^6h^6}{6!}y^{(vi)}$$
$$-\frac{3^7h^7}{7!}y^{(vii)}+\frac{3^8h^8}{8!}y^{(viii)}-\frac{3^9h^9}{9!}y^{(ix)}+\frac{3^{10}h^{10}}{10!}y^{(x)}-\frac{3^{11}h^{11}}{11!}y^{(xi)}+\frac{3^{12}h^{12}}{12!}y^{(xii)}$$

$$-\frac{3^{13}h^{13}}{13!}y^{(xiii)}+\frac{3^{14}h^{14}}{14!}y^{(xiv)}-\frac{3^{15}h^{15}}{15!}y^{(xv)}+\frac{3^{16}h^{16}}{16!}y^{(xvi)}-\frac{3^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{3^{18}h^{18}}{18!}y^{(xviii)}-\frac{3^{19}h^{19}}{19!}y^{(xix)}+\frac{3^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\delta_2[y-2hy'+\frac{2^2h^2}{2!}y''-\frac{2^3h^3}{3!}y'''+\frac{2^4h^6}{4!}y^{(iv)}-\frac{2^5h^5}{5!}y^{(v)}+\frac{2^6h^6}{6!}y^{(vi)}$$
$$-\frac{2^7h^7}{7!}y^{(vii)}+\frac{2^8h^8}{8!}y^{(viii)}-\frac{2^9h^9}{9!}y^{(ix)}+\frac{2^{10}h^{10}}{10!}y^{(x)}-\frac{2^{11}h^{11}}{11!}y^{(xi)}+\frac{2^{12}h^{12}}{12!}y^{(xii)}$$
$$-\frac{2^{13}h^{13}}{13!}y^{(xiii)}+\frac{2^{14}h^{14}}{14!}y^{(xiv)}-\frac{2^{15}h^{15}}{15!}y^{(xv)}+\frac{2^{16}h^{16}}{16!}y^{(xvi)}-\frac{2^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{2^{18}h^{18}}{18!}y^{(xviii)}-\frac{2^{19}h^{19}}{19!}y^{(xix)}+\frac{2^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\delta_3[y-hy'+\frac{h^2}{2!}y''-\frac{h^3}{3!}y'''+\frac{h^4}{4!}y^{(iv)}-\frac{h^5}{5!}y^{(v)}+\frac{h^6}{6!}y^{(vi)}-\frac{h^7}{7!}y^{(vii)}$$
$$+\frac{h^8}{8!}y^{(viii)}-\frac{h^9}{9!}y^{(ix)}+\frac{h^{10}}{10!}y^{(x)}-\frac{h^{11}}{11!}y^{(xi)}+\frac{h^{12}}{12!}y^{(xii)}-\frac{h^{13}}{13!}y^{(xiii)}+\frac{h^{14}}{14!}y^{(xiv)}$$
$$-\frac{h^{15}}{15!}y^{(xv)}+\frac{h^{16}}{16!}y^{(xvi)}-\frac{h^{17}}{17!}y^{(xvii)}+\frac{h^{18}}{18!}y^{(xviii)}-\frac{h^{19}}{19!}y^{(xix)}+\frac{h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\delta_4y^{(x)}$$
$$+h^{10}\delta_5[y+hy'+\frac{h^2}{2!}y''+\frac{h^3}{3!}y'''+\frac{h^4}{4!}y^{(iv)}+\frac{h^5}{5!}y^{(v)}+\frac{h^6}{6!}y^{(vi)}+\frac{h^7}{7!}y^{(vii)}$$
$$+\frac{h^8}{8!}y^{(viii)}+\frac{h^9}{9!}y^{(ix)}+\frac{h^{10}}{10!}y^{(x)}+\frac{h^{11}}{11!}y^{(xi)}+\frac{h^{12}}{12!}y^{(xii)}+\frac{h^{13}}{13!}y^{(xiii)}+\frac{h^{14}}{14!}y^{(xiv)}$$
$$+\frac{h^{15}}{15!}y^{(xv)}+\frac{h^{16}}{16!}y^{(xvi)}+\frac{h^{17}}{17!}y^{(xvii)}+\frac{h^{18}}{18!}y^{(xviii)}+\frac{h^{19}}{19!}y^{(xix)}+\frac{h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\delta_6[y+2hy'+\frac{2^2h^2}{2!}y''+\frac{2^3h^3}{3!}y'''+\frac{2^4h^6}{4!}y^{(iv)}+\frac{2^5h^5}{5!}y^{(v)}+\frac{2^6h^6}{6!}y^{(vi)}$$
$$+\frac{2^7h^7}{7!}y^{(vii)}+\frac{2^8h^8}{8!}y^{(viii)}+\frac{2^9h^9}{9!}y^{(ix)}+\frac{2^{10}h^{10}}{10!}y^{(x)}+\frac{2^{11}h^{11}}{11!}y^{(xi)}+\frac{2^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{2^{13}h^{13}}{13!}y^{(xiii)}+\frac{2^{14}h^{14}}{14!}y^{(xiv)}+\frac{2^{15}h^{15}}{15!}y^{(xv)}+\frac{2^{16}h^{16}}{16!}y^{(xvi)}+\frac{2^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{2^{18}h^{18}}{18!}y^{(xviii)}+\frac{2^{19}h^{19}}{19!}y^{(xix)}+\frac{2^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\delta_7[y+3hy'+\frac{3^2h^2}{2!}y''+\frac{3^3h^3}{3!}y'''+\frac{3^4h^4}{4!}y^{(iv)}+\frac{3^5h^5}{5!}y^{(v)}+\frac{3^6h^6}{6!}y^{(vi)}$$
$$+\frac{3^7h^7}{7!}y^{(vii)}+\frac{3^8h^8}{8!}y^{(viii)}+\frac{3^9h^9}{9!}y^{(ix)}+\frac{3^{10}h^{10}}{10!}y^{(x)}+\frac{3^{11}h^{11}}{11!}y^{(xi)}+\frac{3^{12}h^{12}}{12!}y^{(xii)}$$
$$+\frac{3^{13}h^{13}}{13!}y^{(xiii)}+\frac{3^{14}h^{14}}{14!}y^{(xiv)}+\frac{3^{15}h^{15}}{15!}y^{(xv)}+\frac{3^{16}h^{16}}{16!}y^{(xvi)}+\frac{3^{17}h^{17}}{17!}y^{(xvii)}$$
$$+\frac{3^{18}h^{18}}{18!}y^{(xviii)}+\frac{3^{19}h^{19}}{19!}y^{(xix)}+\frac{3^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\delta_8[y+4hy'+\frac{4^2h^2}{2!}y''+\frac{4^3h^3}{3!}y^{(''')}+\frac{4^4h^4}{4!}y^{(iv)}+\frac{4^5h^5}{5!}y^{(v)}+\frac{4^6h^6}{6!}y^{(vi)}$$
$$+\frac{4^7h^7}{7!}y^{(vii)}+\frac{4^8h^8}{8!}y^{(viii)}+\frac{4^9h^9}{9!}y^{(ix)}+\frac{4^{10}h^{10}}{10!}y^{(x)}+\frac{4^{11}h^{11}}{11!}y^{(xi)}$$
$$+\frac{4^{12}h^{12}}{12!}y^{(xii)}+\frac{4^{13}h^{13}}{13!}y^{(xiii)}+\frac{4^{14}h^{14}}{14!}y^{(xiv)}+\frac{4^{15}h^{15}}{15!}y^{(xv)}+\frac{4^{16}h^{16}}{16!}y^{(xvi)}$$
$$+\frac{4^{17}h^{17}}{17!}y^{(xvii)}+\frac{4^{18}h^{18}}{18!}y^{(xviii)}+\frac{4^{19}h^{19}}{19!}y^{(xix)}+\frac{4^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$
$$+h^{10}\delta_9[y+5hy'+\frac{5^2h^2}{2!}y''+\frac{5^3h^3}{3!}y'''+\frac{5^4h^4}{4!}y^{(iv)}+\frac{5^5h^5}{5!}y^{(v)}+\frac{5^6h^6}{6!}y^{(vi)}$$
$$+\frac{5^7h^7}{7!}y^{(vii)}+\frac{5^8h^8}{8!}y^{(viii)}+\frac{5^9h^9}{9!}y^{(ix)}+\frac{5^{10}h^{10}}{10!}y^{(x)}+\frac{5^{11}h^{11}}{11!}y^{(xi)}$$
$$+\frac{5^{12}h^{12}}{12!}y^{(xii)}+\frac{5^{13}h^{13}}{13!}y^{(xiii)}+\frac{5^{14}h^{14}}{14!}y^{(xiv)}+\frac{5^{15}h^{15}}{15!}y^{(xv)}+\frac{5^{16}h^{16}}{16!}y^{(xvi)}$$
$$+\frac{5^{17}h^{17}}{17!}y^{(xvii)}+\frac{5^{18}h^{18}}{18!}y^{(xviii)}+\frac{5^{19}h^{19}}{19!}y^{(xix)}+\frac{5^{20}h^{20}}{20!}y^{(xx)}+\ldots]$$

$$+h^{10}\delta_{10}[y + 6hy' + \frac{6^2h^2}{2!}y'' + \frac{6^3h^3}{3!}y''' + \frac{6^4h^4}{4!}y^{(iv)} + \frac{6^5h^5}{5!}y^{(v)} + \frac{6^6h^6}{6!}y^{(vi)}$$
$$+\frac{6^7h^7}{7!}y^{(vii)} + \frac{6^8h^8}{8!}y^{(viii)} + \frac{6^9h^9}{9!}y^{(ix)} + \frac{6^{10}h^{10}}{10!}y^{(x)} + \frac{6^{11}h^{11}}{11!}y^{(xi)}$$
$$+\frac{6^{12}h^{12}}{12!}y^{(xii)} + \frac{6^{13}h^{13}}{13!}y^{(xiii)} + \frac{6^{14}h^{14}}{14!}y^{(xiv)} + \frac{6^{15}h^{15}}{15!}y^{(xv)} + \frac{6^{16}h^{16}}{16!}y^{(xvi)}$$
$$+\frac{6^{17}h^{17}}{17!}y^{(xvii)} + \frac{6^{18}h^{18}}{18!}y^{(xviii)} + \frac{6^{19}h^{19}}{19!}y^{(xix)} + \frac{6^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$
$$+h^{10}\delta_{11}[y + 7hy' + \frac{7^2h^2}{2!}y'' + \frac{7^3h^3}{3!}y''' + \frac{7^4h^4}{4!}y^{(iv)} + \frac{7^5h^5}{5!}y^{(v)} + \frac{7^6h^6}{6!}y^{(vi)}$$
$$+\frac{7^7h^7}{7!}y^{(vii)} + \frac{7^8h^8}{8!}y^{(viii)} + \frac{7^9h^9}{9!}y^{(ix)} + \frac{7^{10}h^{10}}{10!}y^{(x)} + \frac{7^{11}h^{11}}{11!}y^{(xi)}$$
$$+\frac{7^{12}h^{12}}{12!}y^{(xii)} + \frac{7^{13}h^{13}}{13!}y^{(xiii)} + \frac{7^{14}h^{14}}{14!}y^{(xiv)} + \frac{7^{15}h^{15}}{15!}y^{(xv)} + \frac{7^{16}h^{16}}{16!}y^{(xvi)}$$
$$+\frac{7^{17}h^{17}}{17!}y^{(xvii)} + \frac{7^{18}h^{18}}{18!}y^{(xviii)} + \frac{7^{19}h^{19}}{19!}y^{(xix)} + \frac{7^{20}h^{20}}{20!}y^{(xx)} + \ldots]$$
$$+h^{10}\delta_{12}[y + 8hy' + \frac{8^2h^2}{2!}y'' + \frac{8^3h^3}{3!}y''' + \frac{8^4h^4}{4!}y^{(iv)} + \frac{8^5h^5}{5!}y^{(v)} + \frac{8^6h^6}{6!}y^{(vi)}$$
$$+\frac{8^7h^7}{7!}y^{(vii)} + \frac{8^8h^8}{8!}y^{(viii)} + \frac{8^9h^9}{9!}y^{(ix)} + \frac{8^{10}h^{10}}{10!}y^{(x)} + \frac{8^{11}h^{11}}{11!}y^{(xi)}$$
$$+\frac{8^{12}h^{12}}{12!}y^{(xii)} + \frac{8^{13}h^{13}}{13!}y^{(xiii)} + \frac{8^{14}h^{14}}{14!}y^{(xiv)} + \frac{8^{15}h^{15}}{15!}y^{(xv)} + \frac{8^{16}h^{16}}{16!}y^{(xvi)}$$
$$+\frac{8^{17}h^{17}}{17!}y^{(xvii)} + \frac{8^{18}h^{18}}{18!}y^{(xviii)} + \frac{8^{19}h^{19}}{19!}y^{(xix)} + \frac{8^{20}h^{20}}{20!}y^{(xx)} + \ldots]. \tag{2.65}$$

Consider (2.65) and equate the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}$ to those in (2.8). This gives the system

$$\delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 = \frac{1814399}{1814400}, \tag{2.66}$$

$$-4\delta_0 - 3\delta_1 - 2\delta_2 = -\frac{122753}{9979200}, \tag{2.67}$$

$$4^2\frac{\delta_0}{2!} + 3^2\frac{\delta_1}{2!} + 2^2\frac{\delta_2}{2!} + \frac{\delta_3}{2!} = \frac{14255849}{34214400} - \frac{5}{12}, \tag{2.68}$$

$$-4^3\frac{\delta_0}{3!} - 3^3\frac{\delta_1}{3!} - 2^3\frac{\delta_2}{3!} - \frac{\delta_3}{3!} = -\frac{68891}{222393600}, \tag{2.69}$$

$$4^4\frac{\delta_0}{4!} - 3^4\frac{\delta_1}{4!} - 2^4\frac{\delta_2}{4!} + \frac{\delta_3}{4!} = \frac{3632171087}{43589145600} - \frac{1}{12}, \tag{2.70}$$

the solution of which gives the parameters of the fourth end-point formula (i.e. $x = x_4$) for the second-order method. It is noted that the parameters

δ_i $(i = 5, 6, 7, \ldots, 12)$ may then be arbitrarily given the value zero . Thus

$$\left.\begin{aligned} \delta_0 &= -\frac{185681143}{52306974720}, \\ \delta_1 &= \frac{608520391}{32691859200}, \\ \delta_2 &= -\frac{70958431}{1743565824}, \\ \delta_3 &= \frac{68068867}{1307674368}, \\ \delta_4 &= \frac{254624963293}{261534873600}. \end{aligned}\right\}. \tag{2.71}$$

The special end point formulae for the points $x_{N-3}, x_{N-2}, x_{N-1}, x_N$ may then be written down from those for x_4, x_3, x_2, x_1, respectively (because of symmetry).

The set of parameter values in (2.36), (2.44), (2.45), (2.53), (2.54), (2.62), (2.63) and (2.71) give c_{12} as the first non-zero constant in (2.9). Global extrapolation on two grids, with p=2 in (2.29), and on three grids, with p=2 in (2.33), gives the numerical methods

$$\mathbf{Y}^{(E)} = \frac{4}{3} I^h_{\frac{1}{2}h} \mathbf{Y}^{(2)} - \frac{1}{3} \mathbf{Y}^{(1)}. \tag{2.72}$$

$$\mathbf{Y}^{(E)} = \frac{243}{120} I^h_{\frac{1}{3}h} \mathbf{Y}^{(3)} - \frac{128}{120} I^h_{\frac{1}{2}h} \mathbf{Y}^{(2)} + \frac{5}{120} \mathbf{Y}^{(1)}. \tag{2.73}$$

2.7 CONSTRUCTION OF A FOURTH-ORDER METHOD

Choosing $\alpha = \beta = \gamma = \delta = 0$ as before and writing $\epsilon = \frac{5}{12}$ in (2.3) gives a fourth-order method. The first two non-zero constants in (2.9) then become

$$c_{14} = \frac{-7}{144}, \quad c_{16} = \frac{-617}{420}, \tag{2.74}$$

with $c_{11} = c_{13} = c_{15} = \ldots = 0$, because of symmetry. Taking the parameters $a_i, b_i, c_i, d_i (i = 0, 2, 4, 6, 8)$ as given in section 2.6 with the parameters $\alpha_i, \beta_i, \gamma_i, \delta_i$ (i= 0,1,2,3,4,5) calculated as follows, ensures the same first non-zero constants in (2.9) is obtained for the end-point formulae (2.12)—(2.19) associated with the fourth-order method.

For the point $x = x_1$, consider (2.37). Then equating the coefficients of the derivatives $y^{(x)}, y^{(xi)}, y^{(xii)}, y^{(xiii)}, y^{(xiv)}$ in (2.38) gives the system

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = \frac{655177}{907200}, \tag{2.75}$$

$$-\alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 = \frac{252023}{907200}, \tag{2.76}$$

$$\frac{\alpha_0}{2!} + \alpha_2 + 2^2\frac{\alpha_3}{2!} + 3^2\frac{\alpha_4}{2!} + 4^2\frac{\alpha_5}{2!} = \frac{27438979}{119750400}, \tag{2.77}$$

$$-\frac{\alpha_0}{3!} + \alpha_2 + 2^3\frac{\alpha_3}{3!} + 3^3\frac{\alpha_4}{3!} + 4^3\frac{\alpha_5}{3!} = \frac{11368009}{119750400}, \tag{2.78}$$

$$\frac{\alpha_0}{4!} + \frac{\alpha_2}{4!} + 2^4\frac{\alpha_3}{4!} + 3^4\frac{\alpha_4}{4!} + 4^4\frac{\alpha_5}{4!} = \frac{131904163}{3113510400} - \frac{7}{144}, \tag{2.79}$$

$$-\frac{\alpha_0}{5!} + \frac{\alpha_2}{5!} + 2^5\frac{\alpha_3}{5!} + 3^5\frac{\alpha_4}{5!} + 4^5\frac{\alpha_5}{5!} = \frac{723798697}{46702656000}. \tag{2.80}$$

Solving this system, we get the parameters of the first end-point formula (i.e. $x = x_1$) for the fourth-order method. It is noted that the parameters α_i $(i = 6, 7, 8, \ldots, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\alpha_0 &= \tfrac{-40637579}{691891200}, \\
\alpha_1 &= \tfrac{1150015783}{1334361600}, \\
\alpha_2 &= \tfrac{-1674003}{28744003}, \\
\alpha_3 &= \tfrac{234778903}{311351040}, \\
\alpha_4 &= \tfrac{-5661511673}{18681062400}, \\
\alpha_5 &= \tfrac{43035359}{849139200}.
\end{aligned}\right\}. \tag{2.81}$$

It can be shown using the method of undetermined coefficients for the point $x = x_2$, that, taking the parameter values b_0, b_2, b_4, b_6, b_8 in (2.45) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}$, $y^{(xv)}$ in (2.47) gives the system

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 = \frac{882773}{907200}, \tag{2.82}$$

$$-2\beta_0 - \beta_1 + \beta_3 + 2\beta_4 + 3\beta_5 = \frac{24427}{453600}, \tag{2.83}$$

$$2^2\frac{\beta_0}{2!} + \frac{\beta_1}{2!} + \frac{\beta_3}{2!} + 2^2\frac{\beta_4}{2!} + 3^2\frac{\beta_5}{2!} = \frac{43202009}{119750400}, \tag{2.84}$$

$$-2^3\frac{\beta_0}{3!} - \frac{\beta_1}{3!} + \frac{\beta_3}{3!} + 2^3\frac{\beta_4}{3!} + 3^3\frac{\beta_5}{3!} = \frac{2394839}{59875200}, \tag{2.85}$$

$$2^4\frac{\beta_0}{4!} + \frac{\beta_1}{4!} + \frac{\beta_3}{4!} + 2^4\frac{\beta_4}{4!} + 3^4\frac{\beta_5}{4!} = \frac{190486607}{3113510400} - \frac{7}{144}, \tag{2.86}$$

$$-2^5\frac{\beta_0}{5!} + \frac{\beta_1}{5!} + \frac{\beta_3}{5!} + 2^5\frac{\beta_4}{5!} + 3^5\frac{\beta_5}{5!} = \frac{21489493}{2122848000}, \tag{2.87}$$

the solution of which give the parameters of the second end-point formula (i.e. $x = x_2$) for the fourth-order method. It is noted that the parameters β_i $(i = 6, 7, 8, \ldots, 12)$ may then be arbitrarily given the value zero . Thus

$$\left.\begin{aligned}
\beta_0 &= \tfrac{-24163651}{691891200}, \\
\beta_1 &= \tfrac{118607251}{266872320}, \\
\beta_2 &= \tfrac{91527613}{718502400}, \\
\beta_3 &= \tfrac{694056739}{1556755200}, \\
\beta_4 &= \tfrac{-43253933}{3736212480}, \\
\beta_5 &= \tfrac{17921741}{9340531200}.
\end{aligned}\right\} . \tag{2.88}$$

Next, it can be shown using the method of undetermined coefficients for the point $x = x_3$, that, taking the parameter values c_0, c_2, c_4, c_6, c_8 given in (2.54), together with the parameters calculated below, guarantees the same first non-zero constant in the local error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}$, $y^{(xv)}$, in (2.56) gives

$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 = \frac{302231}{302400}, \tag{2.89}$$

$$-3\gamma_0 - 2\gamma_1 - \gamma_2 + \gamma_4 + 2\gamma_5 = \frac{169}{100800}, \tag{2.90}$$

$$3^2\frac{\gamma_0}{2!} + 2^2\frac{\gamma_1}{2!} + \frac{\gamma_2}{2!} + \frac{\gamma_4}{2!} + 2^2\frac{\gamma_5}{2!} = \frac{5510311}{13305600}, \tag{2.91}$$

$$-3^3\frac{\gamma_0}{3!} - 2^3\frac{\gamma_1}{3!} - \frac{\gamma_2}{3!} + \frac{\gamma_4}{3!} + 2^3\frac{\gamma_5}{3!} = \frac{11381}{4435200}, \tag{2.92}$$

$$3^4\frac{\gamma_0}{4!}+2^4\frac{\gamma_1}{4!}+\frac{\gamma_2}{4!}+\frac{\gamma_4}{4!}+2^4\frac{\gamma_5}{4!}=\frac{591141643}{7264857600}-\frac{5}{12}, \tag{2.93}$$

$$-3^5\frac{\gamma_0}{5!}-2^5\frac{\gamma_1}{5!}-\frac{\gamma_2}{5!}+\frac{\gamma_4}{5!}+2^5\frac{\gamma_5}{5!}=\frac{14645899}{12108096000}. \tag{2.94}$$

Solving this system we get the parameters of the third end-point formula (i.e. $x = x_3$) for the fourth-order method; they are

$$\left.\begin{aligned}\gamma_0 &= \tfrac{-1007339}{1614412800},\\ \gamma_1 &= \tfrac{46537}{207567360},\\ \gamma_2 &= \tfrac{232672519}{558835200},\\ \gamma_3 &= \tfrac{202081057}{1210809600},\\ \gamma_4 &= \tfrac{1210545577}{2905943040},\\ \gamma_5 &= \tfrac{108743}{7264857600}.\end{aligned}\right\}. \tag{2.95}$$

It is noted that the parameters γ_i $(i = 6, 7, \ldots, 12)$ may be arbitrarily assigned the value zero.

Finally, it can be shown using the method of undetermined coefficients for the point $x = x_4$ that, taking the parameter values d_0, d_2, d_4, d_6, d_8 in (2.63), together with the parameters calculated below, yields the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}$, $y^{(xv)}$, in (2.65) gives

$$\delta_0+\delta_1+\delta_2+\delta_3+\delta_4+\delta_5=\frac{1814399}{1814400}, \tag{2.96}$$

$$-4\delta_0-3\delta_1-2\delta_2-\delta_3+\delta_5=\frac{122753}{9979200}, \tag{2.97}$$

$$4^2\frac{\delta_0}{2!}+3^2\frac{\delta_1}{2!}+2^2\frac{\delta_2}{2!}+\frac{\delta_3}{2!}+\frac{\delta_5}{2!}=\frac{14255849}{34214400}, \tag{2.98}$$

$$-4^3\frac{\delta_0}{3!}-3^3\frac{\delta_1}{3!}-2^3\frac{\delta_2}{3!}-\frac{\delta_3}{3!}+\frac{\delta_5}{3!}=\frac{68891}{222393600}, \tag{2.99}$$

$$4^4\frac{\delta_0}{4!}+3^4\frac{\delta_1}{4!}+2^4\frac{\delta_2}{4!}+\frac{\delta_3}{4!}+\frac{\delta_5}{4!}=\frac{363217187}{435891145600}-\frac{7}{144}, \tag{2.100}$$

$$-4^5\frac{\delta_0}{5!}-3^5\frac{\delta_1}{5!}-2^5\frac{\delta_2}{5!}-\frac{\delta_3}{5!}+\frac{\delta_5}{5!}=\frac{413849}{326918592000}. \tag{2.101}$$

Solving this system we get the parameters of the fourth end-point formula (i.e. $x = x_4$) for the fourth-order method. It is noted that the parameters $\delta_i(i = 6, 7, \dots, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned} \delta_0 &= \frac{-4206679}{7925299200}, \\ \delta_1 &= \frac{230059147}{65383718400}, \\ \delta_2 &= \frac{-274791157}{261534873600}, \\ \delta_3 &= \frac{5689027}{12972960}, \\ \delta_4 &= \frac{4062716183}{26134873600}, \\ \delta_5 &= \frac{27045819673}{65383718400}. \end{aligned}\right\} . \tag{2.102}$$

Because of symmetry, the special end-point fomulae for the points x_N, x_{N-1}, x_{N-2}, x_{N-3} may be written down directly from those for x_1, x_2, x_3, x_4, respectively.

The set of parameter values in (2.36), (2.81), (2.45), (2.88), (2.54), (2.95), (2.63) and (2.102) give c_{14} as the first non-zero constant and $c_{15} = 0$ in (2.9). Global extrapolation on two grids, with p=4 in (2.29), gives the numerical method.

$$\mathbf{Y}^{(E)} = \frac{16}{15} I^{h}_{\frac{1}{2}h} \mathbf{Y}^{(2)} - \frac{1}{15} \mathbf{Y}^{(1)}. \tag{2.103}$$

2.8 CONSTRUCTION OF A SIXTH-ORDER METHOD

Choosing $\alpha = \beta = \gamma = 0$ as before and writing $\epsilon = \frac{2}{9}$, $\delta = \frac{7}{144}$ so that $1 - 2\alpha - 2\beta - 2\gamma - 2\delta - 2\epsilon = \frac{11}{24}$ in (2.3) gives a sixth-order method. The first non-zero constant in (2.9) then becomes

$$c_{16} = \frac{-17}{12096}, \tag{2.104}$$

with $c_{11} = c_{13} = c_{15} = \dots = 0$,because of symmetry. Taking the parameters a_i, b_i, c_i, d_i $(i = 0, 2, 4, 6, 8)$ as given in section 2.6 with the parameters $\alpha_i, \beta_i, \gamma_i$, δ_i $(i = 0, 1, \dots, 7)$ calculated as follows, ensures that the same first non-zero constant in (2.9) is obtained for the end-point formulae (2.12)—(2.19) associated with the sixth-order method.

For the point $x = x_1$, consider (2.37). Then equating the coefficients of the derivatives $y^{(x)}, y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xvii)}$ in (2.38) gives the system

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 = \frac{655177}{907200}, \tag{2.105}$$

$$-\alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 6\alpha_7 = \frac{252023}{907200}, \tag{2.106}$$

$$\frac{\alpha_0}{2!} + \alpha_2 + 2^2\frac{\alpha_3}{2!} + 3^2\frac{\alpha_4}{2!} + 4^2\frac{\alpha_5}{2!} + 5^2\frac{\alpha_6}{2!} + 6^2\frac{\alpha_7}{2!} = \frac{27438979}{119750400}, \tag{2.107}$$

$$-\frac{\alpha_0}{3!} + \alpha_2 + 2^3\frac{\alpha_3}{3!} + 3^3\frac{\alpha_4}{3!} + 4^3\frac{\alpha_5}{3!} + 5^3\frac{\alpha_6}{3!} + 6^3\frac{\alpha_7}{3!} = \frac{11368009}{119750400}, \tag{2.108}$$

$$\frac{\alpha_0}{4!} + \frac{\alpha_2}{4!} + 2^4\frac{\alpha_3}{4!} + 3^4\frac{\alpha_4}{4!} + 4^4\frac{\alpha_5}{4!} + 5^4\frac{\alpha_6}{4!} + 6^4\frac{\alpha_7}{4!} = \frac{131904163}{3113510400}, \tag{2.109}$$

$$-\frac{\alpha_0}{5!} + \frac{\alpha_2}{5!} + 2^5\frac{\alpha_3}{5!} + 3^5\frac{\alpha_4}{5!} + 4^5\frac{\alpha_5}{5!} + 5^5\frac{\alpha_6}{5!} + 6^5\frac{\alpha_7}{5!} = \frac{723798697}{46702656000}, \tag{2.110}$$

$$\frac{\alpha_0}{6!} + \frac{\alpha_2}{6!} + 2^6\frac{\alpha_3}{6!} + 3^6\frac{\alpha_4}{6!} + 4^6\frac{\alpha_5}{6!} + 5^6\frac{\alpha_6}{6!} + 6^6\frac{\alpha_7}{6!} = \frac{2541132023}{475517952000} - \frac{17}{12096}, \tag{2.111}$$

$$-\frac{\alpha_0}{7!} + \frac{\alpha_2}{7!} + 2^7\frac{\alpha_3}{7!} + 3^7\frac{\alpha_4}{7!} + 4^7\frac{\alpha_5}{7!} + 5^7\frac{\alpha_6}{7!} + 6^7\frac{\alpha_7}{7!} = \frac{8768652467}{5230697472000}. \tag{2.112}$$

Solving this system, we get the parameters of the first end-point formula (i.e. $x = x_1$) for the sixth-order method. It is noted that the parameters α_i $(i = 8, 9, 10, 11, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\alpha_0 &= \frac{121680539023}{3923023104000}, \\
\alpha_1 &= \frac{825553878671}{1743565824000}, \\
\alpha_2 &= \frac{49899297233}{871782912000}, \\
\alpha_3 &= \frac{180529065817}{627683696640}, \\
\alpha_4 &= \frac{-9140697491}{43589145600}, \\
\alpha_5 &= \frac{194540768657}{1743565824000}, \\
\alpha_6 &= \frac{-261610352587}{7846046208000}, \\
\alpha_7 &= \frac{192774481}{44706816000}.
\end{aligned}\right\} . \tag{2.113}$$

It can be shown using the method of undetermined coefficients for the point $x = x_2$, that, taking the parameter values $b_0, b_2, b_4, b_6. b_8$ in (2.45) together

with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xvii)}$ gives the system

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 = \frac{882773}{907200}, \tag{2.114}$$

$$-2\beta_0 - \beta_1 + \beta_3 + 2\beta_4 + 3\beta_5 + 4\beta_6 + 5\beta_7 = \frac{24427}{453600}, \tag{2.115}$$

$$2^2\frac{\beta_0}{2!} + \frac{\beta_1}{2!} + \frac{\beta_3}{2!} + 2^2\frac{\beta_4}{2!} + 3^2\frac{\beta_5}{2!} + 4^2\frac{\beta_6}{2!} + 5^2\frac{\beta_7}{2!} = \frac{43202009}{119750400}, \tag{2.116}$$

$$-2^3\frac{\beta_0}{3!} - \frac{\beta_1}{3!} + \frac{\beta_3}{3!} + 2^3\frac{\beta_4}{3!} + 3^3\frac{\beta_5}{3!} + 4^3\frac{\beta_6}{3!} + 5^3\frac{\beta_7}{3!} = \frac{2394839}{59875200}, \tag{2.117}$$

$$2^4\frac{\beta_0}{4!} + \frac{\beta_1}{4!} + \frac{\beta_3}{4!} + 2^4\frac{\beta_4}{4!} + 3^4\frac{\beta_5}{4!} + 4^4\frac{\beta_6}{4!} + 5^4\frac{\beta_7}{4!} = \frac{190486607}{3113510400}, \tag{2.118}$$

$$-2^5\frac{\beta_0}{5!} + \frac{\beta_1}{5!} + \frac{\beta_3}{5!} + 2^5\frac{\beta_4}{5!} + 3^5\frac{\beta_5}{5!} + 4^5\frac{\beta_6}{5!} + 5^5\frac{\beta_7}{5!} = \frac{21489493}{2122848000}, \tag{2.119}$$

$$2^6\frac{\beta_0}{6!} + \frac{\beta_1}{6!} + \frac{\beta_3}{6!} + 2^6\frac{\beta_4}{6!} + 3^6\frac{\beta_5}{6!} + 4^6\frac{\beta_6}{6!} + 5^6\frac{\beta_7}{6!} = \frac{34992742353}{5230697472000} - \frac{17}{12096}, \tag{2.120}$$

$$-2^7\frac{\beta_0}{7!} + \frac{\beta_1}{7!} + \frac{\beta_3}{7!} + 2^7\frac{\beta_4}{7!} + 3^7\frac{\beta_5}{7!} + 4^7\frac{\beta_6}{7!} + 5^7\frac{\beta_7}{7!} = \frac{327962597}{237758976000}, \tag{2.121}$$

the solution of which give the parameters of the second end-point formula (i.e. $x = x_2$) for the sixth-order method. It is noted that the parameters β_i $(i = 8, 9, 10, 11, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\beta_0 &= \frac{121680539023}{3923023104000}, \\
\beta_1 &= \frac{82555387871}{1743565824000}, \\
\beta_2 &= \frac{49899297233}{871782912000}, \\
\beta_3 &= \frac{180529065817}{627683696640}, \\
\beta_4 &= \frac{-9140697491}{43589145600}, \\
\beta_5 &= \frac{194540768657}{1743565824000}, \\
\beta_6 &= \frac{-261610352587}{784604628000}, \\
\beta_7 &= \frac{192774481}{44706816000}.
\end{aligned}\right\} . \tag{2.122}$$

Next, it can be shown using the method of undetermined coefficients for the point $x = x_3$, that, taking the parameter values c_0, c_2, c_4, c_6, c_8 in (2.54) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xvii)}$ in (2.56) gives

$$\gamma_0+\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6+\gamma_7=\frac{302231}{302400}, \tag{2.123}$$

$$-3\gamma_0-2\gamma_1-\gamma_2+\gamma_4+2\gamma_5+3\gamma_6+4\gamma_7=\frac{169}{100800}, \tag{2.124}$$

$$3^2\frac{\gamma_0}{2!}+2^2\frac{\gamma_1}{2!}+\frac{\gamma_2}{2!}+\frac{\gamma_4}{2!}+2^2\frac{\gamma_5}{2!}+3^2\frac{\gamma_6}{2!}+4^2\frac{\gamma_7}{2!}=\frac{5510311}{13305600}, \tag{2.125}$$

$$-3^3\frac{\gamma_0}{3!}-2^3\frac{\gamma_1}{3!}-\frac{\gamma_2}{3!}+\frac{\gamma_4}{3!}+2^3\frac{\gamma_5}{3!}+3^3\frac{\gamma_6}{3!}+4^3\frac{\gamma_7}{3!}=\frac{11381}{4435200}, \tag{2.126}$$

$$3^4\frac{\gamma_0}{4!}+2^4\frac{\gamma_1}{4!}+\frac{\gamma_2}{4!}+\frac{\gamma_4}{4!}+2^4\frac{\gamma_5}{4!}+3^4\frac{\gamma_6}{4!}+4^4\frac{\gamma_7}{4!}=\frac{591141643}{7264857600}, \tag{2.127}$$

$$-3^5\frac{\gamma_0}{5!}-2^5\frac{\gamma_1}{5!}-\frac{\gamma_2}{5!}+\frac{\gamma_4}{5!}+2^5\frac{\gamma_5}{5!}+3^5\frac{\gamma_6}{5!}+4^5\frac{\gamma_7}{5!}=\frac{14645899}{12108096000}, \tag{2.128}$$

$$3^6\frac{\gamma_0}{6!}+2^6\frac{\gamma_1}{6!}+\frac{\gamma_2}{6!}+\frac{\gamma_4}{6!}+2^6\frac{\gamma_5}{6!}+3^6\frac{\gamma_6}{6!}+4^6\frac{\gamma_7}{6!}=\frac{1346510087}{134120448000}-\frac{17}{12096}, \tag{2.129}$$

$$3^7\frac{\gamma_0}{7!}+2^7\frac{\gamma_1}{7!}+\frac{\gamma_2}{7!}+\frac{\gamma_4}{7!}+2^7\frac{\gamma_5}{7!}+3^7\frac{\gamma_6}{7!}+4^7\frac{\gamma_7}{7!}=\frac{162013909}{581188608000}. \tag{2.130}$$

Solving this system we get the parameters of the third end-point formula (i.e. $x = x_3$) for the sixth-order method; they are

$$\left.\begin{aligned}
\gamma_0 &= \frac{-21838081}{33530112000},\\
\gamma_1 &= \frac{1356454837}{27675648000},\\
\gamma_2 &= \frac{7149219919}{3288256000},\\
\gamma_3 &= \frac{160167409321}{348713164800},\\
\gamma_4 &= \frac{27501631}{124185600},\\
\gamma_5 &= \frac{9490656173}{193729536000},\\
\gamma_6 &= \frac{-13324169}{124540416000},\\
\gamma_7 &= \frac{2571931}{193729536000}.
\end{aligned}\right\} \tag{2.131}$$

It is noted that the parameters γ_i $(i = 8, 9, 10, 11, 12)$ may be arbitrarily assigned the value zero.

Finally, it can be shown using the method of undetermined coefficients for the point $x = x_4$ that, taking the parameter values d_0, d_2, d_4, d_6, d_8 in (2.63) together with parameters calculated below, yields the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xvii)}$ in (2.65) gives

$$\delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + \delta_7 = \frac{1814399}{1814400}, \tag{2.132}$$

$$-4\delta_0 - 3\delta_1 - 2\delta_2 - \delta_3 + \delta_5 + 2\delta_6 + 3\delta_7 = \frac{122753}{9979200}, \tag{2.133}$$

$$4^2\frac{\delta_0}{2!} + 3^2\frac{\delta_1}{2!} + 2^2\frac{\delta_2}{2!} + \frac{\delta_3}{2!} + \frac{\delta_5}{2!} + 2^2\frac{\delta_6}{2!} + 3^2\frac{\delta_7}{2!} = \frac{14255849}{34214400}, \tag{2.134}$$

$$-4^3\frac{\delta_0}{3!} - 3^3\frac{\delta_1}{3!} - 2^3\frac{\delta_2}{3!} - \frac{\delta_3}{3!} + \frac{\delta_5}{3!} + 2^3\frac{\delta_6}{3!} + 3^3\frac{\delta_7}{3!} + = \frac{68891}{222393600}, \tag{2.135}$$

$$4^4\frac{\delta_0}{4!} + 3^4\frac{\delta_1}{4!} + 2^4\frac{\delta_2}{4!} + \frac{\delta_3}{4!} + \frac{\delta_5}{4!} + 2^4\frac{\delta_6}{4!} + 3^4\frac{\delta_7}{4!} = \frac{363217187}{43589145600}, \tag{2.136}$$

$$-4^5\frac{\delta_0}{5!} - 3^5\frac{\delta_1}{5!} - 2^5\frac{\delta_2}{5!} - \frac{\delta_3}{5!} + \frac{\delta_5}{5!} + 2^5\frac{\delta_6}{5!} + 3^5\frac{\delta_7}{5!} = \frac{413849}{326918592000}, \tag{2.137}$$

$$4^6\frac{\delta_0}{6!} + 3^6\frac{\delta_1}{6!} + 2^6\frac{\delta_2}{6!} + \frac{\delta_3}{6!} + \frac{\delta_5}{6!} + 2^6\frac{\delta_6}{6!} + 3^6\frac{\delta_7}{6!} = \frac{10139471581}{951035904000} - \frac{17}{12096}, \tag{2.138}$$

$$-4^7\frac{\delta_0}{7!} - 3^7\frac{\delta_1}{7!} - 2^7\frac{\delta_2}{7!} - \frac{\delta_3}{7!} + \frac{\delta_5}{7!} + 2^7\frac{\delta_7}{7!} + 3^7\frac{\delta_7}{7!} = \frac{154643851}{88921857024000}. \tag{2.139}$$

Solving this system we get the parameters of the fourth end-point formula (i.e. $x = x_4$) for the sixth-order method. It is noted that the parameters δ_i $(i = 8, 9, 10, 11, 12)$ may then be arbitrarily given the value zero . Thus

$$\left.\begin{aligned}
\delta_0 &= \frac{19195006261}{266765571072000},\\
\delta_1 &= \frac{118864463057}{177843714048000},\\
\delta_2 &= \frac{337681410533}{7410154752000},\\
\delta_3 &= \frac{996423583781}{4268249137152},\\
\delta_4 &= \frac{8106735502457}{177843714048000},\\
\delta_5 &= \frac{12744987460013}{59281238016000},\\
\delta_6 &= \frac{6622887628141}{133382785536000},\\
\delta_7 &= \frac{-17289181267}{177843714048000}.
\end{aligned}\right\}. \tag{2.140}$$

Because of symmetry, the special end-point formulae for the points $x_N, x_{N-1}, x_{N-2}, x_{N-3}$ may be written down directly from those for x_1, x_2, x_3, x_4, respectively.

The set of parameter values in (2.36), (2.113), (2.45), (2.122), (2.54), (2.131), (2.63) and (2.140) give c_{16} as the first non-zero constant and $c_{17} = 0$ in (2.9). Global extrapolation on two grids, with p=6 in (2.29), gives the numerical method.

$$\mathbf{Y}^{(E)} = \frac{64}{63} I^{h}_{\frac{1}{2}h} \mathbf{Y}^{(2)} - \frac{1}{63} \mathbf{Y}^{(1)}. \tag{2.141}$$

2.9 CONSTRUCTION OF AN EIGHTH-ORDER METHOD

Writing $\alpha = \beta = 0$ as before $\gamma = \frac{17}{12096}$, $\delta = \frac{9}{224}$, $\epsilon = \frac{109}{448}$ so that $\sum = 1 - 2\alpha - 2\beta - 2\gamma - 2\delta - 2\epsilon = \frac{1301}{3024}$ in (2.3) gives an eighth-order method. The first non-zero constant in (2.9) then becomes

$$c_{18} = \frac{-1}{362880}, \tag{2.142}$$

with $c_{11} = c_{13} = c_{15} = \ldots\ldots = 0$, because of symmetry. Taking the parameters a_i, b_i, c_i, d_i $(i = 0, 2, 4, 6, 8)$ as given in section 2.6 with the parameters $\alpha_i, \beta_i, \gamma_i, \delta_i$ $(i = 0, 1, \ldots, 9)$ calculated as follows, ensures that the same non-zero constant in (2.9) is obtained for the end-point formulae (2.12)—(2.19) associated with the eighth-order method.

For the point $x = x_1$, consider (2.37). Then equating the coefficients of the derivatives $y^{(x)}, y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xix)}$ in (2.38) gives the system

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 = \frac{655177}{907200}, \tag{2.143}$$

$$-\alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 6\alpha_7 + 7\alpha_8 + 8\alpha_9 = \frac{252023}{907200}, \tag{2.144}$$

$$\frac{\alpha_0}{2!} + \alpha_2 + 2^2\frac{\alpha_3}{2!} + 3^2\frac{\alpha_4}{2!} + 4^2\frac{\alpha_5}{2!} + 5^2\frac{\alpha_6}{2!} + 6^2\frac{\alpha_7}{2!} + 7^2\frac{\alpha_8}{2!} + 8^2\frac{\alpha_9}{2!} = \frac{27438979}{119750400}, \tag{2.145}$$

$$-\frac{\alpha_0}{3!}+\alpha_2+2^3\frac{\alpha_3}{3!}+3^3\frac{\alpha_4}{3!}+4^3\frac{\alpha_5}{3!}+5^3\frac{\alpha_6}{3!}+6^3\frac{\alpha_7}{3!}+7^3\frac{\alpha_8}{3!}+8^3\frac{\alpha_9}{3!}=\frac{11368009}{119750400}, \tag{2.146}$$

$$\frac{\alpha_0}{4!}+\frac{\alpha_2}{4!}+2^4\frac{\alpha_3}{4!}+3^4\frac{\alpha_4}{4!}+4^4\frac{\alpha_5}{4!}+5^4\frac{\alpha_6}{4!}+6^4\frac{\alpha_7}{4!}+7^4\frac{\alpha_8}{4!}+8^4\frac{\alpha_9}{4!}=\frac{131904163}{3113510400}, \tag{2.147}$$

$$-\frac{\alpha_0}{5!}+\frac{\alpha_2}{5!}+2^5\frac{\alpha_3}{5!}+3^5\frac{\alpha_4}{5!}+4^5\frac{\alpha_5}{5!}+5^5\frac{\alpha_6}{5!}+6^5\frac{\alpha_7}{5!}+7^5\frac{\alpha_8}{5!}+8^5\frac{\alpha_9}{5!}=\frac{723798697}{46702656000}, \tag{2.148}$$

$$\frac{\alpha_0}{6!}+\frac{\alpha_2}{6!}+2^6\frac{\alpha_3}{6!}+3^6\frac{\alpha_4}{6!}+4^6\frac{\alpha_5}{6!}+5^6\frac{\alpha_6}{6!}+6^6\frac{\alpha_7}{6!}+7^6\frac{\alpha_8}{6!}+8^6\frac{\alpha_9}{6!}=\frac{2541132023}{475517952000}, \tag{2.149}$$

$$-\frac{\alpha_0}{7!}+\frac{\alpha_2}{7!}+2^7\frac{\alpha_3}{7!}+3^7\frac{\alpha_4}{7!}+4^7\frac{\alpha_5}{7!}+5^7\frac{\alpha_6}{7!}+6^7\frac{\alpha_7}{7!}+7^7\frac{\alpha_8}{7!}+8^7\frac{\alpha_9}{7!}=\frac{8768652467}{5230697472000}, \tag{2.150}$$

$$\frac{\alpha_0}{8!}+\frac{\alpha_2}{8!}+2^8\frac{\alpha_3}{8!}+3^8\frac{\alpha_4}{8!}+4^8\frac{\alpha_5}{8!}+5^8\frac{\alpha_6}{8!}+6^8\frac{\alpha_7}{8!}+7^8\frac{\alpha_8}{8!}+8^8\frac{\alpha_9}{8!}=\frac{14042390777}{28582025472000}-\frac{1}{362880}, \tag{2.151}$$

$$-\frac{\alpha_0}{9!}+\frac{\alpha_2}{9!}+2^9\frac{\alpha_3}{9!}+3^9\frac{\alpha_4}{9!}+4^9\frac{\alpha_5}{9!}+5^9\frac{\alpha_6}{9!}+6^9\frac{\alpha_7}{9!}+7^9\frac{\alpha_8}{9!}+8^9\frac{\alpha_9}{9!}=\frac{2762162653}{20520428544000}. \tag{2.152}$$

Solving this system, we get the parameters of the first end-point formula (i.e. $x = x_1$) for the eighth-order method. It is noted that the parameters α_i $(i = 10, 11, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\alpha_0 &= \frac{20992458112397}{640237370572800}, \\
\alpha_1 &= \frac{93157191139}{199874304000}, \\
\alpha_2 &= \frac{735619884037}{12312257126400}, \\
\alpha_3 &= \frac{4063603106641}{12312257126400}, \\
\alpha_4 &= \frac{-520840519849037}{1600593426432000}, \\
\alpha_5 &= \frac{41785406610919}{160059342643200}, \\
\alpha_6 &= \frac{-115840187113411}{800296713216000}, \\
\alpha_7 &= \frac{1699033289519}{32011868528640}, \\
\alpha_8 &= \frac{-7421743667363}{640237370572800}, \\
\alpha_9 &= \frac{915081921001}{800296713216000}.
\end{aligned}\right\} \tag{2.153}$$

It can be shown using the method of undetermined coefficients for the point

$x = x_2$, that, taking the parameter values b_0, b_2, b_4, b_6, b_8 in (2.45) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xix)}$ gives the system

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 + \beta_9 = \frac{882773}{907200}, \tag{2.154}$$

$$-2\beta_0 - \beta_1 + \beta_3 + 2\beta_4 + 3\beta_5 + 4\beta_6 + 5\beta_7 + 6\beta_8 + 7\beta_9 = \frac{24427}{453600}, \tag{2.155}$$

$$2^2\frac{\beta_0}{2!}+\frac{\beta_1}{2!}+\frac{\beta_3}{2!}+2^2\frac{\beta_4}{2!}+3^2\frac{\beta_5}{2!}+4^2\frac{\beta_6}{2!}+5^2\frac{\beta_7}{2!}+6^2\frac{\beta_8}{2!}+7^2\frac{\beta_9}{2!} = \frac{43202009}{119750400}, \tag{2.156}$$

$$-2^3\frac{\beta_0}{3!}-\frac{\beta_1}{3!}+\frac{\beta_3}{3!}+2^3\frac{\beta_4}{3!}+3^3\frac{\beta_5}{3!}+4^3\frac{\beta_6}{3!}+5^3\frac{\beta_7}{3!}+6^3\frac{\beta_8}{3!}+7^3\frac{\beta_9}{3!} = \frac{2394839}{59875200}, \tag{2.157}$$

$$2^4\frac{\beta_0}{4!}+\frac{\beta_1}{4!}+\frac{\beta_3}{4!}+2^4\frac{\beta_4}{4!}+3^4\frac{\beta_5}{4!}+4^4\frac{\beta_6}{4!}+5^4\frac{\beta_7}{4!}+6^4\frac{\beta_8}{4!}+7^4\frac{\beta_9}{4!} = \frac{190486607}{3113510400}, \tag{2.158}$$

$$-2^5\frac{\beta_0}{5!}+\frac{\beta_1}{5!}+\frac{\beta_3}{5!}+2^5\frac{\beta_4}{5!}+3^5\frac{\beta_5}{5!}+4^5\frac{\beta_6}{5!}+5^5\frac{\beta_7}{5!}+6^5\frac{\beta_8}{5!}+7^5\frac{\beta_9}{5!} = \frac{21489493}{2122848000}, \tag{2.159}$$

$$2^6\frac{\beta_0}{6!}+\frac{\beta_1}{6!}+\frac{\beta_3}{6!}+2^6\frac{\beta_4}{6!}+3^6\frac{\beta_5}{6!}+4^6\frac{\beta_6}{6!}+5^6\frac{\beta_7}{6!}+6^6\frac{\beta_8}{6!}+7^6\frac{\beta_9}{6!} = \frac{34992742353}{5230697472000}-\frac{17}{12096}, \tag{2.160}$$

$$-2^7\frac{\beta_0}{7!}-\frac{\beta_1}{7!}+\frac{\beta_3}{7!}+2^7\frac{\beta_4}{7!}+3^7\frac{\beta_5}{7!}+4^7\frac{\beta_6}{7!}+5^7\frac{\beta_7}{7!}+6^7\frac{\beta_5}{7!}+7^7\frac{\beta_9}{7!} = \frac{327962597}{237758976000}, \tag{2.161}$$

$$2^8\frac{\beta_0}{8!}+\frac{\beta_1}{8!}+\frac{\beta_3}{8!}+2^8\frac{\beta_4}{8!}+3^8\frac{\beta_5}{8!}+4^8\frac{\beta_6}{8!}+5^8\frac{\beta_7}{8!}+6^8\frac{\beta_5}{8!}+7^8\frac{\beta_9}{8!} = \frac{881182516553}{1600593426432000}-\frac{1}{362880}, \tag{2.162}$$

$$-2^9\frac{\beta_0}{9!}-\frac{\beta_1}{9!}+\frac{\beta_3}{9!}+2^9\frac{\beta_4}{9!}+3^9\frac{\beta_5}{9!}+4^9\frac{\beta_6}{9!}+5^9\frac{\beta_7}{9!}+6^9\frac{\beta_5}{9!}+7^9\frac{\beta_9}{9!} = \frac{2542651289}{20520428544000}, \tag{2.163}$$

the solution of which give the parameters of the second end-point formula (i.e. $x = x_2$) for the eighth-order method. It is noted that the parameters

β_i (i = 10, 11, 12) may then be arbitrarily given the value zero . Thus

$$\left.\begin{aligned}
\beta_0 &= \frac{7750281368173}{640237370572800},\\
\beta_1 &= \frac{95833355799}{3637712332800},\\
\beta_2 &= \frac{304812120880213}{800296713216000},\\
\beta_3 &= \frac{259595936667337}{800296713216000},\\
\beta_4 &= \frac{-3403568201269}{64023737057280},\\
\beta_5 &= \frac{1120702421821}{14550849331200},\\
\beta_6 &= \frac{-616046074277}{14550849331200},\\
\beta_7 &= \frac{12513016249567}{800296713216000},\\
\beta_8 &= \frac{-10991111981903}{3201186852864000},\\
\beta_9 &= \frac{54435448549}{160059342643200}.
\end{aligned}\right\} . \tag{2.164}$$

Next, it can be shown using the method of undetermined coefficients for the point $x = x_3$, that, taking the parameter values c_0, c_2, c_4, c_6, c_8 in (2.54) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xix)}$ in (2.56) gives

$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_8 + \gamma_9 = \frac{302231}{302400}, \tag{2.165}$$

$$-3\gamma_0 - 2\gamma_1 - \gamma_2 + \gamma_4 + 2\gamma_5 + 3\gamma_6 + 4\gamma_7 + 5\gamma_8 + 6\gamma_9 = \frac{169}{100800}, \tag{2.166}$$

$$3^2\frac{\gamma_0}{2!} + 2^2\frac{\gamma_1}{2!} + \frac{\gamma_2}{2!} + \frac{\gamma_4}{2!} + 2^2\frac{\gamma_5}{2!} + 3^2\frac{\gamma_6}{2!} + 4^2\frac{\gamma_7}{2!} + 5^2\frac{\gamma_8}{2!} + 6^2\frac{\gamma_9}{2!} = \frac{5510311}{13305600}, \tag{2.167}$$

$$-3^3\frac{\gamma_0}{3!} - 2^3\frac{\gamma_1}{3!} - \frac{\gamma_2}{3!} + \frac{\gamma_4}{3!} + 2^3\frac{\gamma_5}{3!} + 3^3\frac{\gamma_6}{3!} + 4^3\frac{\gamma_7}{3!} + 5^3\frac{\gamma_8}{3!} + 6^3\frac{\gamma_9}{3!} = \frac{11381}{4435200}, \tag{2.168}$$

$$3^4\frac{\gamma_0}{4!} + 2^4\frac{\gamma_1}{4!} + \frac{\gamma_2}{4!} + \frac{\gamma_4}{4!} + 2^4\frac{\gamma_5}{4!} + 3^4\frac{\gamma_6}{4!} + 4^4\frac{\gamma_7}{4!} + 5^4\frac{\gamma_8}{4!} + 6^4\frac{\gamma_9}{4!} = \frac{591141643}{7264857600}, \tag{2.169}$$

$$-3^5\frac{\gamma_0}{5!} - 2^5\frac{\gamma_1}{5!} - \frac{\gamma_2}{5!} + \frac{\gamma_4}{5!} + 2^5\frac{\gamma_5}{5!} + 3^5\frac{\gamma_6}{5!} + 4^5\frac{\gamma_7}{5!} + 5^5\frac{\gamma_8}{5!} + 6^5\frac{\gamma_9}{5!} = \frac{14645899}{12108096000}, \tag{2.170}$$

$$3^6\frac{\gamma_0}{6!} + 2^6\frac{\gamma_1}{6!} + \frac{\gamma_2}{6!} + \frac{\gamma_4}{6!} + 2^6\frac{\gamma_5}{6!} + 3^6\frac{\gamma_6}{6!} + 4^6\frac{\gamma_7}{6!} + 5^6\frac{\gamma_8}{6!} + 6^6\frac{\gamma_9}{6!} = \frac{1346510087}{134120448000}, \tag{2.171}$$

$$-3^7\frac{\gamma_0}{7!}-2^7\frac{\gamma_1}{7!}-\frac{\gamma_2}{7!}+\frac{\gamma_4}{7!}+2^7\frac{\gamma_5}{7!}+3^7\frac{\gamma_6}{7!}+4^7\frac{\gamma_7}{7!}+5^7\frac{\gamma_8}{7!}+6^7\frac{\gamma_9}{7!}=\frac{162013909}{581188608000}, \quad (2.172)$$

$$3^8\frac{\gamma_0}{8!}+2^8\frac{\gamma_1}{8!}+\frac{\gamma_2}{8!}+\frac{\gamma_4}{8!}+2^8\frac{\gamma_5}{8!}+3^8\frac{\gamma_6}{8!}+4^8\frac{\gamma_7}{8!}+5^8\frac{\gamma_8}{8!}+6^8\frac{\gamma_9}{8!}=\frac{19405166329}{22230464256000}-\frac{1}{362880}, \quad (2.173)$$

$$-3^9\frac{\gamma_0}{9!}-2^9\frac{\gamma_1}{9!}-\frac{\gamma_2}{9!}+\frac{\gamma_4}{9!}+2^9\frac{\gamma_5}{9!}+3^9\frac{\gamma_6}{9!}+4^9\frac{\gamma_7}{9!}+5^9\frac{\gamma_8}{9!}+6^9\frac{\gamma_9}{9!}=\frac{163046441}{4234374144000}. \quad (2.174)$$

Solving this system we get the parameters of the third end-point formula (i.e. $x = x_3$) for the eighth-order method; they are

$$\left.\begin{aligned}
\gamma_0 &= \frac{51893722057}{71137485619200}, \\
\gamma_1 &= \frac{2355227971}{57741465600}, \\
\gamma_2 &= \frac{21493633966657}{88921857024000}, \\
\gamma_3 &= \frac{38495892458893}{88921857024000}, \\
\gamma_4 &= \frac{8541426756427}{35568742809600}, \\
\gamma_5 &= \frac{760794282539}{17784371404800}, \\
\gamma_6 &= \frac{-165940141}{2540624486400}, \\
\gamma_7 &= \frac{48667536763}{88921857024000}, \\
\gamma_8 &= \frac{-6143191781}{50812489728000}, \\
\gamma_9 &= \frac{1222783}{101624979456}.
\end{aligned}\right\} \quad (2.175)$$

It is noted that the parameters γ_i $(i = 10, 11, 12)$ may be arbitrarily assigned the value zero.

Finally, it can be shown using the method of undetermined coefficients for the point $x = x_4$ that, taking the parameter values d_0, d_2, d_4, d_6, d_8 in (2.63) together with parameters calculated below, yields the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xix)}$ in (2.65) gives

$$\delta_0+\delta_1+\delta_2+\delta_3+\delta_4+\delta_5+\delta_6+\delta_7+\delta_8+\delta_9=\frac{1814399}{1814400}, \quad (2.176)$$

$$-4\delta_0 - 3\delta_1 - 2\delta_2 - \delta_3 + \delta_5 + 2\delta_6 + 3\delta_7 + 4\delta_8 + 5\delta_9 = \frac{122753}{9979200}, \tag{2.177}$$

$$4^2\frac{\delta_0}{2!} + 3^2\frac{\delta_1}{2!} + 2^2\frac{\delta_2}{2!} + \frac{\delta_3}{2!} + \frac{\delta_5}{2!} + 2^2\frac{\delta_6}{2!} + 3^2\frac{\delta_7}{2!} + 4^2\frac{\delta_8}{2!} + 5^2\frac{\delta_9}{2!} = \frac{14255849}{34214400}, \tag{2.178}$$

$$-4^3\frac{\delta_0}{3!} - 3^3\frac{\delta_1}{3!} - 2^3\frac{\delta_2}{3!} - \frac{\delta_3}{3!} + \frac{\delta_5}{3!} + 2^3\frac{\delta_6}{3!} + 3^3\frac{\delta_7}{3!} + 4^3\frac{\delta_8}{3!} + 5^3\frac{\delta_9}{3!} = \frac{68891}{222393600}, \tag{2.179}$$

$$4^4\frac{\delta_0}{4!} + 3^4\frac{\delta_1}{4!} + 2^4\frac{\delta_2}{4!} + \frac{\delta_3}{4!} + \frac{\delta_5}{4!} + 2^4\frac{\delta_6}{4!} + 3^4\frac{\delta_7}{4!} + 4^4\frac{\delta_8}{4!} + 5^4\frac{\delta_9}{4!} = \frac{363217187}{43589145600}, \tag{2.180}$$

$$-4^5\frac{\delta_0}{5!} - 3^5\frac{\delta_1}{5!} - 2^5\frac{\delta_2}{5!} - \frac{\delta_3}{5!} + \frac{\delta_5}{5!} + 2^5\frac{\delta_6}{5!} + 3^5\frac{\delta_7}{5!} + 4^5\frac{\delta_8}{5!} + 5^5\frac{\delta_9}{5!} = \frac{413849}{326918592000}, \tag{2.181}$$

$$4^6\frac{\delta_0}{6!} + 3^6\frac{\delta_1}{6!} + 2^6\frac{\delta_2}{6!} + \frac{\delta_3}{6!} + \frac{\delta_5}{6!} + 2^6\frac{\delta_6}{6!} + 3^6\frac{\delta_7}{6!} + 4^6\frac{\delta_8}{6!} + 5^6\frac{\delta_9}{6!} = \frac{10139471581}{951035904000}, \tag{2.182}$$

$$-4^7\frac{\delta_0}{7!} - 3^7\frac{\delta_1}{7!} - 2^7\frac{\delta_2}{7!} - \frac{\delta_3}{7!} + \frac{\delta_5}{7!} + 2^7\frac{\delta_7}{7!} + 3^7\frac{\delta_7}{7!} + 4^7\frac{\delta_8}{7!} + 2^7\frac{\delta_9}{7!} = \frac{154643851}{88921857024000}, \tag{2.183}$$

$$4^8\frac{\delta_0}{8!} + 3^8\frac{\delta_1}{8!} + 2^8\frac{\delta_2}{8!} + \frac{\delta_3}{8!} + \frac{\delta_5}{8!} + 2^8\frac{\delta_6}{8!} + 3^8\frac{\delta_7}{8!} + 4^8\frac{\delta_8}{8!} + 5^8\frac{\delta_9}{8!} = \frac{3141960414959}{3201186852864000} - \frac{1}{362880}, \tag{2.184}$$

$$-4^9\frac{\delta_0}{9!} - 3^9\frac{\delta_1}{9!} - 2^9\frac{\delta_2}{9!} - \frac{\delta_3}{9!} + \frac{\delta_5}{9!} + 2^7\frac{\delta_7}{9!} + 3^9\frac{\delta_7}{9!} + 4^9\frac{\delta_8}{9!} + 2^9\frac{\delta_9}{9!} = \frac{4165158373}{10137091700736000}. \tag{2.185}$$

Solving this system we get the parameters of the fourth end-point formula (i.e. $x = x_4$) for the eighth-order method. It is noted that the parameters δ_i $(i = 10, 11, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\delta_0 &= \frac{499069556333}{2432902008176640 0}, \\
\delta_1 &= \frac{9104056156831}{5529322745856000}, \\
\delta_2 &= \frac{235407175152137}{6082255020441600}, \\
\delta_3 &= \frac{305584340173897}{1216451004088320}, \\
\delta_4 &= \frac{154635757309157}{3577797070848000}, \\
\delta_5 &= \frac{27146722126679}{116966442700800}, \\
\delta_6 &= \frac{120382318113107}{2764661372928000}, \\
\delta_7 &= \frac{2770984913471}{6082255020441600}, \\
\delta_8 &= \frac{47446323377}{267351869030400}, \\
\delta_9 &= \frac{-12443589337}{78990324940800 0}.
\end{aligned}\right\} \tag{2.186}$$

Because of symmetry, the special end-point formulae for the points x_N, x_{N-1}, x_{N-2}, x_{N-3} may be written down directly from those for x_1, x_2, x_3, x_4, respectively.

The set of parameter values in (2.36), (2.153), (2.45), (2.164), (2.54), (2.175), (2.63) and (2.186) give c_{18} as the first non-zero constant and $c_{19} = 0$ in (2.9). Global extrapolation on two grids, with p=8 in (2.29), gives the numerical method.

$$\mathbf{Y}^{(E)} = \frac{256}{255} I^{h}_{\frac{1}{2}h} \mathbf{Y}^{(2)} - \frac{1}{255} \mathbf{Y}^{(1)}. \tag{2.187}$$

2.10 CONSTRUCTION OF A TENTH-ORDER METHOD

Equation (2.3) attains tenth-order accuracy by writing $\alpha = 0$ as before and then by choosing $\beta = \frac{1}{362880}$, $\gamma = \frac{251}{181440}$, $\delta = \frac{913}{22680}$ and $\epsilon = \frac{44117}{181440}$ so that $\sum = 1 - 2(\alpha + \beta + \gamma + \delta + \epsilon) = \frac{15619}{36288}$. The first non-zero constant in (2.9) then becomes as

$$c_{20} = \frac{-1}{47900160}, \tag{2.188}$$

with $c_{11} = c_{13} = c_{15} = \ldots\ldots = 0$, because of symmetry. Choosing the parameters a_i, b_i, c_i, d_i $(i = 0, 2, 4, 6, 8)$ as given in section 2.6 with the parameters $\alpha_i, \beta_i, \gamma_i, \delta_i$ $(i = 0, 1, \ldots, 10)$ calculated as follows, ensures that the same is obtained for the end-point formulae (2.12)—(2.19) associated with the tenth-order method.

For the point $x = x_1$, consider (2.37). Then equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}$, $y^{(xii)}, \ldots\ldots, y^{(xx)}$ in (2.38) gives the system

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} = \frac{655177}{907200}, \tag{2.189}$$

$$-\alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 6\alpha_7 + 7\alpha_8 + 8\alpha_9 + 9\alpha_{10} = \frac{252023}{907200}, \tag{2.190}$$

$$\frac{\alpha_0}{2!} + \alpha_2 + 2^2\frac{\alpha_3}{2!} + 3^2\frac{\alpha_4}{2!} + 4^2\frac{\alpha_5}{2!} + 5^2\frac{\alpha_6}{2!} + 6^2\frac{\alpha_7}{2!} + 7^2\frac{\alpha_8}{2!} + 8^2\frac{\alpha_9}{2!} + 9^2\frac{\alpha_{10}}{2!} = \frac{27438979}{119750400}, \tag{2.191}$$

$$-\frac{\alpha_0}{3!} + \alpha_2 + 2^3\frac{\alpha_3}{3!} + 3^3\frac{\alpha_4}{3!} + 4^3\frac{\alpha_5}{3!} + 5^3\frac{\alpha_6}{3!} + 6^3\frac{\alpha_7}{3!} + 7^3\frac{\alpha_8}{3!} + 8^3\frac{\alpha_9}{3!} + 9^3\frac{\alpha_{10}}{3!} = \frac{11368009}{119750400}, \tag{2.192}$$

$$\frac{\alpha_0}{4!} + \frac{\alpha_2}{4!} + 2^4\frac{\alpha_3}{4!} + 3^4\frac{\alpha_4}{4!} + 4^4\frac{\alpha_5}{4!} + 5^4\frac{\alpha_6}{4!} + 6^4\frac{\alpha_7}{4!} + 7^4\frac{\alpha_8}{4!} + 8^4\frac{\alpha_9}{4!} + 9^4\frac{\alpha_{10}}{4!} = \frac{131904163}{3113510400}, \tag{2.193}$$

$$-\frac{\alpha_0}{5!} + \frac{\alpha_2}{5!} + 2^5\frac{\alpha_3}{5!} + 3^5\frac{\alpha_4}{5!} + 4^5\frac{\alpha_5}{5!} + 5^5\frac{\alpha_6}{5!} + 6^5\frac{\alpha_7}{5!} + 7^5\frac{\alpha_8}{5!} + 8^5\frac{\alpha_9}{5!} + 9^5\frac{\alpha_{10}}{5!} = \frac{723798697}{46702656000}, \tag{2.194}$$

$$\frac{\alpha_0}{6!} + \frac{\alpha_2}{6!} + 2^6\frac{\alpha_3}{6!} + 3^6\frac{\alpha_4}{6!} + 4^6\frac{\alpha_5}{6!} + 5^6\frac{\alpha_6}{6!} + 6^6\frac{\alpha_7}{6!} + 7^6\frac{\alpha_8}{6!} + 8^6\frac{\alpha_9}{6!} + 9^6\frac{\alpha_{10}}{6!} = \frac{2541132023}{475517952000}, \tag{2.195}$$

$$-\frac{\alpha_0}{7!} + \frac{\alpha_2}{7!} + 2^7\frac{\alpha_3}{7!} + 3^7\frac{\alpha_4}{7!} + 4^7\frac{\alpha_5}{7!} + 5^7\frac{\alpha_6}{7!} + 6^7\frac{\alpha_7}{7!} + 7^7\frac{\alpha_8}{7!} + 8^7\frac{\alpha_9}{7!} + 9^7\frac{\alpha_{10}}{7!} = \frac{8768652467}{5230697472000}, \tag{2.196}$$

$$\frac{\alpha_0}{8!} + \frac{\alpha_2}{8!} + 2^8\frac{\alpha_3}{8!} + 3^8\frac{\alpha_4}{8!} + 4^8\frac{\alpha_5}{8!} + 5^8\frac{\alpha_6}{8!} + 6^8\frac{\alpha_7}{8!} + 7^8\frac{\alpha_8}{8!} + 8^8\frac{\alpha_9}{8!} + 9^8\frac{\alpha_{10}}{8!} = \frac{14042390777}{28582025472000}, \tag{2.197}$$

$$-\frac{\alpha_0}{9!} + \frac{\alpha_2}{9!} + 2^9\frac{\alpha_3}{9!} + 3^9\frac{\alpha_4}{9!} + 4^9\frac{\alpha_5}{9!} + 5^9\frac{\alpha_6}{9!} + 6^9\frac{\alpha_7}{9!} + 7^9\frac{\alpha_8}{9!} + 8^9\frac{\alpha_9}{9!} + 9^9\frac{\alpha_{10}}{9!} = \frac{2762162653}{20520428544000}, \tag{2.198}$$

$$\frac{\alpha_0}{10!} + \frac{\alpha_2}{10!} + 2^{10}\frac{\alpha_3}{10!} + 3^{10}\frac{\alpha_4}{10!} + 4^{10}\frac{\alpha_5}{10!} + 5^{10}\frac{\alpha_6}{10!} + 6^{10}\frac{\alpha_7}{10!} + 7^{10}\frac{\alpha_8}{10!} + 8^{10}\frac{\alpha_9}{10!} + 9^{10}\frac{\alpha_{10}}{10!} = \frac{3522018283439}{101370917007360000} - \frac{1}{47900160}. \tag{2.199}$$

Solving this system we get the parameters of the first end-point formula (i.e. $x = x_1$) for the tenth-order method. It is noted that the parameters

α_i $(i = 11, 12)$ may then be arbitrarily given the value zero . Thus

$$\left.\begin{aligned}
\alpha_0 &= \frac{116040349955470841}{3649353012264960000}, \\
\alpha_1 &= \frac{157203913989739}{330258191155200}, \\
\alpha_2 &= \frac{40179841536173}{2673518690304000}, \\
\alpha_3 &= \frac{5622449804159}{12509779968000}, \\
\alpha_4 &= \frac{-867092203321867}{1621934672117760}, \\
\alpha_5 &= \frac{1558230209576339}{3041127510220800000}, \\
\alpha_6 &= \frac{-2874137423864459}{8109673360588800}, \\
\alpha_7 &= \frac{5261248047297509}{30411275102208000}, \\
\alpha_8 &= \frac{01252013974567247}{22117290983424000}, \\
\alpha_9 &= \frac{814229640791783}{72987060245299200}, \\
\alpha_{10} &= \frac{-281299064581543}{2807194624819200000}.
\end{aligned}\right\} . \tag{2.200}$$

It can be shown using the method of undetermined coefficients for the point $x = x_2$, that, taking the parameter values b_0, b_2, b_4, b_6, b_8 in (2.45) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xx)}$ gives the system

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 + \beta_9 + \beta_{10} = \frac{882773}{907200}, \tag{2.201}$$

$$\begin{aligned} &-2\beta_0 - \beta_1 + \beta_3 + 2\beta_4 + 3\beta_5 + 4\beta_6 + 5\beta_7 + 6\beta_8 + 7\beta_9 + 8\beta_{10} \\ &= \frac{24427}{453600}, \end{aligned} \tag{2.202}$$

$$\begin{aligned} &2^2\tfrac{\beta_0}{2!} + \tfrac{\beta_1}{2!} + \tfrac{\beta_3}{2!} + 2^2\tfrac{\beta_4}{2!} + 3^2\tfrac{\beta_5}{2!} + 4^2\tfrac{\beta_6}{2!} + 5^2\tfrac{\beta_7}{2!} + 6^2\tfrac{\beta_8}{2!} + 7^2\tfrac{\beta_9}{2!} \\ &+8^2\tfrac{\beta_{10}}{2!} = \frac{43202009}{119750400}, \end{aligned} \tag{2.203}$$

$$\begin{aligned} &-2^3\tfrac{\beta_0}{3!} - \tfrac{\beta_1}{3!} + \tfrac{\beta_3}{3!} + 2^3\tfrac{\beta_4}{3!} + 3^3\tfrac{\beta_5}{3!} + 4^3\tfrac{\beta_6}{3!} + 5^3\tfrac{\beta_7}{3!} + 6^3\tfrac{\beta_8}{3!} + 7^3\tfrac{\beta_9}{3!} \\ &+8^3\tfrac{\beta_{10}}{3!} = \frac{2394839}{59875200}, \end{aligned} \tag{2.204}$$

$$\begin{aligned} &2^4\tfrac{\beta_0}{4!} + \tfrac{\beta_1}{4!} + \tfrac{\beta_3}{4!} + 2^4\tfrac{\beta_4}{4!} + 3^4\tfrac{\beta_5}{4!} + 4^4\tfrac{\beta_6}{4!} + 5^4\tfrac{\beta_7}{4!} + 6^4\tfrac{\beta_8}{4!} + 7^4\tfrac{\beta_9}{4!} \\ &+8^4\tfrac{\beta_{10}}{4!} = \frac{190486607}{3113510400}, \end{aligned} \tag{2.205}$$

$$-2^5\frac{\beta_0}{5!}+\frac{\beta_1}{5!}+\frac{\beta_3}{5!}+2^5\frac{\beta_4}{5!}+3^5\frac{\beta_5}{5!}+4^5\frac{\beta_6}{5!}+5^5\frac{\beta_7}{5!}+6^5\frac{\beta_8}{5!}+7^5\frac{\beta_9}{5!}$$
$$+8^5\frac{\beta_{10}}{5!}=\frac{21489493}{2122848000}, \tag{2.206}$$

$$2^6\frac{\beta_0}{6!}+\frac{\beta_1}{6!}+\frac{\beta_3}{6!}+2^6\frac{\beta_4}{6!}+3^6\frac{\beta_5}{6!}+4^6\frac{\beta_6}{6!}+5^6\frac{\beta_7}{6!}+6^6\frac{\beta_8}{6!}+7^6\frac{\beta_9}{6!}$$
$$+8^6\frac{\beta_{10}}{6!}=\frac{34992742353}{5230697472000}, \tag{2.207}$$

$$-2^7\frac{\beta_0}{7!}-\frac{\beta_1}{7!}+\frac{\beta_3}{7!}+2^7\frac{\beta_4}{7!}+3^7\frac{\beta_5}{7!}+4^7\frac{\beta_6}{7!}+5^7\frac{\beta_7}{7!}+6^7\frac{\beta_5}{7!}+7^7\frac{\beta_9}{7!}$$
$$+8^7\frac{\beta_{10}}{7!}=\frac{327962597}{237758976000}, \tag{2.208}$$

$$2^8\frac{\beta_0}{8!}+\frac{\beta_1}{8!}+\frac{\beta_3}{8!}+2^8\frac{\beta_4}{8!}+3^8\frac{\beta_5}{8!}+4^8\frac{\beta_6}{8!}+5^8\frac{\beta_7}{8!}+6^8\frac{\beta_5}{8!}+7^8\frac{\beta_9}{8!}$$
$$+8^8\frac{\beta_{10}}{8!}=\frac{881182516553}{1600593426432000}, \tag{2.209}$$

$$-2^9\frac{\beta_0}{9!}-\frac{\beta_1}{9!}+\frac{\beta_3}{9!}+2^9\frac{\beta_4}{9!}+3^9\frac{\beta_5}{9!}+4^9\frac{\beta_6}{9!}+5^9\frac{\beta_7}{9!}+6^9\frac{\beta_5}{9!}+7^9\frac{\beta_9}{9!}$$
$$+8^9\frac{\beta_{10}}{9!}=\frac{2542651289}{20520428544000}, \tag{2.210}$$

$$2^{10}\frac{\beta_0}{10!}+\frac{\beta_1}{10!}+\frac{\beta_3}{10!}+2^{10}\frac{\beta_4}{10!}+3^{10}\frac{\beta_5}{10!}+4^9\frac{\beta_6}{10!}+5^{10}\frac{\beta_7}{10!}+6^{10}\frac{\beta_5}{10!}+7^9\frac{\beta_9}{10!}$$
$$+8^{10}\frac{\beta_{10}}{10!}=\frac{7404524487683}{202741834014720000}-\frac{1}{47900160}, \tag{2.211}$$

the solution of which give the parameters of the second end-point formula (i.e. $x = x_2$) for the tenth-order method. It is noted that the parameters β_i $(i = 11, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\beta_0 &= \frac{43096055908784881}{3649353012264960000},\\
\beta_1 &= \frac{96572492798993699}{364935301226496000},\\
\beta_2 &= \frac{17879555117626619}{48658040163532800},\\
\beta_3 &= \frac{87652728055181}{243290200817664},\\
\beta_4 &= \frac{-47112876557436 11}{40548366802944000},\\
\beta_5 &= \frac{46489634142652499}{30411275102208 0000},\\
\beta_6 &= \frac{-4286488815953951}{40548366802944000},\\
\beta_7 &= \frac{315901599466553}{608225020441600},\\
\beta_8 &= \frac{-830947903694617}{48650163532800},\\
\beta_9 &= \frac{948322538888503}{2807194624819 2000},\\
\beta_{10} &= \frac{015867397222531 7}{5213361446092 80000}.
\end{aligned}\right\} \tag{2.212}$$

Next, it can be shown using the method of undetermined coefficients for the point $x = x_3$, that, taking the parameter values c_0, c_2, c_4, c_6, c_8 in (2.54) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xx)}$ in (2.56) gives

$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_8 + \gamma_9 + \gamma_{10} = \tfrac{302231}{302400}, \qquad (2.213)$$

$$-3\gamma_0 - 2\gamma_1 - \gamma_2 + \gamma_4 + 2\gamma_5 + 3\gamma_6 + 4\gamma_7 + 5\gamma_8 + 6\gamma_9 + 7\gamma_{10} = \tfrac{169}{100800}, \qquad (2.214)$$

$$3^2\tfrac{\gamma_0}{2!} + 2^2\tfrac{\gamma_1}{2!} + \tfrac{\gamma_2}{2!} + \tfrac{\gamma_4}{2!} + 2^2\tfrac{\gamma_5}{2!} + 3^2\tfrac{\gamma_6}{2!} + 4^2\tfrac{\gamma_7}{2!} + 5^2\tfrac{\gamma_8}{2!} + 6^2\tfrac{\gamma_9}{2!} + 7^2\tfrac{\gamma_{10}}{2!} = \tfrac{5510311}{13305600}, \qquad (2.215)$$

$$-3^3\tfrac{\gamma_0}{3!} - 2^3\tfrac{\gamma_1}{3!} - \tfrac{\gamma_2}{3!} + \tfrac{\gamma_4}{3!} + 2^3\tfrac{\gamma_5}{3!} + 3^3\tfrac{\gamma_6}{3!} + 4^3\tfrac{\gamma_7}{3!} + 5^3\tfrac{\gamma_8}{3!} + 6^3\tfrac{\gamma_9}{3!} + 7^3\tfrac{\gamma_{10}}{3!} = \tfrac{11381}{4435200}, \qquad (2.216)$$

$$3^4\tfrac{\gamma_0}{4!} + 2^4\tfrac{\gamma_1}{4!} + \tfrac{\gamma_2}{4!} + \tfrac{\gamma_4}{4!} + 2^4\tfrac{\gamma_5}{4!} + 3^4\tfrac{\gamma_6}{4!} + 4^4\tfrac{\gamma_7}{4!} + 5^4\tfrac{\gamma_8}{4!} + 6^4\tfrac{\gamma_9}{4!} + 7^4\tfrac{\gamma_{10}}{4!} = \tfrac{591141643}{7264857600}, \qquad (2.217)$$

$$-3^5\tfrac{\gamma_0}{5!} - 2^5\tfrac{\gamma_1}{5!} - \tfrac{\gamma_2}{5!} + \tfrac{\gamma_4}{5!} + 2^5\tfrac{\gamma_5}{5!} + 3^5\tfrac{\gamma_6}{5!} + 4^5\tfrac{\gamma_7}{5!} + 5^5\tfrac{\gamma_8}{5!} + 6^5\tfrac{\gamma_9}{5!} + 7^5\tfrac{\gamma_{10}}{5!} = \tfrac{14645899}{12108096000}, \qquad (2.218)$$

$$3^6\tfrac{\gamma_0}{6!} + 2^6\tfrac{\gamma_1}{6!} + \tfrac{\gamma_2}{6!} + \tfrac{\gamma_4}{6!} + 2^6\tfrac{\gamma_5}{6!} + 3^6\tfrac{\gamma_6}{6!} + 4^6\tfrac{\gamma_7}{6!} + 5^6\tfrac{\gamma_8}{6!} + 6^6\tfrac{\gamma_9}{6!} + 7^6\tfrac{\gamma_{10}}{6!} = \tfrac{1346510087}{134120448000}, \qquad (2.219)$$

$$-3^7\tfrac{\gamma_0}{7!} - 2^7\tfrac{\gamma_1}{7!} - \tfrac{\gamma_2}{7!} + \tfrac{\gamma_4}{7!} + 2^7\tfrac{\gamma_5}{7!} + 3^7\tfrac{\gamma_6}{7!} + 4^7\tfrac{\gamma_7}{7!} + 5^7\tfrac{\gamma_8}{7!} + 6^7\tfrac{\gamma_9}{7!} + 7^7\tfrac{\gamma_{10}}{7!} = \tfrac{162013909}{581188608000}, \qquad (2.220)$$

$$3^8\tfrac{\gamma_0}{8!} + 2^8\tfrac{\gamma_1}{8!} + \tfrac{\gamma_2}{8!} + \tfrac{\gamma_4}{8!} + 2^8\tfrac{\gamma_5}{8!} + 3^8\tfrac{\gamma_6}{8!} + 4^8\tfrac{\gamma_7}{8!} + 5^8\tfrac{\gamma_8}{8!} + 6^8\tfrac{\gamma_9}{8!} + 7^8\tfrac{\gamma_{10}}{8!} = \tfrac{19405166329}{22230464256000}, \qquad (2.221)$$

$$-3^9\tfrac{\gamma_0}{9!} - 2^9\tfrac{\gamma_1}{9!} - \tfrac{\gamma_2}{9!} + \tfrac{\gamma_4}{9!} + 2^9\tfrac{\gamma_5}{9!} + 3^9\tfrac{\gamma_6}{9!} + 4^9\tfrac{\gamma_7}{9!} + 5^9\tfrac{\gamma_8}{9!} + 6^9\tfrac{\gamma_9}{9!} + 7^9\tfrac{\gamma_{10}}{9!} = \tfrac{163046441}{4234374144000}, \qquad (2.222)$$

$$3^{10}\frac{\gamma_0}{10!}+2^{10}\frac{\gamma_1}{9!}+\frac{\gamma_2}{10!}+\frac{\gamma_4}{9!}+2^{10}\frac{\gamma_5}{10!}+3^{10}\frac{\gamma_6}{10!}+4^{10}\frac{\gamma_7}{10!}+5^{10}\frac{\gamma_8}{9!}+6^{10}\frac{\gamma_9}{10!}$$
$$+7^{10}\frac{\gamma_{10}}{10!}=\frac{5800069899419}{101370917007360000}-\frac{1}{4700960}. \tag{2.223}$$

Solving this system we get the parameters of the third end-point formula (i.e. $x = x_3$) for the tenth-order method; they are

$$\left.\begin{aligned}
\gamma_0 &= \frac{2061298229437523}{283838567620608000},\\
\gamma_1 &= \frac{11589154862126857}{283838567620608000},\\
\gamma_2 &= \frac{9139422280088273}{37845142349144000},\\
\gamma_3 &= \frac{683712457758821}{1576880931225600},\\
\gamma_4 &= \frac{3228137601455083}{13516122267648000},\\
\gamma_5 &= \frac{1503276660462431}{33790305669120000},\\
\gamma_6 &= \frac{-21616099543697}{13516122267648000},\\
\gamma_7 &= \frac{5080973291}{3435470438400},\\
\gamma_8 &= \frac{-3695100594191}{7569028469882880},\\
\gamma_9 &= \frac{27537990586177}{283838567620608000},\\
\gamma_{10} &= \frac{-1915764666829}{218337359708160000}.
\end{aligned}\right\} \tag{2.224}$$

It is noted that the parameters γ_i $(i = 11, 12)$ may be arbitrarily assigned the value zero.

Finally, it can be shown using the method of undetermined coefficients for the point $x = x_4$ that, taking the parameter values d_0, d_2, d_4, d_6, d_8 in (2.63) together with parameters calculated below, yields the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xx)}$ in (2.65) gives

$$\delta_0+\delta_1+\delta_2+\delta_3+\delta_4+\delta_5+\delta_6+\delta_7+\delta_8+\delta_9+\delta_{10}=\frac{1814399}{1814400}, \tag{2.225}$$

$$-4\delta_0-3\delta_1-2\delta_2-\delta_3+\delta_5+2\delta_6+3\delta_7+4\delta_8+5\delta_9+6\delta_{10}=\frac{122753}{9979200}, \tag{2.226}$$

$$4^2\frac{\delta_0}{2!}+3^2\frac{\delta_1}{2!}+2^2\frac{\delta_2}{2!}+\frac{\delta_3}{2!}+\frac{\delta_5}{2!}+2^2\frac{\delta_6}{2!}+3^2\frac{\delta_7}{2!}+4^2\frac{\delta_8}{2!}+5^2\frac{\delta_9}{2!}$$
$$+6^2\frac{\delta_{10}}{2!}=\frac{14255849}{34214400}, \tag{2.227}$$

$$-4^3\frac{\delta_0}{3!} - 3^3\frac{\delta_1}{3!} - 2^3\frac{\delta_2}{3!} - \frac{\delta_3}{3!} + \frac{\delta_5}{3!} + 2^3\frac{\delta_6}{3!} + 3^3\frac{\delta_7}{3!} + 4^3\frac{\delta_8}{3!} + 5^3\frac{\delta_9}{3!}$$
$$+6^3\frac{\delta_{10}}{3!} = \frac{68891}{222393600}, \tag{2.228}$$

$$4^4\frac{\delta_0}{4!} + 3^4\frac{\delta_1}{4!} + 2^4\frac{\delta_2}{4!} + \frac{\delta_3}{4!} + \frac{\delta_5}{4!} + 2^4\frac{\delta_6}{4!} + 3^4\frac{\delta_7}{4!} + 4^4\frac{\delta_8}{4!} + 5^4\frac{\delta_9}{4!}$$
$$+6^4\frac{\delta_{10}}{4!} = \frac{363217187}{43589145600}, \tag{2.229}$$

$$-4^5\frac{\delta_0}{5!} - 3^5\frac{\delta_1}{5!} - 2^5\frac{\delta_2}{5!} - \frac{\delta_3}{5!} + \frac{\delta_5}{5!} + 2^5\frac{\delta_6}{5!} + 3^5\frac{\delta_7}{5!} + 4^5\frac{\delta_8}{5!} + 5^5\frac{\delta_9}{5!}$$
$$+6^5\frac{\delta_{10}}{5!} = \frac{413849}{326918592000}, \tag{2.230}$$

$$4^6\frac{\delta_0}{6!} + 3^6\frac{\delta_1}{6!} + 2^6\frac{\delta_2}{6!} + \frac{\delta_3}{6!} + \frac{\delta_5}{6!} + 2^6\frac{\delta_6}{6!} + 3^6\frac{\delta_7}{6!} + 4^6\frac{\delta_8}{6!} + 5^6\frac{\delta_9}{6!}$$
$$+6^6\frac{\delta_{10}}{6!} = \frac{10139471581}{951035904000}, \tag{2.231}$$

$$-4^7\frac{\delta_0}{7!} - 3^7\frac{\delta_1}{7!} - 2^7\frac{\delta_2}{7!} - \frac{\delta_3}{7!} + \frac{\delta_5}{7!} + 2^7\frac{\delta_7}{7!} + 3^7\frac{\delta_7}{7!} + 4^7\frac{\delta_8}{7!} + 5^7\frac{\delta_9}{7!}$$
$$+6^7\frac{\delta_{10}}{7!} = \frac{154643851}{88921857024000}, \tag{2.232}$$

$$4^8\frac{\delta_0}{8!} + 3^8\frac{\delta_1}{8!} + 2^8\frac{\delta_2}{8!} + \frac{\delta_3}{8!} + \frac{\delta_5}{8!} + 2^8\frac{\delta_6}{8!} + 3^8\frac{\delta_7}{8!} + 4^8\frac{\delta_8}{8!} + 5^8\frac{\delta_9}{8!}$$
$$+6^8\frac{\delta_{10}}{8!} = \frac{3141960414959}{3201186852864000}, \tag{2.233}$$

$$-4^9\frac{\delta_0}{9!} - 3^9\frac{\delta_1}{9!} - 2^9\frac{\delta_2}{9!} - \frac{\delta_3}{9!} + \frac{\delta_5}{9!} + 2^9\frac{\delta_7}{9!} + 3^9\frac{\delta_7}{9!} + 4^9\frac{\delta_8}{9!} + 5^9\frac{\delta_9}{9!}$$
$$+6^9\frac{\delta_{10}}{9!} = \frac{4165158373}{10137091700736000}, \tag{2.234}$$

$$4^{10}\frac{\delta_0}{10!} + 3^{10}\frac{\delta_1}{9!} + 2^{10}\frac{\delta_2}{10!} + \frac{\delta_3}{10!} + \frac{\delta_5}{10!} + 2^{10}\frac{\delta_7}{10!} + 3^{10}\frac{\delta_7}{10!} + 4^{10}\frac{\delta_8}{10!}$$
$$+5^{10}\frac{\delta_9}{10!} + 6^{10}\frac{\delta_{10}}{10!} = \frac{28108982850101}{405483668029440000} - \frac{1}{47900160}. \tag{2.235}$$

Solving this system we get the parameters of the fourth end-point formula (i.e. $x = x_4$) for the tenth-order method. It is noted that the parameters

δ_i $(i = 11, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\delta_0 &= \frac{-504886766892491}{51090942171709440000}, \\
\delta_1 &= \frac{1579429435112527}{102181884343418880 0}, \\
\delta_2 &= \frac{705680560899513}{179266463760384000}, \\
\delta_3 &= \frac{10648644932761 0741}{425757851430912000}, \\
\delta_4 &= \frac{41412396984 8707}{954079218892800}, \\
\delta_5 &= \frac{12714726728652943}{552932274585600 00}, \\
\delta_6 &= \frac{4196107713185}{92681981263872}, \\
\delta_7 &= \frac{-217783613195039}{425757851430912000}, \\
\delta_8 &= \frac{1820951439198607}{340662811447296000}, \\
\delta_9 &= \frac{-19314059878021}{204363768683 7760}, \\
\delta_{10} &= \frac{23668609477577}{3005349539512320000}.
\end{aligned}\right\} \qquad (2.236)$$

Because of symmetry, the special end-point formulae for the points $x_N, x_{N-1}, x_{N-2}, x_{N-3}$ may be written down directly from those for x_1, x_2, x_3, x_4, respectively.

The set of parameter values in (2.36), (2.200), (2.45), (2.212), (2.54), (2.224), (2.63) and (2.236) give c_{20} as the first non-zero constant in (2.9). Global extrapolation on two grids, with p=10 in (2.29), gives the numerical method.

$$\mathbf{Y}^{(E)} = \frac{1024}{1023} I^h_{\frac{1}{2}h} \mathbf{Y}^{(2)} - \frac{1}{1023} \mathbf{Y}^{(1)}. \qquad (2.237)$$

2.11 CONSTRUCTION OF A TWELFTH-ORDER METHOD

Writing $\alpha = \frac{1}{47900160}$, $\beta = \frac{61}{23950080}$, $\gamma = \frac{22103}{15966720}$, $\delta = \frac{11477}{285120}$ and $\epsilon = \frac{215687}{887040}$ so that $\sum = 1 - 2(\alpha - \beta - \gamma - \delta - \epsilon) = \frac{1718069}{3991680}$, in (2.3), gives the unique twelfth-order method of the family (2.3) for $n = 1, 2, 3, 4, N-3, N-2, N-1$, or N . The first non-zero constant in (2.9) then becomes as

$$c_{22} = \frac{691}{23775897600}, \tag{2.238}$$

with $c_{13} = c_{15} = c_{17} = c_{19} = c_{21} = \ldots\ldots = 0$, because of symmetry.

One can obtain the same values of c_i $(i = 11, 12, 13, \ldots\ldots, 22)$ for the end points $n = 1, 2, 3, 4, N-3, N-2, N-1, N$ by choosing the parameters a_i, b_i, c_i, d_i $(i = 0, 2, 4, 6, 8)$ as given in section 2.6 and assigning the remaining parameters in (2.12)—(2.19) as follows.

For the point $x = x_1$, consider the scheme (2.37). Then equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}$, $y^{(xii)}, \ldots\ldots, y^{(xxii)}$ in (2.38) gives the system

$$\begin{aligned}&\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}\\&+\alpha_{11} + \alpha_{12} = \tfrac{655177}{907200},\end{aligned} \tag{2.239}$$

$$\begin{aligned}&-\alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 6\alpha_7 + 7\alpha_8 + 8\alpha_9\\&+9\alpha_{10} + 10\alpha_{11} + 11\alpha_{12} = \tfrac{252023}{907200},\end{aligned} \tag{2.240}$$

$$\begin{aligned}&\tfrac{\alpha_0}{2!} + \alpha_2 + 2^2\tfrac{\alpha_3}{2!} + 3^2\tfrac{\alpha_4}{2!} + 4^2\tfrac{\alpha_5}{2!} + 5^2\tfrac{\alpha_6}{2!} + 6^2\tfrac{\alpha_7}{2!} + 7^2\tfrac{\alpha_8}{2!} + 8^2\tfrac{\alpha_9}{2!}\\&+9^2\tfrac{\alpha_{10}}{2!} + 10^2\tfrac{\alpha_{11}}{2!} + 11^2\tfrac{\alpha_{12}}{2!} = \tfrac{27438979}{119750400},\end{aligned} \tag{2.241}$$

$$\begin{aligned}&-\tfrac{\alpha_0}{3!} + \alpha_2 + 2^3\tfrac{\alpha_3}{3!} + 3^3\tfrac{\alpha_4}{3!} + 4^3\tfrac{\alpha_5}{3!} + 5^3\tfrac{\alpha_6}{3!} + 6^3\tfrac{\alpha_7}{3!} + 7^3\tfrac{\alpha_8}{3!} + 8^3\tfrac{\alpha_9}{3!}\\&+9^3\tfrac{\alpha_{10}}{3!} + 10^3\tfrac{\alpha_{11}}{3!} + 11^3\tfrac{\alpha_{12}}{3!} = \tfrac{11368009}{119750400},\end{aligned} \tag{2.242}$$

$$\begin{aligned}&\tfrac{\alpha_0}{4!} + \tfrac{\alpha_2}{4!} + 2^4\tfrac{\alpha_3}{4!} + 3^4\tfrac{\alpha_4}{4!} + 4^4\tfrac{\alpha_5}{4!} + 5^4\tfrac{\alpha_6}{4!} + 6^4\tfrac{\alpha_7}{4!} + 7^4\tfrac{\alpha_8}{4!} + 8^4\tfrac{\alpha_9}{4!}\\&+9^4\tfrac{\alpha_{10}}{4!} + 10^4\tfrac{\alpha_{11}}{4!} + 11^4\tfrac{\alpha_{12}}{4!} = \tfrac{131904163}{3113510400},\end{aligned} \tag{2.243}$$

$$\begin{aligned}&-\tfrac{\alpha_0}{5!} + \tfrac{\alpha_2}{5!} + 2^5\tfrac{\alpha_3}{5!} + 3^5\tfrac{\alpha_4}{5!} + 4^5\tfrac{\alpha_5}{5!} + 5^5\tfrac{\alpha_6}{5!} + 6^5\tfrac{\alpha_7}{5!} + 7^5\tfrac{\alpha_8}{5!} + 8^5\tfrac{\alpha_9}{5!}\\&+9^5\tfrac{\alpha_{10}}{5!} + 10^5\tfrac{\alpha_{11}}{5!} + 11^5\tfrac{\alpha_{12}}{5!} = \tfrac{723798697}{46702656000},\end{aligned} \tag{2.244}$$

$$\begin{aligned}&\tfrac{\alpha_0}{6!} + \tfrac{\alpha_2}{6!} + 2^6\tfrac{\alpha_3}{6!} + 3^6\tfrac{\alpha_4}{6!} + 4^6\tfrac{\alpha_5}{6!} + 5^6\tfrac{\alpha_6}{6!} + 6^6\tfrac{\alpha_7}{6!} + 7^6\tfrac{\alpha_8}{6!} + 8^6\tfrac{\alpha_9}{6!}\\&+9^6\tfrac{\alpha_{10}}{6!} + 10^6\tfrac{\alpha_{11}}{6!} + 11^6\tfrac{\alpha_{12}}{6!} = \tfrac{2541132023}{475517952000},\end{aligned} \tag{2.245}$$

$$\begin{aligned}&-\tfrac{\alpha_0}{7!} + \tfrac{\alpha_2}{7!} + 2^7\tfrac{\alpha_3}{7!} + 3^7\tfrac{\alpha_4}{7!} + 4^7\tfrac{\alpha_5}{7!} + 5^7\tfrac{\alpha_6}{7!} + 6^7\tfrac{\alpha_7}{7!} + 7^7\tfrac{\alpha_8}{7!} + 8^7\tfrac{\alpha_9}{7!}\\&+9^7\tfrac{\alpha_{10}}{7!} + 10^7\tfrac{\alpha_{11}}{7!} + 11^7\tfrac{\alpha_{12}}{7!} = \tfrac{8768652467}{523069747200},\end{aligned} \tag{2.246}$$

$$\begin{aligned}&\tfrac{\alpha_0}{8!}+\tfrac{\alpha_2}{8!}+2^8\tfrac{\alpha_3}{8!}+3^8\tfrac{\alpha_4}{8!}+4^8\tfrac{\alpha_5}{8!}+5^8\tfrac{\alpha_6}{8!}+6^8\tfrac{\alpha_7}{8!}+7^8\tfrac{\alpha_8}{8!}+8^8\tfrac{\alpha_9}{8!}\\&+9^8\tfrac{\alpha_{10}}{8!}+10^8\tfrac{\alpha_{11}}{8!}+11^8\tfrac{\alpha_{12}}{8!}=\tfrac{14042390777}{28582025472000},\end{aligned}\tag{2.247}$$

$$\begin{aligned}&-\tfrac{\alpha_0}{9!}+\tfrac{\alpha_2}{9!}+2^9\tfrac{\alpha_3}{9!}+3^9\tfrac{\alpha_4}{9!}+4^9\tfrac{\alpha_5}{9!}+5^9\tfrac{\alpha_6}{9!}+6^9\tfrac{\alpha_7}{9!}+7^9\tfrac{\alpha_8}{9!}+8^9\tfrac{\alpha_9}{9!}\\&+9^9\tfrac{\alpha_{10}}{9!}+10^9\tfrac{\alpha_{11}}{9!}+11^9\tfrac{\alpha_{12}}{9!}=\tfrac{2762162653}{20520428544000},\end{aligned}\tag{2.248}$$

$$\begin{aligned}&\tfrac{\alpha_0}{10!}+\tfrac{\alpha_2}{10!}+2^{10}\tfrac{\alpha_3}{10!}+3^{10}\tfrac{\alpha_4}{10!}+4^{10}\tfrac{\alpha_5}{10!}+5^{10}\tfrac{\alpha_6}{10!}+6^{10}\tfrac{\alpha_7}{10!}+7^{10}\tfrac{\alpha_8}{10!}\\&+8^{10}\tfrac{\alpha_9}{10!}+9^{10}\tfrac{\alpha_{10}}{10!}+10^{10}\tfrac{\alpha_{11}}{10!}+11^{10}\tfrac{\alpha_{12}}{10!}=\tfrac{3522018283439}{1013709170073600000},\end{aligned}\tag{2.249}$$

$$\begin{aligned}&-\tfrac{\alpha_0}{11!}+\tfrac{\alpha_2}{11!}+2^{11}\tfrac{\alpha_3}{11!}+3^{11}\tfrac{\alpha_4}{11!}+4^{11}\tfrac{\alpha_5}{11!}+5^{11}\tfrac{\alpha_6}{11!}+6^{11}\tfrac{\alpha_7}{11!}+7^{11}\tfrac{\alpha_8}{11!}\\&+8^{11}\tfrac{\alpha_9}{11!}+9^{11}\tfrac{\alpha_{10}}{11!}+10^{11}\tfrac{\alpha_{11}}{11!}+11^{11}\tfrac{\alpha_{12}}{11!}=\tfrac{368462718776}{4344467817440000},\end{aligned}\tag{2.250}$$

$$\begin{aligned}&-\tfrac{\alpha_0}{12!}+\tfrac{\alpha_2}{12!}+2^{12}\tfrac{\alpha_3}{12!}+3^{12}\tfrac{\alpha_4}{12!}+4^{12}\tfrac{\alpha_5}{12!}+5^{12}\tfrac{\alpha_6}{12!}+6^{12}\tfrac{\alpha_7}{12!}+7^{12}\tfrac{\alpha_8}{12!}\\&+8^{12}\tfrac{\alpha_9}{12!}+9^{12}\tfrac{\alpha_{10}}{12!}+10^{12}\tfrac{\alpha_{11}}{12!}+11^{12}\tfrac{\alpha_{12}}{12!}=\tfrac{30689602988243}{15611121219133440000}+\tfrac{691}{23775897600}.\end{aligned}\tag{2.251}$$

Solving this system, we get the parameters of the first end-point formula (i.e. $x = x_1$) for the twelfth-order method. They are

$$\left.\begin{aligned}
\alpha_0 &= \frac{5971246575268812433}{19835306978131200000},\\
\alpha_1 &= \frac{3663586639878490261}{7397416301158400000},\\
\alpha_2 &= \frac{-4440754776771707783}{5109094217170944000 0},\\
\alpha_3 &= \frac{5923300186040234237}{766364132576416000},\\
\alpha_4 &= \frac{-927652722807756367}{756902846988000},\\
\alpha_5 &= \frac{3318929166936347}{2128789257154560000},\\
\alpha_6 &= \frac{-5498940528394228543}{3649353012264960000},\\
\alpha_7 &= \frac{2345698766262862649}{2128789257154560000},\\
\alpha_8 &= \frac{-4544095735229393 51}{75690284698828000},\\
\alpha_9 &= \frac{1812148033564250279}{766364132575641600 0},\\
\alpha_{10} &= \frac{-3253745190725366999}{51090942171709440000},\\
\alpha_{11} &= \frac{-1475894306383821109}{140500090972200960000},\\
\alpha_{12} &= \frac{-2701027397950574263}{337200218333282304000 0}.
\end{aligned}\right\}.\tag{2.252}$$

It can be shown using the method of undetermined coefficients for the point $x = x_2$, that, taking the parameter values b_0, b_2, b_4, b_6, b_8 in (2.45) together

with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}, y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xxii)}$ gives the system

$$\begin{aligned}&\beta_0+\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6+\beta_7+\beta_8+\beta_9+\beta_{10}\\&+\beta_{11}+\beta_{12}=\tfrac{882773}{907200},\end{aligned} \tag{2.253}$$

$$\begin{aligned}&-2\beta_0-\beta_1+\beta_3+2\beta_4+3\beta_5+4\beta_6+5\beta_7+6\beta_8+7\beta_9+8\beta_{10}\\&+9\beta_9+10\beta_{12}=\tfrac{24427}{453600},\end{aligned} \tag{2.254}$$

$$\begin{aligned}&2^2\tfrac{\beta_0}{2!}+\tfrac{\beta_1}{2!}+\tfrac{\beta_3}{2!}+2^2\tfrac{\beta_4}{2!}+3^2\tfrac{\beta_5}{2!}+4^2\tfrac{\beta_6}{2!}+5^2\tfrac{\beta_7}{2!}+6^2\tfrac{\beta_8}{2!}+7^2\tfrac{\beta_9}{2!}\\&+8^2\tfrac{\beta_{10}}{2!}+9^2\tfrac{\beta_{11}}{2!}+10^2\tfrac{\beta_{12}}{2!}=\tfrac{43202009}{119750400},\end{aligned} \tag{2.255}$$

$$\begin{aligned}&-2^3\tfrac{\beta_0}{3!}-\tfrac{\beta_1}{3!}+\tfrac{\beta_3}{3!}+2^3\tfrac{\beta_4}{3!}+3^3\tfrac{\beta_5}{3!}+4^3\tfrac{\beta_6}{3!}+5^3\tfrac{\beta_7}{3!}+6^3\tfrac{\beta_8}{3!}+7^3\tfrac{\beta_9}{3!}\\&+8^3\tfrac{\beta_{10}}{3!}+9^3\tfrac{\beta_{11}}{3!}+10^3\tfrac{\beta_{12}}{3!}=\tfrac{2394839}{59875200},\end{aligned} \tag{2.256}$$

$$\begin{aligned}&2^4\tfrac{\beta_0}{4!}+\tfrac{\beta_1}{4!}+\tfrac{\beta_3}{4!}+2^4\tfrac{\beta_4}{4!}+3^4\tfrac{\beta_5}{4!}+4^4\tfrac{\beta_6}{4!}+5^4\tfrac{\beta_7}{4!}+6^4\tfrac{\beta_8}{4!}+7^4\tfrac{\beta_9}{4!}\\&+8^4\tfrac{\beta_{10}}{4!}+9^4\tfrac{\beta_{11}}{4!}+10^4\tfrac{\beta_{12}}{4!}=\tfrac{190486607}{3113510400},\end{aligned} \tag{2.257}$$

$$\begin{aligned}&-2^5\tfrac{\beta_0}{5!}+\tfrac{\beta_1}{5!}+\tfrac{\beta_3}{5!}+2^5\tfrac{\beta_4}{5!}+3^5\tfrac{\beta_5}{5!}+4^5\tfrac{\beta_6}{5!}+5^5\tfrac{\beta_7}{5!}+6^5\tfrac{\beta_8}{5!}+7^5\tfrac{\beta_9}{5!}\\&+8^5\tfrac{\beta_{10}}{5!}+9^5\tfrac{\beta_{11}}{5!}+10^5\tfrac{\beta_{12}}{5!}=\tfrac{21489493}{2122848000},\end{aligned} \tag{2.258}$$

$$\begin{aligned}&2^6\tfrac{\beta_0}{6!}+\tfrac{\beta_1}{6!}+\tfrac{\beta_3}{6!}+2^6\tfrac{\beta_4}{6!}+3^6\tfrac{\beta_5}{6!}+4^6\tfrac{\beta_6}{6!}+5^6\tfrac{\beta_7}{6!}+6^6\tfrac{\beta_8}{6!}+7^6\tfrac{\beta_9}{6!}\\&+8^6\tfrac{\beta_{10}}{6!}+9^6\tfrac{\beta_{11}}{6!}+10^6\tfrac{\beta_{12}}{6!}=\tfrac{34992742353}{5230697472000},\end{aligned} \tag{2.259}$$

$$\begin{aligned}&-2^7\tfrac{\beta_0}{7!}-\tfrac{\beta_1}{7!}+\tfrac{\beta_3}{7!}+2^7\tfrac{\beta_4}{7!}+3^7\tfrac{\beta_5}{7!}+4^7\tfrac{\beta_6}{7!}+5^7\tfrac{\beta_7}{7!}+6^7\tfrac{\beta_5}{7!}+7^7\tfrac{\beta_9}{7!}\\&+8^7\tfrac{\beta_{10}}{7!}+9^7\tfrac{\beta_{11}}{7!}+10^7\tfrac{\beta_{12}}{7!}=\tfrac{327962597}{237758976000},\end{aligned} \tag{2.260}$$

$$\begin{aligned}&2^8\tfrac{\beta_0}{8!}+\tfrac{\beta_1}{8!}+\tfrac{\beta_3}{8!}+2^8\tfrac{\beta_4}{8!}+3^8\tfrac{\beta_5}{8!}+4^8\tfrac{\beta_6}{8!}+5^8\tfrac{\beta_7}{8!}+6^8\tfrac{\beta_5}{8!}+7^8\tfrac{\beta_9}{8!}\\&+8^8\tfrac{\beta_{10}}{8!}+9^8\tfrac{\beta_{11}}{8!}+10^8\tfrac{\beta_{12}}{8!}=\tfrac{881182516553}{1600593426432000},\end{aligned} \tag{2.261}$$

$$\begin{aligned}&-2^9\tfrac{\beta_0}{9!}-\tfrac{\beta_1}{9!}+\tfrac{\beta_3}{9!}+2^9\tfrac{\beta_4}{9!}+3^9\tfrac{\beta_5}{9!}+4^9\tfrac{\beta_6}{9!}+5^9\tfrac{\beta_7}{9!}+6^9\tfrac{\beta_5}{9!}+7^9\tfrac{\beta_9}{9!}\\&+8^9\tfrac{\beta_{10}}{9!}+9^9\tfrac{\beta_{11}}{9!}+10^9\tfrac{\beta_{12}}{9!}=\tfrac{2542651289}{20520428544000},\end{aligned} \tag{2.262}$$

$$2^{10}\frac{\beta_0}{10!}+\frac{\beta_1}{10!}+\frac{\beta_3}{10!}+2^{10}\frac{\beta_4}{10!}+3^{10}\frac{\beta_5}{10!}+4^{10}\frac{\beta_6}{10!}+5^{10}\frac{\beta_7}{10!}+6^{10}\frac{\beta_5}{10!}+7^{10}\frac{\beta_9}{10!}$$
$$+8^{10}\frac{\beta_{10}}{10!}+9^{10}\frac{\beta_{11}}{10!}+10^{10}\frac{\beta_{12}}{10!}=\frac{7404524487683}{202741834014720000}, \tag{2.263}$$

$$-2^{11}\frac{\beta_0}{11!}-\frac{\beta_1}{11!}+\frac{\beta_3}{11!}+2^{11}\frac{\beta_4}{11!}+3^{11}\frac{\beta_5}{11!}+4^{11}\frac{\beta_6}{11!}+5^{11}\frac{\beta_7}{11!}+6^{11}\frac{\beta_5}{11!}+7^{11}\frac{\beta_9}{11!}$$
$$+8^{11}\frac{\beta_{10}}{11!}+9^{11}\frac{\beta_{11}}{11!}+10^{11}\frac{\beta_{12}}{11!}=\frac{2496498203783}{3041127510220800000}, \tag{2.264}$$

$$-2^{12}\frac{\beta_0}{12!}-\frac{\beta_1}{12!}+\frac{\beta_3}{12!}+2^{12}\frac{\beta_4}{12!}+3^{12}\frac{\beta_5}{12!}+4^{12}\frac{\beta_6}{12!}+5^{12}\frac{\beta_7}{12!}+6^{12}\frac{\beta_5}{12!}+7^{12}\frac{\beta_9}{12!}$$
$$+8^{12}\frac{\beta_{10}}{12!}+9^{12}\frac{\beta_{11}}{12!}+10^{12}\frac{\beta_{12}}{12!}=\frac{20863491928843}{1040741414608896000 0}+\frac{691}{23775897600}, \tag{2.265}$$

the solution of which give the parameters of the second end-point formula (i.e. $x = x_2$) for the twelfth-order method. Writing

$$\left.\begin{aligned}
\beta_0 &= \frac{29247693780518024 29}{25938478333329408 0000},\\
\beta_1 &= \frac{380407114022636328 71}{140500090972200960000},\\
\beta_2 &= \frac{17131080986560697393}{51090942171709440000},\\
\beta_3 &= \frac{3543210612727803301}{7663641325756416000},\\
\beta_4 &= \frac{-506142686347310 51}{151380569397657600},\\
\beta_5 &= \frac{1029422989100344339}{2128789257154560000},\\
\beta_6 &= \frac{-17156193648318083 27}{3649353012264960000},\\
\beta_7 &= \frac{736404847800877441}{2128789257154560000},\\
\beta_8 &= \frac{-2514334152940439}{13278997315584000},\\
\beta_9 &= \frac{114714019591716851}{1532728265151283200},\\
\beta_{10} &= \frac{-10327518393117899 83}{51090942171709440000},\\
\beta_{11} &= \frac{46952036408009398 1}{140500090972200960000},\\
\beta_{12} &= \frac{-86087859967348300 7}{33720021833328230400 00}.
\end{aligned}\right\} \tag{2.266}$$

Next, it can be shown using the method of undetermined coefficients for the point $x = x_3$, that, taking the parameter values c_0, c_2, c_4, c_6, c_8 in (2.54) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xxii)}$ in

(2.56) gives

$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_8 + \gamma_9 + \gamma_{10} + \gamma_{11} + \gamma_{12} = \tfrac{302231}{302400}, \tag{2.267}$$

$$-3\gamma_0 - 2\gamma_1 - \gamma_2 + \gamma_4 + 2\gamma_5 + 3\gamma_6 + 4\gamma_7 + 5\gamma_8 + 6\gamma_9 + 7\gamma_{10} + 8\gamma_{11} + 9\gamma_{12} = \tfrac{169}{100800}, \tag{2.268}$$

$$3^2\tfrac{\gamma_0}{2!} + 2^2\tfrac{\gamma_1}{2!} + \tfrac{\gamma_2}{2!} + \tfrac{\gamma_4}{2!} + 2^2\tfrac{\gamma_5}{2!} + 3^2\tfrac{\gamma_6}{2!} + 4^2\tfrac{\gamma_7}{2!} + 5^2\tfrac{\gamma_8}{2!} + 6^2\tfrac{\gamma_9}{2!} + 7^2\tfrac{\gamma_{10}}{2!} + 8^2\tfrac{\gamma_{11}}{2!} + 9^2\tfrac{\gamma_{12}}{2!} = \tfrac{5510311}{13305600}, \tag{2.269}$$

$$-3^3\tfrac{\gamma_0}{3!} - 2^3\tfrac{\gamma_1}{3!} - \tfrac{\gamma_2}{3!} + \tfrac{\gamma_4}{3!} + 2^3\tfrac{\gamma_5}{3!} + 3^3\tfrac{\gamma_6}{3!} + 4^3\tfrac{\gamma_7}{3!} + 5^3\tfrac{\gamma_8}{3!} + 6^3\tfrac{\gamma_9}{3!} + 7^3\tfrac{\gamma_{10}}{3!} + 8^3\tfrac{\gamma_{11}}{3!} + 9^3\tfrac{\gamma_{12}}{3!} = \tfrac{11381}{4435200}, \tag{2.270}$$

$$3^4\tfrac{\gamma_0}{4!} + 2^4\tfrac{\gamma_1}{4!} + \tfrac{\gamma_2}{4!} + \tfrac{\gamma_4}{4!} + 2^4\tfrac{\gamma_5}{4!} + 3^4\tfrac{\gamma_6}{4!} + 4^4\tfrac{\gamma_7}{4!} + 5^4\tfrac{\gamma_8}{4!} + 6^4\tfrac{\gamma_9}{4!} + 7^4\tfrac{\gamma_{10}}{4!} + 8^4\tfrac{\gamma_{11}}{4!} + 9^4\tfrac{\gamma_{12}}{4!} = \tfrac{591141643}{7264857600}, \tag{2.271}$$

$$-3^5\tfrac{\gamma_0}{5!} - 2^5\tfrac{\gamma_1}{5!} - \tfrac{\gamma_2}{5!} + \tfrac{\gamma_4}{5!} + 2^5\tfrac{\gamma_5}{5!} + 3^5\tfrac{\gamma_6}{5!} + 4^5\tfrac{\gamma_7}{5!} + 5^5\tfrac{\gamma_8}{5!} + 6^5\tfrac{\gamma_9}{5!} + 7^5\tfrac{\gamma_{10}}{5!} + 8^5\tfrac{\gamma_{11}}{5!} + 9^5\tfrac{\gamma_{12}}{5!} = \tfrac{14645899}{12108096000}, \tag{2.272}$$

$$3^6\tfrac{\gamma_0}{6!} + 2^6\tfrac{\gamma_1}{6!} + \tfrac{\gamma_2}{6!} + \tfrac{\gamma_4}{6!} + 2^6\tfrac{\gamma_5}{6!} + 3^6\tfrac{\gamma_6}{6!} + 4^6\tfrac{\gamma_7}{6!} + 5^6\tfrac{\gamma_8}{6!} + 6^6\tfrac{\gamma_9}{6!} + 7^6\tfrac{\gamma_{10}}{6!} + 8^6\tfrac{\gamma_{11}}{6!} + 9^6\tfrac{\gamma_{12}}{6!} = \tfrac{1346510087}{134120448000}, \tag{2.273}$$

$$-3^7\tfrac{\gamma_0}{7!} - 2^7\tfrac{\gamma_1}{7!} - \tfrac{\gamma_2}{7!} + \tfrac{\gamma_4}{7!} + 2^7\tfrac{\gamma_5}{7!} + 3^7\tfrac{\gamma_6}{7!} + 4^7\tfrac{\gamma_7}{7!} + 5^7\tfrac{\gamma_8}{7!} + 6^7\tfrac{\gamma_9}{7!} + 7^7\tfrac{\gamma_{10}}{7!} + 8^7\tfrac{\gamma_{11}}{7!} + 9^7\tfrac{\gamma_{12}}{7!} = \tfrac{162013909}{581188608000}, \tag{2.274}$$

$$3^8\tfrac{\gamma_0}{8!} + 2^8\tfrac{\gamma_1}{8!} + \tfrac{\gamma_2}{8!} + \tfrac{\gamma_4}{8!} + 2^8\tfrac{\gamma_5}{8!} + 3^8\tfrac{\gamma_6}{8!} + 4^8\tfrac{\gamma_7}{8!} + 5^8\tfrac{\gamma_8}{8!} + 6^8\tfrac{\gamma_9}{8!} + 7^8\tfrac{\gamma_{10}}{8!} + 8^8\tfrac{\gamma_{11}}{8!} + 9^8\tfrac{\gamma_{12}}{8!} = \tfrac{19405166329}{22230464256000}, \tag{2.275}$$

$$-3^9\tfrac{\gamma_0}{9!} - 2^9\tfrac{\gamma_1}{9!} - \tfrac{\gamma_2}{9!} + \tfrac{\gamma_4}{9!} + 2^9\tfrac{\gamma_5}{9!} + 3^9\tfrac{\gamma_6}{9!} + 4^9\tfrac{\gamma_7}{9!} + 5^9\tfrac{\gamma_8}{9!} + 6^9\tfrac{\gamma_9}{9!} + 7^9\tfrac{\gamma_{10}}{9!} + 8^9\tfrac{\gamma_{11}}{9!} + 9^9\tfrac{\gamma_{12}}{9!} = \tfrac{163046441}{4234374144000}, \tag{2.276}$$

$$3^{10}\tfrac{\gamma_0}{10!} + 2^{10}\tfrac{\gamma_1}{9!} + \tfrac{\gamma_2}{10!} + \tfrac{\gamma_4}{9!} + 2^{10}\tfrac{\gamma_5}{10!} + 3^{10}\tfrac{\gamma_6}{10!} + 4^{10}\tfrac{\gamma_7}{10!} + 5^{10}\tfrac{\gamma_8}{10!} + 6^{10}\tfrac{\gamma_{10}}{10!} + 7^{10}\tfrac{\gamma_{10}}{10!} + 8^{10}\tfrac{\gamma_{11}}{10!} + 9^{10}\tfrac{\gamma_{12}}{10!} = \tfrac{5800069899419}{101370917007360000} \tag{2.277}$$

$$3^{11}\tfrac{\gamma_0}{11!} + 2^{11}\tfrac{\gamma_1}{11!} + \tfrac{\gamma_2}{11!} + \tfrac{\gamma_4}{11!} + 2^{11}\tfrac{\gamma_5}{11!} + 3^{11}\tfrac{\gamma_6}{11!} + 4^{11}\tfrac{\gamma_7}{11!} + 5^{11}\tfrac{\gamma_8}{11!} + 6^{11}\tfrac{\gamma_9}{11!} + 7^{11}\tfrac{\gamma_{10}}{11!} + 8^{11}\tfrac{\gamma_{11}}{11!} + 9^{11}\tfrac{\gamma_{12}}{11!} = \tfrac{847167156811}{23653213968384000 0}, \tag{2.278}$$

$$3^{12}\frac{\gamma_0}{11!} + 2^{11}\frac{\gamma_1}{12!} + \frac{\gamma_2}{12!} + \frac{\gamma_4}{12!} + 2^{12}\frac{\gamma_5}{12!} + 3^{12!}\frac{\gamma_6}{12!} + 4^{12}\frac{\gamma_7}{12!} + 5^{12}\frac{\gamma_8}{12!} + 6^{12}\frac{\gamma_9}{12!}$$
$$+7^{12}\frac{\gamma_{10}}{12!} + 8^{12}\frac{\gamma_{11}}{12!} + 9^{12}\frac{\gamma_{12}}{12!} = \frac{8172140843813}{27549037445529600000} + \frac{691}{23775897600}. \tag{2.279}$$

Solving this system we get the parameters of the third end-point formula (i.e. $x = x_3$) for the tenth-order method; they are

$$\left.\begin{aligned}
\gamma_0 &= \frac{20464729968383761}{288205314814771200000}, \\
\gamma_1 &= \frac{2634978111642113}{64243297198080000}, \\
\gamma_2 &= \frac{151708742486039533}{630752372490240000}, \\
\gamma_3 &= \frac{1779153894068441}{4074237812736000}, \\
\gamma_4 &= \frac{35151347430156497}{151380569397657600}, \\
\gamma_5 &= \frac{12903409227110351}{236532139683840000}, \\
\gamma_6 &= \frac{-5151286358526083}{405483668029440000}, \\
\gamma_7 &= \frac{2472096523139189}{236532139683840000}, \\
\gamma_8 &= \frac{-4355674222843283}{756902846988288000}, \\
\gamma_9 &= \frac{15571578934727}{6812125622894592}, \\
\gamma_{10} &= \frac{-1172953595222249}{1892257117470720000}, \\
\gamma_{11} &= \frac{535052750986283}{5203707073044480000}, \\
\gamma_{12} &= \frac{-2951463594930203}{374666909259202560000}.
\end{aligned}\right\} . \tag{2.280}$$

Finally, it can be shown using the method of undetermined coefficients for the point $x = x_4$ that, taking the parameter values d_0, d_2, d_4, d_6, d_8 in (2.63) together with parameters calculated below, yields the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xxii)}$ in (2.65) gives

$$\delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + \delta_7 + \delta_8 + \delta_9 + \delta_{10} + \delta_{11} + \delta_{12}$$
$$= \frac{1814399}{1814400}, \tag{2.281}$$

$$-4\delta_0 - 3\delta_1 - 2\delta_2 - \delta_3 + \delta_5 + 2\delta_6 + 3\delta_7 + 4\delta_8 + 5\delta_9 + 6\delta_{10}$$
$$+7\delta_{11} + 8\delta_{12} = \frac{122753}{9979200}, \tag{2.282}$$

$$4^2\tfrac{\delta_0}{2!}+3^2\tfrac{\delta_1}{2!}+2^2\tfrac{\delta_2}{2!}+\tfrac{\delta_3}{2!}+\tfrac{\delta_5}{2!}+2^2\tfrac{\delta_6}{2!}+3^2\tfrac{\delta_7}{2!}+4^2\tfrac{\delta_8}{2!}+5^2\tfrac{\delta_9}{2!}$$
$$+6^2\tfrac{\delta_{10}}{2!}+7^2\tfrac{\delta_{11}}{2!}+8^2\tfrac{\delta_{12}}{2!}=\tfrac{14255849}{34214400}, \tag{2.283}$$

$$-4^3\tfrac{\delta_0}{3!}-3^3\tfrac{\delta_1}{3!}-2^3\tfrac{\delta_2}{3!}-\tfrac{\delta_3}{3!}+\tfrac{\delta_5}{3!}+2^3\tfrac{\delta_6}{3!}+3^3\tfrac{\delta_7}{3!}+4^3\tfrac{\delta_8}{3!}+5^3\tfrac{\delta_9}{3!}$$
$$+6^3\tfrac{\delta_{10}}{3!}+7^3\tfrac{\delta_{11}}{3!}+8^3\tfrac{\delta_{12}}{3!}=\tfrac{68891}{222393600}, \tag{2.284}$$

$$4^4\tfrac{\delta_0}{4!}+3^4\tfrac{\delta_1}{4!}+2^4\tfrac{\delta_2}{4!}+\tfrac{\delta_3}{4!}+\tfrac{\delta_5}{4!}+2^4\tfrac{\delta_6}{4!}+3^4\tfrac{\delta_7}{4!}+4^4\tfrac{\delta_8}{4!}+5^4\tfrac{\delta_9}{4!}$$
$$+6^4\tfrac{\delta_{10}}{4!}+7^4\tfrac{\delta_{11}}{4!}+8^4\tfrac{\delta_{12}}{4!}=\tfrac{363217187}{43589145600}, \tag{2.285}$$

$$-4^5\tfrac{\delta_0}{5!}-3^5\tfrac{\delta_1}{5!}-2^5\tfrac{\delta_2}{5!}-\tfrac{\delta_3}{5!}+\tfrac{\delta_5}{5!}+2^5\tfrac{\delta_6}{5!}+3^5\tfrac{\delta_7}{5!}+4^5\tfrac{\delta_8}{5!}+5^5\tfrac{\delta_9}{5!}$$
$$+6^5\tfrac{\delta_{10}}{5!}+7^5\tfrac{\delta_{11}}{5!}+8^5\tfrac{\delta_{12}}{5!}=\tfrac{413849}{326918592000}, \tag{2.286}$$

$$4^6\tfrac{\delta_0}{6!}+3^6\tfrac{\delta_1}{6!}+2^6\tfrac{\delta_2}{6!}+\tfrac{\delta_3}{6!}+\tfrac{\delta_5}{6!}+2^6\tfrac{\delta_6}{6!}+3^6\tfrac{\delta_7}{6!}+4^6\tfrac{\delta_8}{6!}+5^6\tfrac{\delta_9}{6!}$$
$$+6^6\tfrac{\delta_{10}}{6!}+7^6\tfrac{\delta_{11}}{6!}+8^6\tfrac{\delta_{12}}{6!}=\tfrac{10139471581}{951035904000}, \tag{2.287}$$

$$-4^7\tfrac{\delta_0}{7!}-3^7\tfrac{\delta_1}{7!}-2^7\tfrac{\delta_2}{7!}-\tfrac{\delta_3}{7!}+\tfrac{\delta_5}{7!}+2^7\tfrac{\delta_7}{7!}+3^7\tfrac{\delta_7}{7!}+4^7\tfrac{\delta_8}{7!}+5^7\tfrac{\delta_9}{7!}$$
$$+6^7\tfrac{\delta_{10}}{7!}+7^7\tfrac{\delta_{11}}{7!}+8^7\tfrac{\delta_{12}}{7!}=\tfrac{154643851}{88921857024000}, \tag{2.288}$$

$$4^8\tfrac{\delta_0}{8!}+3^8\tfrac{\delta_1}{8!}+2^8\tfrac{\delta_2}{8!}+\tfrac{\delta_3}{8!}+\tfrac{\delta_5}{8!}+2^8\tfrac{\delta_6}{8!}+3^8\tfrac{\delta_7}{8!}+4^8\tfrac{\delta_8}{8!}+5^8\tfrac{\delta_9}{8!}$$
$$+6^8\tfrac{\delta_{10}}{8!}+7^8\tfrac{\delta_{11}}{8!}+8^8\tfrac{\delta_{12}}{8!}=\tfrac{3141960414959}{3201186852864000}, \tag{2.289}$$

$$-4^9\tfrac{\delta_0}{9!}-3^9\tfrac{\delta_1}{9!}-2^9\tfrac{\delta_2}{9!}-\tfrac{\delta_3}{9!}+\tfrac{\delta_5}{9!}+2^9\tfrac{\delta_7}{9!}+3^9\tfrac{\delta_7}{9!}+4^9\tfrac{\delta_8}{9!}+5^9\tfrac{\delta_9}{9!}$$
$$+6^9\tfrac{\delta_{10}}{9!}+7^9\tfrac{\delta_{11}}{9!}+8^9\tfrac{\delta_{12}}{9!}=\tfrac{4165158373}{10137091700736000}, \tag{2.290}$$

$$4^{10}\tfrac{\delta_0}{10!}+3^{10}\tfrac{\delta_1}{10!}+2^{10}\tfrac{\delta_2}{10!}+\tfrac{\delta_3}{10!}+\tfrac{\delta_5}{10!}+2^{10}\tfrac{\delta_7}{10!}+3^{10}\tfrac{\delta_7}{10!}+4^{10}\tfrac{\delta_8}{10!}$$
$$+5^{10}\tfrac{\delta_9}{10!}+6^{10}\tfrac{\delta_{10}}{10!}+7^{10}\tfrac{\delta_{11}}{10!}+8^{10}\tfrac{\delta_{12}}{10!}=\tfrac{28108982850101}{405483668029440000}, \tag{2.291}$$

$$4^{11}\tfrac{\delta_0}{11!}+3^{11}\tfrac{\delta_1}{11!}+2^{11}\tfrac{\delta_2}{11!}+\tfrac{\delta_3}{11!}+\tfrac{\delta_5}{11!}+2^{11}\tfrac{\delta_7}{11!}+3^{11}\tfrac{\delta_7}{11!}+4^{11}\tfrac{\delta_8}{11!}$$
$$+5^{11}\tfrac{\delta_9}{11!}+6^{11}\tfrac{\delta_{10}}{11!}+7^{11}\tfrac{\delta_{11}}{11!}+8^{11}\tfrac{\delta_{12}}{11!}=\tfrac{259687418609}{4257578514309120000}, \tag{2.292}$$

$$4^{12}\tfrac{\delta_0}{12!}+3^{12}\tfrac{\delta_1}{12!}+2^{12}\tfrac{\delta_2}{12!}+\tfrac{\delta_3}{12!}+\tfrac{\delta_5}{12!}+2^{12}\tfrac{\delta_7}{12!}+3^{12}\tfrac{\delta_7}{12!}+4^{12}\tfrac{\delta_8}{12!}$$
$$+5^{12}\tfrac{\delta_9}{12!}+6^{12}\tfrac{\delta_{10}}{12!}+7^{12}\tfrac{\delta_{11}}{12!}+8^{12}\tfrac{\delta_{12}}{12!}=\tfrac{4984415723143}{1274377242378240000}+\tfrac{691}{23775897600}. \tag{2.293}$$

Solving this system we get the parameters of the fourth end-point formula (i.e. $x = x_4$) for the twelfth-order method. They are

$$\left.\begin{aligned}
\delta_0 &= \frac{-20111634850897253}{67440043666656460800000}, \\
\delta_1 &= \frac{10850134190213011}{7397416301115840000}, \\
\delta_2 &= \frac{577222659467368697}{14597412049059840000}, \\
\delta_3 &= \frac{112174364942641021}{450802430926848000}, \\
\delta_4 &= \frac{60105119162462761}{137618699452416000}, \\
\delta_5 &= \frac{480950075796503597}{2128789257154560000}, \\
\delta_6 &= \frac{27866487234499003}{561438924963840000}, \\
\delta_7 &= \frac{-84329739335516631}{2128789257154560000}, \\
\delta_8 &= \frac{12673160847528O1}{504601897992192000}, \\
\delta_9 &= \frac{-978231278605993}{1094805903679488000}, \\
\delta_{10} &= \frac{228598710556O3727}{10218188434188800000}, \\
\delta_{11} &= \frac{-4904760768458891}{140500090972200960000}, \\
\delta_{12} &= \frac{17185081040673019}{67440043666656460800000}.
\end{aligned}\right\} . \tag{2.294}$$

Because of symmetry, the special end-point formulae for the points x_N, x_{N-1}, x_{N-2}, x_{N-3} may be written down directly from those for x_1, x_2, x_3, x_4, respectively.

The set of parameter values in (2.36), (2.252), (2.45), (2.266), (2.54), (2.280), (2.63) and (2.294) give c_{22} as the first non-zero constant in (2.9). Global extrapolation on two grids, with p=12 in (2.29), gives the numerical method.

$$\mathbf{Y}^{(E)} = \frac{4096}{4095} I^{h}_{\frac{1}{2}h} \mathbf{Y}^{(2)} - \frac{1}{4095} \mathbf{Y}^{(1)}. \tag{2.295}$$

2.12 NUMERICAL RESULTS

To compare the accuracy of the methods developed in this chapter, they were tested on the following problem.

In the computer program a one-point iteration function, analogous to the Gauss-Seidel method for solving linear algebraic systems, was used to obtain the solution vector.

PROBLEM.

$$y^{(10)}(x) = y^{(x)}(x) = 9!e^{-10y(x)} - 2(9!)(1+x)^{-10}, 0 < x < e^{\frac{1}{2}} - 1,$$

with boundary conditions

$$\left.\begin{array}{c} y(0) = 0, \quad y(e^{\frac{1}{2}} - 1) = \frac{1}{2}, \quad y^{(2i)}(0) = -(2i-1)! \\ \text{and} \\ y^{(2i)}(e^{\frac{1}{2}} - 1) = y^{(2i)}(0)e^{(-i)}, \quad i = 1, 2, 3, 4. \end{array}\right\} \quad (2.296)$$

The theoretical solution is given by

$$y(x) = \ln(1+x). \quad (2.297)$$

The interval $0 \leq x \leq e^{\frac{1}{2}} - 1$ for the problem was divided into N+1 equal subintervals each of width $h = 2^{-i}(e^{\frac{1}{2}} - 1)$ for $i = 4,\ 5,\ 6$. The corresponding values of N are then given by $N = 2^i - 1$.

The value of $||\mathbf{y} - \mathbf{Y}||_\infty$, where $\mathbf{Y}$ is some numerical solution, was computed for each value of N. The results for the second-, fourth-, sixth-, eighth- and twelfth-order methods are given in Table 2.1. Table 2.2 includes results for the extrapolation on two grids and the extrapolation on three grids (for the second-order method only).

It is evident from Table 2.2 that extrapolation on two and three grids does not improve accuracy. Overall, there is evidence in Tables 2.1 and 2.2 that decreasing the grid-size does not necessarily produce the desired effect of a considerable improvement in accuracy when using a higher order-method. This is due to the small value of h, raised to a large power, having little bearing on the calculation. This observation is also applicable to the extrapolation procedures which use fine grids.

Table 2.1: Error norms

N→ / Methods↓	15	31	63
Second-order	0.3109D-01	0.3113D-01	0.3114D-01
Fourth-order	0.3109D-01	0.3113D-01	0.3114D-01
Sixth-order	0.3109D-01	0.3113D-01	0.3114D-01
Eighth-order	0.3109D-01	0.3113D-01	0.3114D-01
Tenth-order	0.3109D-01	0.3113D-01	0.3114D-01
Twelfth-order	0.3109D-01	0.3113D-01	0.3114D-01

Table 2.2: Error norms for the extrapolation on two and three grids

Extrapolation→ / **Methods↓**	G_1	Two grids	Three grids
Second-order	0.3109D-01	0.3108D-01	0.3109D-01
Fourth-order	0.3109D-01	0.3108D-01	–
Sixth-order	0.3109D-01	0.3108D-01	–
Eighth-order	0.3109D-01	0.3108D-01	–
Tenth-order	0.3109D-01	0.3108D-01	–
Twelfth-order	0.3109D-01	0.3108D-01	–

Chapter 3

SPECIAL LINEAR TENTH-ORDER BOUNDARY-VALUE PROBLEMS

3.1 A FAMILY OF NUMERICAL METHODS

Consider the problem

$$y^{(x)}(x) = q(x)y(x) + r(x), \quad a < x < b; \tag{3.1}$$

$$y(a)^{(2i)} = A_{2i}, \quad y^{(2i)}(b) = B_{2i} \quad (i = 0, 1, 2, 3, 4). \tag{3.2}$$

It is assumed that the functions q(x) and r(x) are continuous on [a,b] and that A_{2i}, B_{2i} $(i = 0, 1, 2, 3, 4)$ are real finite constants; it will further be assumed that q(x), r(x) and y(x) are sufficiently-often differentiable on [a,b].

Consider first the mesh G, obtained by descretizing the interval $a \leq x \leq b$ into N+1 subintervals each of width $h = \frac{(b-a)}{N+1}$ where $N \geq 9$ is an integer. The solution y(x) will be computed at the mesh points $x_n = a + nh$ $(n = 1, 2, 3, 4$

$5, 6, \ldots\ldots, N)$ of mesh G and the notation y_n will be adopted to denote the solution of an approximating difference scheme at the grid point x_n. It is clear that, according to (3.2),

$$y_0 = A_0 \quad \text{and} \quad y_{N+1} = B_0.$$

A general family of symmetric numerical methods using is given by equation (2.3) of Chapter 2. Noting that $y^{(x)}(x_n) = q_n y_n + r_n$, where $q_n = q(x_n)$ and $r_n = r(x_n)$ for $n = 0, 1, 2, \ldots, N, N+1$ it is easy to show that

$$\begin{aligned} & y_{n-5} - 10y_{n-4} + 45y_{n-3} - 120y_{n-2} + 210y_{n-1} - 252y_n \\ & +210y_{n+1} - 120y_{n+2} + 45y_{n+3} - 10y_{n+4} + y_{n+5} \\ = \; & h^{10}[\alpha(q_{n-5}y_{n-5} + r_{n-5}) + \beta(q_{n-4}y_{n-4} + r_{n-4}) + \gamma(q_{n-3}y_{n-3} + r_{n-3}) \\ & +\delta(q_{n-2}y_{n-2} + r_{n-2}) + \epsilon(q_{n-1}y_{n-1} + r_{n-1}) + \sum(q_n y_n + r_n) \\ & +\epsilon(q_{n+1}y_{n+1} + r_{n+1}) + \delta(q_{n+2}y_{n+2} + r_{n+2}) + \gamma(q_{n+3}y_{n+3} + r_{n+3}) \\ & +\beta(q_{n+4}y_{n+4} + r_{n+4}) + \alpha(q_{n+5}y_{n+5} + r_{n+5})], \end{aligned} \tag{3.3}$$

in which α, β, γ, δ, ϵ are parameters chosen to ensure consistency as a minimum requirement and $\sum = 1 - 2(\alpha + \beta + \gamma + \delta + \epsilon)$.

3.2 SECOND-ORDER METHOD

Consider the second-order approximation

$$\begin{aligned} y^{(x)} = \; & h^{-10}[y_{n-5} - 10y_{n-4} + 45y_{n-3} - 120y_{n-2} \\ & +210y_{n-1} - 252y_n + 210y_{n+1} - 120y_{n+2} \\ & +45y_{n+3} - 10y_{n+4} + y_{n+5}] + O(h^2). \end{aligned} \tag{3.4}$$

Given the ordinary differential equation $y^{(x)} = f(x, y) = q(x)y(x) + r(x)$, at point n of the discretization $x_1, x_2, x_3, \quad \dots, x_n$, we have

$$\begin{aligned} &y_{n-5} - 10y_{n-4} + 45y_{n-3} - 120y_{n-2} + 210y_{n-1} - 252y_n \\ &+210y_{n+1} - 120y_{n+2} + 45y_{n+3} - 10y_{n+4} + y_{n+5} \\ &= h^{10}f_n = h^{10}(q_n y_n + r_n). \end{aligned} \tag{3.5}$$

Equation (3.5) may be written as

$$\begin{aligned} &-y_{n-5} + 10y_{n-4} - 45y_{n-3} + 120y_{n-2} - 210y_{n-1} + 252y_n \\ &-210y_{n+1} + 120y_{n+2} - 45y_{n+3} - 10y_{n+4} - y_{n+5} + h^{10}f_n \\ &= 0, \text{for} \quad n = 5, 6, 7, \quad \dots, N-5, N-4. \end{aligned} \tag{3.6}$$

The local truncation error (l.t.e.) of this numerical method is given by

$$\begin{aligned} L[y(x); h] = \; & -y(x-5h) + 10y(x-4h) - 45y(x-3h) + 120y(x-2h) \\ & -210y(x-h) + 252y(x) - 210y(x+h) + 120y(x+2h) \\ & -45y(x+3h) - 10y(x+4h) - y(x+5h) + h^{10}y^{(x)}(x). \end{aligned} \tag{3.7}$$

Writing (3.7) as a Taylor series about y(x) gives

$$L[y(x); h] = -\tfrac{5}{12}h^{12}y^{(xii)}(x) - \tfrac{1}{12}h^{14}y^{(xiv)}(x) - \tfrac{43}{4032}h^{16}y^{(xvi)}(x) - \dots. \tag{3.8}$$

The local truncation error t_n at the point x_n is then given by

$$\begin{aligned} t_n = \; & c_{11}h^{11}y^{(xi)}(x_n) + c_{12}h^{12}y^{(xii)}(x_n) + c_{13}h^{13}y^{(xiii)}(x_n) \\ & +c_{14}h^{14}y^{(xiv)}(x_n) + \; \dots\dots; \end{aligned} \tag{3.9}$$

in (3.9) the $c_{11}, c_{12}, c_{13}, c_{14}, \dots$ are constants with $c_{11} = c_{13} = c_{15} = c_{17} = c_{19} = c_{21} = \quad \dots \dots \dots = 0$ because of symmetry.

Equation (3.3) is applicable only to the N-8 mesh points x_n ($n = 5, 6, 7, 8, 9, 10, \dots, N-6, N-5, N-4$) of G. In order to be able to implement global extrapolation procedures special formulae are needed for the other mesh points n=1,2,3,4 and $n = N-3, N-2, N-1, N$ which also have local truncation error with principal part $\frac{-5}{12}h^{12}y^{(xii)}(x)$ in (3.8). These formulae will be assumed to be consistent.

It will be convenient in the convergence analysis on grid G again to introduce the matrix J of order N given by

$$J = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix}, \tag{3.10}$$

for which it is known that

$$||J^{-1}||_\infty = \frac{(N+1)^2}{8}. \tag{3.11}$$

In order to use the powers of the matrix J, these special end-point formulae will be assumed to be of the forms (3.12)—(3.19), as follows

$$\begin{aligned} &132y_1 - 165y_2 + 110y_3 - 44y_4 + 10y_5 - y_6 + a_0y_0 + a_2h^2y_0'' \\ &+a_4h^4y_0^{(iv)} + a_6h^6y_0^{(vi)} + a_8h^8y_0^{(viii)} \\ &+h^{10}[\alpha_0(q_0y_0 + r_0) + \alpha_1(q_1y_1 + r_1) + \alpha_2(q_2y_2 + r_2) + \quad \ldots + \alpha_{12}(q_{12}y_{12} + r_{12})] \\ &= 0, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &-165y_1 + 242y_2 - 209y_3 + 120y_4 - 45y_5 + 10y_6 - y_7 + b_0y_0 \\ &+b_2h^2y_0'' + b_4h^4y_0^{(iv)} + b_6h^6y_0^{(vi)} + b_8h^8y_0^{(viii)} \\ &+h^{10}[\beta_0(q_0y_0 + r_0) + \beta_1(q_1y_1 + r_1) + \beta_2(q_2y_2 + r_2) + \quad \ldots + \beta_{12}(q_{12}y_{12} + r_{12})] \\ &= 0, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &110y_1 - 209y_2 + 252y_3 - 210y_4 + 120y_5 - 45y_6 + 10y_7 - y_8 \\ &+c_0y_0 + c_2h^2y_0'' + c_4h^4y_0^{(iv)} + c_6h^6y_0^{(vi)} + c_8h^8y_0^{(viii)} \\ &+h^{10}[\gamma_0(q_0y_0 + r_0) + \gamma_1(q_1y_1 + r_1) + \gamma_2(q_2y_2 + r_2) + \quad \ldots + \gamma_{12}(q_{12}y_{12} + r_{12})] \\ &= 0, \end{aligned} \tag{3.14}$$

$$-44y_1 + 120y_2 - 210y_3 + 252y_4 - 210y_5 + 120y_6 - 45y_7 + 10y_8$$
$$-y_9 + d_0y_0 + d_2h^2y_0'' + d_4h^4y_0^{(iv)} + d_6h^6y_0^{(vi)} + d_8h^8y_0^{(viii)}$$
$$+h^{10}[\delta_0(q_0y_0 + r_0) + \delta_1(q_1y_1 + r_1) + \delta_2(q_2y_2 + r_2) + \quad \ldots + \delta_{12}(q_{12}y_{12} + r_{12})]$$
$$= 0. \tag{3.15}$$

At the other end of the array, the special end-point formula are as follows

$$-y_{N-8} + 10y_{N-7} - 45y_{N-6} + 120y_{N-5} - 210y_{N-4} + 252y_{N-3} - 210y_{N-2}$$
$$+120y_{N-1} - 44y_N + d_0y_{N+1} + d_2h^2y_{N+1}^{(ii)} + d_4h^4y_{N+1}^{(iv)} + d_6h^6y_{N+1}^{(vi)} + d_8h^8y_{N+1}^{(viii)}$$
$$+h^{10}[\delta_0(q_{N+1}y_{N+1} + r_{N+1}) + \delta_1(q_Ny_N + r_N) + \delta_2(q_{N-1}y_{N-1} + r_{N-1}) + \quad \ldots$$
$$+\delta_{12}(q_{N-11}y_{N-11} + r_{N-11})]$$
$$= 0, \tag{3.16}$$

$$-y_{N-7} + 10y_{N-6} - 45y_{N-5} + 120y_{N-4} - 210y_{N-3} + 252y_{N-2} - 209y_{N-1}$$
$$+110y_N + c_0y_{N+1} + c_2h^2y_{N+1}^{(ii)} + c_4h^4y_{N+1}^{(iv)} + c_6h^6y_{N+1}^{(vi)} + c_8h^8y_{N+1}^{(viii)}$$
$$+h^{10}[\gamma_0(q_{N+1}y_{N+1} + r_{N+1}) + \gamma_1(q_Ny_N + r_N) + \gamma_2(q_{N-1}y_{N-1} + r_{N-1}) + \quad \ldots$$
$$+\gamma_{12}(q_{N-11}y_{N-11} + r_{N-11})]$$
$$= 0, \tag{3.17}$$

$$-y_{N-6} + 10y_{N-5} - 45y_{N-4} + 120y_{N-3} - 209y_{N-2} + 242y_{N-1} - 165y_N$$
$$+b_0y_{N+1} + b_2h^2y_{N+1}^{(ii)} + b_4h^4y_{N+1}^{(iv)} + b_6h^6y_{N+1}^{(vi)} + b_8h^8y_{N+1}^{(viii)}$$
$$+h^{10}[\beta_0(q_{N+1}y_{N+1} + r_{N+1}) + \beta_1(q_Ny_N + r_N) + \beta_2(q_{N-1}y_{N-1} + r_{N-1}) + \quad \ldots$$
$$+\beta_{12}(q_{N-11}y_{N-11} + r_{N-11})]$$
$$= 0, \tag{3.18}$$

$$-y_{N-5} + 10y_{N-4} - 44y_{N-3} + 110y_{N-2} - 165y_{N-1} + 132y_N$$
$$+a_0y_{N+1} + a_2h^2y_{N+1}^{(ii)} + a_4h^4y_{N+1}^{(iv)} + a_6h^6y_{N+1}^{(vi)} + a_8h^8y_{N+1}^{(viii)}$$
$$+h^{10}[\alpha_0(q_{N+1}y_{N+1} + r_{N+1}) + \alpha_1(q_Ny_N + r_N) + \alpha_2(q_{N-1}y_{N-1} + r_{N-1}) + \quad \ldots$$
$$+\alpha_{12}(q_{N-11}y_{N-11} + r_{N-11})]$$
$$= 0. \tag{3.19}$$

The a_i, b_i, c_i, d_i $(i = 0, 2, 4, 6, 8)$ and $\alpha_i, \beta_i, \gamma_i, \delta_i$ $(i = 0, 1, 2, 3, \ldots, 12)$ are parameters which must be chosen so that the local truncation errors of (3.12)—(3.19) are identical with the (3.9) to the order required in sections 3.3, 3.4.

Clearly, the family of numerical methods is described by the set of equations $\{(3.12), (3.13), (3.14), (3.15), (3.16), (3.17), (3.18), (3.19)\}$ and the solution vector $\mathbf{Y} = [y_1, y_2, y_3, y_4, \ldots, y_N]^T$, T denoting transpose, is obtained by solving a linear algebraic system of order N which has the form

$$(J^5 + h^{10}MQ)\mathbf{Y} = \mathbf{b} - h^{10}M\mathbf{r}. \tag{3.20}$$

The matrix M in (3.20), of order N, is given by

$$M = \begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \alpha_9 & \alpha_{10} & \alpha_{11} & \alpha_{12} & & & \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 & \beta_{10} & \beta_{11} & \beta_{12} & & & \\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & & & \\
\delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 & \delta_7 & \delta_8 & \delta_9 & \delta_{10} & \delta_{11} & \delta_{12} & & & \\
\beta & \gamma & \delta & \epsilon & \sum & \epsilon & \delta & \gamma & \beta & \alpha & & & & & \\
\alpha & \beta & \gamma & \delta & \epsilon & \sum & \epsilon & \delta & \gamma & \beta & \alpha & & & & \\
 & \alpha & \beta & \gamma & \delta & \epsilon & \sum & \epsilon & \delta & \gamma & \beta & \alpha & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & \alpha & \beta & \gamma & \delta & \epsilon & \sum & \epsilon & \delta & \gamma & \beta & \alpha & \\
 & & & & \alpha & \beta & \gamma & \delta & \epsilon & \sum & \epsilon & \delta & \gamma & \beta & \alpha \\
 & & & & & \alpha & \beta & \gamma & \delta & \epsilon & \sum & \epsilon & \delta & \gamma & \beta \\
 & & & \delta_{12} & \delta_{11} & \delta_{10} & \delta_9 & \delta_8 & \delta_7 & \delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\
 & & & \gamma_{12} & \gamma_{11} & \gamma_{10} & \gamma_9 & \gamma_8 & \gamma_7 & \gamma_6 & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\
 & & & \beta_{12} & \beta_{11} & \beta_{10} & \beta_9 & \beta_8 & \beta_7 & \beta_6 & \beta_5 & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\
 & & & \alpha_{12} & \alpha_{11} & \alpha_{10} & \alpha_9 & \alpha_8 & \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1
\end{bmatrix}, \tag{3.21}$$

the vector $\mathbf{r}$ of order N has the form

$$\mathbf{r} = [r_1, r_2, r_3, r_4, r_5, \ldots, r_N]^T,$$

$Q = \text{diag}\{q_n\}$ is a diagonal matrix of order N, and the constant vector **b** of order N is given by

$$\mathbf{b} = \begin{bmatrix} a_0A_0 + a_2h^2A_2 + a_4h^4A_4 + a_6h^6A_6 + a_8h^8A_8 + a_{10}h^{10}y_0^{(x)} \\ b_0A_0 + b_2h^2A_2 + b_4h^4A_4 + b_6h^6A_6 + b_8h^8A_8 + b_{10}h^{10}y_0^{(x)} \\ c_0A_0 + c_2h^2A_2 + c_4h^4A_4 + c_6h^6A_6 + c_8h^8A_8 + c_{10}h^{10}y_0^{(x)} \\ d_0A_0 + d_2h^2A_2 + d_4h^4A_4 + d_6h^6A_6 + d_8h^8A_8 + d_{10}h^{10}y_0^{(x)} \\ -A_0 + h^{10}\alpha_0(q_0A_0 + r_0) \\ 0 \\ \vdots \\ 0 \\ -B_0 + h^{10}\alpha_0(q_{N+1}B_0 + r_{N+1}) \\ d_0B_0 + d_2h^2B_2 + d_4h^4B_4 + d_6h^6B_6 + d_8h^8B_8 + d_{10}h^{10}y_{N+1}^{(x)} \\ c_0B_0 + c_2h^2B_2 + c_4h^4B_4 + c_6h^6B_6 + c_8h^8B_8 + c_{10}h^{10}y_{N+1}^{(x)} \\ b_0B_0 + b_2h^2B_2 + b_4h^4B_4 + b_6h^6B_6 + b_8h^8B_8 + b_{10}h^{10}y_{N+1}^{(x)} \\ a_0B_0 + a_2h^2B_2 + a_4h^4B_4 + a_6h^6B_6 + a_8h^8B_8 + a_{10}h^{10}y_{N+1}^{(x)} \end{bmatrix}, \quad (3.22)$$

The exact solution vector $\mathbf{y} = [y(x_1), y(x_2), \ldots, y(x_N)]^T$ satisfies the equation

$$(J^5 + h^{10}MQ)\mathbf{y} = \mathbf{b} - h^{10}M\mathbf{r} + \mathbf{t} \quad (3.23)$$

where, $\mathbf{t} = [t_1, t_2, t_3, \ldots, t_N]^T$ is the vector of local truncation errors.

3.3 CONVERGENCE ANALYSIS OF THE SECOND-ORDER METHOD

For the convergence analysis we must obtain a bound on $||\mathbf{z}||_\infty$, where $\mathbf{z} = \mathbf{y} - \mathbf{Y}$. Equations (3.20) and (3.23) give

$$(J^5 + h^{10}MQ)\mathbf{z} = \mathbf{t}, \quad (3.24)$$

from which it follows (see Chapter 2) that

$$||z|| \leq \frac{(b-a)^{10}}{32768-(b-a)^{10}M^*Q^*}[|c_{12}|h^2V_{12} + |c_{14}|h^4V_{14} + |c_{16}|h^6V_{16} + \ldots] \quad (3.25)$$

where

$$V_i = \max_{a \leq x \leq b} |\frac{d^iy(x)}{dx^i}|, \quad M^* = ||M||_\infty,$$

and

$$Q^* = \max_n |q_n|,$$

provided

$$Q^* < \frac{32768}{(b-a)^{10}M^*}$$

and the parameters in (3.12)—(3.19) are chosen to ensure that $c_{11} = c_{13} = 0$. The order of convergence of a numerical method is p if c_{p+10} is the first non-vanishing constant on the right-hand side of (3.9).

3.4 THE PARAMETERS OF THE SECOND-ORDER METHOD

Writing $\alpha = \beta = \gamma = \delta = \epsilon = 0$ in (3.3) gives, as has already been seen,

$$c_{12} = \frac{-5}{12}, c_{14} = \frac{-1}{12} \quad (3.26)$$

in (3.9), thus verifying that (3.3) is a second-order method. To be able to implement global extrapolation on two and three grids the parameters in the special end-point formulae (3.12)—(3.19) must be chosen so that $c_{11} = c_{13} = 0$ in (3.9) and so that c_{12} and c_{14} in (3.9), with $n = 1, 2, 3, 4, N-3, N-2, N-1$, or N agree with (3.26).

Using the method of undetermined coefficients reveals that, for the point $x = x_1$ this is achieved provided

$$a_0 = -42, a_2 = 14, a_4 = \frac{-23}{6}, a_6 = \frac{217}{180}, a_8 = \frac{-809}{1440}, \quad (3.27)$$

together with parameters $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{12}$ calculated from the local truncation error (3.12) which, with $x = x_1$, becomes

$$\left.\begin{aligned} L[y(x);h] = {} & 132y(x) - 165y(x+h) + 110y(x+2h) - 44y(x+3h) \\ & +10y(x+4h) - y(x+5h) - 42y(x-h) + 14h^2y''(x-h) \\ & -\tfrac{23}{6}h^4y^{(iv)}(x-h) + \tfrac{217}{180}h^6y^{(vi)}(x-h) \\ & -\tfrac{809}{1440}h^8y^{(viii)}(x-h) + h^{10}[\alpha_0y^{(x)}(x-h) \\ & +\alpha_1y^{(x)}(x) + \alpha_2y^{(x)}(x+h) + \alpha_3y^{(x)}(x+2h) \\ & +\alpha_4y^{(x)}(x+3h) + \alpha_5y^{(x)}(x+4h) + \alpha_6y^{(x)}(x+5h) \\ & +\alpha_7y^{(x)}(x+6h) + \alpha_8y^{(x)}(x+7h) + \alpha_9y^{(x)}(x+8h) \\ & +\alpha_{10}y^{(x)}(x+9h) + \alpha_{11}y^{(x)}(x+10h) + \alpha_{12}y^{(x)}(x+11h)] \end{aligned}\right\} . \tag{3.28}$$

Expanding the function terms and their derivatives in (3.28) by the Taylor expansion gives, at the point $x = x_1$,

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \frac{655177}{907200}, \tag{3.29}$$

$$\alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 = \frac{252023}{907200}, \tag{3.30}$$

$$\frac{\alpha_0}{2!} + \alpha_2 + 2^2\frac{\alpha_3}{2!} + 3^2\frac{\alpha_4}{2!} = \frac{27438979}{119750400} - \frac{5}{12}, \tag{3.31}$$

$$-\frac{\alpha_0}{3!} + \alpha_2 + 2^3\frac{\alpha_3}{3!} + 3^3\frac{\alpha_4}{3!} = \frac{11368009}{119750400}, \tag{3.32}$$

$$\frac{\alpha_0}{4!} + \frac{\alpha_2}{4!} + 2^4\frac{\alpha_3}{4!} + 3^4\frac{\alpha_4}{4!} = \frac{131904163}{3113510400} - \frac{1}{12}. \tag{3.33}$$

Solving this system we get the parameters of the first end-point formula (i.e. $x = x_1$) for the second-order method. They are

$$\left.\begin{aligned} \alpha_0 &= -\tfrac{1586842547}{3736212480}, \\ \alpha_1 &= \tfrac{1683367717}{1167566400}, \\ \alpha_2 &= -\tfrac{306653299}{622702080}, \\ \alpha_3 &= \tfrac{2886847}{11675664}, \\ \alpha_4 &= -\tfrac{927622183}{18681062400}, \end{aligned}\right\} \tag{3.34}$$

and it is noted that the parameters α_i (i = 5,6,7, ... , 12) may then be arbitrarily given the value zero.

Using the method of undetermined coefficients reveals that for the point $x = x_2$ the first two non-vanishing terms in the local truncation error have the values given in (3.8) provided

$$b_0 = 48, b_2 = -14, b_4 = \frac{17}{6}, b_6 = \frac{-67}{180}, b_8 = \frac{-809}{1440}, \tag{3.35}$$

together with parameters β_i $(i = 1, 2, \ldots, 12)$ calculated from the expression

$$\left.\begin{aligned} L[y(x);h] = \quad & -165y(x-h) + 242y(x) - 209y(x+h) + 120y(x+2h) \\ & -45y(x+3h) + 10y(x+4h) - y(x+5h) \\ & +48y(x-2h) - 14h^2y''(x-2h) + \tfrac{17}{6}h^4y^{(iv)}(x-2h) \\ & -\tfrac{67}{180}h^6y^{(vi)}(x-2h) - \tfrac{289}{1440}h^8y^{(viii)}(x-2h) \\ & +h^{10}[\beta_0y^{(x)}(x-2h) + \beta_1y^{(x)}(x-h) + \beta_2y^{(x)}(x) \\ & +\beta_3y^{(x)}(x+h) + \beta_4y^{(x)}(x+2h) + \beta_5y^{(x)}(x+3h) \\ & +\beta_6y^{(x)}(x+4h) + \beta_7y^{(x)}(x+5h) + \beta_8y^{(x)}(x+6h) \\ & +\beta_9y^{(x)}(x+7h) + \beta_{10}y^{(x)}(x+8h) + \beta_{11}y^{(x)}(x+9h) \\ & +\beta_{12}y^{(x)}(x+10h) + \ldots] \end{aligned}\right\} \tag{3.36}$$

in which $x = x_2$.

Expanding the function terms and their derivatives in (3.36), and equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}$, $y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}$ gives

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = \frac{882773}{907200}, \tag{3.37}$$

$$-2\beta_0 - \beta_1 + \beta_3 + 2\beta_4 = \frac{24427}{453600}, \tag{3.38}$$

$$2^2\frac{\beta_0}{2!} + \frac{\beta_1}{2!} + \frac{\beta_3}{2!} + 2^2\frac{\beta_4}{2!} = \frac{43202009}{119750400} - \frac{5}{12}, \tag{3.39}$$

$$-2^3\frac{\beta_0}{3!} - \frac{\beta_1}{3!} + \frac{\beta_3}{3!} + 2^3\frac{\beta_4}{3!} = \frac{2394839}{59875200}, \tag{3.40}$$

$$2^4\frac{\beta_0}{4!} + \frac{\beta_1}{4!} + \frac{\beta_3}{4!} + 2^4\frac{\beta_4}{4!} = \frac{190486607}{3113510400} - \frac{1}{12}. \tag{3.41}$$

Solving this system, we get the parameters of the second end-point formula (i.e. $x = x_2$) for the second-order method. It is noted that the parameters β_i $(i = 5, 6, 7, \ldots, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\beta_0 &= -\frac{123315019}{3736212480},\\
\beta_1 &= \frac{4243927}{233513280},\\
\beta_2 &= \frac{277359163}{283046400},\\
\beta_3 &= \frac{5827189}{583783200},\\
\beta_4 &= -\frac{7410451}{3736212480}.
\end{aligned}\right\} \tag{3.42}$$

Using the method of undetermined coefficients reveals that for the point $x = x_3$ the first two non-vanishing terms in the local truncation error have the values given in (3.8) provided

$$c_0 = -27, c_2 = 6, c_4 = \frac{-1}{2}, c_6 = \frac{-3}{20}, c_8 = \frac{-41}{3360}, \tag{3.43}$$

together with parameters γ_i $(i = 0, 1, 2, \ldots, 12)$ calculated from the expression

$$\left.\begin{aligned}
L[y(x);h] = {} & 110y(x-2h) - 209y(x-h) + 252y(x) - 210y(x+h)\\
& +120y(x+2h) - 45y(x+3h) + 10y(x+4h) - y(x+5h)\\
& -27y(x-3h) + 6h^2y''(x-3h) - \tfrac{1}{2}h^4y^{(iv)}(x-3h)\\
& -\tfrac{3}{20}h^6y^{(vi)}(x-3h) - \tfrac{41}{3360}h^8y^{(viii)}(x-3h)\\
& +h^{10}[\gamma_0y^{(x)}(x-3h) + \gamma_1y^{(x)}(x-2h) + \gamma_2y^{(x)}(x-h)\\
& +\gamma_3y^{(x)}(x) + \gamma_4y^{(x)}(x+h) + \gamma_5y^{(x)}(x+2h)\\
& +\gamma_6y^{(x)}(x+3h) + \gamma_7y^{(x)}(x+4h) + \gamma_8y^{(x)}(x+5h)\\
& +\gamma_9y^{(x)}(x+6h) + \gamma_{10}y^{(x)}(x+7h) + \gamma_{11}y^{(x)}(x+8h)\\
& +\gamma_{12}y^{(x)}(x+9h) + \ldots]
\end{aligned}\right\} \tag{3.44}$$

in which $x = x_3$.

Expanding the terms in (3.44) about $y(x)$ and its derivatives, at the point $x = x_3$ and then equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}$, $y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}$ gives

$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = \frac{302231}{302400}, \tag{3.45}$$

$$-3\gamma_0 - 2\gamma_1 - \gamma_2 + 2\gamma_4 = \frac{169}{100800}, \tag{3.46}$$

$$3^2\frac{\gamma_0}{2!} + 2^2\frac{\gamma_1}{2!} + \frac{\gamma_2}{2!} + \frac{\gamma_4}{2!} = \frac{5510311}{13305600} - \frac{5}{12}, \tag{3.47}$$

$$-3^3\frac{\gamma_0}{3!} - 3^2\frac{\gamma_1}{3!} - \frac{\gamma_2}{3!} + \frac{\gamma_4}{3!} = \frac{11381}{4435200}, \tag{3.48}$$

$$3^4\frac{\gamma_0}{4!} + 2^4\frac{\gamma_1}{4!} + \frac{\gamma_2}{4!} + \frac{\gamma_4}{4!} = \frac{591141643}{7264857600} - \frac{1}{12}. \tag{3.49}$$

Solving this system, we get the parameters of the third end-point formula (i.e. $x = x_3$) for the second-order method. It is noted that the parameters γ_i $(i = 5, 6, 7, \ldots, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned} \gamma_0 &= -\frac{160883}{264176640}, \\ \gamma_1 &= \frac{27127}{181621440}, \\ \gamma_2 &= \frac{132647}{807206400}, \\ \gamma_3 &= \frac{14190326}{14189175}, \\ \gamma_4 &= -\frac{46537}{2905943040}. \end{aligned}\right\} \tag{3.50}$$

Using the method of undetermined coefficients reveals that for the point $x = x_4$ the first two nonvanishing terms in the local truncation error have the values given in (3.8) provided

$$d_0 = 8, d_2 = -1, d_4 = \frac{-1}{12}, d_6 = \frac{-1}{360}, d_8 = \frac{-1}{20160}, \tag{3.51}$$

together with parameters δ_i $(i = 0, 1, 2 \ldots, 12)$ calculated from the expression

$$\left.\begin{aligned} L[y(x);h] = \; & -44y(x-3h) + 120y(x-2h) - 210y(x-h) \\ & +252y(x) - 210y(x+h) + 120y(x+2h) \\ & -45y(x+3h) + 10y(x+4h) - y(x+5h) \\ & +8y(x-4h) - h^2y''(x-4h) - \tfrac{1}{12}h^4y^{(iv)}(x-4h) \\ & -\tfrac{1}{360}h^6y^{(vi)}(x-4h) - \tfrac{1}{20160}h^8y^{(viii)}(x-4h) \\ & +h^{10}[\delta_0y^{(x)}(x-4h) + \delta_1y^{(x)}(x-3h) \\ & +\delta_2y^{(x)}(x-2h) + \delta_3y^{(x)}(x-h) + \delta_4y^{(x)}(x) \\ & +\delta_5y^{(x)}(x+h) + \delta_6y^{(x)}(x+2h) + \delta_7y^{(x)}(x+3h) \\ & +\delta_8y^{(x)}(x+4h) + \delta_9y^{(x)}(x+5h) + \delta_{10}y^{(x)}(x+6h) \\ & +\delta_{11}y^{(x)}(x+7h) + \delta_{12}y^{(x)}(x+8h) + \ldots] \end{aligned}\right\} \tag{3.52}$$

in which $x = x_4$.

Expanding the terms in (3.52) about y(x) and its derivatives, at the point $x = x_4$ and then equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}$, $y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}$ gives the system

$$\delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 = \frac{1814399}{1814400}, \tag{3.53}$$

$$-4\delta_0 - 3\delta_1 - 2\delta_2 = -\frac{122753}{9979200}, \tag{3.54}$$

$$4^2\frac{\delta_0}{2!} + 3^2\frac{\delta_1}{2!} + 2^2\frac{\delta_2}{2!} + \frac{\delta_3}{2!} = \frac{14255849}{34214400} - \frac{5}{12}, \tag{3.55}$$

$$-4^3\frac{\delta_0}{3!} - 3^3\frac{\delta_1}{3!} - 2^3\frac{\delta_2}{3!} - \frac{\delta_3}{3!} = -\frac{68891}{222393600}, \tag{3.56}$$

$$4^4\frac{\delta_0}{4!} - 3^4\frac{\delta_1}{4!} - 2^4\frac{\delta_2}{4!} + \frac{\delta_3}{4!} = \frac{3632171087}{43589145600} - \frac{1}{12}, \tag{3.57}$$

the solution of which gives the parameters of the fourth end-point formula (i.e. $x = x_4$) for the second-order method. It is noted that the parameters δ_i $(i = 5, 6, 7, \ldots, 12)$ may then be arbitrarily given the value zero . Thus

$$\left.\begin{aligned} \delta_0 &= -\frac{185681143}{52306974720}, \\ \delta_1 &= \frac{608520391}{32691859200}, \\ \delta_2 &= -\frac{70958431}{1743565824}, \\ \delta_3 &= \frac{68068867}{1307674368}, \\ \delta_4 &= \frac{254624963293}{261534873600}. \end{aligned}\right\} \tag{3.58}$$

The special end point formulae for the points $x_{N-3}, x_{N-2}, x_{N-1}, x_N$ may then be written down from those for x_4, x_3, x_2, x_1, respectively (because of symmetry).

The set of parameter values in (3.27), (3.34), (3.35), (3.42), (3.43), (3.50), (3.51) and (3.58) give c_{12} as the first non-zero constant in (3.9). Global extrapolation on two grids, with p=2 in (2.29), and on three grids, with p=2 in (2.33), gives, using the notation of Chapter 2, the numerical methods

$$\mathbf{Y}^{(E)} = \frac{4}{3} I^{h}_{\frac{1}{2}h} \mathbf{Y}^{(2)} - \frac{1}{3}\mathbf{Y}^{(1)} \tag{3.59}$$

$$\mathbf{Y}^{(E)} = \frac{243}{120} I^{h}_{\frac{1}{3}h} \mathbf{Y}^{(3)} - \frac{128}{120} I^{h}_{\frac{1}{2}h} \mathbf{Y}^{(2)} + \frac{5}{120}\mathbf{Y}^{(1)}. \tag{3.60}$$

3.5 CONSTRUCTION OF A FOURTH-ORDER METHOD

Choosing $\alpha = \beta = \gamma = \delta = 0$ as before and writing $\epsilon = \frac{5}{12}$ in (3.3) gives a fourth-order method. The first non-zero constant in (3.9) then becomes

$$c_{14} = \frac{-7}{144}, \tag{3.61}$$

with $c_{11} = c_{13} = c_{15} = \ldots = 0$, because of symmetry. Taking the parameters a_i, b_i, c_i, d_i $(i = 0, 2, 4, 6, 8)$ as given in section 3.4 with the parameters $\alpha_i, \beta_i, \gamma_i, \delta_i$ (i= 0,1,2,3,4,5) calculated as follows, ensures that the same leading non-zero constant in (3.9) is obtained for the end-point formulae (3.12)—(3.19) associated with the fourth-order method.

For the point $x = x_1$, consider (3.28). Then equating the coefficients of the derivatives $y^{(x)}, y^{(xi)}, y^{(xii)}, y^{(xiii)}, y^{(xiv)}, y^{(xv)}$ gives the system

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = \frac{655177}{907200}, \tag{3.62}$$

$$-\alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 = \frac{252023}{907200}, \tag{3.63}$$

$$\frac{\alpha_0}{2!} + \alpha_2 + 2^2\frac{\alpha_3}{2!} + 3^2\frac{\alpha_4}{2!} + 4^2\frac{\alpha_5}{2!} = \frac{27438979}{119750400}, \tag{3.64}$$

$$-\frac{\alpha_0}{3!} + \alpha_2 + 2^3\frac{\alpha_3}{3!} + 3^3\frac{\alpha_4}{3!} + 4^3\frac{\alpha_5}{3!} = \frac{11368009}{119750400}, \tag{3.65}$$

$$\frac{\alpha_0}{4!} + \frac{\alpha_2}{4!} + 2^4\frac{\alpha_3}{4!} + 3^4\frac{\alpha_4}{4!} + 4^4\frac{\alpha_5}{4!} = \frac{131904163}{3113510400} - \frac{7}{144}, \tag{3.66}$$

$$-\frac{\alpha_0}{5!} + \frac{\alpha_2}{5!} + 2^5\frac{\alpha_3}{5!} + 3^5\frac{\alpha_4}{5!} + 4^5\frac{\alpha_5}{5!} = \frac{723798697}{46702656000}. \tag{3.67}$$

Solving this system, we get the parameters of the first end-point formula (i.e. $x = x_1$) for the fourth-order method. It is noted that the parameters α_i $(i = 6, 7, 8, \ldots, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\alpha_0 &= \tfrac{-40637579}{691891200}, \\
\alpha_1 &= \tfrac{1150015783}{1334361600}, \\
\alpha_2 &= \tfrac{-1674003}{28744003}, \\
\alpha_3 &= \tfrac{234778903}{311351040}, \\
\alpha_4 &= \tfrac{-5661511673}{18681062400}, \\
\alpha_5 &= \tfrac{43035359}{849139200}.
\end{aligned}\right\} \tag{3.68}$$

It can be shown using the method of undetermined coefficients for the point $x = x_2$, that, taking the parameter values b_0, b_2, b_4, b_6, b_8 in (3.35) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}, y^{(xv)}$ in (3.36) gives the system

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 = \frac{882773}{907200}, \tag{3.69}$$

$$-2\beta_0 - \beta_1 + \beta_3 + 2\beta_4 + 3\beta_5 = \frac{24427}{453600}, \tag{3.70}$$

$$2^2\frac{\beta_0}{2!} + \frac{\beta_1}{2!} + \frac{\beta_3}{2!} + 2^2\frac{\beta_4}{2!} + 3^2\frac{\beta_5}{2!} = \frac{43202009}{119750400}, \tag{3.71}$$

$$-2^3\frac{\beta_0}{3!} - \frac{\beta_1}{3!} + \frac{\beta_3}{3!} + 2^3\frac{\beta_4}{3!} + 3^3\frac{\beta_5}{3!} = \frac{2394839}{59875200}, \tag{3.72}$$

$$2^4\frac{\beta_0}{4!} + \frac{\beta_1}{4!} + \frac{\beta_3}{4!} + 2^4\frac{\beta_4}{4!} + 3^4\frac{\beta_5}{4!} = \frac{190486607}{3113510400} - \frac{7}{144}, \tag{3.73}$$

$$-2^5\frac{\beta_0}{5!} + \frac{\beta_1}{5!} + \frac{\beta_3}{5!} + 2^5\frac{\beta_4}{5!} + 3^5\frac{\beta_5}{5!} = \frac{21489493}{2122848000}, \tag{3.74}$$

the solution of which give the parameters of the second end-point formula (i.e. $x = x_2$) for the fourth-order method. It is noted that the parameters β_i $(i = 6, 7, 8, \ldots, 12)$ may then be arbitrarily given the value zero . Thus

$$\left.\begin{aligned} \beta_0 &= \tfrac{-24163651}{691891200}, \\ \beta_1 &= \tfrac{118607251}{266872320}, \\ \beta_2 &= \tfrac{91527613}{718502400}, \\ \beta_3 &= \tfrac{694056739}{1556755200}, \\ \beta_4 &= \tfrac{-43253933}{3736212480}, \\ \beta_5 &= \tfrac{17921741}{9340531200}. \end{aligned}\right\} \tag{3.75}$$

Next, it can be shown using the method of undetermined coefficients for the point $x = x_3$, that, taking the parameter values c_0, c_2, c_4, c_6, c_8 given in (3.43), together with the parameters calculated below, guarantees the same first non-zero constant in the local error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}$, $y^{(xv)}$, in (3.44) gives

$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 = \frac{302231}{302400}, \tag{3.76}$$

$$-3\gamma_0 - 2\gamma_1 - \gamma_2 + \gamma_4 + 2\gamma_5 = \frac{169}{100800}, \tag{3.77}$$

$$3^2\frac{\gamma_0}{2!} + 2^2\frac{\gamma_1}{2!} + \frac{\gamma_2}{2!} + \frac{\gamma_4}{2!} + 2^2\frac{\gamma_5}{2!} = \frac{5510311}{13305600}, \tag{3.78}$$

$$-3^3\frac{\gamma_0}{3!} - 2^3\frac{\gamma_1}{3!} - \frac{\gamma_2}{3!} + \frac{\gamma_4}{3!} + 2^3\frac{\gamma_5}{3!} = \frac{11381}{4435200}, \tag{3.79}$$

$$3^4\frac{\gamma_0}{4!} + 2^4\frac{\gamma_1}{4!} + \frac{\gamma_2}{4!} + \frac{\gamma_4}{4!} + 2^4\frac{\gamma_5}{4!} = \frac{591141643}{7264857600} - \frac{5}{12}, \tag{3.80}$$

$$-3^5\frac{\gamma_0}{5!} - 2^5\frac{\gamma_1}{5!} - \frac{\gamma_2}{5!} + \frac{\gamma_4}{5!} + 2^5\frac{\gamma_5}{5!} = \frac{14645899}{12108096000}. \tag{3.81}$$

Solving this system we get the parameters of the third end-point formula (i.e. $x = x_3$) for the fourth-order method; they are

$$\left.\begin{aligned} \gamma_0 &= \tfrac{-1007339}{1614412800}, \\ \gamma_1 &= \tfrac{46537}{207567360}, \\ \gamma_2 &= \tfrac{232672519}{558835200}, \\ \gamma_3 &= \tfrac{202081057}{1210809600}, \\ \gamma_4 &= \tfrac{1210545577}{2905943040}, \\ \gamma_5 &= \tfrac{108743}{7264857600}. \end{aligned}\right\} \tag{3.82}$$

It is noted that the parameters γ_i $(i = 6, 7, \ldots, 12)$ may be arbitrarily assigned the value zero.

Finally, it can be shown using the method of undetermined coefficients for the point $x = x_4$ that, taking the parameters d_0, d_2, d_4, d_6, d_8 given (3.51), together with the parameters calculated below, yields the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}y^{(xii)}$, $y^{(xiii)}$, $y^{(xiv)}$, $y^{(xv)}$, in (3.52) gives

$$\delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 = \frac{1814399}{1814400}, \tag{3.83}$$

$$-4\delta_0 - 3\delta_1 - 2\delta_2 - \delta_3 + \delta_5 = \frac{122753}{9979200}, \tag{3.84}$$

$$4^2\frac{\delta_0}{2!} + 3^2\frac{\delta_1}{2!} + 2^2\frac{\delta_2}{2!} + \frac{\delta_3}{2!} + \frac{\delta_5}{2!} = \frac{14255849}{34214400}, \tag{3.85}$$

$$-4^3\frac{\delta_0}{3!} - 3^3\frac{\delta_1}{3!} - 2^3\frac{\delta_2}{3!} - \frac{\delta_3}{3!} + \frac{\delta_5}{3!} = \frac{68891}{222393600}, \tag{3.86}$$

$$4^4\frac{\delta_0}{4!} + 3^4\frac{\delta_1}{4!} + 2^4\frac{\delta_2}{4!} + \frac{\delta_3}{4!} + \frac{\delta_5}{4!} = \frac{363217187}{435891145600} - \frac{7}{144}, \tag{3.87}$$

$$-4^5\frac{\delta_0}{5!} - 3^5\frac{\delta_1}{5!} - 2^5\frac{\delta_2}{5!} - \frac{\delta_3}{5!} + \frac{\delta_5}{5!} = \frac{413849}{326918592000}. \tag{3.88}$$

Solving this system we get the parameters of the fourth end-point formula (i.e. $x = x_4$) for the fourth-order method. It is noted that the parameters δ_i $(i = 6, 7, \ldots, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned} \delta_0 &= \tfrac{-4206679}{7925299200}, \\ \delta_1 &= \tfrac{230059147}{65383718400}, \\ \delta_2 &= \tfrac{-274791157}{26153487360}, \\ \delta_3 &= \tfrac{5689027}{12972960}, \\ \delta_4 &= \tfrac{4062716183}{26134873600}, \\ \delta_5 &= \tfrac{27045819673}{65383718400} \end{aligned}\right\}. \tag{3.89}$$

Because of symmetry, the special end-point fomulae for the points $x_N, x_{N-1}, x_{N-2}, x_{N-3}$ may be written down directly from those for x_1, x_2, x_3, x_4, respectively.

The set of parameter values in (3.27), (3.68), (3.35), (3.75), (3.43), (3.82), (3.51) and (3.89) give c_{14} as the first non-zero constant in (3.9). Global extrapolation on two grids, with p=4 in (2.29), gives, using the notation of Chapter 2, the numerical method

$$\mathbf{Y}^{(E)} = \frac{16}{15} I^{h}_{\frac{1}{2}h} \mathbf{Y}^{(2)} - \frac{1}{15}\mathbf{Y}^{(1)}. \tag{3.90}$$

3.6 CONSTRUCTION OF A SIXTH-ORDER METHOD

Choosing $\alpha = \beta = \gamma = 0$ as before and writing $\epsilon = \frac{2}{9}$, $\delta = \frac{7}{144}$ so that $1 - 2(\alpha+\beta+\gamma+\delta+\epsilon) = \frac{11}{24}$ in (3.3) gives a sixth-order method. The first non-zero constant in (3.9) then becomes

$$c_{16} = \frac{-17}{12096}, \tag{3.91}$$

with $c_{11} = c_{13} = c_{15} = \ldots = 0$,because of symmetry. Taking the parameters a_i, b_i, c_i, d_i $(i = 0, 2, 4, 6, 8)$ as given in section 2.6 with the parameters α_i, β_i, γ_i, δ_i $(i = 0, 1, \ldots, 7)$ calculated as follows, ensures that the same leading non-zero constant in (3.9) is obtained for the end-point formulae (3.12)—(3.19) associated with the sixth-order method.

For the point $x = x_1$, consider (3.36). Equating the coefficients of the derivatives $y^{(x)}, y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xvii)}$ gives the system

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 = \frac{655177}{907200}, \tag{3.92}$$

$$-\alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 6\alpha_7 = \frac{252023}{907200}, \tag{3.93}$$

$$\frac{\alpha_0}{2!} + \alpha_2 + 2^2\frac{\alpha_3}{2!} + 3^2\frac{\alpha_4}{2!} + 4^2\frac{\alpha_5}{2!} + 5^2\frac{\alpha_6}{2!} + 6^2\frac{\alpha_7}{2!} = \frac{27438979}{119750400}, \tag{3.94}$$

$$-\frac{\alpha_0}{3!} + \alpha_2 + 2^3\frac{\alpha_3}{3!} + 3^3\frac{\alpha_4}{3!} + 4^3\frac{\alpha_5}{3!} + 5^3\frac{\alpha_6}{3!} + 6^3\frac{\alpha_7}{3!} = \frac{11368009}{119750400}, \tag{3.95}$$

$$\frac{\alpha_0}{4!} + \frac{\alpha_2}{4!} + 2^4\frac{\alpha_3}{4!} + 3^4\frac{\alpha_4}{4!} + 4^4\frac{\alpha_5}{4!} + 5^4\frac{\alpha_6}{4!} + 6^4\frac{\alpha_7}{4!} = \frac{131904163}{3113510400}, \tag{3.96}$$

$$-\frac{\alpha_0}{5!} + \frac{\alpha_2}{5!} + 2^5\frac{\alpha_3}{5!} + 3^5\frac{\alpha_4}{5!} + 4^5\frac{\alpha_5}{5!} + 5^5\frac{\alpha_6}{5!} + 6^5\frac{\alpha_7}{5!} = \frac{723798697}{46702656000}, \tag{3.97}$$

$$\frac{\alpha_0}{6!} + \frac{\alpha_2}{6!} + 2^6\frac{\alpha_3}{6!} + 3^6\frac{\alpha_4}{6!} + 4^6\frac{\alpha_5}{6!} + 5^6\frac{\alpha_6}{6!} + 6^6\frac{\alpha_7}{6!} = \frac{2541132023}{475517952000} - \frac{17}{12096}, \tag{3.98}$$

$$-\frac{\alpha_0}{7!} + \frac{\alpha_2}{7!} + 2^7\frac{\alpha_3}{7!} + 3^7\frac{\alpha_4}{7!} + 4^7\frac{\alpha_5}{7!} + 5^7\frac{\alpha_6}{7!} + 6^7\frac{\alpha_7}{7!} = \frac{8768652467}{5230697472000}. \tag{3.99}$$

Solving this system, we get the parameters of the first end-point formula (i.e. $x = x_1$) for the sixth-order method. It is noted that the parameters

α_i $(i = 8, 9, 10, 11, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\alpha_0 &= \frac{121680539023}{3923023104000},\\
\alpha_1 &= \frac{825553878671}{1743565824000},\\
\alpha_2 &= \frac{49899297233}{871782912000},\\
\alpha_3 &= \frac{180529065817}{627683696640},\\
\alpha_4 &= \frac{-9140697491}{43589145600},\\
\alpha_5 &= \frac{194540768657}{1743565824000},\\
\alpha_6 &= \frac{-261610352587}{7846046208000},\\
\alpha_7 &= \frac{192774481}{44706816000}.
\end{aligned}\right\} \cdot \tag{3.100}$$

It can be shown using the method of undetermined coefficients for the point $x = x_2$, that, taking the parameter values b_0, b_2, b_4, b_6, b_8 in (3.35) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xvii)}$ in (3.36) gives the system

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 = \frac{882773}{907200}, \tag{3.101}$$

$$-2\beta_0 - \beta_1 + \beta_3 + 2\beta_4 + 3\beta_5 + 4\beta_6 + 5\beta_7 = \frac{24427}{453600}, \tag{3.102}$$

$$2^2\frac{\beta_0}{2!} + \frac{\beta_1}{2!} + \frac{\beta_3}{2!} + 2^2\frac{\beta_4}{2!} + 3^2\frac{\beta_5}{2!} + 4^2\frac{\beta_6}{2!} + 5^2\frac{\beta_7}{2!} = \frac{43202009}{119750400}, \tag{3.103}$$

$$-2^3\frac{\beta_0}{3!} - \frac{\beta_1}{3!} + \frac{\beta_3}{3!} + 2^3\frac{\beta_4}{3!} + 3^3\frac{\beta_5}{3!} + 4^3\frac{\beta_6}{3!} + 5^3\frac{\beta_7}{3!} = \frac{2394839}{59875200}, \tag{3.104}$$

$$2^4\frac{\beta_0}{4!} + \frac{\beta_1}{4!} + \frac{\beta_3}{4!} + 2^4\frac{\beta_4}{4!} + 3^4\frac{\beta_5}{4!} + 4^4\frac{\beta_6}{4!} + 5^4\frac{\beta_7}{4!} = \frac{190486607}{3113510400}, \tag{3.105}$$

$$-2^5\frac{\beta_0}{5!} + \frac{\beta_1}{5!} + \frac{\beta_3}{5!} + 2^5\frac{\beta_4}{5!} + 3^5\frac{\beta_5}{5!} + 4^5\frac{\beta_6}{5!} + 5^5\frac{\beta_7}{5!} = \frac{21489493}{2122848000}, \tag{3.106}$$

$$2^6\frac{\beta_0}{6!} + \frac{\beta_1}{6!} + \frac{\beta_3}{6!} + 2^6\frac{\beta_4}{6!} + 3^6\frac{\beta_5}{6!} + 4^6\frac{\beta_6}{6!} + 5^6\frac{\beta_7}{6!} = \frac{34992742353}{5230697472000} - \frac{17}{12096}, \tag{3.107}$$

$$-2^7\frac{\beta_0}{7!} + \frac{\beta_1}{7!} + \frac{\beta_3}{7!} + 2^7\frac{\beta_4}{7!} + 3^7\frac{\beta_5}{7!} + 4^7\frac{\beta_6}{7!} + 5^7\frac{\beta_7}{7!} = \frac{327962597}{237758976000}, \tag{3.108}$$

the soluion of which give the parameters of the second end-point formula (i.e. $x = x_2$) for the sixth-order method. It is noted that the parameters β_i $(i = 8, 9, 10, 11, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\beta_0 &= \frac{121680539023}{3923023104000}, \\
\beta_1 &= \frac{82555387871}{1743565824000}, \\
\beta_2 &= \frac{49899297233}{871782912000}, \\
\beta_3 &= \frac{180529065817}{627683696640}, \\
\beta_4 &= \frac{-9140697491}{43589145600}, \\
\beta_5 &= \frac{194540768657}{1743565824000}, \\
\beta_6 &= \frac{-261610352587}{784604628000}, \\
\beta_7 &= \frac{192774481}{44706816000}.
\end{aligned}\right\} . \tag{3.109}$$

Next, it can be shown using the method of undetermined coefficients for the point $x = x_3$, that, taking the parameter values c_0, c_2, c_4, c_6, c_8 in (3.43) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xvii)}$, in (3.44) gives

$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 = \frac{302231}{302400}, \tag{3.110}$$

$$-3\gamma_0 - 2\gamma_1 - \gamma_2 + \gamma_4 + 2\gamma_5 + 3\gamma_6 + 4\gamma_7 = \frac{169}{100800}, \tag{3.111}$$

$$3^2\frac{\gamma_0}{2!} + 2^2\frac{\gamma_1}{2!} + \frac{\gamma_2}{2!} + \frac{\gamma_4}{2!} + 2^2\frac{\gamma_5}{2!} + 3^2\frac{\gamma_6}{2!} + 4^2\frac{\gamma_7}{2!} = \frac{5510311}{13305600}, \tag{3.112}$$

$$-3^3\frac{\gamma_0}{3!} - 2^3\frac{\gamma_1}{3!} - \frac{\gamma_2}{3!} + \frac{\gamma_4}{3!} + 2^3\frac{\gamma_5}{3!} + 3^3\frac{\gamma_6}{3!} + 4^3\frac{\gamma_7}{3!} = \frac{11381}{4435200}, \tag{3.113}$$

$$3^4\frac{\gamma_0}{4!} + 2^4\frac{\gamma_1}{4!} + \frac{\gamma_2}{4!} + \frac{\gamma_4}{4!} + 2^4\frac{\gamma_5}{4!} + 3^4\frac{\gamma_6}{4!} + 4^4\frac{\gamma_7}{4!} = \frac{591141643}{7264857600}, \tag{3.114}$$

$$-3^5\frac{\gamma_0}{5!} - 2^5\frac{\gamma_1}{5!} - \frac{\gamma_2}{5!} + \frac{\gamma_4}{5!} + 2^5\frac{\gamma_5}{5!} + 3^5\frac{\gamma_6}{5!} + 4^5\frac{\gamma_7}{5!} = \frac{14645899}{12108096000}, \tag{3.115}$$

$$3^6\frac{\gamma_0}{6!} + 2^6\frac{\gamma_1}{6!} + \frac{\gamma_2}{6!} + \frac{\gamma_4}{6!} + 2^6\frac{\gamma_5}{6!} + 3^6\frac{\gamma_6}{6!} + 4^6\frac{\gamma_7}{6!} = \frac{1346510087}{134120448000} - \frac{17}{12096}, \tag{3.116}$$

$$3^7\frac{\gamma_0}{7!} + 2^7\frac{\gamma_1}{7!} + \frac{\gamma_2}{7!} + \frac{\gamma_4}{7!} + 2^7\frac{\gamma_5}{7!} + 3^7\frac{\gamma_6}{7!} + 4^7\frac{\gamma_7}{7!} = \frac{162013909}{581188608000}. \tag{3.117}$$

Solving this system we get the parameters of the third end-point formula (i.e. $x = x_3$) for the sixth-order method; they are

$$\left.\begin{aligned}
\gamma_0 &= \tfrac{-21838081}{33530112000}, \\
\gamma_1 &= \tfrac{1356454837}{27675648000}, \\
\gamma_2 &= \tfrac{7149219919}{3288256000}, \\
\gamma_3 &= \tfrac{160167409321}{348713164800}, \\
\gamma_4 &= \tfrac{27501631}{124185600}, \\
\gamma_5 &= \tfrac{9490656173}{193729536000}, \\
\gamma_6 &= \tfrac{-13324169}{124540416000}, \\
\gamma_7 &= \tfrac{2571931}{193729536000}.
\end{aligned}\right\} . \tag{3.118}$$

It is noted that the parameters γ_i $(i = 8, 9, 10, 11, 12)$ may be arbitrarily assigned the value zero.

Finally, it can be shown using the method of undetermined coefficients for the point $x = x_4$ that, taking the parameters d_0, d_2, d_4, d_6, d_8 given (3.51), and using the parameters calculated below, yields the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xvii)}$, in (2.52) gives

$$\delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + \delta_7 = \frac{1814399}{1814400}, \tag{3.119}$$

$$-4\delta_0 - 3\delta_1 - 2\delta_2 - \delta_3 + \delta_5 + 2\delta_6 + 3\delta_7 = \frac{122753}{9979200}, \tag{3.120}$$

$$4^2\frac{\delta_0}{2!} + 3^2\frac{\delta_1}{2!} + 2^2\frac{\delta_2}{2!} + \frac{\delta_3}{2!} + \frac{\delta_5}{2!} + 2^2\frac{\delta_6}{2!} + 3^2\frac{\delta_7}{2!} = \frac{14255849}{34214400}, \tag{3.121}$$

$$-4^3\frac{\delta_0}{3!} - 3^3\frac{\delta_1}{3!} - 2^3\frac{\delta_2}{3!} - \frac{\delta_3}{3!} + \frac{\delta_5}{3!} + 2^3\frac{\delta_6}{3!} + 3^3\frac{\delta_7}{3!} + = \frac{68891}{222393600}, \tag{3.122}$$

$$4^4\frac{\delta_0}{4!} + 3^4\frac{\delta_1}{4!} + 2^4\frac{\delta_2}{4!} + \frac{\delta_3}{4!} + \frac{\delta_5}{4!} + 2^4\frac{\delta_6}{4!} + 3^4\frac{\delta_7}{4!} = \frac{363217187}{43589145600}, \tag{3.123}$$

$$-4^5\frac{\delta_0}{5!} - 3^5\frac{\delta_1}{5!} - 2^5\frac{\delta_2}{5!} - \frac{\delta_3}{5!} + \frac{\delta_5}{5!} + 2^5\frac{\delta_6}{5!} + 3^5\frac{\delta_7}{5!} = \frac{413849}{326918592000}, \tag{3.124}$$

$$4^6\frac{\delta_0}{6!} + 3^6\frac{\delta_1}{6!} + 2^6\frac{\delta_2}{6!} + \frac{\delta_3}{6!} + \frac{\delta_5}{6!} + 2^6\frac{\delta_6}{6!} + 3^6\frac{\delta_7}{6!} = \frac{10139471581}{951035904000} - \frac{17}{12096}, \tag{3.125}$$

$$-4^7\frac{\delta_0}{7!}-3^7\frac{\delta_1}{7!}-2^7\frac{\delta_2}{7!}-\frac{\delta_3}{7!}+\frac{\delta_5}{7!}+2^7\frac{\delta_7}{7!}+3^7\frac{\delta_7}{7!}=\frac{154643851}{88921857024000}. \quad (3.126)$$

Solving this system we get the parameters of the fourth end-point formula (i.e. $x = x_4$) for the sixth-order method. It is noted that the parameters δ_i $(i = 8, 9, 10, 11, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\delta_0 &= \tfrac{19195006261}{266765571072000},\\
\delta_1 &= \tfrac{118864463057}{177843714048000},\\
\delta_2 &= \tfrac{337681410533}{7410154752000},\\
\delta_3 &= \tfrac{996423583781}{4268249137152},\\
\delta_4 &= \tfrac{8106735502457}{177843714048000},\\
\delta_5 &= \tfrac{12744987460013}{59281238016000},\\
\delta_6 &= \tfrac{6622887628141}{133382785536000},\\
\delta_7 &= \tfrac{-17289181267}{177843714048000}.
\end{aligned}\right\}. \quad (3.127)$$

Because of symmetry, the special end-point formulae for the points x_N, x_{N-1}, x_{N-2}, x_{N-3} may be written down directly from those for x_1, x_2, x_3, x_4, respectively.

The set of parameter values in (3.27), (3.100), (3.35), (3.109), (3.43), (3.118), (3.51) and (3.127) give c_{16} as the first non-zero constant in (3.9). Global extrapolation on two grids, with p=6 in (2.29), gives, using the notation of Chapter 2, the numerical method

$$\mathbf{Y}^{(E)}=\frac{64}{63}I^{h}_{\frac{1}{2}h}\mathbf{Y}^{(2)}-\frac{1}{63}\mathbf{Y}^{(1)}. \quad (3.128)$$

3.7 CONSTRUCTION OF AN EIGHTH-ORDER METHOD

Writing $\alpha = \beta = 0$ as before $\gamma = \frac{17}{12096}$, $\delta = \frac{9}{224}$, $\epsilon = \frac{109}{448}$ so that $\sum = 1-2(\alpha+\beta+\gamma+\delta+\epsilon) = \frac{1301}{3024}$ in (3.3) gives an eighth-order method. The first non-zero constant in (3.9) then becomes

$$c_{18}=\frac{-1}{362880}, \quad (3.129)$$

with $c_{11} = c_{13} = c_{15} = \ldots\ldots = 0$, because of symmetry. Taking the parameters a_i, b_i, c_i, d_i $(i = 0, 2, 4, 6, 8)$ as given in section 3.4 with the parameters $\alpha_i, \beta_i, \gamma_i, \delta_i$ $(i = 0, 1, \ldots, 9)$ calculated as follows, ensures that the same leading non-zero constant in (3.9) is obtained for the end-point formulae (3.12)—(3.19) associated with the eighth-order method.

For the point $x = x_1$, consider (3.28). Then equating the coefficients of the derivatives $y^{(x)}, y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xix)}$ gives the system

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 = \frac{655177}{907200}, \tag{3.130}$$

$$-\alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 6\alpha_7 + 7\alpha_8 + 8\alpha_9 = \frac{252023}{907200}, \tag{3.131}$$

$$\frac{\alpha_0}{2!} + \alpha_2 + 2^2\frac{\alpha_3}{2!} + 3^2\frac{\alpha_4}{2!} + 4^2\frac{\alpha_5}{2!} + 5^2\frac{\alpha_6}{2!} + 6^2\frac{\alpha_7}{2!} + 7^2\frac{\alpha_8}{2!} + 8^2\frac{\alpha_9}{2!} = \frac{27438979}{119750400}, \tag{3.132}$$

$$-\frac{\alpha_0}{3!} + \alpha_2 + 2^3\frac{\alpha_3}{3!} + 3^3\frac{\alpha_4}{3!} + 4^3\frac{\alpha_5}{3!} + 5^3\frac{\alpha_6}{3!} + 6^3\frac{\alpha_7}{3!} + 7^3\frac{\alpha_8}{3!} + 8^3\frac{\alpha_9}{3!} = \frac{11368009}{119750400}, \tag{3.133}$$

$$\frac{\alpha_0}{4!} + \frac{\alpha_2}{4!} + 2^4\frac{\alpha_3}{4!} + 3^4\frac{\alpha_4}{4!} + 4^4\frac{\alpha_5}{4!} + 5^4\frac{\alpha_6}{4!} + 6^4\frac{\alpha_7}{4!} + 7^4\frac{\alpha_8}{4!} + 8^4\frac{\alpha_9}{4!} = \frac{131904163}{3113510400}, \tag{3.134}$$

$$-\frac{\alpha_0}{5!} + \frac{\alpha_2}{5!} + 2^5\frac{\alpha_3}{5!} + 3^5\frac{\alpha_4}{5!} + 4^5\frac{\alpha_5}{5!} + 5^5\frac{\alpha_6}{5!} + 6^5\frac{\alpha_7}{5!} + 7^5\frac{\alpha_8}{5!} + 8^5\frac{\alpha_9}{5!} = \frac{723798697}{46702656000}, \tag{3.135}$$

$$\frac{\alpha_0}{6!} + \frac{\alpha_2}{6!} + 2^6\frac{\alpha_3}{6!} + 3^6\frac{\alpha_4}{6!} + 4^6\frac{\alpha_5}{6!} + 5^6\frac{\alpha_6}{6!} + 6^6\frac{\alpha_7}{6!} + 7^6\frac{\alpha_8}{6!} + 8^6\frac{\alpha_9}{6!} = \frac{2541132023}{475517952000}, \tag{3.136}$$

$$-\frac{\alpha_0}{7!} + \frac{\alpha_2}{7!} + 2^7\frac{\alpha_3}{7!} + 3^7\frac{\alpha_4}{7!} + 4^7\frac{\alpha_5}{7!} + 5^7\frac{\alpha_6}{7!} + 6^7\frac{\alpha_7}{7!} + 7^7\frac{\alpha_8}{7!} + 8^7\frac{\alpha_9}{7!} = \frac{8768652467}{5230697472000}, \tag{3.137}$$

$$\frac{\alpha_0}{8!} + \frac{\alpha_2}{8!} + 2^8\frac{\alpha_3}{8!} + 3^8\frac{\alpha_4}{8!} + 4^8\frac{\alpha_5}{8!} + 5^8\frac{\alpha_6}{8!} + 6^8\frac{\alpha_7}{8!} + 7^8\frac{\alpha_8}{8!} + 8^8\frac{\alpha_9}{8!} = \frac{14042390777}{28582025472000} - \frac{1}{362880}, \tag{3.138}$$

$$-\frac{\alpha_0}{9!} + \frac{\alpha_2}{9!} + 2^9\frac{\alpha_3}{9!} + 3^9\frac{\alpha_4}{9!} + 4^9\frac{\alpha_5}{9!} + 5^9\frac{\alpha_6}{9!} + 6^9\frac{\alpha_7}{9!} + 7^9\frac{\alpha_8}{9!} + 8^9\frac{\alpha_9}{9!} = \frac{2762162653}{20520428544000}. \tag{3.139}$$

Solving this system, we get the parameters of the first end-point formula (i.e. $x = x_1$) for the eighth-order method. It is noted that the parameters α_i

(i = 10,11,12) may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\alpha_0 &= \frac{20992458112397}{640237370572800}, \\
\alpha_1 &= \frac{93157191139}{199874304000}, \\
\alpha_2 &= \frac{735619884037}{12312257126400}, \\
\alpha_3 &= \frac{4063603106641}{12312257126400}, \\
\alpha_4 &= \frac{-520840519849037}{1600593426432000}, \\
\alpha_5 &= \frac{41785406610919}{160059342643200}, \\
\alpha_6 &= \frac{-115840187113411}{800296713216000}, \\
\alpha_7 &= \frac{1699033289519}{32011868528640}, \\
\alpha_8 &= \frac{-7421743667363}{640237370572800}, \\
\alpha_9 &= \frac{915081921001}{800296713216000}.
\end{aligned}\right\} . \tag{3.140}$$

It can be shown using the method of undetermined coefficients for the point $x = x_2$, that, taking the parameter values b_0, b_2, b_4, b_6, b_8 in (3.35) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xix)}$ in (3.36) gives the system

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 + \beta_9 = \frac{882773}{907200}, \tag{3.141}$$

$$-2\beta_0 - \beta_1 + \beta_3 + 2\beta_4 + 3\beta_5 + 4\beta_6 + 5\beta_7 + 6\beta_8 + 7\beta_9 = \frac{24427}{453600}, \tag{3.142}$$

$$2^2\frac{\beta_0}{2!}+\frac{\beta_1}{2!}+\frac{\beta_3}{2!}+2^2\frac{\beta_4}{2!}+3^2\frac{\beta_5}{2!}+4^2\frac{\beta_6}{2!}+5^2\frac{\beta_7}{2!}+6^2\frac{\beta_8}{2!}+7^2\frac{\beta_9}{2!} = \frac{43202009}{119750400}, \tag{3.143}$$

$$-2^3\frac{\beta_0}{3!}-\frac{\beta_1}{3!}+\frac{\beta_3}{3!}+2^3\frac{\beta_4}{3!}+3^3\frac{\beta_5}{3!}+4^3\frac{\beta_6}{3!}+5^3\frac{\beta_7}{3!}+6^3\frac{\beta_8}{3!}+7^3\frac{\beta_9}{3!} = \frac{2394839}{59875200}, \tag{3.144}$$

$$2^4\frac{\beta_0}{4!} + \frac{\beta_1}{4!} + \frac{\beta_3}{4!} + 2^4\frac{\beta_4}{4!} + 3^4\frac{\beta_5}{4!} + 4^4\frac{\beta_6}{4!} + 5^4\frac{\beta_7}{4!} + 6^4\frac{\beta_8}{4!} + 7^4\frac{\beta_9}{4!} = \frac{190486607}{3113510400}, \tag{3.145}$$

$$-2^5\frac{\beta_0}{5!} + \frac{\beta_1}{5!} + \frac{\beta_3}{5!} + 2^5\frac{\beta_4}{5!} + 3^5\frac{\beta_5}{5!} + 4^5\frac{\beta_6}{5!} + 5^5\frac{\beta_7}{5!} + 6^5\frac{\beta_8}{5!} + 7^5\frac{\beta_9}{5!} = \frac{21489493}{2122848000}. \tag{3.146}$$

$$2^6\frac{\beta_0}{6!}+\frac{\beta_1}{6!}+\frac{\beta_3}{6!}+2^6\frac{\beta_4}{6!}+3^6\frac{\beta_5}{6!}+4^6\frac{\beta_6}{6!}+5^6\frac{\beta_7}{6!}+6^6\frac{\beta_8}{6!}+7^6\frac{\beta_9}{6!}=\frac{34992742353}{5230697472000}-\frac{17}{12096}, \tag{3.147}$$

$$-2^7\frac{\beta_0}{7!}-\frac{\beta_1}{7!}+\frac{\beta_3}{7!}+2^7\frac{\beta_4}{7!}+3^7\frac{\beta_5}{7!}+4^7\frac{\beta_6}{7!}+5^7\frac{\beta_7}{7!}+6^7\frac{\beta_5}{7!}+7^7\frac{\beta_9}{7!}=\frac{327962597}{237758976000}, \tag{3.148}$$

$$2^8\frac{\beta_0}{8!}+\frac{\beta_1}{8!}+\frac{\beta_3}{8!}+2^8\frac{\beta_4}{8!}+3^8\frac{\beta_5}{8!}+4^8\frac{\beta_6}{8!}+5^8\frac{\beta_7}{8!}+6^8\frac{\beta_5}{8!}+7^8\frac{\beta_9}{8!}=\frac{881182516553}{1600593426432000}-\frac{1}{362880}, \tag{3.149}$$

$$-2^9\frac{\beta_0}{9!}-\frac{\beta_1}{9!}+\frac{\beta_3}{9!}+2^9\frac{\beta_4}{9!}+3^9\frac{\beta_5}{9!}+4^9\frac{\beta_6}{9!}+5^9\frac{\beta_7}{9!}+6^9\frac{\beta_5}{9!}+7^9\frac{\beta_9}{9!}=\frac{2542651289}{20520428544000}, \tag{3.150}$$

the solution of which give the parameters of the second end-point formula (i.e. $x = x_2$) for the eighth-order method. It is noted that the parameters β_i $(i = 10, 11, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\beta_0 &= \frac{7750281368173}{640237370572800}, \\
\beta_1 &= \frac{95833355799}{3637712332800}, \\
\beta_2 &= \frac{304812120880213}{800296713216000}, \\
\beta_3 &= \frac{259595936667337}{800296713216000}, \\
\beta_4 &= \frac{-3403568201269}{64023737057280}, \\
\beta_5 &= \frac{1120702421821}{14550849331200}, \\
\beta_6 &= \frac{-616046074277}{14550849331200}, \\
\beta_7 &= \frac{12513016249567}{800296713216000}, \\
\beta_8 &= \frac{-10991111981903}{3201186852864000}, \\
\beta_9 &= \frac{54435448549}{160059342643200}.
\end{aligned}\right\} . \tag{3.151}$$

Next, it can be shown using the method of undetermined coefficients for the point $x = x_3$, that, taking the parameter values c_0, c_2, c_4, c_6, c_8 in (3.43) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xix)}$ in (2.44) gives

$$\gamma_0+\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6+\gamma_7+\gamma_8+\gamma_9=\frac{302231}{302400}, \tag{3.152}$$

$$-3\gamma_0 - 2\gamma_1 - \gamma_2 + \gamma_4 + 2\gamma_5 + 3\gamma_6 + 4\gamma_7 + 5\gamma_8 + 6\gamma_9 = \frac{169}{100800}, \quad (3.153)$$

$$3^2\frac{\gamma_0}{2!}+2^2\frac{\gamma_1}{2!}+\frac{\gamma_2}{2!}+\frac{\gamma_4}{2!}+2^2\frac{\gamma_5}{2!}+3^2\frac{\gamma_6}{2!}+4^2\frac{\gamma_7}{2!}+5^2\frac{\gamma_8}{2!}+6^2\frac{\gamma_9}{2!} = \frac{5510311}{13305600}, \quad (3.154)$$

$$-3^3\frac{\gamma_0}{3!}-2^3\frac{\gamma_1}{3!}-\frac{\gamma_2}{3!}+\frac{\gamma_4}{3!}+2^3\frac{\gamma_5}{3!}+3^3\frac{\gamma_6}{3!}+4^3\frac{\gamma_7}{3!}+5^3\frac{\gamma_8}{3!}+6^3\frac{\gamma_9}{3!} = \frac{11381}{4435200}, \quad (3.155)$$

$$3^4\frac{\gamma_0}{4!}+2^4\frac{\gamma_1}{4!}+\frac{\gamma_2}{4!}+\frac{\gamma_4}{4!}+2^4\frac{\gamma_5}{4!}+3^4\frac{\gamma_6}{4!}+4^4\frac{\gamma_7}{4!}+5^4\frac{\gamma_8}{4!}+6^4\frac{\gamma_9}{4!} = \frac{591141643}{7264857600}, \quad (3.156)$$

$$-3^5\frac{\gamma_0}{5!}-2^5\frac{\gamma_1}{5!}-\frac{\gamma_2}{5!}+\frac{\gamma_4}{5!}+2^5\frac{\gamma_5}{5!}+3^5\frac{\gamma_6}{5!}+4^5\frac{\gamma_7}{5!}+5^5\frac{\gamma_8}{5!}+6^5\frac{\gamma_9}{5!} = \frac{14645899}{12108096000}, \quad (3.157)$$

$$3^6\frac{\gamma_0}{6!}+2^6\frac{\gamma_1}{6!}+\frac{\gamma_2}{6!}+\frac{\gamma_4}{6!}+2^6\frac{\gamma_5}{6!}+3^6\frac{\gamma_6}{6!}+4^6\frac{\gamma_7}{6!}+5^6\frac{\gamma_8}{6!}+6^6\frac{\gamma_9}{6!} = \frac{1346510087}{134120448000}, \quad (3.158)$$

$$-3^7\frac{\gamma_0}{7!}-2^7\frac{\gamma_1}{7!}-\frac{\gamma_2}{7!}+\frac{\gamma_4}{7!}+2^7\frac{\gamma_5}{7!}+3^7\frac{\gamma_6}{7!}+4^7\frac{\gamma_7}{7!}+5^7\frac{\gamma_8}{7!}+6^7\frac{\gamma_9}{7!} = \frac{162013909}{581188608000}, \quad (3.159)$$

$$3^8\frac{\gamma_0}{8!}+2^8\frac{\gamma_1}{8!}+\frac{\gamma_2}{8!}+\frac{\gamma_4}{8!}+2^8\frac{\gamma_5}{8!}+3^8\frac{\gamma_6}{8!}+4^8\frac{\gamma_7}{8!}+5^8\frac{\gamma_8}{8!}+6^8\frac{\gamma_9}{8!} = \frac{19405166329}{22230464256000}-\frac{1}{362880}, \quad (3.160)$$

$$-3^9\frac{\gamma_0}{9!}-2^9\frac{\gamma_1}{9!}-\frac{\gamma_2}{9!}+\frac{\gamma_4}{9!}+2^9\frac{\gamma_5}{9!}+3^9\frac{\gamma_6}{9!}+4^9\frac{\gamma_7}{9!}+5^9\frac{\gamma_8}{9!}+6^9\frac{\gamma_9}{9!} = \frac{163046441}{4234374144000}. \quad (3.161)$$

Solving this system we get the parameters of the third end-point formula (i.e. $x = x_3$) for the eighth-order method; they are

$$\left.\begin{aligned}
\gamma_0 &= \frac{51893722057}{71137485619200}, \\
\gamma_1 &= \frac{2355227971}{57741465600}, \\
\gamma_2 &= \frac{21493633966657}{88921857024000}, \\
\gamma_3 &= \frac{38495892458893}{88921857024000}, \\
\gamma_4 &= \frac{8541426756427}{35568742809600}, \\
\gamma_5 &= \frac{760794282539}{17784371404800}, \\
\gamma_6 &= \frac{-165940141}{2540624486400}, \\
\gamma_7 &= \frac{48667536763}{88921857024000}, \\
\gamma_8 &= \frac{-6143191781}{50812489728000}, \\
\gamma_9 &= \frac{1222783}{101624979456}.
\end{aligned}\right\} . \quad (3.162)$$

It is noted that the parameters γ_i (i = 10, 11, 12) may be arbitrarily assigned the value zero.

Finally, it can be shown using the method of undetermined coefficients for the point $x = x_4$ that, taking the parameters d_0, d_2, d_4, d_6, d_8 given (3.51), together with the parameters calculated below, yields the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xix)}$ in (3.52) gives

$$\delta_0+\delta_1+\delta_2+\delta_3+\delta_4+\delta_5+\delta_6+\delta_7+\delta_8+\delta_9=\frac{1814399}{1814400}, \tag{3.163}$$

$$-4\delta_0-3\delta_1-2\delta_2-\delta_3+\delta_5+2\delta_6+3\delta_7+4\delta_8+5\delta_9=\frac{122753}{9979200}, \tag{3.164}$$

$$4^2\frac{\delta_0}{2!}+3^2\frac{\delta_1}{2!}+2^2\frac{\delta_2}{2!}+\frac{\delta_3}{2!}+\frac{\delta_5}{2!}+2^2\frac{\delta_6}{2!}+3^2\frac{\delta_7}{2!}+4^2\frac{\delta_8}{2!}+5^2\frac{\delta_9}{2!}=\frac{14255849}{34214400}, \tag{3.165}$$

$$-4^3\frac{\delta_0}{3!}-3^3\frac{\delta_1}{3!}-2^3\frac{\delta_2}{3!}-\frac{\delta_3}{3!}+\frac{\delta_5}{3!}+2^3\frac{\delta_6}{3!}+3^3\frac{\delta_7}{3!}+4^3\frac{\delta_8}{3!}+5^3\frac{\delta_9}{3!}=\frac{68891}{222393600}, \tag{3.166}$$

$$4^4\frac{\delta_0}{4!}+3^4\frac{\delta_1}{4!}+2^4\frac{\delta_2}{4!}+\frac{\delta_3}{4!}+\frac{\delta_5}{4!}+2^4\frac{\delta_6}{4!}+3^4\frac{\delta_7}{4!}+4^4\frac{\delta_8}{4!}+5^4\frac{\delta_9}{4!}=\frac{363217187}{43589145600}, \tag{3.167}$$

$$-4^5\frac{\delta_0}{5!}-3^5\frac{\delta_1}{5!}-2^5\frac{\delta_2}{5!}-\frac{\delta_3}{5!}+\frac{\delta_5}{5!}+2^5\frac{\delta_6}{5!}+3^5\frac{\delta_7}{5!}+4^5\frac{\delta_8}{5!}+5^5\frac{\delta_9}{5!}=\frac{413849}{326918592000}, \tag{3.168}$$

$$4^6\frac{\delta_0}{6!}+3^6\frac{\delta_1}{6!}+2^6\frac{\delta_2}{6!}+\frac{\delta_3}{6!}+\frac{\delta_5}{6!}+2^6\frac{\delta_6}{6!}+3^6\frac{\delta_7}{6!}+4^6\frac{\delta_8}{6!}+5^6\frac{\delta_9}{6!}=\frac{10139471581}{951035904000}, \tag{3.169}$$

$$-4^7\frac{\delta_0}{7!}-3^7\frac{\delta_1}{7!}-2^7\frac{\delta_2}{7!}-\frac{\delta_3}{7!}+\frac{\delta_5}{7!}+2^7\frac{\delta_7}{7!}+3^7\frac{\delta_7}{7!}+4^7\frac{\delta_8}{7!}+2^7\frac{\delta_9}{7!}=\frac{154643851}{88921857024000}, \tag{3.170}$$

$$4^8\frac{\delta_0}{8!}+3^8\frac{\delta_1}{8!}+2^8\frac{\delta_2}{8!}+\frac{\delta_3}{8!}+\frac{\delta_5}{8!}+2^8\frac{\delta_6}{8!}+3^8\frac{\delta_7}{8!}+4^8\frac{\delta_8}{8!}+5^8\frac{\delta_9}{8!}=\frac{3141960414959}{3201186852864000}-\frac{1}{362880}, \tag{3.171}$$

$$-4^9\frac{\delta_0}{9!}-3^9\frac{\delta_1}{9!}-2^9\frac{\delta_2}{9!}-\frac{\delta_3}{9!}+\frac{\delta_5}{9!}+2^7\frac{\delta_7}{9!}+3^9\frac{\delta_7}{9!}+4^9\frac{\delta_8}{9!}+2^9\frac{\delta_9}{9!}=\frac{4165158373}{10137091700736000}. \tag{3.172}$$

Solving this system we get the parameters of the fourth end-point formula (i.e. $x = x_4$) for the eighth-order method. It is noted that the parameters

δ_i ($i = 10, 11, 12$) may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\delta_0 &= \tfrac{499069556333}{2432902008176640 0}, \\
\delta_1 &= \tfrac{9104056156831}{5529322745856000}, \\
\delta_2 &= \tfrac{235407175152137}{6082255020441600}, \\
\delta_3 &= \tfrac{305584340173897}{1216451004088320}, \\
\delta_4 &= \tfrac{154635757309157}{3577797070848000}, \\
\delta_5 &= \tfrac{27146722126679}{116966442700800}, \\
\delta_6 &= \tfrac{120382318113107}{2764661372928000}, \\
\delta_7 &= \tfrac{2770984913471}{6082255020441600}, \\
\delta_8 &= \tfrac{47446323377}{267351869030400}, \\
\delta_9 &= \tfrac{-12443589337}{789903249408000}.
\end{aligned}\right\}. \tag{3.173}$$

Because of symmetry, the special end-point formulae for the points $x_N, x_{N-1}, x_{N-2}, x_{N-3}$ may be written down directly from those for x_1, x_2, x_3, x_4, respectively.

The set of parameter values in (3.27), (3.140), (3.35), (3.151), (3.43), (3.162), (3.51) and (3.173) give c_{18} as the first non-zero constant in (3.9). Global extrapolation on two grids, with p=8 in (2.29), gives, using the notation of Chapter 2, the numerical method

$$\mathbf{Y}^{(E)} = \frac{256}{255} I^{h}_{\frac{1}{2}h} \mathbf{Y}^{(2)} - \frac{1}{255} \mathbf{Y}^{(1)}. \tag{3.174}$$

3.8 CONSTRUCTION OF A TENTH-ORDER METHOD

Equation (3.3) attains tenth-order accuracy by writing $\alpha = 0$ as before and then by choosing $\beta = \frac{1}{362880}$, $\gamma = \frac{251}{181440}$, $\delta = \frac{913}{22680}$ and $\epsilon = \frac{44117}{181440}$ so that $\sum = 1 - 2(\alpha + \beta + \gamma + \delta + \epsilon) = \frac{15619}{36288}$. The first non-zero constant in (3.9) then becomes as

$$c_{20} = \frac{-1}{47900160}, \tag{3.175}$$

with $c_{11} = c_{13} = c_{15} = \ldots\ldots = 0$ because of symmetry. Choosing the parameters a_i, b_i, c_i, d_i $(i = 0, 2, 4, 6, 8)$ as given in section 3.4 with the parameters $\alpha_i, \beta_i, \gamma_i, \delta_i$ $(i = 0, \ldots, 10)$ calculated as follows, ensures that the same leading non-zero constant is obtained for the end-point formulae (3.12)—(3.19) associated with the tenth-order method.

For the point $x = x_1$, consider (3.28). Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}$, $y^{(xii)}, \ldots\ldots, y^{(xx)}$ gives the system

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} = \tfrac{655177}{907200}, \tag{3.176}$$

$$-\alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 6\alpha_7 + 7\alpha_8 + 8\alpha_9 + 9\alpha_{10} = \tfrac{252023}{907200}, \tag{3.177}$$

$$\tfrac{\alpha_0}{2!} + \alpha_2 + 2^2\tfrac{\alpha_3}{2!} + 3^2\tfrac{\alpha_4}{2!} + 4^2\tfrac{\alpha_5}{2!} + 5^2\tfrac{\alpha_6}{2!} + 6^2\tfrac{\alpha_7}{2!} + 7^2\tfrac{\alpha_8}{2!} + 8^2\tfrac{\alpha_9}{2!} + 9^2\tfrac{\alpha_{10}}{2!} = \tfrac{27438979}{119750400}, \tag{3.178}$$

$$-\tfrac{\alpha_0}{3!} + \alpha_2 + 2^3\tfrac{\alpha_3}{3!} + 3^3\tfrac{\alpha_4}{3!} + 4^3\tfrac{\alpha_5}{3!} + 5^3\tfrac{\alpha_6}{3!} + 6^3\tfrac{\alpha_7}{3!} + 7^3\tfrac{\alpha_8}{3!} + 8^3\tfrac{\alpha_9}{3!} + 9^3\tfrac{\alpha_{10}}{3!} = \tfrac{11368009}{119750400}, \tag{3.179}$$

$$\tfrac{\alpha_0}{4!} + \tfrac{\alpha_2}{4!} + 2^4\tfrac{\alpha_3}{4!} + 3^4\tfrac{\alpha_4}{4!} + 4^4\tfrac{\alpha_5}{4!} + 5^4\tfrac{\alpha_6}{4!} + 6^4\tfrac{\alpha_7}{4!} + 7^4\tfrac{\alpha_8}{4!} + 8^4\tfrac{\alpha_9}{4!} + 9^4\tfrac{\alpha_{10}}{4!} = \tfrac{131904163}{3113510400}, \tag{3.180}$$

$$-\tfrac{\alpha_0}{5!} + \tfrac{\alpha_2}{5!} + 2^5\tfrac{\alpha_3}{5!} + 3^5\tfrac{\alpha_4}{5!} + 4^5\tfrac{\alpha_5}{5!} + 5^5\tfrac{\alpha_6}{5!} + 6^5\tfrac{\alpha_7}{5!} + 7^5\tfrac{\alpha_8}{5!} + 8^5\tfrac{\alpha_9}{5!} + 9^5\tfrac{\alpha_{10}}{5!} = \tfrac{723798697}{46702656000}, \tag{3.181}$$

$$\tfrac{\alpha_0}{6!} + \tfrac{\alpha_2}{6!} + 2^6\tfrac{\alpha_3}{6!} + 3^6\tfrac{\alpha_4}{6!} + 4^6\tfrac{\alpha_5}{6!} + 5^6\tfrac{\alpha_6}{6!} + 6^6\tfrac{\alpha_7}{6!} + 7^6\tfrac{\alpha_8}{6!} + 8^6\tfrac{\alpha_9}{6!} + 9^6\tfrac{\alpha_{10}}{6!} = \tfrac{2541132023}{475517952000}, \tag{3.182}$$

$$-\tfrac{\alpha_0}{7!} + \tfrac{\alpha_2}{7!} + 2^7\tfrac{\alpha_3}{7!} + 3^7\tfrac{\alpha_4}{7!} + 4^7\tfrac{\alpha_5}{7!} + 5^7\tfrac{\alpha_6}{7!} + 6^7\tfrac{\alpha_7}{7!} + 7^7\tfrac{\alpha_8}{7!} + 8^7\tfrac{\alpha_9}{7!} + 9^7\tfrac{\alpha_{10}}{7!} = \tfrac{8768652467}{5230697472000}, \tag{3.183}$$

$$\tfrac{\alpha_0}{8!} + \tfrac{\alpha_2}{8!} + 2^8\tfrac{\alpha_3}{8!} + 3^8\tfrac{\alpha_4}{8!} + 4^8\tfrac{\alpha_5}{8!} + 5^8\tfrac{\alpha_6}{8!} + 6^8\tfrac{\alpha_7}{8!} + 7^8\tfrac{\alpha_8}{8!} + 8^8\tfrac{\alpha_9}{8!} + 9^8\tfrac{\alpha_{10}}{8!} = \tfrac{14042390777}{28582025472000}, \tag{3.184}$$

$$-\frac{\alpha_0}{9!} + \frac{\alpha_2}{9!} + 2^9\frac{\alpha_3}{9!} + 3^9\frac{\alpha_4}{9!} + 4^9\frac{\alpha_5}{9!} + 5^9\frac{\alpha_6}{9!} + 6^9\frac{\alpha_7}{9!} + 7^9\frac{\alpha_8}{9!} + 8^9\frac{\alpha_9}{9!} + 9^9\frac{\alpha_{10}}{9!} = \frac{2762162653}{20520428544000}, \tag{3.185}$$

$$\frac{\alpha_0}{10!} + \frac{\alpha_2}{10!} + 2^{10}\frac{\alpha_3}{10!} + 3^{10}\frac{\alpha_4}{10!} + 4^{10}\frac{\alpha_5}{10!} + 5^{10}\frac{\alpha_6}{10!} + 6^{10}\frac{\alpha_7}{10!} + 7^{10}\frac{\alpha_8}{10!} + 8^{10}\frac{\alpha_9}{10!} + 9^{10}\frac{\alpha_{10}}{10!} = \frac{3522018283439}{101370917007360000} - \frac{1}{47900160}. \tag{3.186}$$

Solving this system we get the parameters of the first end-point formula (i.e. $x = x_1$) for the tenth-order method. It is noted that the parameters α_i $(i = 11, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\alpha_0 &= \frac{116040349955470841}{3649353012264960000}, \\
\alpha_1 &= \frac{157203913989739}{330258191155200}, \\
\alpha_2 &= \frac{40179841536173}{2673518690304000}, \\
\alpha_3 &= \frac{5622449804159}{12509779968000}, \\
\alpha_4 &= \frac{-867092203321867}{1621934672117760}, \\
\alpha_5 &= \frac{1558230209576339}{3041127510220800000}, \\
\alpha_6 &= \frac{-2874137423864459}{8109673360588800}, \\
\alpha_7 &= \frac{5261248047297509}{30411275102208000}, \\
\alpha_8 &= \frac{01252013974567247}{22117290983424000}, \\
\alpha_9 &= \frac{814229640791783}{7298706024529920 0}, \\
\alpha_{10} &= \frac{-281299064581543}{2807194624819200000}.
\end{aligned}\right\} \tag{3.187}$$

It can be shown using the method of undetermined coefficients for the point $x = x_2$, that, taking the parameter values b_0, b_2, b_4, b_6, b_8 in (3.35) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots,$ $y^{(xx)}$ in (3.36) gives the system

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 + \beta_9 + \beta_{10} = \frac{882773}{907200}, \tag{3.188}$$

$$-2\beta_0 - \beta_1 + \beta_3 + 2\beta_4 + 3\beta_5 + 4\beta_6 + 5\beta_7 + 6\beta_8 + 7\beta_9 + 8\beta_{10} = \frac{24427}{453600}, \tag{3.189}$$

$$2^2\frac{\beta_0}{2!}+\frac{\beta_1}{2!}+\frac{\beta_3}{2!}+2^2\frac{\beta_4}{2!}+3^2\frac{\beta_5}{2!}+4^2\frac{\beta_6}{2!}+5^2\frac{\beta_7}{2!}+6^2\frac{\beta_8}{2!}+7^2\frac{\beta_9}{2!}$$
$$+8^2\frac{\beta_{10}}{2!}=\frac{43202009}{119750400}, \tag{3.190}$$

$$-2^3\frac{\beta_0}{3!}-\frac{\beta_1}{3!}+\frac{\beta_3}{3!}+2^3\frac{\beta_4}{3!}+3^3\frac{\beta_5}{3!}+4^3\frac{\beta_6}{3!}+5^3\frac{\beta_7}{3!}+6^3\frac{\beta_8}{3!}+7^3\frac{\beta_9}{3!}$$
$$+8^3\frac{\beta_{10}}{3!}=\frac{2394839}{59875200}, \tag{3.191}$$

$$2^4\frac{\beta_0}{4!}+\frac{\beta_1}{4!}+\frac{\beta_3}{4!}+2^4\frac{\beta_4}{4!}+3^4\frac{\beta_5}{4!}+4^4\frac{\beta_6}{4!}+5^4\frac{\beta_7}{4!}+6^4\frac{\beta_8}{4!}+7^4\frac{\beta_9}{4!}$$
$$+8^4\frac{\beta_{10}}{4!}=\frac{190486607}{3113510400}, \tag{3.192}$$

$$-2^5\frac{\beta_0}{5!}+\frac{\beta_1}{5!}+\frac{\beta_3}{5!}+2^5\frac{\beta_4}{5!}+3^5\frac{\beta_5}{5!}+4^5\frac{\beta_6}{5!}+5^5\frac{\beta_7}{5!}+6^5\frac{\beta_8}{5!}+7^5\frac{\beta_9}{5!}$$
$$+8^5\frac{\beta_{10}}{5!}=\frac{21489493}{2122848000}, \tag{3.193}$$

$$2^6\frac{\beta_0}{6!}+\frac{\beta_1}{6!}+\frac{\beta_3}{6!}+2^6\frac{\beta_4}{6!}+3^6\frac{\beta_5}{6!}+4^6\frac{\beta_6}{6!}+5^6\frac{\beta_7}{6!}+6^6\frac{\beta_8}{6!}+7^6\frac{\beta_9}{6!}$$
$$+8^6\frac{\beta_{10}}{6!}=\frac{34992742353}{5230697472000}, \tag{3.194}$$

$$-2^7\frac{\beta_0}{7!}-\frac{\beta_1}{7!}+\frac{\beta_3}{7!}+2^7\frac{\beta_4}{7!}+3^7\frac{\beta_5}{7!}+4^7\frac{\beta_6}{7!}+5^7\frac{\beta_7}{7!}+6^7\frac{\beta_5}{7!}+7^7\frac{\beta_9}{7!}$$
$$+8^7\frac{\beta_{10}}{7!}=\frac{327962597}{237758976000}, \tag{3.195}$$

$$2^8\frac{\beta_0}{8!}+\frac{\beta_1}{8!}+\frac{\beta_3}{8!}+2^8\frac{\beta_4}{8!}+3^8\frac{\beta_5}{8!}+4^8\frac{\beta_6}{8!}+5^8\frac{\beta_7}{8!}+6^8\frac{\beta_5}{8!}+7^8\frac{\beta_9}{8!}$$
$$+8^8\frac{\beta_{10}}{8!}=\frac{881182516553}{16005934264320000}, \tag{3.196}$$

$$-2^9\frac{\beta_0}{9!}-\frac{\beta_1}{9!}+\frac{\beta_3}{9!}+2^9\frac{\beta_4}{9!}+3^9\frac{\beta_5}{9!}+4^9\frac{\beta_6}{9!}+5^9\frac{\beta_7}{9!}+6^9\frac{\beta_5}{9!}+7^9\frac{\beta_9}{9!}$$
$$+8^9\frac{\beta_{10}}{9!}=\frac{2542651289}{20520428544000}, \tag{3.197}$$

$$2^{10}\frac{\beta_0}{10!}+\frac{\beta_1}{10!}+\frac{\beta_3}{10!}+2^{10}\frac{\beta_4}{10!}+3^{10}\frac{\beta_5}{10!}+4^9\frac{\beta_6}{10!}+5^{10}\frac{\beta_7}{10!}+6^{10}\frac{\beta_5}{10!}+7^9\frac{\beta_9}{10!}$$
$$+8^{10}\frac{\beta_{10}}{10!}=\frac{7404524487683}{202741834014720000}-\frac{1}{47900160}, \tag{3.198}$$

the solution of which give the parameters of the second end-point formula (i.e. $x = x_2$) for the tenth-order method. It is noted that the parameters β_i (i =

11,12) may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}
\beta_0 &= \frac{4309605590878488 1}{3649353012264960000},\\
\beta_1 &= \frac{96572492798993699}{364935301226496000},\\
\beta_2 &= \frac{17879555117626619}{48658040163532800},\\
\beta_3 &= \frac{87652728055181}{243290200817664},\\
\beta_4 &= \frac{-4711287655743611}{40548366802944000},\\
\beta_5 &= \frac{46489634142652499}{304112751022080000},\\
\beta_6 &= \frac{-4286488815953951}{40548366802944000},\\
\beta_7 &= \frac{315901599466553}{608225020441600},\\
\beta_8 &= \frac{-830947903694617}{48650163532800},\\
\beta_9 &= \frac{94832253888503}{280719462481920000},\\
\beta_{10} &= \frac{0158673972225317}{5213361446092800000}.
\end{aligned}\right\} \tag{3.199}$$

Next, it can be shown using the method of undetermined coefficients for the point $x = x_3$, that, taking the parameter values c_0, c_2, c_4, c_6, c_8 in (3.43), together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xx)}$ in (3.44) gives

$$\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_8 + \gamma_9 + \gamma_{10} = \frac{302231}{302400}, \tag{3.200}$$

$$\begin{aligned}&-3\gamma_0 - 2\gamma_1 - \gamma_2 + \gamma_4 + 2\gamma_5 + 3\gamma_6 + 4\gamma_7 + 5\gamma_8 + 6\gamma_9 + 7\gamma_{10}\\ &= \frac{169}{100800},\end{aligned} \tag{3.201}$$

$$\begin{aligned}&3^2\tfrac{\gamma_0}{2!} + 2^2\tfrac{\gamma_1}{2!} + \tfrac{\gamma_2}{2!} + \tfrac{\gamma_4}{2!} + 2^2\tfrac{\gamma_5}{2!} + 3^2\tfrac{\gamma_6}{2!} + 4^2\tfrac{\gamma_7}{2!} + 5^2\tfrac{\gamma_8}{2!} + 6^2\tfrac{\gamma_9}{2!}\\ &+7^2\tfrac{\gamma_{10}}{2!} = \frac{5510311}{13305600},\end{aligned} \tag{3.202}$$

$$\begin{aligned}&-3^3\tfrac{\gamma_0}{3!} - 2^3\tfrac{\gamma_1}{3!} - \tfrac{\gamma_2}{3!} + \tfrac{\gamma_4}{3!} + 2^3\tfrac{\gamma_5}{3!} + 3^3\tfrac{\gamma_6}{3!} + 4^3\tfrac{\gamma_7}{3!} + 5^3\tfrac{\gamma_8}{3!} + 6^3\tfrac{\gamma_9}{3!}\\ &+7^3\tfrac{\gamma_{10}}{3!} = \frac{11381}{4435200},\end{aligned} \tag{3.203}$$

$$\begin{aligned}&3^4\tfrac{\gamma_0}{4!} + 2^4\tfrac{\gamma_1}{4!} + \tfrac{\gamma_2}{4!} + \tfrac{\gamma_4}{4!} + 2^4\tfrac{\gamma_5}{4!} + 3^4\tfrac{\gamma_6}{4!} + 4^4\tfrac{\gamma_7}{4!} + 5^4\tfrac{\gamma_8}{4!} + 6^4\tfrac{\gamma_9}{4!}\\ &+7^4\tfrac{\gamma_{10}}{4!} = \frac{591141643}{7264857600},\end{aligned} \tag{3.204}$$

$$-3^5\frac{\gamma_0}{5!}-2^5\frac{\gamma_1}{5!}-\frac{\gamma_2}{5!}+\frac{\gamma_4}{5!}+2^5\frac{\gamma_5}{5!}+3^5\frac{\gamma_6}{5!}+4^5\frac{\gamma_7}{5!}+5^5\frac{\gamma_8}{5!}+6^5\frac{\gamma_9}{5!}$$
$$+7^5\frac{\gamma_{10}}{5!}=\frac{14645899}{12108096000}, \qquad (3.205)$$

$$3^6\frac{\gamma_0}{6!}+2^6\frac{\gamma_1}{6!}+\frac{\gamma_2}{6!}+\frac{\gamma_4}{6!}+2^6\frac{\gamma_5}{6!}+3^6\frac{\gamma_6}{6!}+4^6\frac{\gamma_7}{6!}+5^6\frac{\gamma_8}{6!}+6^6\frac{\gamma_9}{6!}$$
$$+7^6\frac{\gamma_{10}}{6!}=\frac{1346510087}{134120448000}, \qquad (3.206)$$

$$-3^7\frac{\gamma_0}{7!}-2^7\frac{\gamma_1}{7!}-\frac{\gamma_2}{7!}+\frac{\gamma_4}{7!}+2^7\frac{\gamma_5}{7!}+3^7\frac{\gamma_6}{7!}+4^7\frac{\gamma_7}{7!}+5^7\frac{\gamma_8}{7!}+6^7\frac{\gamma_9}{7!}$$
$$+7^7\frac{\gamma_{10}}{7!}=\frac{162013909}{581188608000}, \qquad (3.207)$$

$$3^8\frac{\gamma_0}{8!}+2^8\frac{\gamma_1}{8!}+\frac{\gamma_2}{8!}+\frac{\gamma_4}{8!}+2^8\frac{\gamma_5}{8!}+3^8\frac{\gamma_6}{8!}+4^8\frac{\gamma_7}{8!}+5^8\frac{\gamma_8}{8!}+6^8\frac{\gamma_9}{8!}$$
$$+7^8\frac{\gamma_{10}}{8!}=\frac{19405166329}{22230464256000}, \qquad (3.208)$$

$$-3^9\frac{\gamma_0}{9!}-2^9\frac{\gamma_1}{9!}-\frac{\gamma_2}{9!}+\frac{\gamma_4}{9!}+2^9\frac{\gamma_5}{9!}+3^9\frac{\gamma_6}{9!}+4^9\frac{\gamma_7}{9!}+5^9\frac{\gamma_8}{9!}+6^9\frac{\gamma_9}{9!}$$
$$+7^9\frac{\gamma_{10}}{9!}=\frac{163046441}{4234374144000}, \qquad (3.209)$$

$$3^{10}\frac{\gamma_0}{10!}+2^{10}\frac{\gamma_1}{9!}+\frac{\gamma_2}{10!}+\frac{\gamma_4}{9!}+2^{10}\frac{\gamma_5}{10!}+3^{10}\frac{\gamma_6}{10!}+4^{10}\frac{\gamma_7}{10!}+5^{10}\frac{\gamma_8}{9!}+6^{10}\frac{\gamma_9}{10!}$$
$$+7^{10}\frac{\gamma_{10}}{10!}=\frac{5800069899419}{101370917007360000}-\frac{1}{4700960}. \qquad (3.210)$$

Solving this system we get the parameters of the third end-point formula (i.e. $x = x_3$) for the tenth-order method; they are

$$\left.\begin{aligned}
\gamma_0 &= \frac{2061298229437523}{28383856762060800000},\\
\gamma_1 &= \frac{11589154862126857}{28383856762060800000},\\
\gamma_2 &= \frac{9139422280088273}{3784514234914400},\\
\gamma_3 &= \frac{683712457758821}{1576880931225600},\\
\gamma_4 &= \frac{3228137601455083}{13516122267648000},\\
\gamma_5 &= \frac{1503276660462431}{33790305669120000},\\
\gamma_6 &= \frac{-21616099543697}{13516122267648000},\\
\gamma_7 &= \frac{5080973291}{3435470438400},\\
\gamma_8 &= \frac{-3695100594191}{7569028469882880},\\
\gamma_9 &= \frac{27537990586177}{28383856762060800000},\\
\gamma_{10} &= \frac{-1915764666829}{2183373597081600000}.
\end{aligned}\right\} \qquad (3.211)$$

It is noted that the parameters γ_i $(i = 11, 12)$ may be arbitrarily assigned the value zero.

Finally, it can be shown using the method of undetermined coefficients for the point $x = x_4$ that, taking the parameters d_0, d_2, d_4, d_6, d_8 given (3.51), and using the parameters calculated below, yields the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xx)}$ in (3.52) gives

$$\delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + \delta_7 + \delta_8 + \delta_9 + \delta_{10} = \frac{1814399}{1814400}, \tag{3.212}$$

$$-4\delta_0 - 3\delta_1 - 2\delta_2 - \delta_3 + \delta_5 + 2\delta_6 + 3\delta_7 + 4\delta_8 + 5\delta_9 + 6\delta_{10} = \frac{122753}{9979200}, \tag{3.213}$$

$$\begin{aligned} &4^2\tfrac{\delta_0}{2!} + 3^2\tfrac{\delta_1}{2!} + 2^2\tfrac{\delta_2}{2!} + \tfrac{\delta_3}{2!} + \tfrac{\delta_5}{2!} + 2^2\tfrac{\delta_6}{2!} + 3^2\tfrac{\delta_7}{2!} + 4^2\tfrac{\delta_8}{2!} + 5^2\tfrac{\delta_9}{2!} \\ &+6^2\tfrac{\delta_{10}}{2!} = \tfrac{14255849}{34214400}, \end{aligned} \tag{3.214}$$

$$\begin{aligned} &-4^3\tfrac{\delta_0}{3!} - 3^3\tfrac{\delta_1}{3!} - 2^3\tfrac{\delta_2}{3!} - \tfrac{\delta_3}{3!} + \tfrac{\delta_5}{3!} + 2^3\tfrac{\delta_6}{3!} + 3^3\tfrac{\delta_7}{3!} + 4^3\tfrac{\delta_8}{3!} + 5^3\tfrac{\delta_9}{3!} \\ &+6^3\tfrac{\delta_{10}}{3!} = \tfrac{68891}{222393600}, \end{aligned} \tag{3.215}$$

$$\begin{aligned} &4^4\tfrac{\delta_0}{4!} + 3^4\tfrac{\delta_1}{4!} + 2^4\tfrac{\delta_2}{4!} + \tfrac{\delta_3}{4!} + \tfrac{\delta_5}{4!} + 2^4\tfrac{\delta_6}{4!} + 3^4\tfrac{\delta_7}{4!} + 4^4\tfrac{\delta_8}{4!} + 5^4\tfrac{\delta_9}{4!} \\ &+6^4\tfrac{\delta_{10}}{4!} = \tfrac{363217187}{43589145600}, \end{aligned} \tag{3.216}$$

$$\begin{aligned} &-4^5\tfrac{\delta_0}{5!} - 3^5\tfrac{\delta_1}{5!} - 2^5\tfrac{\delta_2}{5!} - \tfrac{\delta_3}{5!} + \tfrac{\delta_5}{5!} + 2^5\tfrac{\delta_6}{5!} + 3^5\tfrac{\delta_7}{5!} + 4^5\tfrac{\delta_8}{5!} + 5^5\tfrac{\delta_9}{5!} \\ &+6^5\tfrac{\delta_{10}}{5!} = \tfrac{413849}{326918592000}, \end{aligned} \tag{3.217}$$

$$\begin{aligned} &4^6\tfrac{\delta_0}{6!} + 3^6\tfrac{\delta_1}{6!} + 2^6\tfrac{\delta_2}{6!} + \tfrac{\delta_3}{6!} + \tfrac{\delta_5}{6!} + 2^6\tfrac{\delta_6}{6!} + 3^6\tfrac{\delta_7}{6!} + 4^6\tfrac{\delta_8}{6!} + 5^6\tfrac{\delta_9}{6!} \\ &+6^6\tfrac{\delta_{10}}{6!} = \tfrac{10139471581}{951035904000}, \end{aligned} \tag{3.218}$$

$$\begin{aligned} &-4^7\tfrac{\delta_0}{7!} - 3^7\tfrac{\delta_1}{7!} - 2^7\tfrac{\delta_2}{7!} - \tfrac{\delta_3}{7!} + \tfrac{\delta_5}{7!} + 2^7\tfrac{\delta_7}{7!} + 3^7\tfrac{\delta_7}{7!} + 4^7\tfrac{\delta_8}{7!} + 5^7\tfrac{\delta_9}{7!} \\ &+6^7\tfrac{\delta_{10}}{7!} = \tfrac{154643851}{88921857024000}, \end{aligned} \tag{3.219}$$

$$\begin{aligned} &4^8\tfrac{\delta_0}{8!} + 3^8\tfrac{\delta_1}{8!} + 2^8\tfrac{\delta_2}{8!} + \tfrac{\delta_3}{8!} + \tfrac{\delta_5}{8!} + 2^8\tfrac{\delta_6}{8!} + 3^8\tfrac{\delta_7}{8!} + 4^8\tfrac{\delta_8}{8!} + 5^8\tfrac{\delta_9}{8!} \\ &+6^8\tfrac{\delta_{10}}{8!} = \tfrac{3141960414959}{3201186852864000}, \end{aligned} \tag{3.220}$$

$$\begin{aligned} &-4^9\tfrac{\delta_0}{9!} - 3^9\tfrac{\delta_1}{9!} - 2^9\tfrac{\delta_2}{9!} - \tfrac{\delta_3}{9!} + \tfrac{\delta_5}{9!} + 2^9\tfrac{\delta_7}{9!} + 3^9\tfrac{\delta_7}{9!} + 4^9\tfrac{\delta_8}{9!} + 5^9\tfrac{\delta_9}{9!} \\ &+6^9\tfrac{\delta_{10}}{9!} = \tfrac{4165158373}{101370917007360000}, \end{aligned} \tag{3.221}$$

$$\begin{aligned}&4^{10}\tfrac{\delta_0}{10!}+3^{10}\tfrac{\delta_1}{9!}+2^{10}\tfrac{\delta_2}{10!}+\tfrac{\delta_3}{10!}+\tfrac{\delta_5}{10!}+2^{10}\tfrac{\delta_7}{10!}+3^{10}\tfrac{\delta_7}{10!}+4^{10}\tfrac{\delta_8}{10!}\\&+5^{10}\tfrac{\delta_9}{10!}+6^{10}\tfrac{\delta_{10}}{10!}=\tfrac{28108982850101}{405483668029440000}-\tfrac{1}{47900160}.\end{aligned} \tag{3.222}$$

Solving this system we get the parameters of the fourth end-point formula (i.e. $\mathrm{x} = \mathrm{x}_4$) for the tenth-order method. It is noted that the parameters δ_{i} $(\mathrm{i} = 11, 12)$ may then be arbitrarily given the value zero. Thus

$$\left.\begin{aligned}\delta_0 &= \tfrac{-504886766892491}{5109094217170944000},\\ \delta_1 &= \tfrac{1579429435112527}{1021818843434188800},\\ \delta_2 &= \tfrac{705680560899513}{179266463760384000},\\ \delta_3 &= \tfrac{106486449327610741}{425757851430912000},\\ \delta_4 &= \tfrac{414123969848707}{954079218892800},\\ \delta_5 &= \tfrac{12714726728652943}{55293227458560000},\\ \delta_6 &= \tfrac{4196107713185}{92681981263872},\\ \delta_7 &= \tfrac{-217783613195039}{425757851430912000},\\ \delta_8 &= \tfrac{1820951439198607}{340662811447296000},\\ \delta_9 &= \tfrac{-19314059878021}{2043637686837760},\\ \delta_{10} &= \tfrac{23668609477577}{300534953951232000}.\end{aligned}\right\}. \tag{3.223}$$

Because of symmetry, the special end-point formulae for the points $x_N, x_{N-1}, x_{N-2}, x_{N-3}$ may be written down directly from those for $\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4$, respectively.

The set of parameter values in (3.27), (3.187), (3.35), (3.199), (3.43), (3.211), (3.51) and (3.223) give c_{20} as the first non-zero constant in (3.9). Global extrapolation on two grids, with p=10 in (2.29), gives, using the notation of Chapter 2, the numerical method

$$\mathbf{Y}^{(\mathrm{E})} = \frac{1024}{1023}\mathrm{I}^{\mathrm{h}}_{\frac{1}{2}\mathrm{h}}\mathbf{Y}^{(2)} - \frac{1}{1023}\mathbf{Y}^{(1)}. \tag{3.224}$$

3.9 CONSTRUCTION OF A TWELFTH-ORDER METHOD

Writing $\alpha = \frac{1}{47900160}$, $\beta = \frac{61}{23950080}$, $\gamma = \frac{22103}{15966720}$, $\delta = \frac{11477}{285120}$ and $\epsilon = \frac{215687}{887040}$ so that $\sum = 1 - 2(\alpha + \beta + \gamma + \delta + \epsilon) = \frac{1718069}{3991680}$, in (3.3), gives the unique twelfth-order method of the family (3.3) for $n \neq 1, 2, 3, 4, N-3, N-2, N-1$, or N. The first non-zero constant in (3.9) then becomes as

$$c_{22} = \frac{691}{23775897600}, \tag{3.225}$$

with $c_{13} = c_{15} = c_{17} = c_{19} = c_{21} = \ldots\ldots = 0$, because of symmetry.

One can obtain the same values of c_i $(i = 11, 12, 13, \ldots\ldots, 22)$ for the end points $n = 1, 2, 3, 4, N-3, N-2, N-1, N$ by choosing the parameters a_i, b_i, c_i, d_i $(i = 0, 2, 4, 6, 8)$ as given in section 3.4 and assigning the remaining parameters in (3.12)—(3.19) respectively, in the following way.

For the point $x = x_1$, consider the scheme (3.28). Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}$, $y^{(xii)}, \ldots\ldots, y^{(xxii)}$ gives the system

$$\begin{aligned}&\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}\\ &+\alpha_{11} + \alpha_{12} = \tfrac{655177}{907200},\end{aligned} \tag{3.226}$$

$$\begin{aligned}&-\alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 6\alpha_7 + 7\alpha_8 + 8\alpha_9\\ &+9\alpha_{10} + 10\alpha_{11} + 11\alpha_{12} = \tfrac{252023}{907200},\end{aligned} \tag{3.227}$$

$$\begin{aligned}&\tfrac{\alpha_0}{2!} + \alpha_2 + 2^2\tfrac{\alpha_3}{2!} + 3^2\tfrac{\alpha_4}{2!} + 4^2\tfrac{\alpha_5}{2!} + 5^2\tfrac{\alpha_6}{2!} + 6^2\tfrac{\alpha_7}{2!} + 7^2\tfrac{\alpha_8}{2!} + 8^2\tfrac{\alpha_9}{2!}\\ &+9^2\tfrac{\alpha_{10}}{2!} + 10^2\tfrac{\alpha_{11}}{2!} + 11^2\tfrac{\alpha_{12}}{2!} = \tfrac{27438979}{119750400},\end{aligned} \tag{3.228}$$

$$\begin{aligned}&-\tfrac{\alpha_0}{3!} + \alpha_2 + 2^3\tfrac{\alpha_3}{3!} + 3^3\tfrac{\alpha_4}{3!} + 4^3\tfrac{\alpha_5}{3!} + 5^3\tfrac{\alpha_6}{3!} + 6^3\tfrac{\alpha_7}{3!} + 7^3\tfrac{\alpha_8}{3!} + 8^3\tfrac{\alpha_9}{3!}\\ &+9^3\tfrac{\alpha_{10}}{3!} + 10^3\tfrac{\alpha_{11}}{3!} + 11^3\tfrac{\alpha_{12}}{3!} = \tfrac{11368009}{119750400},\end{aligned} \tag{3.229}$$

$$\begin{aligned}&\tfrac{\alpha_0}{4!} + \tfrac{\alpha_2}{4!} + 2^4\tfrac{\alpha_3}{4!} + 3^4\tfrac{\alpha_4}{4!} + 4^4\tfrac{\alpha_5}{4!} + 5^4\tfrac{\alpha_6}{4!} + 6^4\tfrac{\alpha_7}{4!} + 7^4\tfrac{\alpha_8}{4!} + 8^4\tfrac{\alpha_9}{4!}\\ &+9^4\tfrac{\alpha_{10}}{4!} + 10^4\tfrac{\alpha_{11}}{4!} + 11^4\tfrac{\alpha_{12}}{4!} = \tfrac{131904163}{3113510400},\end{aligned} \tag{3.230}$$

$$\begin{aligned}&-\tfrac{\alpha_0}{5!} + \tfrac{\alpha_2}{5!} + 2^5\tfrac{\alpha_3}{5!} + 3^5\tfrac{\alpha_4}{5!} + 4^5\tfrac{\alpha_5}{5!} + 5^5\tfrac{\alpha_6}{5!} + 6^5\tfrac{\alpha_7}{5!} + 7^5\tfrac{\alpha_8}{5!} + 8^5\tfrac{\alpha_9}{5!}\\ &+9^5\tfrac{\alpha_{10}}{5!} + 10^5\tfrac{\alpha_{11}}{5!} + 11^5\tfrac{\alpha_{12}}{5!} = \tfrac{723798697}{46702656000},\end{aligned} \tag{3.231}$$

$$\begin{aligned}&\tfrac{\alpha_0}{6!} + \tfrac{\alpha_2}{6!} + 2^6\tfrac{\alpha_3}{6!} + 3^6\tfrac{\alpha_4}{6!} + 4^6\tfrac{\alpha_5}{6!} + 5^6\tfrac{\alpha_6}{6!} + 6^6\tfrac{\alpha_7}{6!} + 7^6\tfrac{\alpha_8}{6!} + 8^6\tfrac{\alpha_9}{6!}\\ &+9^6\tfrac{\alpha_{10}}{6!} + 10^6\tfrac{\alpha_{11}}{6!} + 11^6\tfrac{\alpha_{12}}{6!} = \tfrac{2541132023}{475517952000},\end{aligned} \tag{3.232}$$

$$\begin{aligned}&-\tfrac{\alpha_0}{7!} + \tfrac{\alpha_2}{7!} + 2^7\tfrac{\alpha_3}{7!} + 3^7\tfrac{\alpha_4}{7!} + 4^7\tfrac{\alpha_5}{7!} + 5^7\tfrac{\alpha_6}{7!} + 6^7\tfrac{\alpha_7}{7!} + 7^7\tfrac{\alpha_8}{7!} + 8^7\tfrac{\alpha_9}{7!}\\ &+9^7\tfrac{\alpha_{10}}{7!} + 10^7\tfrac{\alpha_{11}}{7!} + 11^7\tfrac{\alpha_{12}}{7!} = \tfrac{8768652467}{5230697472000},\end{aligned} \tag{3.233}$$

$$\frac{\alpha_0}{8!}+\frac{\alpha_2}{8!}+2^8\frac{\alpha_3}{8!}+3^8\frac{\alpha_4}{8!}+4^8\frac{\alpha_5}{8!}+5^8\frac{\alpha_6}{8!}+6^8\frac{\alpha_7}{8!}+7^8\frac{\alpha_8}{8!}+8^8\frac{\alpha_9}{8!}$$
$$+9^8\frac{\alpha_{10}}{8!}+10^8\frac{\alpha_{11}}{8!}+11^8\frac{\alpha_{12}}{8!}=\frac{14042390777}{28582025472000}, \tag{3.234}$$

$$-\frac{\alpha_0}{9!}+\frac{\alpha_2}{9!}+2^9\frac{\alpha_3}{9!}+3^9\frac{\alpha_4}{9!}+4^9\frac{\alpha_5}{9!}+5^9\frac{\alpha_6}{9!}+6^9\frac{\alpha_7}{9!}+7^9\frac{\alpha_8}{9!}+8^9\frac{\alpha_9}{9!}$$
$$+9^9\frac{\alpha_{10}}{9!}+10^9\frac{\alpha_{11}}{9!}+11^9\frac{\alpha_{12}}{9!}=\frac{2762162653}{20520428544000}, \tag{3.235}$$

$$\frac{\alpha_0}{10!}+\frac{\alpha_2}{10!}+2^{10}\frac{\alpha_3}{10!}+3^{10}\frac{\alpha_4}{10!}+4^{10}\frac{\alpha_5}{10!}+5^{10}\frac{\alpha_6}{10!}+6^{10}\frac{\alpha_7}{10!}+7^{10}\frac{\alpha_8}{10!}$$
$$+8^{10}\frac{\alpha_9}{10!}+9^{10}\frac{\alpha_{10}}{10!}+10^{10}\frac{\alpha_{11}}{10!}+11^{10}\frac{\alpha_{12}}{10!}=\frac{3522018283439}{101370917007360000}, \tag{3.236}$$

$$-\frac{\alpha_0}{11!}+\frac{\alpha_2}{11!}+2^{11}\frac{\alpha_3}{11!}+3^{11}\frac{\alpha_4}{11!}+4^{11}\frac{\alpha_5}{11!}+5^{11}\frac{\alpha_6}{11!}+6^{11}\frac{\alpha_7}{11!}+7^{11}\frac{\alpha_8}{11!}$$
$$+8^{11}\frac{\alpha_9}{11!}+9^{11}\frac{\alpha_{10}}{11!}+10^{11}\frac{\alpha_{11}}{11!}+11^{11}\frac{\alpha_{12}}{11!}=\frac{368462718776}{4344467817440000}, \tag{3.237}$$

$$-\frac{\alpha_0}{12!}+\frac{\alpha_2}{12!}+2^{12}\frac{\alpha_3}{12!}+3^{12}\frac{\alpha_4}{12!}+4^{12}\frac{\alpha_5}{12!}+5^{12}\frac{\alpha_6}{12!}+6^{12}\frac{\alpha_7}{12!}+7^{12}\frac{\alpha_8}{12!}$$
$$+8^{12}\frac{\alpha_9}{12!}+9^{12}\frac{\alpha_{10}}{12!}+10^{12}\frac{\alpha_{11}}{12!}+11^{12}\frac{\alpha_{12}}{12!}=\frac{30689602988243}{1561112121913344000}+\frac{691}{23775897600}. \tag{3.238}$$

Solving this system, we get the parameters of the first end-point formula (i.e. $x = x_1$) for the twelfth-order method. They are

$$\left.\begin{aligned}
\alpha_0 &= \frac{5971246575268812433}{1983530697813120000},\\
\alpha_1 &= \frac{3663586639878490261}{739741630115840000},\\
\alpha_2 &= \frac{-4440754776771707783}{5109094217170944000},\\
\alpha_3 &= \frac{5923300186040234237}{766364132576416000},\\
\alpha_4 &= \frac{-9276527228077563 67}{756902846988000},\\
\alpha_5 &= \frac{331892916693634 7}{2128789257154560000},\\
\alpha_6 &= \frac{-5498940528394228543}{3649353012264960000},\\
\alpha_7 &= \frac{2345698766262862649}{2128789257154560000},\\
\alpha_8 &= \frac{-4544095735229393 51}{75690284698828000},\\
\alpha_9 &= \frac{1812148033564250279}{766364132575641 6000},\\
\alpha_{10} &= \frac{-3253745190725366999}{5109094217170944000},\\
\alpha_{11} &= \frac{-1475894306383821109}{14050009097220096000},\\
\alpha_{12} &= \frac{-2701027397950574263}{337200218333282304000}.
\end{aligned}\right\} \tag{3.239}$$

It can be shown using the method of undetermined coefficients for the point $x = x_2$, that, taking the parameter values b_0, b_2, b_4, b_6, b_8 in (3.35) together

with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xxii)}$ in (3.36) gives the system

$$\beta_0+\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6+\beta_7+\beta_8+\beta_9+\beta_{10} +\beta_{11}+\beta_{12}=\tfrac{882773}{907200}, \tag{3.240}$$

$$-2\beta_0-\beta_1+\beta_3+2\beta_4+3\beta_5+4\beta_6+5\beta_7+6\beta_8+7\beta_9+8\beta_{10} +9\beta_9+10\beta_{12}=\tfrac{24427}{453600}, \tag{3.241}$$

$$2^2\tfrac{\beta_0}{2!}+\tfrac{\beta_1}{2!}+\tfrac{\beta_3}{2!}+2^2\tfrac{\beta_4}{2!}+3^2\tfrac{\beta_5}{2!}+4^2\tfrac{\beta_6}{2!}+5^2\tfrac{\beta_7}{2!}+6^2\tfrac{\beta_8}{2!}+7^2\tfrac{\beta_9}{2!} +8^2\tfrac{\beta_{10}}{2!}+9^2\tfrac{\beta_{11}}{2!}+10^2\tfrac{\beta_{12}}{2!}=\tfrac{43202009}{119750400}, \tag{3.242}$$

$$-2^3\tfrac{\beta_0}{3!}-\tfrac{\beta_1}{3!}+\tfrac{\beta_3}{3!}+2^3\tfrac{\beta_4}{3!}+3^3\tfrac{\beta_5}{3!}+4^3\tfrac{\beta_6}{3!}+5^3\tfrac{\beta_7}{3!}+6^3\tfrac{\beta_8}{3!}+7^3\tfrac{\beta_9}{3!} +8^3\tfrac{\beta_{10}}{3!}+9^3\tfrac{\beta_{11}}{3!}+10^3\tfrac{\beta_{12}}{3!}=\tfrac{2394839}{59875200}, \tag{3.243}$$

$$2^4\tfrac{\beta_0}{4!}+\tfrac{\beta_1}{4!}+\tfrac{\beta_3}{4!}+2^4\tfrac{\beta_4}{4!}+3^4\tfrac{\beta_5}{4!}+4^4\tfrac{\beta_6}{4!}+5^4\tfrac{\beta_7}{4!}+6^4\tfrac{\beta_8}{4!}+7^4\tfrac{\beta_9}{4!} +8^4\tfrac{\beta_{10}}{4!}+9^4\tfrac{\beta_{11}}{4!}+10^4\tfrac{\beta_{12}}{4!}=\tfrac{190486607}{3113510400}, \tag{3.244}$$

$$-2^5\tfrac{\beta_0}{5!}+\tfrac{\beta_1}{5!}+\tfrac{\beta_3}{5!}+2^5\tfrac{\beta_4}{5!}+3^5\tfrac{\beta_5}{5!}+4^5\tfrac{\beta_6}{5!}+5^5\tfrac{\beta_7}{5!}+6^5\tfrac{\beta_8}{5!}+7^5\tfrac{\beta_9}{5!} +8^5\tfrac{\beta_{10}}{5!}+9^5\tfrac{\beta_{11}}{5!}+10^5\tfrac{\beta_{12}}{5!}=\tfrac{21489493}{2122848000}, \tag{3.245}$$

$$2^6\tfrac{\beta_0}{6!}+\tfrac{\beta_1}{6!}+\tfrac{\beta_3}{6!}+2^6\tfrac{\beta_4}{6!}+3^6\tfrac{\beta_5}{6!}+4^6\tfrac{\beta_6}{6!}+5^6\tfrac{\beta_7}{6!}+6^6\tfrac{\beta_8}{6!}+7^6\tfrac{\beta_9}{6!} +8^6\tfrac{\beta_{10}}{6!}+9^6\tfrac{\beta_{11}}{6!}+[10]^6\tfrac{\beta_{12}}{6!}=\tfrac{34992742353}{5230697472000}, \tag{3.246}$$

$$-2^7\tfrac{\beta_0}{7!}-\tfrac{\beta_1}{7!}+\tfrac{\beta_3}{7!}+2^7\tfrac{\beta_4}{7!}+3^7\tfrac{\beta_5}{7!}+4^7\tfrac{\beta_6}{7!}+5^7\tfrac{\beta_7}{7!}+6^7\tfrac{\beta_5}{7!}+7^7\tfrac{\beta_9}{7!} +8^7\tfrac{\beta_{10}}{7!}+9^7\tfrac{\beta_{11}}{7!}+10^7\tfrac{\beta_{12}}{7!}=\tfrac{327962597}{237758976000}, \tag{3.247}$$

$$2^8\tfrac{\beta_0}{8!}+\tfrac{\beta_1}{8!}+\tfrac{\beta_3}{8!}+2^8\tfrac{\beta_4}{8!}+3^8\tfrac{\beta_5}{8!}+4^8\tfrac{\beta_6}{8!}+5^8\tfrac{\beta_7}{8!}+6^8\tfrac{\beta_5}{8!}+7^8\tfrac{\beta_9}{8!} +8^8\tfrac{\beta_{10}}{8!}+9^8\tfrac{\beta_{11}}{8!}+10^8\tfrac{\beta_{12}}{8!}=\tfrac{881182516553}{1600593426432000}, \tag{3.248}$$

$$-2^9\tfrac{\beta_0}{9!}-\tfrac{\beta_1}{9!}+\tfrac{\beta_3}{9!}+2^9\tfrac{\beta_4}{9!}+3^9\tfrac{\beta_5}{9!}+4^9\tfrac{\beta_6}{9!}+5^9\tfrac{\beta_7}{9!}+6^9\tfrac{\beta_5}{9!}+7^9\tfrac{\beta_9}{9!} +8^9\tfrac{\beta_{10}}{9!}+9^9\tfrac{\beta_{11}}{9!}+10^9\tfrac{\beta_{12}}{9!}=\tfrac{2542651289}{20520428544000}, \tag{3.249}$$

$$2^{10}\frac{\beta_0}{10!}+\frac{\beta_1}{10!}+\frac{\beta_3}{10!}+2^{10}\frac{\beta_4}{10!}+3^{10}\frac{\beta_5}{10!}+4^{10}\frac{\beta_6}{10!}+5^{10}\frac{\beta_7}{10!}+6^{10}\frac{\beta_5}{10!}+7^{10}\frac{\beta_9}{10!}$$
$$+8^{10}\frac{\beta_{10}}{10!}+9^{10}\frac{\beta_{11}}{10!}+10^{10}\frac{\beta_{12}}{10!}=\frac{7404524487683}{202741834014720000}, \tag{3.250}$$

$$-2^{11}\frac{\beta_0}{11!}-\frac{\beta_1}{11!}+\frac{\beta_3}{11!}+2^{11}\frac{\beta_4}{11!}+3^{11}\frac{\beta_5}{11!}+4^{11}\frac{\beta_6}{11!}+5^{11}\frac{\beta_7}{11!}+6^{11}\frac{\beta_5}{11!}+7^{11}\frac{\beta_9}{11!}$$
$$+8^{11}\frac{\beta_{10}}{11!}+9^{11}\frac{\beta_{11}}{11!}+10^{11}\frac{\beta_{12}}{11!}=\frac{2496498203783}{3041127510220800000}, \tag{3.251}$$

$$-2^{12}\frac{\beta_0}{12!}-\frac{\beta_1}{12!}+\frac{\beta_3}{12!}+2^{12}\frac{\beta_4}{12!}+3^{12}\frac{\beta_5}{12!}+4^{12}\frac{\beta_6}{12!}+5^{12}\frac{\beta_7}{12!}+6^{12}\frac{\beta_5}{12!}+7^{12}\frac{\beta_9}{12!}$$
$$+8^{12}\frac{\beta_{10}}{12!}+9^{12}\frac{\beta_{11}}{12!}+10^{12}\frac{\beta_{12}}{12!}=\frac{20863491928843}{104074141460889600000}+\frac{691}{23775897600}, \tag{3.252}$$

the solution of which give the parameters of the second end-point formula (i.e. $x = x_2$) for the twelfth-order method. They are

$$\left.\begin{aligned}
\beta_0 &= \frac{29247693780518024 29}{259384783333294080000},\\
\beta_1 &= \frac{38040711402263632871}{140500090972200960000},\\
\beta_2 &= \frac{17131080986560697393}{51090942171709440000},\\
\beta_3 &= \frac{3543210612727803301}{7663641325756416000},\\
\beta_4 &= \frac{-506142686347310 51}{151380569397657600},\\
\beta_5 &= \frac{1029422989100344339}{2128789257154560000},\\
\beta_6 &= \frac{-17156193648318083 27}{3649353012264960000},\\
\beta_7 &= \frac{736404847800877441}{2128789257154560000},\\
\beta_8 &= \frac{-25143341529404 39}{13278997315584000},\\
\beta_9 &= \frac{11471401959171685 1}{153272826515128320 0},\\
\beta_{10} &= \frac{-10327518393117899 83}{51090942171709440000},\\
\beta_{11} &= \frac{4695203640800939 81}{140500090972200960000},\\
\beta_{12} &= \frac{-8608785996734830 07}{337200218333282304 0000}.
\end{aligned}\right\} \tag{3.253}$$

Next, it can be shown using the method of undetermined coefficients for the point $x = x_3$, that, taking the parameter values c_0, c_2, c_4, c_6, c_8 given in (3.43) together with parameters calculated as follows, ensures the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xxii)}$ in

(3.44) gives

$$\begin{aligned}&\gamma_0+\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6+\gamma_7+\gamma_8+\gamma_9+\gamma_{10}+\gamma_{11}+\gamma_{12}\\&=\tfrac{302231}{302400},\end{aligned}\tag{3.254}$$

$$\begin{aligned}&-3\gamma_0-2\gamma_1-\gamma_2+\gamma_4+2\gamma_5+3\gamma_6+4\gamma_7+5\gamma_8+6\gamma_9+7\gamma_{10}\\&+8\gamma_{11}+9\gamma_{12}=\tfrac{169}{100800},\end{aligned}\tag{3.255}$$

$$\begin{aligned}&3^2\tfrac{\gamma_0}{2!}+2^2\tfrac{\gamma_1}{2!}+\tfrac{\gamma_2}{2!}+\tfrac{\gamma_4}{2!}+2^2\tfrac{\gamma_5}{2!}+3^2\tfrac{\gamma_6}{2!}+4^2\tfrac{\gamma_7}{2!}+5^2\tfrac{\gamma_8}{2!}+6^2\tfrac{\gamma_9}{2!}\\&+7^2\tfrac{\gamma_{10}}{2!}+8^2\tfrac{\gamma_{11}}{2!}+9^2\tfrac{\gamma_{12}}{2!}=\tfrac{5510311}{13305600},\end{aligned}\tag{3.256}$$

$$\begin{aligned}&-3^3\tfrac{\gamma_0}{3!}-2^3\tfrac{\gamma_1}{3!}-\tfrac{\gamma_2}{3!}+\tfrac{\gamma_4}{3!}+2^3\tfrac{\gamma_5}{3!}+3^3\tfrac{\gamma_6}{3!}+4^3\tfrac{\gamma_7}{3!}+5^3\tfrac{\gamma_8}{3!}+6^3\tfrac{\gamma_9}{3!}\\&+7^3\tfrac{\gamma_{10}}{3!}+8^3\tfrac{\gamma_{11}}{3!}+9^3\tfrac{\gamma_{12}}{3!}=\tfrac{11381}{4435200},\end{aligned}\tag{3.257}$$

$$\begin{aligned}&3^4\tfrac{\gamma_0}{4!}+2^4\tfrac{\gamma_1}{4!}+\tfrac{\gamma_2}{4!}+\tfrac{\gamma_4}{4!}+2^4\tfrac{\gamma_5}{4!}+3^4\tfrac{\gamma_6}{4!}+4^4\tfrac{\gamma_7}{4!}+5^4\tfrac{\gamma_8}{4!}+6^4\tfrac{\gamma_9}{4!}\\&+7^4\tfrac{\gamma_{10}}{4!}+8^4\tfrac{\gamma_{11}}{4!}+9^4\tfrac{\gamma_{12}}{4!}=\tfrac{591141643}{7264857600},\end{aligned}\tag{3.258}$$

$$\begin{aligned}&-3^5\tfrac{\gamma_0}{5!}-2^5\tfrac{\gamma_1}{5!}-\tfrac{\gamma_2}{5!}+\tfrac{\gamma_4}{5!}+2^5\tfrac{\gamma_5}{5!}+3^5\tfrac{\gamma_6}{5!}+4^5\tfrac{\gamma_7}{5!}+5^5\tfrac{\gamma_8}{5!}+6^5\tfrac{\gamma_9}{5!}\\&+7^5\tfrac{\gamma_{10}}{5!}+8^5\tfrac{\gamma_{11}}{5!}+9^5\tfrac{\gamma_{12}}{5!}=\tfrac{14645899}{12108096000},\end{aligned}\tag{3.259}$$

$$\begin{aligned}&3^6\tfrac{\gamma_0}{6!}+2^6\tfrac{\gamma_1}{6!}+\tfrac{\gamma_2}{6!}+\tfrac{\gamma_4}{6!}+2^6\tfrac{\gamma_5}{6!}+3^6\tfrac{\gamma_6}{6!}+4^6\tfrac{\gamma_7}{6!}+5^6\tfrac{\gamma_8}{6!}+6^6\tfrac{\gamma_9}{6!}\\&+7^6\tfrac{\gamma_{10}}{6!}+8^6\tfrac{\gamma_{11}}{6!}+9^6\tfrac{\gamma_{12}}{6!}=\tfrac{1346510087}{134120448000},\end{aligned}\tag{3.260}$$

$$\begin{aligned}&-3^7\tfrac{\gamma_0}{7!}-2^7\tfrac{\gamma_1}{7!}-\tfrac{\gamma_2}{7!}+\tfrac{\gamma_4}{7!}+2^7\tfrac{\gamma_5}{7!}+3^7\tfrac{\gamma_6}{7!}+4^7\tfrac{\gamma_7}{7!}+5^7\tfrac{\gamma_8}{7!}+6^7\tfrac{\gamma_9}{7!}\\&+7^7\tfrac{\gamma_{10}}{7!}+8^7\tfrac{\gamma_{11}}{7!}+9^7\tfrac{\gamma_{12}}{7!}=\tfrac{162013909}{581188608000},\end{aligned}\tag{3.261}$$

$$\begin{aligned}&3^8\tfrac{\gamma_0}{8!}+2^8\tfrac{\gamma_1}{8!}+\tfrac{\gamma_2}{8!}+\tfrac{\gamma_4}{8!}+2^8\tfrac{\gamma_5}{8!}+3^8\tfrac{\gamma_6}{8!}+4^8\tfrac{\gamma_7}{8!}+5^8\tfrac{\gamma_8}{8!}+6^8\tfrac{\gamma_9}{8!}\\&+7^8\tfrac{\gamma_{10}}{8!}+8^8\tfrac{\gamma_{11}}{8!}+9^8\tfrac{\gamma_{12}}{8!}=\tfrac{19405166329}{22230464256000},\end{aligned}\tag{3.262}$$

$$\begin{aligned}&-3^9\tfrac{\gamma_0}{9!}-2^9\tfrac{\gamma_1}{9!}-\tfrac{\gamma_2}{9!}+\tfrac{\gamma_4}{9!}+2^9\tfrac{\gamma_5}{9!}+3^9\tfrac{\gamma_6}{9!}+4^9\tfrac{\gamma_7}{9!}+5^9\tfrac{\gamma_8}{9!}+6^9\tfrac{\gamma_9}{9!}\\&+7^9\tfrac{\gamma_{10}}{9!}+8^9\tfrac{\gamma_{11}}{9!}+9^9\tfrac{\gamma_{12}}{9!}=\tfrac{163046441}{4234374144000},\end{aligned}\tag{3.263}$$

$$\begin{aligned}&3^{10}\tfrac{\gamma_0}{10!}+2^{10}\tfrac{\gamma_1}{9!}+\tfrac{\gamma_2}{10!}+\tfrac{\gamma_4}{9!}+2^{10}\tfrac{\gamma_5}{10!}+3^{10}\tfrac{\gamma_6}{10!}+4^{10}\tfrac{\gamma_7}{10!}+5^{10}\tfrac{\gamma_8}{10!}+6^{10}\tfrac{\gamma_{10}}{10!}\\&+7^{10}\tfrac{\gamma_{10}}{10!}+8^{10}\tfrac{\gamma_{11}}{10!}+9^{10}\tfrac{\gamma_{12}}{10!}=\tfrac{5800069899419}{101370917007360000},\end{aligned}\tag{3.264}$$

$$\begin{aligned}&3^{11}\tfrac{\gamma_0}{11!}+2^{11}\tfrac{\gamma_1}{11!}+\tfrac{\gamma_2}{11!}+\tfrac{\gamma_4}{11!}+2^{11}\tfrac{\gamma_5}{11!}+3^{11}\tfrac{\gamma_6}{11!}+4^{11}\tfrac{\gamma_7}{11!}+5^{11}\tfrac{\gamma_8}{11!}+6^{11}\tfrac{\gamma_9}{11!}\\&+7^{11}\tfrac{\gamma_{10}}{11!}+8^{11}\tfrac{\gamma_{11}}{11!}+9^{11}\tfrac{\gamma_{12}}{11!}=\tfrac{847167156811}{236532139683840000},\end{aligned}\tag{3.265}$$

$$3^{12}\frac{\gamma_0}{11!} + 2^{11}\frac{\gamma_1}{12!} + \frac{\gamma_2}{12!} + \frac{\gamma_4}{12!} + 2^{12}\frac{\gamma_5}{12!} + 3^{12!}\frac{\gamma_6}{12!} + 4^{12}\frac{\gamma_7}{12!} + 5^{12}\frac{\gamma_8}{12!} + 6^{12}\frac{\gamma_9}{12!}$$
$$+7^{12}\frac{\gamma_{10}}{12!} + 8^{12}\frac{\gamma_{11}}{12!} + 9^{12}\frac{\gamma_{12}}{12!} = \frac{8172140843813}{275490374455296000} + \frac{691}{23775897600}. \quad (3.266)$$

Solving this system we get the parameters of the third end-point formula (i.e. x = x_3) for the tenth-order method; they are

$$\left.\begin{aligned}
\gamma_0 &= \frac{20464729968383761}{288205314814771200000},\\
\gamma_1 &= \frac{2634978111642113}{64243297198080000},\\
\gamma_2 &= \frac{151708742486039533}{630752372490240000},\\
\gamma_3 &= \frac{1779153894068441}{4074237812736000},\\
\gamma_4 &= \frac{35151347430156497}{151380569397657600},\\
\gamma_5 &= \frac{12903409227110351}{236532139683840000},\\
\gamma_6 &= \frac{-5151286358526083}{405483668029440000},\\
\gamma_7 &= \frac{24720965231391 89}{236532139683840000},\\
\gamma_8 &= \frac{-4355674222843283}{756902846988288000},\\
\gamma_9 &= \frac{15571578934727}{6812125622894592},\\
\gamma_{10} &= \frac{-1172953595222249}{1892257117470720000},\\
\gamma_{11} &= \frac{535052750986283}{5203707073044480000},\\
\gamma_{12} &= \frac{-2951463594930203}{374666909259202560000}.
\end{aligned}\right\} . \quad (3.267)$$

Finally, it can be shown using the method of undetermined coefficients for the point x = x_4 that, taking the parameters d_0, d_2, d_4, d_6, d_8, given in (3.51), calculated below, yields the same first non-zero constant in the local truncation error associated with this point.

Equating the coefficients of the derivatives $y^{(x)}$, $y^{(xi)}, y^{(xii)}, \ldots\ldots, y^{(xxii)}$ in (3.52) gives

$$\delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + \delta_7 + \delta_8 + \delta_9 + \delta_{10} + \delta_{11} + \delta_{12} = \frac{1814399}{1814400}, \quad (3.268)$$

$$-4\delta_0 - 3\delta_1 - 2\delta_2 - \delta_3 + \delta_5 + 2\delta_6 + 3\delta_7 + 4\delta_8 + 5\delta_9 + 6\delta_{10} + 7\delta_{11} + 8\delta_{12} = \frac{122753}{9979200}, \quad (3.269)$$

$$4^2\frac{\delta_0}{2!}+3^2\frac{\delta_1}{2!}+2^2\frac{\delta_2}{2!}+\frac{\delta_3}{2!}+\frac{\delta_5}{2!}+2^2\frac{\delta_6}{2!}+3^2\frac{\delta_7}{2!}+4^2\frac{\delta_8}{2!}+5^2\frac{\delta_9}{2!} +6^2\frac{\delta_{10}}{2!}+7^2\frac{\delta_{11}}{2!}+8^2\frac{\delta_{12}}{2!}=\frac{14255849}{34214400}, \tag{3.270}$$

$$-4^3\frac{\delta_0}{3!}-3^3\frac{\delta_1}{3!}-2^3\frac{\delta_2}{3!}-\frac{\delta_3}{3!}+\frac{\delta_5}{3!}+2^3\frac{\delta_6}{3!}+3^3\frac{\delta_7}{3!}+4^3\frac{\delta_8}{3!}+5^3\frac{\delta_9}{3!} +6^3\frac{\delta_{10}}{3!}+7^3\frac{\delta_{11}}{3!}+8^3\frac{\delta_{12}}{3!}=\frac{68891}{222393600}, \tag{3.271}$$

$$4^4\frac{\delta_0}{4!}+3^4\frac{\delta_1}{4!}+2^4\frac{\delta_2}{4!}+\frac{\delta_3}{4!}+\frac{\delta_5}{4!}+2^4\frac{\delta_6}{4!}+3^4\frac{\delta_7}{4!}+4^4\frac{\delta_8}{4!}+5^4\frac{\delta_9}{4!} +6^4\frac{\delta_{10}}{4!}+7^4\frac{\delta_{11}}{4!}+8^4\frac{\delta_{12}}{4!}=\frac{363217187}{43589145600}, \tag{3.272}$$

$$-4^5\frac{\delta_0}{5!}-3^5\frac{\delta_1}{5!}-2^5\frac{\delta_2}{5!}-\frac{\delta_3}{5!}+\frac{\delta_5}{5!}+2^5\frac{\delta_6}{5!}+3^5\frac{\delta_7}{5!}+4^5\frac{\delta_8}{5!}+5^5\frac{\delta_9}{5!} +6^5\frac{\delta_{10}}{5!}+7^5\frac{\delta_{11}}{5!}+8^5\frac{\delta_{12}}{5!}=\frac{413849}{326918592000}, \tag{3.273}$$

$$4^6\frac{\delta_0}{6!}+3^6\frac{\delta_1}{6!}+2^6\frac{\delta_2}{6!}+\frac{\delta_3}{6!}+\frac{\delta_5}{6!}+2^6\frac{\delta_6}{6!}+3^6\frac{\delta_7}{6!}+4^6\frac{\delta_8}{6!}+5^6\frac{\delta_9}{6!} +6^6\frac{\delta_{10}}{6!}+7^6\frac{\delta_{11}}{6!}+8^6\frac{\delta_{12}}{6!}=\frac{10139471581}{951035904000}, \tag{3.274}$$

$$-4^7\frac{\delta_0}{7!}-3^7\frac{\delta_1}{7!}-2^7\frac{\delta_2}{7!}-\frac{\delta_3}{7!}+\frac{\delta_5}{7!}+2^7\frac{\delta_7}{7!}+3^7\frac{\delta_7}{7!}+4^7\frac{\delta_8}{7!}+5^7\frac{\delta_9}{7!} +6^7\frac{\delta_{10}}{7!}+7^7\frac{\delta_{11}}{7!}+8^7\frac{\delta_{12}}{7!}=\frac{154643851}{88921857024000}, \tag{3.275}$$

$$4^8\frac{\delta_0}{8!}+3^8\frac{\delta_1}{8!}+2^8\frac{\delta_2}{8!}+\frac{\delta_3}{8!}+\frac{\delta_5}{8!}+2^8\frac{\delta_6}{8!}+3^8\frac{\delta_7}{8!}+4^8\frac{\delta_8}{8!}+5^8\frac{\delta_9}{8!} +6^8\frac{\delta_{10}}{8!}+7^8\frac{\delta_{11}}{8!}+8^8\frac{\delta_{12}}{8!}=\frac{3141960414959}{3201186852864000}, \tag{3.276}$$

$$-4^9\frac{\delta_0}{9!}-3^9\frac{\delta_1}{9!}-2^9\frac{\delta_2}{9!}-\frac{\delta_3}{9!}+\frac{\delta_5}{9!}+2^9\frac{\delta_7}{9!}+3^9\frac{\delta_7}{9!}+4^9\frac{\delta_8}{9!}+5^9\frac{\delta_9}{9!} +6^9\frac{\delta_{10}}{9!}+7^9\frac{\delta_{11}}{9!}+8^9\frac{\delta_{12}}{9!}=\frac{4165158373}{10137091700736000}, \tag{3.277}$$

$$4^{10}\frac{\delta_0}{10!}+3^{10}\frac{\delta_1}{10!}+2^{10}\frac{\delta_2}{10!}+\frac{\delta_3}{10!}+\frac{\delta_5}{10!}+2^{10}\frac{\delta_7}{10!}+3^{10}\frac{\delta_7}{10!}+4^{10}\frac{\delta_8}{10!} +5^{10}\frac{\delta_9}{10!}+6^{10}\frac{\delta_{10}}{10!}+7^{10}\frac{\delta_{11}}{10!}+8^{10}\frac{\delta_{12}}{10!}=\frac{28108982850101}{405483668029440000}, \tag{3.278}$$

$$4^{11}\frac{\delta_0}{11!}+3^{11}\frac{\delta_1}{11!}+2^{11}\frac{\delta_2}{11!}+\frac{\delta_3}{11!}+\frac{\delta_5}{11!}+2^{11}\frac{\delta_7}{11!}+3^{11}\frac{\delta_7}{11!}+4^{11}\frac{\delta_8}{11!} +5^{11}\frac{\delta_9}{11!}+6^{11}\frac{\delta_{10}}{11!}+7^{11}\frac{\delta_{11}}{11!}+8^{11}\frac{\delta_{12}}{11!}=\frac{259687418609}{4257578514309120000}, \tag{3.279}$$

$$4^{12}\frac{\delta_0}{12!}+3^{12}\frac{\delta_1}{12!}+2^{12}\frac{\delta_2}{12!}+\frac{\delta_3}{12!}+\frac{\delta_5}{12!}+2^{12}\frac{\delta_7}{12!}+3^{12}\frac{\delta_7}{12!}+4^{12}\frac{\delta_8}{12!} +5^{12}\frac{\delta_9}{12!}+6^{12}\frac{\delta_{10}}{12!}+7^{12}\frac{\delta_{11}}{12!}+8^{12}\frac{\delta_{12}}{12!}=\frac{4984415723143}{1274377242378240000}+\frac{691}{23775897600}. \tag{3.280}$$

Solving this system we get the parameters of the fourth end-point formula (i.e. $x = x_4$) for the tenth-order method.They are

$$\left.\begin{aligned}
\delta_0 &= \frac{-20111634850897253}{6744004366665646080000}, \\
\delta_1 &= \frac{10850134190213011}{739741630111584000}, \\
\delta_2 &= \frac{577222659467368697}{14597412049059840000}, \\
\delta_3 &= \frac{112174364942641021}{450802430926848000}, \\
\delta_4 &= \frac{60105119162462761}{137618699452416000}, \\
\delta_5 &= \frac{480950075796503597}{2128789257154560000}, \\
\delta_6 &= \frac{27866487234499003}{561438924963840000}, \\
\delta_7 &= \frac{-84329739335166 31}{2128789257154560000}, \\
\delta_8 &= \frac{1267316084752801}{504601897992192000}, \\
\delta_9 &= \frac{-978231278605993}{1094805903679488000}, \\
\delta_{10} &= \frac{22859871055603727}{102181884341888 0000}, \\
\delta_{11} &= \frac{-4904760768458891}{14050009097220096 0000}, \\
\delta_{12} &= \frac{1718508104067301 9}{6744004366665646080000}.
\end{aligned}\right\} . \tag{3.281}$$

Because of symmetry, the special end-point formulae for the points $x_N, x_{N-1}, x_{N-2}, x_{N-3}$ may be written down directly from those for x_1, x_2, x_3, x_4, respectively.

The set of parameter values in (3.27), (3.239), (3.35), (3.253), (3.43), (3.267), (3.51) and (3.281) give c_{22} as the first non-zero constant in (3.9). Global extrapolation on two grids, with p=12 in (2.29), gives, using the notation of Chapter 2, the numerical method

$$\mathbf{Y}^{(E)} = \frac{4096}{4095} I^{h}_{\frac{1}{2}h} \mathbf{Y}^{(2)} - \frac{1}{4095} \mathbf{Y}^{(1)}. \tag{3.282}$$

3.10 NUMERICAL RESULTS

To compare the accuracy of the methods developed in this chapter, they were tested on the following problem.

In the computer programs the Gauss-Elimination method with full pivoting for solving linear algebraic systems, was used to obtain the solution vector.

Problem.

$$y^{(x)}(x) = y(x) - (80 + 20x)e^{x}, 0 < x < 1,$$

with boundary conditions

$$\left.\begin{array}{l} y(0) = 0, \quad y''(0) = 0, \quad y^{(iv)}(0) = -8, \\ y^{(vi)}(0) = -24, \quad y^{(viii)}(0) = -48 \\ \qquad\qquad \text{and} \\ y(1) = 0, \quad y^{('')}(1) = -4e, \quad y^{(iv)}(1) = -16, \\ y^{(vi)}(1) = -36e, \quad y^{(viii)}(1) = -64e. \end{array}\right\} \qquad (3.283)$$

The interval $0 \leq x \leq 1$ for the problem was divided in to N+1 equal subintervals each of width $h = 2^{-i}(e^{\frac{1}{2}} - 1)$ for i = 4,5,6. The corresponding values of N are then given by $N = 2^{i} - 1$.

The values of $||\mathbf{y} - \mathbf{Y}||$ were computed for each value of N. The results for the second-, fourth-, sixth-, eighth-, tenth-, and twelfth-order methods are given in Table 3.1. Table 3.2 includes results for the global extrapolation on two grids for all the methods, and on three grids for the second-order method, with N = 15. Table 3.2 shows more improvement after using the extrapolation methods on two grids with N = 15. The global extrapolation on three grids has produced a disappointing result. This is due to small value of h, raised to a large power, having little bearing on the calculation.

Table 3.1: Error norms

N→ / Methods↓	15	31	63
Second-order	0.3331D-04	0.9686D-05	0.3420D-01
Fourth-order	0.3281D-04	0.9942D-05	0.3420D-01
Sixth-order	0.3148D-04	0.9119D-05	0.3420D-01
Eighth-order	0.3162D-04	0.9568d-05	0.3420D-01
Tenth-order	0.3159D-04	0.8836d-05	0.3420D-01
Twelfth-order	0.3282D-04	0.4898D-04	0.3420D-01

Table 3.2: Error norms for the extrapolation on two and three grids

$\frac{N\rightarrow}{Methods\downarrow}$	G_1	Two grids	Three grids
Second-order	0.3331D-04	0.1811D-05	0.6925D-01
Fourth-order	0.3281D-04	0.8418D-05	–
Sixth-order	0.3148D-04	0.8764D-05	–
Eighth-order	0.3162D-04	0.9481D-05	–
Tenth-order	0.3159D-04	0.8814D-05	–
Twelfth-order	0.3282D-04	0.4899D-04	–

Chapter 4

GENERAL TENTH-ORDER LINEAR BOUNDARY-VALUE PROBLEMS

4.1 INTRODUCTION

The general tenth-order two-point boundary-value problems consists of the differential equation

$$\begin{aligned} y^{(x)}(x) = \ & f(x, y(x), y'(x), y''(x), y'''(x), y^{(iv)}(x), y^{(v)}(x), \\ & y^{(vi)}(x), y^{(vii)}(x), y^{(viii)}(x), y^{(ix)}(x)), \end{aligned} \tag{4.1}$$

which holds in some interval $a < x < b$, together with conditions imposed on the dependent variable at the two points $x = a$ and $x = b$. The linear boundary conditions can be written in vector-matrix form as

$$B_a \, \mathbf{Y}(a) + B_b \, \mathbf{Y}(b) = \mathbf{C}, \tag{4.2}$$

where B_a and B_b are two matrices of order 10×10, $\mathbf{Y}(a)$ and$\mathbf{Y}(b)$ are 10×1 vectors defined as

$$\mathbf{Y}(a) = [0, y^{(viii)}(a), 0, y^{(vi)}(a), 0, y^{(iv)}(a), 0, y''(a), 0, y(a)]^T,$$

$$\mathbf{Y}(b) = [0, y^{(viii)}(b), 0, y^{(vi)}(b), 0, y^{(iv)}(b), 0, y''(b), 0, y(b)]^T,$$

and $\mathbf{C}$ is a 10×1 constant vector.

The general boundary-value problem (4.1), (4.2) is linear if f is a linear function of y(x) and its derivatives and is nonlinear otherwise. The linear general problem will be solved in this chapter and the nonlinear problem will be solved in Chapter 5.

4.2 LINEAR TENTH-ORDER BOUNDARY-VALUE PROBLEMS

The general linear tenth-order boundary-value problem consists of the ODE

$$\begin{aligned} y^{(x)}(x) &= \alpha_0(x)y(x) + \alpha_1(x)y'(x) + \alpha_2(x)y''(x) + \alpha_3(x)y'''(x) \\ &\quad + \alpha_4(x)y^{(iv)}(x) + \alpha_5(x)y^{(v)}(x) + \alpha_6(x)y^{(vi)}(x) + \alpha_7(x)y^{(vii)}(x) \qquad (4.3) \\ &\quad + \alpha_8(x)y^{(viii)}(x) + \alpha_9(x)y^{(ix)}(x) + \alpha_{10}(x); \quad a < x < b, \end{aligned}$$

with linear boundary conditions

$$\begin{aligned} y(a) &= A_0, & y(b) &= B_0, \\ y''(a) &= A_2, & y''(b) &= B_2, \\ y^{(iv)}(a) &= A_4, & y^{(iv)}(b) &= B_4, \\ y^{(vi)}(a) &= A_6, & y^{(vi)}(b) &= B_6, \\ y^{(viii)}(a) &= A_8, & y^{(viii)}(b) &= B_8. \end{aligned}$$

Let $w^{(0)} = y(x)$, $w^{(1)} = y'(x)$, $w^{(2)} = y''(x)$,......, $w^{(9)} = y^{(ix)}(x)$. Using this notation, the value of y and its derivatives at the typical mesh point x_n, are given by

$$w_n^{(0)}, \quad w_n^{(1)}, \quad w_n^{(2)}, \quad \ldots, \quad w_n^{(9)}.$$

The single, tenth-order ODE will be transformed to a system of 10 first-order ordinary differential equations; the associated tenth-order boundary-value problem then becomes the following boundary-value problem system (of the first order)

$$(w^{(0)})' = y' = w^{(1)}, \quad a < x < b \ ; \quad w^{(0)}(a) = A_0, \quad w^{(0)}(b) = B_0,$$
$$(w^{(1)})' = y'' = w^{(2)}, \quad a < x < b \ ;$$
$$(w^{(2)})' = y''' = w^{(3)}, \quad a < x < b \ ; \quad w^{(2)}(a) = A_2, \quad w^{(2)}(b) = B_2,$$
$$(w^{(3)})' = y^{(iv)} = w^{(4)}, \quad a < x < b \ ;$$
$$(w^{(4)})' = y^{(v)} = w^{(5)}, \quad a < x < b \ ; \quad w^{(4)}(a) = A_4, \quad w^{(2)}(b) = B_4,$$
$$(w^{(5)})' = y^{(vi)} = w^{(6)}, \quad a < x < b \ ;$$
$$(w^{(6)})' = y^{(vii)} = w^{(7)}, \quad a < x < b \ ; \quad w^{(6)}(a) = A_6, \quad w^{(2)}(b) = B_6,$$
$$(w^{(7)})' = y^{(viii)} = w^{(8)}, \quad a < x < b \ ;$$
$$(w^{(8)})' = y^{(ix)} = w^{(9)}, \quad a < x < b \ ; \quad w^{(8)}(a) = A_8, \quad w^{(2)}(b) = B_8,$$
$$(w^{(9)})' = y^{(x)} = -\alpha_0 w^{(0)} - \alpha_1 w^{(1)} - \alpha_2 w^{(2)} - \alpha_3 w^{(3)} - \alpha_4 w^{(4)} - \alpha_5 w^{(5)}$$
$$-\alpha_6 w^{(6)} - \alpha_7 w^{(7)} - \alpha_8 w^{(8)} - \alpha_9 w^{(9)} - \alpha_{10}$$

These can be written in system form as

$$\mathbf{w}' = \mathrm{A}\mathbf{w} + \mathbf{C}, \tag{4.4}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_6 & -\alpha_7 & -\alpha_8 & -\alpha_9 \end{bmatrix},$$

$$\mathbf{w} = \left[\; w^{(0)}, \; w^{(1)}, \; w^{(2)}, \; w^{(3)}, \; w^{(4)}, \; w^{(5)}, \; w^{(6)}, \; w^{(7)}, \; w^{(8)}, \; w^{(9)} \; \right]^T,$$

and

$$\mathbf{C} = \left[\; 0, \; 0, \; 0, \; 0, \; 0, \; 0, \; 0, \; 0, \; 0, \; -\alpha_{10} \; \right]^T.$$

After transfoming to a first-order system, any numerical method will determine, at every point $x_1, x_2, x_3, \ldots\ldots, x_N$ of the grid not only the value of y but also its first nine derivatives so, a total of ten bits of information will be calculated at each mesh point $x_1, x_2, x_3, \ldots\ldots, x_N$.

Note also that at the boundary x=a and at the boundary x=b $y', y''', y^{(v)}, y^{(vii)}, y^{(ix)}$ (a total of 10) are not given. The numerical method will find these bits of information also (these are $w^{(1)}(a) = w_0^{(1)}$, $w^{(3)}(a) = w_0^{(3)}$, $w^{(5)}(a) = w_0^{(5)}$, $w^{(7)}(a) = w_0^{(7)}$, $w^{(9)}(a) = w_0^{(9)}$; $w_{N+1}^{(1)}$, $w_{N+1}^{(3)}$, $w_{N+1}^{(5)}$, $w_{N+1}^{(7)}$, $w_{N+1}^{(9)}$). Hence, the total number of bits of information to be found is

$$5 + 10N + 5 = 10(N + 1).$$

We apply the system of ten first-order ordinary differential equations to the points x_0, x_1, x_2, $\ldots\ldots$, x_N of the grid. Thus, ten ordinary differential equations are applied to N+1 mesh points; this total of 10(N+1) bits of information will give the 10(N+1) bits of information required.

Let $\mathbf{w}(x) = [w^{(0)}(x), w^{(1)}(x), w^{(2)}(x), \ldots\ldots, w^{(9)}(x)]^T$. Then, the system (4.1) is of the form

$$D\mathbf{w}(x) \equiv \mathbf{w}'(x) = A\mathbf{w}(x) + \mathbf{C}. \tag{4.5}$$

This may be solved using the recurrence relation

$$\mathbf{w}(x+h) = \exp(hD)\mathbf{w}(x). \tag{4.6}$$

In (4.5) and (4.6)

$$D = \begin{bmatrix} \frac{d}{dx} & & & & & & & & & \\ & \frac{d}{dx} & & & & & & & & \\ & & \frac{d}{dx} & & & & & & & \\ & & & \frac{d}{dx} & & & & & & \\ & & & & \frac{d}{dx} & & & & & \\ & & & & & \frac{d}{dx} & & & & \\ & & & & & & \frac{d}{dx} & & & \\ & & & & & & & \frac{d}{dx} & & \\ & & & & & & & & \frac{d}{dx} & \\ & & & & & & & & & \frac{d}{dx} \end{bmatrix} = \mathrm{diag}\{\frac{d}{dx}\},$$

is a matrix of order 10.

4.3 NUMERICAL METHODS

To obtain the solution $\mathbf{w}$, recurrence relation (4.6) is applied to the (N+1) mesh points $x_0, x_1, x_2, \ldots\ldots, x_N$.

Suppose that exp(hD) in (4.6) is replaced by its (1,1) Padé approximant $(I - \frac{1}{2}hD)^{-1}(I + \frac{1}{2}hD)$ where I is the identity matrix of order 10. This gives

$$\mathbf{w}(x+h) = (I - \frac{1}{2}hD)^{-1}(I + \frac{1}{2}hD)\mathbf{w}(x)$$

i.e.

$$(I - \frac{1}{2}hD)\mathbf{w}(x+h) = (I + \frac{1}{2}hD)\mathbf{w}(x). \tag{4.7}$$

The use of this Padé approximant, which is a second-order replacement of the exponential function exp(hD), gives rise to a second-order method for solving the boundary-value problem. Now

$$D\mathbf{w}(x) = A\mathbf{w}(x) + \mathbf{C}$$

and so

$$D\mathbf{w}(x+h) = A\mathbf{w}(x+h) + \mathbf{C}$$

Let us now generalize and suppose that α_0 , α_1 , α_2 , α_3 , α_4 , α_5 , α_6 , α_7 , α_8 , α_9, α_{10} are all functions of x. Then

$$D\mathbf{w}(x) = A\mathbf{w}(x) + \mathbf{C}(x)$$

and

$$D\mathbf{w}(x+h) = A\mathbf{w}(x+h) + \mathbf{C}(x+h)$$

This gives, in (4.7),

$$I\mathbf{w}(x+h) - \frac{1}{2}h[A(x+h)\mathbf{w}(x+h) + \mathbf{C}(x+h)] = I\mathbf{w}(x) + \frac{1}{2}h[A(x)\mathbf{w}(x) + \mathbf{C}(x)]$$

which implies that

$$[I - \frac{1}{2}hA(x+h)]\mathbf{w}(x+h) = [I + \frac{1}{2}hA(x)]\mathbf{w}(x+h) + \frac{1}{2}h[\mathbf{C}(x+h) + \mathbf{C}(x)]$$

i.e.

$$P(x+h)\mathbf{w}(x+h) - \frac{1}{2}h[C(x+h) + \mathbf{C}(x)] = Q(x)\mathbf{w}(x)$$

or

$$Q(x)\mathbf{w}(x) + \frac{1}{2}h\mathbf{C}(x) = P(x+h)\mathbf{w}(x+h) - \frac{1}{2}h\mathbf{C}(x+h), \qquad (4.8)$$

with $x = x_0\ ,\ x_1\ ,\ x_2\ ,\dots\ ,\dots\ ,x_N$. In (4.8)

$$P(x+h) = I - \frac{1}{2}hA(x+h).$$

and

$$Q(x) = I + \frac{1}{2}hA(x)$$

so that

$$P(x+h) = \begin{bmatrix} 1 & -\frac{1}{2}h & & & & & & & & \\ & 1 & -\frac{1}{2}h & & & & & & & \\ & & 1 & -\frac{1}{2}h & & & & & & \\ & & & 1 & -\frac{1}{2}h & & & & & \\ & & & & 1 & -\frac{1}{2}h & & & & \\ & & & & & 1 & -\frac{1}{2}h & & & \\ & & & & & & 1 & -\frac{1}{2}h & & \\ & & & & & & & 1 & -\frac{1}{2}h & \\ & & & & & & & & 1 & -\frac{1}{2}h \\ a & b & c & d & e & f & g & j & k & t \end{bmatrix},$$

with

$a = \frac{1}{2}h\alpha_0(x+h),\ b = \frac{1}{2}h\alpha_1(x+h),\ c = \frac{1}{2}h\alpha_2(x+h),\ d = \frac{1}{2}h\alpha_3(x+h).$

$e = \frac{1}{2}h\alpha_4(x+h)$, $f = \frac{1}{2}h\alpha_5(x+h)$, $g = \frac{1}{2}h\alpha_6(x+h)$, $j = \frac{1}{2}h\alpha_7(x+h)$, $k = \frac{1}{2}h\alpha_8(x+h)$, $t = 1 + \frac{1}{2}h\alpha_9(x+h)$ and

$$\mathbf{w}(x+h) = \left[\ w^{(0)}(x+h),\ \ w^{(1)}(x+h),\ \ w^{(2)}(x+h),\ \ \ldots\ \ \ldots,\ \ w^{(9)}(x+h)\ \right]^T.$$

Similarly

$$Q(x) = \begin{bmatrix} 1 & \frac{1}{2}h & & & & & & & & \\ & 1 & \frac{1}{2}h & & & & & & & \\ & & 1 & \frac{1}{2}h & & & & & & \\ & & & 1 & \frac{1}{2}h & & & & & \\ & & & & 1 & \frac{1}{2}h & & & & \\ & & & & & 1 & \frac{1}{2}h & & & \\ & & & & & & 1 & \frac{1}{2}h & & \\ & & & & & & & 1 & \frac{1}{2}h & \\ & & & & & & & & 1 & \frac{1}{2}h \\ k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & 1-k_9 \end{bmatrix}$$

where

$k_0 = -\frac{1}{2}h\alpha_0(x+h)$, $k_1 = \frac{1}{2}h\alpha_1(x+h)$, $k_2 = -\frac{1}{2}h\alpha_2(x+h)$, $k_3 = \frac{1}{2}h\alpha_3(x+h)$, $k_4 = -\frac{1}{2}h\alpha_4(x+h)$, $k_5 = \frac{1}{2}h\alpha_5(x+h)$, $k_6 = -\frac{1}{2}h\alpha_6(x+h)$, $k_7 = \frac{1}{2}h\alpha_7(x+h)$, $k_8 = -\frac{1}{2}h\alpha_8(x+h)$, $k_9 = 1 - \frac{1}{2}h\alpha_9(x+h)$ and

$$\mathbf{w}(x) = \left[\ w^{(0)}(x),\ \ w^{(1)}(x),\ \ w^{(2)}(x),\ \ \ldots\ \ \ldots,\ \ w^{(9)}(x)\ \right]^T.$$

Consider (4.8) at $x = x_0$; it becomes

$$Q(x_0)\mathbf{w}(x_0) + \frac{1}{2}h\mathbf{C}(x_0) = P(x_1+h)\mathbf{w}(x_1+h) - \frac{1}{2}h\mathbf{C}(x_1+h). \tag{4.9}$$

We rename this equation as follows

$$Q_0\mathbf{w}_0 + \mathbf{r}_0 = P_1\mathbf{w}_1 - \mathbf{r}_1, \tag{4.10}$$

which gives

$$\begin{bmatrix} 1 & z & & & & & & & & \\ & 1 & z & & & & & & & \\ & & 1 & z & & & & & & \\ & & & 1 & z & & & & & \\ & & & & 1 & z & & & & \\ & & & & & 1 & z & & & \\ & & & & & & 1 & z & & \\ & & & & & & & 1 & z & \\ & & & & & & & & 1 & z \\ k_0 & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & 1-k_9 \end{bmatrix} \begin{bmatrix} A_0 \\ w^{(1)}(x_0) \\ A_2 \\ w^{(3)}(x_0) \\ A_4 \\ w^{(5)}(x_0) \\ A_6 \\ w^{(7)}(x_0) \\ A_8 \\ w^{(9)}(x_0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k_{10} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -z & & & & & & & & \\ & 1 & -z & & & & & & & \\ & & 1 & -z & & & & & & \\ & & & 1 & -z & & & & & \\ & & & & 1 & -z & & & & \\ & & & & & 1 & -z & & & \\ & & & & & & 1 & -z & & \\ & & & & & & & 1 & -z & \\ & & & & & & & & 1 & -z \\ d_0 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 & 1+d_9 \end{bmatrix} \begin{bmatrix} w^{(0)}(x_1) \\ w^{(1)}(x_1) \\ w^{(2)}(x_1) \\ w^{(3)}(x_1) \\ w^{(4)}(x_1) \\ w^{(5)}(x_1) \\ w^{(6)}(x_1) \\ w^{(7)}(x_1) \\ w^{(8)}(x_1) \\ w^{(9)}(x_1) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ d_{10} \end{bmatrix}.$$

with

$z = \frac{1}{2}h,\ k_0 = -\frac{1}{2}h\alpha_0(x_0),\ k_1 = -\frac{1}{2}h\alpha_1(x_0),\ k_2 = -\frac{1}{2}h\alpha_2(x_0),\ k_3 = -\frac{1}{2}h\alpha_3(x_0),$
$k_4 = -\frac{1}{2}h\alpha_4(x_0),\ k_5 = -\frac{1}{2}h\alpha_5(x_0),\ k_6 = -\frac{1}{2}h\alpha_6(x_0),\ k_7 = -\frac{1}{2}h\alpha_7(x_0),$
$k_8 = -\frac{1}{2}h\alpha_8(x_0),\ k_9 = 1 - \frac{1}{2}h\alpha_9(x_0),\ k_{10} = -\frac{1}{2}h\alpha_{10}(x_0);\ d_0 = \frac{1}{2}h\alpha_0(x_1),$
$d_1 = \frac{1}{2}h\alpha_1(x_1),\ d_2 = \frac{1}{2}h\alpha_2(x_1),\ d_3 = \frac{1}{2}h\alpha_3(x_1),\ d_4 = \frac{1}{2}h\alpha_4(x_1),$
$d_5 = \frac{1}{2}h\alpha_5(x_1),\ d_6 = \frac{1}{2}h\alpha_6(x_1),\ d_7 = \frac{1}{2}h\alpha_7(x_1),\ d_8 = \frac{1}{2}h\alpha_8(x_1),$
$d_9 = 1 + \frac{1}{2}h\alpha_9(x_1),\ d_{10} = -\frac{1}{2}h\alpha_{10}(x_1).$

For the point $x = x_n$ (with $n = 1, 2, 3, \ldots\ \ldots, N-1$) equation (4.8) becomes

$$Q(x_n)\mathbf{w}(x_n) + \frac{1}{2}h\mathbf{C}(x_n) = P(x_n + h)\mathbf{w}(x_n + h) - \frac{1}{2}h\mathbf{C}(x_n + h). \tag{4.11}$$

We write this equation as follows

$$Q_n\mathbf{w}_n + \mathbf{r}_n = P_{n+1}\mathbf{w}_{n+1} - \mathbf{r}_{n+1}, \tag{4.12}$$

which gives

$$\begin{bmatrix} 1 & z & & & & & & & & \\ & 1 & z & & & & & & & \\ & & 1 & z & & & & & & \\ & & & 1 & z & & & & & \\ & & & & 1 & z & & & & \\ & & & & & 1 & z & & & \\ & & & & & & 1 & z & & \\ & & & & & & & 1 & z & \\ & & & & & & & & 1 & z \\ e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{bmatrix} \begin{bmatrix} w^{(0)}(x_n) \\ w^{(1)}(x_n) \\ w^{(2)}(x_n) \\ w^{(3)}(x_n) \\ w^{(4)}(x_n) \\ w^{(5)}(x_n) \\ w^{(6)}(x_n) \\ w^{(7)}(x_n) \\ w^{(8)}(x_n) \\ w^{(9)}(x_n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ e_{10} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -z & & & & & & & & \\ & 1 & -z & & & & & & & \\ & & 1 & -z & & & & & & \\ & & & 1 & -z & & & & & \\ & & & & 1 & -z & & & & \\ & & & & & 1 & -z & & & \\ & & & & & & 1 & -z & & \\ & & & & & & & 1 & -z & \\ & & & & & & & & 1 & -z \\ c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 \end{bmatrix} \begin{bmatrix} w^{(0)}(x_{n+1}) \\ w^{(1)}(x_{n+1}) \\ w^{(2)}(x_{n+1}) \\ w^{(3)}(x_{n+1}) \\ w^{(4)}(x_{n+1}) \\ w^{(5)}(x_{n+1}) \\ w^{(6)}(x_{n+1}) \\ w^{(7)}(x_{n+1}) \\ w^{(8)}(x_{n+1}) \\ w^{(9)}(x_{n+1}) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c_{10} \end{bmatrix}.$$

with

$z = \frac{1}{2}h$, $e_0 = -\frac{1}{2}h\alpha_0(x_n)$, $e_1 = -\frac{1}{2}h\alpha_1(x_n)$, $e_2 = -\frac{1}{2}h\alpha_2(x_n)$, $e_3 = -\frac{1}{2}h\alpha_3(x_n)$,

$e_4 = -\frac{1}{2}h\alpha_4(x_n),\ e_5 = -\frac{1}{2}h\alpha_5(x_n),\ e_6 = -\frac{1}{2}h\alpha_6(x_n),\ e_7 = -\frac{1}{2}h\alpha_7(x_n),$
$e_8 = -\frac{1}{2}h\alpha_6(x_n),\ e_9 = 1 - \frac{1}{2}h\alpha_7(x_n),\ e_{10} = -\frac{1}{2}h\alpha_{10}(x_n);$
$c_0 = \frac{1}{2}h\alpha_0(x_{n+1}),\ c_1 = \frac{1}{2}h\alpha_1(x_{n+1}),\ c_2 = \frac{1}{2}h\alpha_2(x_{n+1}),\ c_3 = \frac{1}{2}h\alpha_3(x_{n+1}),$
$c_4 = \frac{1}{2}h\alpha_4(x_{n+1}),\ c_5 = \frac{1}{2}h\alpha_5(x_{n+1}),\ c_6 = \frac{1}{2}h\alpha_6(x_{n+1}),\ c_7 = \frac{1}{2}h\alpha_7(x_{n+1}),$
$c_8 = \frac{1}{2}h\alpha_8(x_{n+1}),\ c_9 = 1 + \frac{1}{2}h\alpha_9(x_{n+1}),\ c_{10} = -\frac{1}{2}h\alpha_{10}(x_{n+1})$

Finally for the point $x = x_N$ equation (4.8) becomes as

$$Q(x_N)\mathbf{w}(x_N) + \frac{1}{2}h\mathbf{C}(x_N) = P(x_{N+1})\mathbf{w}(x_{N+1}) - \frac{1}{2}h\mathbf{C}(x_{N+1}), \tag{4.13}$$

which we rewrite as

$$Q_N\mathbf{w}_N + \mathbf{r}_N = P_{N+1}\mathbf{w}_{N+1} - \mathbf{r}_{N+1}, \tag{4.14}$$

giving

$$\begin{bmatrix} 1 & z & & & & & & & & \\ & 1 & z & & & & & & & \\ & & 1 & z & & & & & & \\ & & & 1 & z & & & & & \\ & & & & 1 & z & & & & \\ & & & & & 1 & z & & & \\ & & & & & & 1 & z & & \\ & & & & & & & 1 & z & \\ & & & & & & & & 1 & z \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{bmatrix} \begin{bmatrix} w^{(0)}(x_N) \\ w^{(1)}(x_N) \\ w^{(2)}(x_N) \\ w^{(3)}(x_N) \\ w^{(4)}(x_N) \\ w^{(5)}(x_N) \\ w^{(6)}(x_N) \\ w^{(7)}(x_N) \\ w^{(8)}(x_N) \\ w^{(9)}(x_N) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_{10} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -z & & & & & & & & \\ & 1 & -z & & & & & & & \\ & & 1 & -z & & & & & & \\ & & & 1 & -z & & & & & \\ & & & & 1 & -z & & & & \\ & & & & & 1 & -z & & & \\ & & & & & & 1 & -z & & \\ & & & & & & & 1 & -z & \\ & & & & & & & & 1 & -z \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 \end{bmatrix} \begin{bmatrix} B_0 \\ w^{(1)}(x_{N+1}) \\ B_2 \\ w^{(3)}(x_{N+1}) \\ B_4 \\ w^{(5)}(x_{N+1}) \\ B_6 \\ w^{(7)}(x_{N+1}) \\ B_8 \\ w^{(9)}(x_{N+1}) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ b_{10} \end{bmatrix},$$

with

$z = \frac{1}{2}h,\ a_0 = -\frac{1}{2}h\alpha_0(x_N),\ a_1 = -\frac{1}{2}h\alpha_1(x_N),\ a_2 = -\frac{1}{2}h\alpha_2(x_N),$

$a_3 = -\frac{1}{2}h\alpha_3(x_N),\ a_4 = -\frac{1}{2}h\alpha_4(x_N),\ a_5 = -\frac{1}{2}h\alpha_5(x_N),\ a_6 = -\frac{1}{2}h\alpha_6(x_N),$

$a_7 = -\frac{1}{2}h\alpha_7(x_N),\ a_8 = -\frac{1}{2}h\alpha_6(x_N),\ a_9 = 1 - \frac{1}{2}h\alpha_7(x_N),\ a_{10} = -\frac{1}{2}h\alpha_{10}(x_N);$

$b_0 = \frac{1}{2}h\alpha_0(x_{N+1}),\ b_1 = \frac{1}{2}h\alpha_1(x_{N+1}),\ b_2 = \frac{1}{2}h\alpha_2(x_{N+1}),\ b_3 = \frac{1}{2}h\alpha_3(x_{N+1}),$

$b_4 = \frac{1}{2}h\alpha_4(x_{N+1}),\ b_5 = \frac{1}{2}h\alpha_5(x_{N+1}),\ b_6 = \frac{1}{2}h\alpha_6(x_{N+1}),\ b_7 = \frac{1}{2}h\alpha_7(x_{N+1}),$

$b_8 = \frac{1}{2}h\alpha_8(x_{N+1}),\ b_9 = 1 + \frac{1}{2}h\alpha_9(x_{N+1}),\ b_{10} = -\frac{1}{2}h\alpha_{10}(x_{N+1})$

Recalling

$$P(x+h)\mathbf{w}(x+h) - \mathbf{r}(x+h) = Q(x)\mathbf{w}(x) + \mathbf{r}(x)$$

i.e.

$$P_{m+1}\mathbf{w}_{m+1} - \mathbf{r}_{m+1} = Q_m\mathbf{w}_m + \mathbf{r}_m, \tag{4.15}$$

gives

$$P_1\mathbf{w}_1 - \mathbf{r}_1 = Q_0\mathbf{w}_0 + \mathbf{r}_0, \tag{4.16}$$

for m=0 so that

$$w_1^{(0)} - \frac{1}{2}hw_1^{(1)} = A_0 + \frac{1}{2}hw_0^{(1)} \tag{4.17}$$

$$w_1^{(1)} - \frac{1}{2}hw_1^{(2)} = w_0^{(1)} + \frac{1}{2}hA_2 \tag{4.18}$$

$$w_1^{(2)} - \frac{1}{2}hw_1^{(3)} = A_2 + \frac{1}{2}hw_0^{(3)} \tag{4.19}$$

$$w_1^{(3)} - \frac{1}{2}hw_1^{(4)} = w_0^{(3)} + \frac{1}{2}hA_4 \tag{4.20}$$

$$w_1^{(4)} - \frac{1}{2}hw_1^{(5)} = A_4 + \frac{1}{2}hw_0^{(5)} \tag{4.21}$$

$$w_1^{(5)} - \frac{1}{2}hw_1^{(6)} = w_0^{(5)} + \frac{1}{2}hA_6 \tag{4.22}$$

$$w_1^{(6)} - \frac{1}{2}hw_1^{(7)} = A_6 + \frac{1}{2}hw_0^{(7)} \tag{4.23}$$

$$w_1^{(7)} - \frac{1}{2}hw_1^{(8)} = w_0^{(7)} + \frac{1}{2}hA_8 \tag{4.24}$$

$$w_1^{(8)} - \frac{1}{2}hw_1^{(9)} = A_8 + \frac{1}{2}hw_0^{(9)} \tag{4.25}$$

$$\begin{aligned}
&\tfrac{1}{2}h\alpha_0(x_1)w_1^{(0)} + \tfrac{1}{2}h\alpha_1(x_1)w_1^{(1)} + \tfrac{1}{2}h\alpha_2(x_1)w_1^{(2)} + \tfrac{1}{2}h\alpha_3(x_1)w_1^{(3)} \\
&+\tfrac{1}{2}h\alpha_4(x_1)w_1^{(4)} + \tfrac{1}{2}h\alpha_5(x_1)w_1^{(5)} + \tfrac{1}{2}h\alpha_6(x_1)w_1^{(6)} + \tfrac{1}{2}h\alpha_7(x_1)w_1^{(7)} \\
&+\tfrac{1}{2}h\alpha_8(x_1)w_1^{(8)} + [1 + \tfrac{1}{2}h\alpha_9(x_1)]w_1^{(9)} + \tfrac{1}{2}h\alpha_{10}(x_1) \\
&= -\tfrac{1}{2}h\alpha_0(x_0)A_0 - \tfrac{1}{2}h\alpha_1(x_0)w_0^{(1)} - \tfrac{1}{2}h\alpha_2(x_0)A_2 - \tfrac{1}{2}h\alpha_3(x_0)w_0^{(3)} \\
&-\tfrac{1}{2}h\alpha_4(x_0)A_4 - \tfrac{1}{2}h\alpha_5(x_0)w_0^{(5)} - \tfrac{1}{2}h\alpha_6(x_0)A_6 - \tfrac{1}{2}h\alpha_7(x_0)w_0^{(7)} \\
&-\tfrac{1}{2}h\alpha_8(x_0)A_8 + [1 - \tfrac{1}{2}h\alpha_9(x_0)]w_0^{(9)} - \tfrac{1}{2}\alpha_{10}(x_0)
\end{aligned} \tag{4.26}$$

Now, using (4.17)—(4.26), we develop a vector as follows

$$S_1\mathbf{W} + \mathbf{U},$$

where

$$S_1 = \begin{bmatrix} \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 \\ t_0 & t_1 & t_2 & t_3 & t_4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{W} = \begin{bmatrix} w^{(1)}(x_0) \\ w^{(3)}(x_0) \\ w^{(5)}(x_0) \\ w^{(7)}(x_0) \\ w^{(9)}(x_0) \\ w^{(1)}(x_{N+1}) \\ w^{(3)}(x_{N+1}) \\ w^{(5)}(x_{N+1}) \\ w^{(7)}(x_{N+1}) \\ w^{(9)}(x_{N+1}) \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} A_0 \\ \frac{1}{2}hA_2 \\ A_2 \\ \frac{1}{2}hA_4 \\ A_4 \\ \frac{1}{2}hA_6 \\ A_6 \\ \frac{1}{2}hA_8 \\ A_8 \\ c* \end{bmatrix},$$

with

$t_0 = -\frac{1}{2}h\alpha_1(x_0)$, $t_1 = -\frac{1}{2}h\alpha_3(x_0)$, $t_1 = -\frac{1}{2}h\alpha_5(x_0)$, $t_3 = -\frac{1}{2}h\alpha_7(x_0)$,

$t_4 = 1 - \frac{1}{2}h\alpha_9(x_0)$ and

$$c* = -\frac{1}{2}h\alpha_0(x_0)A_0 - \frac{1}{2}h\alpha_2(x_0)A_2 - \frac{1}{2}h\alpha_4(x_0)A_4 - \frac{1}{2}h\alpha_6(x_0)A_6$$
$$-\frac{1}{2}h\alpha_8(x_0)A_8 - \frac{1}{2}h\alpha_{10}(x_0).$$

Thus, now, (4.16) for m=0 reduces

$$P_1\mathbf{w}_1 - S_1\mathbf{W} = \mathbf{r}_1 + \mathbf{U} \tag{4.27}$$

i.e.

$$-P_1\mathbf{w}_1 + S_1\mathbf{W} = -\mathbf{U} - \mathbf{r}_1. \tag{4.28}$$

For the point x=N the formula is

$$Q_N\mathbf{w}_N + \mathbf{r}_N = P_{N+1}\mathbf{w}_{N+1} - \mathbf{r}_{N+1}, \tag{4.29}$$

which gives

$$w_N^{(0)} + \frac{1}{2}hw_N^{(1)} = B_0 - \frac{1}{2}hw_{N+1}^{(1)} \tag{4.30}$$

$$w_N^{(1)} + \frac{1}{2}hw_N^{(2)} = w_{N+1}^{(1)} - \frac{1}{2}hB_2 \tag{4.31}$$

$$w_N^{(2)} + \frac{1}{2}hw_N^{(3)} = B_2 - \frac{1}{2}hw_{N+1}^{(3)} \tag{4.32}$$

$$w_N^{(3)} + \frac{1}{2}hw_N^{(4)} = w_{N+1}^{(3)} - \frac{1}{2}hB_4 \tag{4.33}$$

$$w_N^{(4)} + \frac{1}{2}hw_N^{(5)} = B_4 - \frac{1}{2}hw_{N+1}^{(5)} \tag{4.34}$$

$$w_N^{(5)} + \frac{1}{2}hw_N^{(6)} = w_{N+1}^{(5)} - \frac{1}{2}hB_6 \tag{4.35}$$

$$w_N^{(6)} + \frac{1}{2}hw_N^{(7)} = B_6 - \frac{1}{2}hw_{N+1}^{(7)} \tag{4.36}$$

$$w_N^{(7)} + \frac{1}{2}hw_N^{(8)} = w_{N+1}^{(7)} - \frac{1}{2}hB_8 \tag{4.37}$$

$$w_N^{(8)} + \frac{1}{2}hw_N^{(9)} = B_8 - \frac{1}{2}hw_{N+1}^{(9)} \tag{4.38}$$

$$\begin{aligned}
&-\tfrac{1}{2}h\alpha_0(x_N)w_N^{(0)} - \tfrac{1}{2}h\alpha_1(x_N)w_N^{(1)} - \tfrac{1}{2}h\alpha_2(x_N)w_N^{(2)} - \tfrac{1}{2}h\alpha_3(x_N)w_N^{(3)}\\
&-\tfrac{1}{2}h\alpha_4(x_N)w_N^{(4)} - \tfrac{1}{2}h\alpha_5(x_N)w_N^{(5)} - \tfrac{1}{2}h\alpha_6(x_N)w_N^{(6)} - \tfrac{1}{2}h\alpha_7(x_N)w_N^{(7)}\\
&-\tfrac{1}{2}h\alpha_8(x_N)w_N^{(8)} + [1 - \tfrac{1}{2}h\alpha_9(x_N)]w_N^{(9)} - \tfrac{1}{2}h\alpha_{10}(x_N)\\
&= \tfrac{1}{2}h\alpha_0(x_{N+1})B_0 + \tfrac{1}{2}h\alpha_1(x_{N+1})w_{N+1}^{(1)} + \tfrac{1}{2}h\alpha_2(x_{N+1})B_2 + \tfrac{1}{2}h\alpha_3(x_{N+1})w_{N+1}^{(3)}\\
&+\tfrac{1}{2}h\alpha_4(x_{N+1})B_4 + \tfrac{1}{2}h\alpha_5(x_{N+1})w_{N+1}^{(5)} + \tfrac{1}{2}h\alpha_6(x_{N+1})B_6 + \tfrac{1}{2}h\alpha_7(x_{N+1})w_{N+1}^{(7)}\\
&+\tfrac{1}{2}h\alpha_8(x_{N+1})B_8 + [1 + \tfrac{1}{2}h\alpha_9(x_{N+1})]w_{N+1}^{(9)} + \tfrac{1}{2}\alpha_{10}(x_{N+1})
\end{aligned} \tag{4.39}$$

Using (4.27)—(4.36), we develop a vector as follows

$$S_{N+1}\mathbf{W} + \mathbf{V} ,$$

where

$$S_{N+1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h \\ 0 & 0 & 0 & 0 & 0 & j_0 & j_1 & j_2 & j_3 & j_4 \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} B_0, & -\frac{1}{2}hA_2, & B_2, & -\frac{1}{2}hA_4, & B_4, & -\frac{1}{2}hA_6, & B_6, & -\frac{1}{2}hA_8, & B_8, & k^* \end{bmatrix}^T,$$

with

$j_0 = -\frac{1}{2}h\alpha_1(x_0)$, $j_1 = -\frac{1}{2}h\alpha_3(x_0)$, $j_2 = -\frac{1}{2}h\alpha_5(x_0)$, $j_3 = -\frac{1}{2}h\alpha_7(x_0)$,

$j_4 = 1 + \frac{1}{2}h\alpha_9$;

$k^* = \frac{1}{2}h\alpha_0(x_0) + \frac{1}{2}h\alpha_2(x_0) + \frac{1}{2}h\alpha_4(x_0) + \frac{1}{2}h\alpha_6(x_0) + \frac{1}{2}h\alpha_8(x_0) + \frac{1}{2}h\alpha_{10}(x_0)$

and $\mathbf{W}$ is already defined. Remember that

$$\mathbf{r} = [0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -\frac{1}{2}h\alpha_{10}(x_1)]^T.$$

For the general mesh points $m = 1, 2, 3, \dots\ \dots, N-1$.

$$Q_m\mathbf{w}_m - P_{m+1}\mathbf{w}_{m+1} = -\mathbf{r}_m - \mathbf{r}_{m+1} , \tag{4.40}$$

with

$$-\mathbf{r}_m - \mathbf{r}_{m+1} = [0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -\frac{1}{2}h\{\alpha_{10}(x_m) + \alpha_{10}(x_{m+1})\}]^T$$

Lastly, for m=N

$$Q_N w_N - S_{N+1}\mathbf{W} = -\mathbf{r}_N + \mathbf{V}. \tag{4.41}$$

Recall,

$P_1, P_2, \ldots\ldots, P_m, P_{m+1}, \ldots, P_N;\ Q_1, Q_1, \ldots\ldots, Q_m, Q_{m+1}, \ldots, Q_N;\ S_1,\ S_{N+1}$ are all 10×10 matrices and the vectors $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3, \ldots\ldots, \mathbf{w}_N,\ \mathbf{W};\ \mathbf{U},\ \mathbf{V};$ $\mathbf{r}_1,\ \mathbf{r}_2, \ldots\ldots\ \mathbf{r}_N$ are all 10×1 vectors.

Using (4.12), (4.15), (4.28), (4.40), (4.41) and on rearranging we have the block matrix-vector product

$$\begin{bmatrix} -P_1 & & & & & & & & S_1 \\ Q_1 & -P_2 & & & & & & & \\ & Q_2 & -P_3 & & & & & & \\ & & Q_3 & -P_4 & & & & & \\ & & & \ddots & \ddots & & & & \\ & & & & Q_m & P_{m+1} & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & & Q_N & -S_{N+1} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \vdots \\ \mathbf{w}_m \\ \mathbf{w}_{m+1} \\ \vdots \\ \mathbf{w}_N \\ \mathbf{W} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \vdots \\ \mathbf{b}_m \\ \mathbf{b}_{m+1} \\ \vdots \\ \mathbf{b}_N \\ \mathbf{b}_{N+1} \end{bmatrix}, \tag{4.42}$$

where

$$\begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \mathbf{b}_4 \\ \vdots \\ \mathbf{b}_{m+1} \\ \vdots \\ \mathbf{b}_N \\ \mathbf{b}_{N+1} \end{bmatrix} = \begin{bmatrix} -\mathbf{U} - \mathbf{r}_1 \\ -\mathbf{r}_1 - \mathbf{r}_2 \\ -\mathbf{r}_2 - \mathbf{r}_3 \\ -\mathbf{r}_3 - \mathbf{r}_4 \\ \vdots \\ -\mathbf{r}_m - \mathbf{r}_{m+1} \\ \vdots \\ -\mathbf{r}_{N-1} - \mathbf{r}_N \\ -\mathbf{r}_N - \mathbf{V} \end{bmatrix}, \tag{4.43}$$

The system (4.42) may be rearranged into the form

$$\begin{bmatrix} -\mathrm{P}_1 & & & & & & & & \hat{\mathrm{S}}_1 \\ & -\mathrm{P}_2 & & & & & & & \hat{\mathrm{S}}_2 \\ & & -\mathrm{P}_3 & & & & & & \hat{\mathrm{S}}_3 \\ & & & -\mathrm{P}_4 & & & & & \hat{\mathrm{S}}_4 \\ & & & & \ddots & & & & \vdots \\ & & & & & -\mathrm{P}_{m+1} & & & \hat{\mathrm{S}}_m \\ & & & & & & \ddots & & \hat{\mathrm{S}}_{m+1} \\ & & & & & & & -\mathrm{P}_N & \hat{\mathrm{S}}_N \\ & & & & & & & & \hat{\mathrm{S}}_{N+1} \end{bmatrix} .$$

$$\begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \vdots \\ \mathbf{w}_m \\ \mathbf{w}_{m+1} \\ \vdots \\ \mathbf{w}_N \\ \mathbf{W} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \\ \vdots \\ \mathbf{g}_m \\ \mathbf{g}_{m+1} \\ \vdots \\ \mathbf{g}_N \\ \mathbf{g}_{N+1} \end{bmatrix}, \qquad (4.44)$$

using the following algorithm

Algorithm

$\hat{S}_1 = S_1$

$\hat{S}_m = Q_m P_m^{-1} \hat{S}_{m-1}$; $\quad m = 2, 3, 4, \ldots \ldots, N$

$\hat{S}_{N+1} = -S_{N+1} + Q_N P_N^{-1} \hat{S}_N$;

$\mathbf{g}_1 = \mathbf{b}_1,$ and

$\mathbf{g}_m = \mathbf{b}_m + Q_{m-1} P_{m-1}^{-1} \mathbf{g}_{m-1}$; $m = 2, 3, 4, \ldots \ldots, N+1$;

The solution may then be computed on an architecture with N processors as follows.

Solve the system $\hat{S}_{N+1}\mathbf{W} = \mathbf{g}_{N+1}$ to find $\mathbf{W}$,

then solve the systems

$$-P_m \mathbf{w}_m + \hat{S}_m \mathbf{W} = \mathbf{g}_m \quad (m = 1, 2, 3, \ldots \ldots, N)$$

to find each $\mathbf{w}_m$ ($m = 1, 2, 3, \ldots \ldots, N$), using N processors, each of which solves a linear system of order 10.

4.4 NUMERICAL RESULTS

The numerical methods in sections (4.1) were tested on the following problems.

Equation (4.42) was solved using an LU-decomposition routine because a multi-processor architecture was not available.

PROBLEM 4.1.

$$\begin{aligned}
& y^{(10)} + (1+x^2)y + (4+2x^2)y^{(1)} - (2+3x^2)y^{(2)} - (3+x^3)y^{(3)} \\
& +(x^3-1)y^{(4)} - (1-3x^3)y^{(5)} + (1+x^4-1)y^{(6)} - (4+x^4)y^{(7)} \\
& +x^5y^{(8)} + (1-2x^5)y^{(9)} \\
= \; & e^x(x^7+20x^6+92x^5+40x^4+59x^3+7x^2+22x-11), \quad -1 < x < 1, \\
& y(-1)=0, \quad y^{(2)}(-1)=2e^{-1}, \quad y^{(4)}(-1)=-4e^{-1}, \\
& y^{(6)}(-1)=-18e^{-1}, \quad y^{(8)}(-1)=-40e^{-1} \\
& \qquad \text{and} \\
& y(1)=0, \quad y^{(2)}(1)=-6e, \quad y^{(4)}(1)=-20e, \\
& y^{(6)}(1)=-42e, \quad y^{(8)}(1)=-72e.
\end{aligned} \tag{4.45}$$

The theoretical solution is given by

$$y(x) = x(1-x)e^x. \tag{4.46}$$

The interval $-1 \leq x \leq 1$ was divided into N+1 equal subintervals each of width $h = \frac{(b-a)}{(N+1)}$. The corresponding values of N are then given by $N = 2^i - 1$; the values i=4,5,6 were used in the calculations. The value of $||\mathbf{y} - \mathbf{Y}||_\infty$ was computed for each value of N and these are given in Table 4.1.

Table 4.1: Error norms

$y^{(\lambda)}$	N = 15	N = 31	N = 63
$\lambda = 0$	0.2298D-02	0.3349D-02	0.3611D-02
$\lambda = 1$	0.1231D-01	0.1170D-01	0.1164D-01
$\lambda = 2$	0.3918D-01	0.3718D-01	0.3667D-01
$\lambda = 3$	0.1256D+00	0.1171D+00	0.1149D+00
$\lambda = 4$	0.3481D+00	0.3572D+00	0.3594D+00
$\lambda = 5$	0.1128D+00	0.1147D+01	0.1153D+01
$\lambda = 6$	0.3523D+01	0.3545D+01	0.3549D+01
$\lambda = 7$	0.1177D+02	0.1208D+02	0.1216D+02
$\lambda = 8$	0.3508D+02	0.3527D+02	0.3529D+02
$\lambda = 9$	0.1516D+03	0.1657D+03	0.1722D+03

PROBLEM 4.2.

$$\begin{aligned} & y^{(10)} + y + y^{(1)} + y^{(2)} + y^{(3)} + y^{(4)} + y^{(5)} + y^{(6)} + y^{(7)} + y^{(8)} + y^{(9)} \\ = \; & \sin x(10x + 54) + \cos x(x^2 + 12x - 41). \end{aligned}$$

subject to the boundary conditions

$$y(-1) = 0, \quad y^{(2)}(-1) = -4\cos(-1) + 2\sin(-1),$$
$$y^{(4)}(-1) = 8\cos(-1) - 12\sin(-1), \quad y^{(6)}(-1) = -12\cos(-1) + 30\sin(-1),$$
$$y^{(8)}(-1) = 16\cos(-1) - 56\sin(-1);$$

and

$$y(1) = 0, \quad y^{(2)}(1) = 4\cos(1) + 2\sin(1), \quad y^{(4)}(1) = -8\cos(1) - 12\sin(1),$$
$$y^{(6)}(-1) = 12\cos(1) + 30\sin(1), \quad y^{(8)}(1) = -16\cos(1) - 56\sin(1). \tag{4.47}$$

The theoretical solution is given by

$$y(x) = (x^2 - 1)\sin x. \tag{4.48}$$

The interval $-1 \leq x \leq 1$ was divided into N+1 equal subintervals each of width $h = \frac{(b-a)}{(N+1)}$. The corresponding values of N are then given by $N = 2^i - 1$; the values i=4,5,6 were used in the calculations. The value of $||\mathbf{y} - \mathbf{Y}||_\infty$ was computed for each value of N and these are given in Table 4.2.

Table 4.2: Error norms

$y^{(\lambda)}$	N = 15	N = 31	N = 63
$\lambda = 0$	0.9967D-02	0.9892D-02	0.9879D-02
$\lambda = 1$	0.6516D-01	0.7120D-01	0.7448D-01
$\lambda = 2$	0.3310D+00	0.3405D+00	0.3452D+00
$\lambda = 3$	0.5780D+00	0.5799D+00	0.5803D+00
$\lambda = 4$	0.3323D+00	0.3411D+00	0.3455D+00
$\lambda = 5$	0.6034D+00	0.6081D+00	0.6090D+00
$\lambda = 6$	0.3818D+00	0.3834D+00	0.3827D+00
$\lambda = 7$	0.1220D+01	0.1246D+01	0.1249D+01
$\lambda = 8$	0.3636D+01	0.3747D+00	0.3760D+01
$\lambda = 9$	0.2435D+02	0.2702D+02	0.2856D+02

Tables 4.1 and 4.2 contain the error norms for these values of N for y and its first nine derivatives. It is noted that the maximum errors in y are small, but that, for any value of N, the errors gradually increase as the higher-order derivatives are considered. It is also seen that, with a small number of exceptions, the error norms for y and its derivatives increase as N increases (or as h gets smaller). This is due to the conditioning of the block matrix in (4.42) being affected by the mesh refinement and to the build-up of round-off errors associated with a large increase in the number of arithmetic operations. This is a common phenomenon as reported by Twizell et al. (1994).

Chapter 5

GENERAL TENTH-ORDER NON-LINEAR BOUNDARY-VALUE PROBLEMS

Consider the general tenth-order non-linear two-point boundary-value problem

$$\begin{aligned} y^{(x)}(x) = \; & f(x, y(x), y'(x), y''(x), y'''(x), y^{(iv)}(x), y^{(v)}(x), \\ & y^{(vi)}(x), y^{(vii)}(x), y^{(viii)}(x), y^{(ix)}(x)), \qquad a < x < b, \end{aligned} \tag{5.1}$$

with the boundary conditions

$$\begin{aligned} y(a) &= A_0, & y(b) &= B_0, \\ y''(a) &= A_2, & y''(b) &= B_2, \\ y^{(iv)}(a) &= A_4, & y(iv)(b) &= B_4, \\ y^{(vi)}(a) &= A_6, & y(vi)(b) &= B_6, \\ y^{(viii)}(a) &= A_8, & y(viii)(b) &= B_8. \end{aligned}$$

Now suppose that

$$y^0 = p \qquad \Rightarrow p' = y' = q,$$

$$
\begin{aligned}
y'(x) &= q & &\Rightarrow q' = y'' = r,\\
y''(x) &= r & &\Rightarrow r' = y''' = s,\\
y'''(x) &= s & &\Rightarrow s' = y^{(iv)} = t,\\
y^{(iv)}(x) &= t & &\Rightarrow t' = y^{(v)} = u,\\
y^{(v)}(x) &= u & &\Rightarrow u' = y^{(vi)} = v,\\
y^{(vi)}(x) &= v & &\Rightarrow v' = y^{(vii)} = \lambda,\\
y^{(vii)}(x) &= \lambda & &\Rightarrow \lambda' = y^{(viii)} = \mu,\\
y^{(viii)}(x) &= \mu & &\Rightarrow \mu' = y^{(ix)} = \xi
\end{aligned}
$$

and

$$y^{(ix)}(x) = \xi \qquad \Rightarrow \xi' = y^{(x)} = f(x, p, q, r, s, t, u, v, \lambda, \mu, \xi)$$

Let $\mathbf{w} = [p,\ q,\ r,\ s, t,\ u,\ v,\ \lambda,\ \mu,\ \xi]^T$ then the system is

$$
\frac{d\mathbf{w}}{dx} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \mathbf{w} + \begin{bmatrix}
0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ f(x, p, q, r, s, t, u, v, \lambda, \mu, \xi)
\end{bmatrix}. \tag{5.2}
$$

The the system of equations (5.2) is of the form

$$D\mathbf{w}(x) \equiv \frac{d\mathbf{w}}{dx} = M\mathbf{w}(x) + \mathbf{c}(x, \mathbf{w}(x)), \quad a < x < b . \tag{5.3}$$

Note: $p = p(x),\ q = q(x),\ r = r(x),\ s = s(x),\ t = t(x),$ $u = u(x),\ v = v(x),\ \lambda = \lambda(x),\ \mu = \mu(x),\ \xi = \xi(x)$ and so $\mathbf{c}=\mathbf{c}(x,\mathbf{w}(x))$.

The associated boundary conditions are

$$\begin{aligned} p(a) &= A_0, & p(b) &= B_0, \\ r(a) &= A_2, & r(b) &= B_2, \\ t(a) &= A_4, & t(b) &= B_4, \\ v(a) &= A_6, & v(b) &= B_6, \\ \mu(a) &= A_8, & \mu(b) &= B_8. \end{aligned} \tag{5.4}$$

System (5.3) will be solved by using the recurrence relation

$$\mathbf{w}(x + h) = [\exp(hD)]\mathbf{w}(x), \tag{5.5}$$

where $D \equiv \text{diag}\{\frac{d}{dx}\}$ is a matrix of order 10 defined in section 4.2. Suppose that $\exp(hD)$ in (5.4) is replaced by the (1,1) Padé approximant $(I - \frac{1}{2}hD)^{-1}(I + \frac{1}{2}hD)$ where I is the identity matrix of order 10. This gives, to second order,

$$\mathbf{w}(x + h) = (I - \frac{1}{2}hD)^{-1}(I + \frac{1}{2}hD)\mathbf{w}(x), \tag{5.6}$$

that is

$$(I - \frac{1}{2}hD)\mathbf{w}(x + h) = (I + \frac{1}{2}hD)\mathbf{w}(x). \tag{5.7}$$

Then

$$\mathbf{w}(x + h) - \frac{1}{2}h[M(x + h)\mathbf{w}(x + h) + \mathbf{c}(x, \mathbf{w}(x + h))] = \mathbf{w}(x) + \frac{1}{2}h[M\mathbf{w}(x) + \mathbf{c}(x, \mathbf{w}(x))], \tag{5.8}$$

giving

$$(I - \frac{1}{2}hM)\mathbf{w}(x + h) - \frac{1}{2}h\mathbf{c}(x + h, \mathbf{w}(x + h)) = (I + \frac{1}{2}hM)\mathbf{w}(x) + \frac{1}{2}h\mathbf{c}(x, \mathbf{w}(x)). \tag{5.9}$$

This is of the form

$$P\mathbf{w}(x + h) - \frac{1}{2}h\mathbf{c}(x + h, \mathbf{w}(x + h)) = Q\mathbf{w}(x) + \frac{1}{2}\mathbf{c}(x, \mathbf{w}(x)), \tag{5.10}$$

where the constant matrix P, the constant matrix Q, and **c** are given by

$$P = \begin{bmatrix} 1 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2}h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{c} = \begin{bmatrix} 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & f(x,p,q,r,s,t,u,v,\lambda,\mu,\xi) \end{bmatrix}^T.$$

Note that P and Q are upper bidiagonal matrices. Recall the mesh points are $x_m = a + mh \quad (m = 0, 1, 2, ..., N, N+1)$.

5.1 NOTATION

The notation used in Chapter 4 will be retained, so that

$$\mathbf{w}_m = \begin{bmatrix} p(x_m) \\ q(x_m) \\ r(x_m) \\ s(x_m) \\ t(x_m) \\ u(x_m) \\ v(x_m) \\ \lambda(x_m) \\ \mu(x_m) \\ \xi(x_m) \end{bmatrix} = \begin{bmatrix} p_m \\ q_m \\ r_m \\ s_m \\ t_m \\ u_m \\ v_m \\ \lambda_m \\ \mu_m \\ \xi_m \end{bmatrix} = \begin{bmatrix} w_m^{(0)} \\ w_m^{(1)} \\ w_m^{(2)} \\ w_m^{(3)} \\ w_m^{(4)} \\ w_m^{(5)} \\ w_m^{(6)} \\ w_m^{(7)} \\ w_m^{(8)} \\ w_m^{(9)} \end{bmatrix}, \tag{5.11}$$

with $m = 1, 2, 3, \ldots, N$;

$$\mathbf{w}_0 = \begin{bmatrix} w_0^{(0)} \\ w_0^{(1)} \\ w_0^{(2)} \\ w_0^{(3)} \\ w_0^{(4)} \\ w_0^{(5)} \\ w_0^{(6)} \\ w_0^{(7)} \\ w_0^{(8)} \\ w_0^{(9)} \end{bmatrix} = \begin{bmatrix} A_0 \\ q_0 \\ A_2 \\ s_0 \\ A_4 \\ u_0 \\ A_6 \\ \lambda_0 \\ A_8 \\ \xi_0 \end{bmatrix}, \tag{5.12}$$

$$\mathbf{w}_{N+1} = \begin{bmatrix} w_{N+1}^{(0)} \\ w_{N+1}^{(1)} \\ w_{N+1}^{(2)} \\ w_{N+1}^{(3)} \\ w_{N+1}^{(4)} \\ w_{N+1}^{(5)} \\ w_{N+1}^{(6)} \\ w_{N+1}^{(7)} \\ w_{N+1}^{(8)} \\ w_{N+1}^{(9)} \end{bmatrix} = \begin{bmatrix} B_0 \\ q_{N+1} \\ B_2 \\ s_{N+1} \\ B_4 \\ u_{N+1} \\ B_6 \\ \lambda_{N+1} \\ B_8 \\ \xi_{N+1} \end{bmatrix}, \tag{5.13}$$

$$\mathbf{c}_m = \Big[\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ f(x_m, p_m, q_m, r_m, s_m, t_m, u_m, v_m, \lambda_m, \mu_m, \xi_m)\ \Big]^T, \tag{5.14}$$

$$\mathbf{c}_0 = \Big[\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ f(a, A_0, q_0, A_2, s_0, A_4, u_0, A_6, \lambda_0, A_8, \xi_0)\ \Big]^T, \tag{5.15}$$

and

$$\mathbf{c}_{N+1} = \Big[\ 0\ 0\ \ldots\ f(b, B_0, q_{N+1}, B_2, s_{N+1}, B_4, u_{N+1}, B_6, \lambda_{N]+1}, B_8, \xi_{N+1})\ \Big]^T. \tag{5.16}$$

5.2 NUMERICAL METHOD

Applying the numerical method to the general mesh point x_m $(m = 0, 1, 2, 3, 4, \ldots \ldots, N)$ gives

$$Q\mathbf{w}_m - P\mathbf{w}_{m+1} + \frac{1}{2}h\mathbf{c}(x_m, \mathbf{w}_m) + \frac{1}{2}h\mathbf{c}(x_{m+1}, \mathbf{w}_{m+1}) = \mathbf{0}, \tag{5.17}$$

which is of the form

$$\mathbf{F}_m = \mathbf{F}(\mathbf{w}_m, \mathbf{w}_{m+1}) = \mathbf{0}, \tag{5.18}$$

in (5.18) $\mathbf{F}_m$ is a vector of order 10.

Consider the point $x = x_0$; then equation (5.17) becomes

$$\mathbf{Q}\mathbf{w}_0 - \mathbf{P}\mathbf{w}_1 + \frac{1}{2}h\mathbf{c}(x_0, \mathbf{w}_0) + \frac{1}{2}h\mathbf{c}(x_1, \mathbf{w}_1) = \mathbf{0} \tag{5.19}$$

or

$$\mathbf{P}\mathbf{w}_1 - \frac{1}{2}h\mathbf{c}(x_1, \mathbf{w}_1) = \mathbf{Q}\mathbf{w}_0 + \frac{1}{2}h\mathbf{c}(x_0, \mathbf{w}_0). \tag{5.20}$$

This gives

$$p_1 - \frac{1}{2}hq_1 = A_0 + \frac{1}{2}hq_0 \tag{5.21}$$

$$q_1 - \frac{1}{2}hr_1 = q_0 + \frac{1}{2}hA_2 \tag{5.22}$$

$$r_1 - \frac{1}{2}hs_1 = A_2 + \frac{1}{2}hs_0 \tag{5.23}$$

$$s_1 - \frac{1}{2}ht_1 = s_0 + \frac{1}{2}hA_4 \tag{5.24}$$

$$t_1 - \frac{1}{2}hu_1 = A_4 + \frac{1}{2}hu_0 \tag{5.25}$$

$$u_1 - \frac{1}{2}hv_1 = u_0 + \frac{1}{2}hA_6 \tag{5.26}$$

$$v_1 - \frac{1}{2}h\lambda_1 = A_6 + \frac{1}{2}h\lambda_0 \tag{5.27}$$

$$\lambda_1 - \frac{1}{2}h\mu_1 = \lambda_0 + \frac{1}{2}hA_8 \tag{5.28}$$

$$\mu_1 - \frac{1}{2}h\xi_1 = A_8 + \frac{1}{2}h\xi_0 \tag{5.29}$$

$$\begin{aligned} & \xi_1 - \tfrac{1}{2}hf(x_1, p_1, q_1, r_1, s_1, t_1, u_1, v_1, \lambda_1, \mu_1, \xi_1) \\ = \quad & \xi_0 + \tfrac{1}{2}hf(a, A_0, q_0, A_2, s_0, A_4, u_0, A_6, \lambda_0, A_8, \xi_0). \end{aligned} \tag{5.30}$$

Taking all terms to the left hand sides of (5.21)—(5.30) enables us to create the vector

$$S_1\mathbf{W}$$

with

$$\mathbf{W} = [q_0,\ s_0,\ u_0,\ \lambda_0,\ \xi_0,\ q_{N+1},\ s_{N+1},\ u_{N+1},\ \lambda_{N+1},\ \xi_{N+1}]^T$$

and

$$S_1 = \begin{bmatrix} \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consider, next, the general point x_m $(m = 1, 2, 3, \ldots, N-1)$ then (5.17) reduces and gives

$$p_m + \frac{1}{2}hq_m - p_{m+1} + \frac{1}{2}hq_{m+1} = 0, \qquad (5.31)$$

$$q_m + \frac{1}{2}hr_m - q_{m+1} + \frac{1}{2}hr_{m+1} = 0, \qquad (5.32)$$

$$r_m + \frac{1}{2}hs_m - r_{m+1} + \frac{1}{2}hs_{m+1} = 0, \qquad (5.33)$$

$$t_m + \frac{1}{2}hu_m - t_{m+1} + \frac{1}{2}hu_{m+1} = 0, \qquad (5.34)$$

$$u_m + \frac{1}{2}hv_m - u_{m+1} + \frac{1}{2}hv_{m+1} = 0, \qquad (5.35)$$

$$v_m + \frac{1}{2}h\lambda_m - v_{m+1} + \frac{1}{2}h\lambda_{m+1} = 0, \qquad (5.36)$$

$$\lambda_m + \frac{1}{2}h\mu_m - \lambda_{m+1} + \frac{1}{2}h\mu_{m+1} = 0, \qquad (5.37)$$

$$\mu_m + \frac{1}{2}h\xi_m - \mu_{m+1} + \frac{1}{2}h\xi_{m+1} = 0, \qquad (5.38)$$

$$\begin{aligned} &\xi_m - \xi_{m+1} + \tfrac{1}{2}hf(x_m, p_m, q_m, r_m, s_m, t_m, u_m, v_m, \lambda_m, \mu_m, \xi_m) \\ &+\tfrac{1}{2}hf(x_{m+1}, p_{m+1}, q_{m+1}, r_{m+1}, s_{m+1}, t_{m+1}, u_{m+1}, v_{m+1}, \lambda_{m+1}, \mu_{m+1}, \xi_{m+1}) \\ = \;& 0 \end{aligned} \qquad (5.39)$$

Consider, finally, the point x_N; then (5.12) becomes

$$Q\mathbf{w}_N - P\mathbf{w}_{N+1} + \frac{1}{2}h\mathbf{c}(x_N, \mathbf{w}_N) + \frac{1}{2}h\mathbf{c}(x_{N+1}, \mathbf{w}_{N+1}) = \mathbf{0}, \tag{5.40}$$

or

$$Q\mathbf{w}_N + \frac{1}{2}h\mathbf{c}(x_N, \mathbf{w}_N) = P\mathbf{w}_{N+1} - \frac{1}{2}h\mathbf{c}(x_{N+1}, \mathbf{w}_{N+1}). \tag{5.41}$$

This gives

$$p_N + \frac{1}{2}hq_N = B_0 - \frac{1}{2}hq_{N+1} = 0 \tag{5.42}$$

$$q_N + \frac{1}{2}hr_N = q_{N+1} - \frac{1}{2}hB_2 = 0 \tag{5.43}$$

$$r_N + \frac{1}{2}hs_N = B_2 - \frac{1}{2}hs_{N+1} = 0 \tag{5.44}$$

$$s_N + \frac{1}{2}ht_N = s_{N+1} - \frac{1}{2}hB_4 = 0 \tag{5.45}$$

$$t_N + \frac{1}{2}hu_N = B_4 - \frac{1}{2}hu_{N+1} = 0 \tag{5.46}$$

$$u_N + \frac{1}{2}hv_N = u_N + 1 - \frac{1}{2}hB_6 = 0 \tag{5.47}$$

$$v_N + \frac{1}{2}h\lambda_N = B_6 - \frac{1}{2}h\lambda_{N+1} = 0 \tag{5.48}$$

$$\lambda_N + \frac{1}{2}h\mu_N = \lambda_{N+1} - \frac{1}{2}hB_8 = 0 \tag{5.49}$$

$$\mu_N + \frac{1}{2}h\xi_N = B_8 - \frac{1}{2}h\xi_{N+1} = 0 \tag{5.50}$$

$$\begin{aligned} & \xi_N + \tfrac{1}{2}hf(x_N, p_N, q_N, r_N, s_N, t_N, u_N, v_N, \lambda_N, \mu_N, \xi_N) \\ = \; & \xi_{N+1} - \tfrac{1}{2}hf(b, B_0, q_{N+1}, B_2, s_{N+1}, B_4, u_{N+1}, B_6, \lambda_{N+1}, B_8, \xi_{N+1}). \end{aligned} \tag{5.51}$$

Equations (5.42)—(5.51) may be written in system form as follows

$$Q_N\mathbf{w}_N = S_{N+1}\mathbf{W}, \tag{5.52}$$

(It may be noted that $\mathbf{W}$ is already defined.)

with

$$S_{N+1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2}h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

The next aim is to find the block matrices. Return to the ten equations (5.21)—(5.30) for the mesh point $x = x_0$ and let these equations be named $E_{0,1}, E_{0,2}, E_{0,3}, E_{0,4}, E_{0,5}, E_{0,6}, E_{0,7}, E_{0,8}, E_{0,9}, E_{0,10}$ respectively; they give the vector

$$\mathbf{E}_0 = \begin{bmatrix} E_{0,1} \\ E_{0,2} \\ E_{0,3} \\ E_{0,4} \\ E_{0,5} \\ E_{0,6} \\ E_{0,7} \\ E_{0,8} \\ E_{0,9} \\ E_{0,10} \end{bmatrix}. \qquad (5.53)$$

For each general point x_m $(m = 1, 2, 3, \dots, N-1)$ these are also ten equations. Let these equations be named $E_{m,1}, E_{m,2}, E_{m,3}, E_{m,4}, E_{m,5}, E_{m,6}, E_{m,7}, E_{m,8}, E_{m,9}$,

$E_{m,10}$ respectively; they give the vector

$$\mathbf{E}_m = \begin{bmatrix} E_{m,1} \\ E_{m,2} \\ E_{m,3} \\ E_{m,4} \\ E_{m,5} \\ E_{m,6} \\ E_{m,7} \\ E_{m,8} \\ E_{m,9} \\ E_{m,10} \end{bmatrix}. \quad (5.54)$$

Similarly for the last point x_N there are also ten (non-linear) algebraic equations. These are given in equations (5.42)—(5.51) which will be renamed $E_{N,1}, E_{N,2}, E_{N,3}, E_{N,4}, E_{N,5}, E_{N,6}, E_{N,7}, E_{N,8}, E_{N,9}, E_{N,10}$ respectively; collectively these give

$$\mathbf{E}_N = \begin{bmatrix} E_{N,1} \\ E_{N,2} \\ E_{N,3} \\ E_{N,4} \\ E_{N,5} \\ E_{N,6} \\ E_{N,7} \\ E_{N,8} \\ E_{N,9} \\ E_{N,10} \end{bmatrix}. \quad (5.55)$$

The unknowns in (5.53)—(5.55) are q_0, s_0, u_0, λ_0, ξ_0, p_m, q_m, r_m, s_m, t_m, u_m, v_m, λ_m, μ_m, ξ_m($m = 1, 2, 3, \ldots, N$), q_{N+1}, s_{N+1}, u_{N+1}, λ_{N+1}, ξ_{N+1}. The total number of unknowns is $5 + 10N + 5 = 10(N+1)$. There are $10 + 10(N-1) + 10 = 10(N+1)$ non-linear algebraic equations in which the unknowns are q_0, s_0, u_0, λ_0, ξ_0, p_m, q_m, r_m, s_m, t_m, u_m, v_m, λ_m, μ_m, ξ_m ($m = 1, 2, 3, \ldots, N$), q_{N+1}, s_{N+1}, u_{N+1}, λ_{N+1}, ξ_{N+1}.

5.3 THE NEWTON-RAPHSON METHOD

Consider the nonlinear algebraic system of M equations given by

$$\mathbf{F}(\mathbf{x}) = \mathbf{0} \tag{5.56}$$

i.e.

$$\begin{aligned}
F_1(x_1, x_2, x_3, \ldots \ldots, x_M) &= 0,\\
F_2(x_1, x_2, x_3, \ldots \ldots, x_M) &= 0,\\
F_3(x_1, x_2, x_3, \ldots \ldots, x_M) &= 0,\\
&\vdots\\
&\vdots\\
&\vdots\\
F_M(x_1, x_2, x_3, \ldots \ldots, x_M) &= 0.
\end{aligned}$$

As was noted in Chapter 1, the Newton-Raphson method becomes

$$\mathbf{X}^{(k+1)} = \mathbf{X}^{k} - [J(\mathbf{X}^{(k)})]^{-1}\mathbf{F}(\mathbf{X}^{(k)}) \ ; \qquad k = 0, 1, 2, \ldots \tag{5.57}$$

with

$$\mathbf{X} = [X_1, X_2, X_3, X_1, \ldots \ldots, X_M]^T.$$

Clearly k is the iteration number and we need to "guess" $\mathbf{X}^{(0)}$ so that the Newton-Raphson method for a system will converge to a fixed point $\mathbf{X}^*$. Let

$$J^{(k)} = J(\mathbf{X}^{(k)}), \tag{5.58}$$

be the Jacobian given by

$$J^{(k)} = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \frac{\partial F_1}{\partial X_3} & \cdots\cdots & \frac{\partial F_1}{\partial X_M} \\ \frac{\partial F_2}{\partial X_1} & \frac{\partial F_2}{\partial X_2} & \frac{\partial F_2}{\partial X_3} & \cdots\cdots & \frac{\partial F_2}{\partial X_M} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial F_M}{\partial X_1} & \frac{\partial F_M}{\partial X_2} & \frac{\partial F_M}{\partial X_3} & \cdots\cdots & \frac{\partial F_M}{\partial X_M} \end{bmatrix} . \tag{5.59}$$

Now, $J^{(k)}$ should be inverted to give $[J^{(k)}]^{-1}$; this is a major problem if M is large. So, to avoid having to invert $J^{(k)}$, let

$$\mathbf{Z}^{(k)} = -J[(\mathbf{X}^{(k)})]^{-1}\mathbf{F}(\mathbf{X}^{(k)}) , \tag{5.60}$$

then the above equation becomes

$$J^{(k)}\mathbf{Z}^{(k)} = -\mathbf{F}(\mathbf{X}^{(k)}). \tag{5.61}$$

This is a linear system in $\mathbf{Z}^{(k)}$ which can be solved using a suitable method (e.g. LU-decomposition, or Gauss-Seidel).

Having found $\mathbf{Z}^{(k)}$ by solving this linear system, the next iterate $\mathbf{X}^{(k+1)}$ is found from the simple equation

$$\mathbf{X}^{(k+1)} = \mathbf{X}^{(k)} + \mathbf{Z}^{(k)} \tag{5.62}$$

The main aim is to get the linear system

$$J^{(k)}\mathbf{Z}^{(k)} = -\mathbf{F}(\mathbf{X}^{(k)}) \tag{5.63}$$

into the blok form

$$\begin{bmatrix} -P_1^{(k)} & & & & & S_1^{(k)} \\ Q_2^{(k)} & -P_2^{(k)} & & & & \\ & Q_3^{(k)} & -P_3^{(k)} & & & \\ & & \ddots & \ddots & & \\ & & & Q_{N-1}^{(k)} & -P_N^{(k)} & \\ & & & & Q_N^{(k)} & -S_{N+1}^{(k)} \end{bmatrix} .\mathbf{Z}^{(k)} = -\mathbf{F}(\mathbf{X}^{(k)}) \tag{5.64}$$

5.4 CLARIFYING THE NOTATION

At any iterate k =0, 1, 2,...

$$\mathbf{X} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{w}_N \\ \mathbf{w}_{N+1} \end{bmatrix}, \tag{5.65}$$

where

$$\mathbf{w}_m = \begin{bmatrix} p_m^{(k)} \\ q_m^{(k)} \\ r_m^{(k)} \\ s_m^{(k)} \\ t_m^{(k)} \\ u_m^{(k)} \\ v_m^{(k)} \\ \lambda_m^{(k)} \\ \mu_m^{(k)} \\ \xi_m^{(k)} \end{bmatrix},$$

m=0,1,2,...,N+1. Clearly

$$\mathbf{F}^{(k)} = \mathbf{F}(\mathbf{X}^{(k)}) = \begin{bmatrix} \mathbf{E}_0^{(k)} \\ \mathbf{E}_1^{(k)} \\ \mathbf{E}_2^{(k)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{E}_{N-1}^{(k)} \\ \mathbf{E}_N^{(k)} \end{bmatrix}. \tag{5.66}$$

To form the Jacobian, we need to find the derivaties $\frac{\partial E_{m,j}}{\partial p_i}$, $\frac{\partial E_{m,j}}{\partial q_i}$, $\frac{\partial E_{m,j}}{\partial r_i}$, $\frac{\partial E_{m,j}}{\partial s_i}$, $\frac{\partial E_{m,j}}{\partial t_i}$, $\frac{\partial E_{m,j}}{\partial u_i}$, $\frac{\partial E_{m,j}}{\partial v_i}$, $\frac{\partial E_{m,j}}{\partial \lambda_i}$, $\frac{\partial E_{m,j}}{\partial \mu_i}$, $\frac{\partial E_{m,j}}{\partial \xi_i}$; for $i = m, m+1$, $j = 1,2,3,4,5$ and $m = 0,1,2,...,N-1,N$. There are 10(N+1) of these derivatives; this, of course, is the order of the square matrix J which we aim to write in the above block form.

5.5 NUMERICAL RESULTS

The numerical method in sections (5.3)—(5.5) for tenth-order non-linear two-point boundary-value problems is tested on the following problem.

Problem 5.1.

$$y^{(x)} = g(x) - y - yy' - y'' + y''' - y^{(iv)} - y^{(v)} - y^{(vi)} - y^{(vii)} - y^{(viii)} - y^{(ix)}$$

where

$$g(x) = e^x(2 - e^{-x}), \quad 0 < x < 1,$$

subject to the boundary conditions

$$y(0) = y''(0) = y^{(iv)}(0) = y^{(vi)}(0) = y^{(viii)}(0) = 1$$

and

$$y(1) = y'(1) = y'''(1) = y^{(v)}(1) = y^{(vii)}(1) = y^{(ix)}(1) = -e^{-1}$$

The theoretical solution is given by

$$y(x) = e^{-x}.$$

The interval $0 \leq x \leq 1$ for the problem was divided in to N+1 equal subintervals each of width $h = \frac{(b-a)}{(N+1)}$. The corresponding values of N are then given by $N = 2^i - 1$; the values i=4,5,6 were used in the calculations. The value of $||\mathbf{y} - \mathbf{Y}||_\infty$ was computed for each value of N. Because of the non-availablity of multi-processor architecture, LU-decomposition was used to solve equation (5.64) at each iterate.

Table 5.1 contains the error norms for these values of N for y and its first nine derivatives. It is noted that the maximum errors in y are small, but that, for any value of N, the errors gradually increase as the higher-order derivatives are considered. It is also seen that, with a small number of exceptions, the error norms for y and its derivatives increase as N increases (or as h gets smaller). This is due to the conditioning of the block matrix in (5.64) being affected by the mesh refinement and to the build-up of round-off errors associated with a large increase in the number of arithmetic operations. This is a common phenomenon as reported by Twizell et al. (1994).

Table 5.1: Error norms

$y^{(\lambda)}$	N = 15	N = 31	N = 63
$\lambda = 0$	0.2435D-04	0.1811D-04	0.2735D-04
$\lambda = 1$	0.2712D-03	0.1456D-03	0.1262D-03
$\lambda = 2$	0.3014D-03	0.2989D-03	0.2961D-03
$\lambda = 3$	0.9150D-03	0.1123D-02	0.1159D-02
$\lambda = 4$	0.2515D-02	0.2836D-02	0.2901D-02
$\lambda = 5$	0.1047D-01	0.1141D-01	0.1154D-01
$\lambda = 6$	0.2483D-01	0.2703D-01	0.2748D-01
$\lambda = 7$	0.1046D+00	0.11336D+00	0.1158D+00
$\lambda = 8$	0.3436D+00	0.3762D+00	0.3964D+00
$\lambda = 9$	0.1197D+01	0.1308D+01	0.1377D+01

Chapter 6

TENTH-ORDER EIGENVALUE PROBLEMS

6.1 INTRODUCTION

Consider a horizontal layer of fluid in which an adverse temperature gradient is maintained by heating the underside. The fluid at the bottom will be lighter than that the top and, in this situation, the layer will be potentially unstable. The *rôle* played by viscosity is to inhibit a tendency on the part of the fluid to redistribute itself. This *rôle* is affected by an additional effect of rotation and the rotation will introduce new factors into the ensuing thermal instability (Chandrasekhar, 1961).

The "top-heavy" state of the fluid gives rise to an eighth-order eigenvalue problem, when instability sets in as overstability, consisting of the ordinary differential equation (ODE)

$$\begin{aligned}&(D^2 - A^2 - p_1\sigma)[(D^2 - A^2 - \sigma)^2(D^2 - A^2) + TD^2]w(x)\\&= -RA^2(D^2 - A^2 - \sigma)w(x),\end{aligned} \tag{6.1}$$

and the boundary conditions

$$w(x) \to 0 \quad \text{as} \, x \to \pm\infty. \tag{6.2}$$

The variables, parameters and constants in (6.1) and (6.2) and the other terms on which they depend are described below

A : wave number, μ : magnetic permeability, H : uniform magnetic field.
$\eta = \frac{1}{(4\pi\mu\sigma)}$: resistivity, $D = \frac{d}{dx}$,
$p_1 = \frac{\upsilon}{\kappa}$ where υ : kinetic viscosity and κ : thermoconductivity ,
σ : time costant (relative to dimensionless time and space coordinates),
$T = \frac{4\Omega^2 d^4}{\upsilon^2}$ = the Taylor number where Ω = angular velocity,
$R = g\frac{\alpha\beta d^4}{\kappa\upsilon}$ = the Rayleigh number where g = acceleration due to gravity,
$\alpha =$ coefficient of volumetric expansion, $\beta =$ adverse temprature gradient,
x : dimensionless vertical coordinate, $p_2 = \frac{\eta}{\kappa}$, d : depthoflayeroffluid,
w= w(x) : vertical coordinate, $Q = \mu\frac{H^2 d^2}{(4\pi\upsilon\eta)}$.

In solving (6.1) and (6.2), the minimum value of R (and A) is sought and the corresponding value of σ, which can be complex. It is apparent from (6.1) that, for an arbitrary complex value value of σ, R will be complex. However, the physical meaning of R requires it to be real and so it will be assumed that σ is pure imaginary.

It may be noted that, when instability sets in as ordinary convection, the marginal state will be characterized by $\sigma = 0$ and the ODE in (6.1) reduces to a sixth-order equation (Chandrasekhar, 1961). This ODE was solved by Baldwin (1987) who used global phase-integral methods, by Twizell and Boutayeb (1990) and Boutayeb and Twizell (1991) using finite difference methods.

The effect of rotation on a horizontal layer of fluid heated from below is known to be similar to the effect of a magnetic field acting under the same conditions in that they both inhibit the onset of thermal instability (Chandrasekhar, 1961). A magnetic field imparts to the fluid certain aspects of viscosity which facilitate the onset of instability when rotation is present. Acting togather, rotation and magnetic field do not reinforce each other, however. In fact, they exhibit conflicting tendencies when applied simultaneously.

In liquid metals such as mercury, instability sets in mostly as overstability when rotation is present, but it sets in as convection under the influence of a magnetic field (Chandrasekhar, 1961). Thus, it has always been illuminating to study thermal instability under the combined effects of rotation and a magnetic field.

Consider, therefore, an infinite horizontal layer of fluid of uniform thickness, heated from below, in a state of uniform rotation subject to a uniform magnetic field acting across the fluid in the same direction as gravity. Chandrasekhar (1961) shows that, when it sets in as ordinary convection, instability may be modelled by the tenth-order eigenvalue problem

$$\begin{aligned}&(D^2 - A^2)[\{(D^2 - A^2)^2 - QD^2\} + TD^2(D^2 - A^2)]w(x)\\&+RA^2[(D^2 - A^2)^2 - QD^2]w(x) = 0,\end{aligned} \tag{6.3}$$

with boundary conditions given by (6.2); the terms in (6.4) are as described above. When instability sets in as overstability, the model differential equation is of order twelve and is given by

$$\begin{aligned}&(D^2 - A^2 - p_1\sigma)[(D^2 - A^2)\{(D^2 - A^2 - \sigma)(D^2 - A^2 - p_2\sigma) - QD^2\}^2\\&+TD^2(D^2 - A^2 - p_2\sigma)^2]w(x)\\&+RA^2[(D^2 - A^2 - \sigma)(D^2 - A^2 - p_2\sigma) - QD^2](D^2 - A^2 - p_2\sigma)w(x) = 0,\end{aligned} \tag{6.4}$$

with boundary conditions given by (6.2); the terms in (6.4) are as described before.

The numerical analysis literature on the solution of eighth-, tenth-, and twelfth-order boundary-value problem is extrely small. Such problems are contained implicitly in the paper by Chawala and Katti (1979); however, the emphasis is on fourth-order problems in that paper. Agarwal (1986), in his book, gives theorems which emphasize the conditions for existence and uniqueness of solutions of such higher-order problems, but no numerical experiments are reported therein. Numerical methods for the solution of higher-order boundary-value problems are also considered by Acher et al. (1988), Doedal (1979).

Esser (1980), Keller (1975), Kreiss (1972), Lynch and Rice (1980) and Osborne (1967). Boutayeb and Twizell (1991) have solved the special, nonlinear, twelfth-order, boundary-value problem

$$w^{(xii)}(x) = f(x, w), \quad a < x < b; \quad a, b, x \in \Re; \tag{6.5}$$

$$w^{(2i)}(a) = A_{2i}, \quad w^{(2i)}(b) = B_{2i}, \quad i = 0, 1, \ldots, 5, \tag{6.6}$$

in which w(x) and f(x,w) are real and as many times differentiable as required, and A_{2i}, B_{2i} ($i = 0, 1, \ldots, 5$) are real and finite contants. The approach followed involved the transformation of (6.5) into a system of six second-order differential equations first of all. Thereafter, a well-known second-order finite difference method was employed to obtain the solution and global extrapolation on two grids was carried out to improve accuracy.

An alternative approach is to solve the problems $\{(6.1), (6.2)\}$, $\{(6.3), (6.2)\}$ and $\{(6.4), (6.2)\}$ directly, as higher-order problems rather than reduce the differential equations to lower-order systems. This approach was followed successfully in Twizell et al. (1994) to compute critical values of R, a and σ in (6.1), R and a in (6.3), and R, a and $\sigma_1 = \frac{\sigma}{p^2}$ in (6.4).

6.2 ROTATION AND A MAGNETIC FIELD: INSTABILITY AS ORDINARY CONVECTION

Numerical methods for solving high-order eigenvalue problems directly may suffer word-length problems due to the high condition numbers involved. This was experienced in earlier chapters of this thesis. One way of minimizing such difficulties is to transform the given differential equation into a system of lower-order equations and then to use appropriate techniques for solving low-order boundary-value problems.

6.3 TENTH-ORDER EIGENVALUE PROBLEMS

Consider again the general tenth-order differential equation

$$\begin{aligned}&(D^2 - A^2)[\{(D^2 - A^2)^2 - QD^2\} + TD^2(D^2 - A^2)]w(x)\\&+RA^2[(D^2 - A^2)^2 - QD^2]w(x) = 0,\end{aligned} \tag{6.7}$$

This is clearly a linear ODE which may be written in the form

$$\begin{aligned}&(D^{10} - k_1D^8 + k_2D^6 - k_3D^4 + k_4D^2 - A^{10})w(x)\\&+RA^2(D^4 - k_5D^2 + A^4)w(x) = 0,\end{aligned} \tag{6.8}$$

with

$$k_1 = 5A^2 + 2Q, \quad k_2 = 10A^4 + 6A^2Q + T + Q^2,$$

$$k_3 = 10A^6 + 6A^4Q + 2A^2T + A^2Q^2,$$

$$k_4 = 5A^8 + 2A^6Q + A^4T, \quad k_5 = 2A^2 + Q.$$

It will be assumed that, in (6.7), $0 < x < 1$ and, following Chandrasekhar, the free-free boundary conditions

$$w^{(2i)}(0) = w^{(2i)}(1) = 0; \quad i = 0, 1, 2, 3, 4, \tag{6.9}$$

will be imposed and a first-order numerical method will be used to estimate the critical values of R associated with the eigenvalue problem (6.7)—(6.9) for the different values of A, T and Q.

6.4 FIRST-ORDER SYSTEM

Consider the linear eigenvalue problem (6.8) written in the form

$$\begin{aligned}&D^{10}w = k_1D^8w - k_2D^6w + k_3D^4w - k_4D^2w + A^{10}w\\&-RA^2D^4w + RA^2k_5D^2w - RA^6w.\end{aligned} \tag{6.10}$$

Let

$$w = p,$$

$$w'(x) = q,$$

$$w''(x) = r,$$

$$w'''(x) = s,$$

$$w^{(iv)}(x) = t,$$

$$w^{(v)}(x) = u,$$

$$w^{(vi)}(x) = v,$$

$$w^{(vii)}(x) = \lambda,$$

$$w^{(viii)}(x) = \eta,$$

$$w^{(ix)}(x) = \xi,$$

so that

$$p' = w' = q,$$

$$q' = w'' = r,$$

$$r' = w''' = s,$$

$$s' = w^{(iv)} = t,$$

$$t' = w^{(v)} = u,$$

$$u' = w^{(vi)} = v,$$

$$v' = w^{(vii)} = \lambda,$$

$$\lambda' = w^{(viii)} = \eta,$$

$$\eta' = w^{(ix)} = \xi$$

and

$$\xi'(x) = k_1\eta - k_2 v + k_3 t - k_4 r + A^{10} p + RA^2(-t + k_5 r - A^4).$$

Twizell et al. (1994) solve (6.8), (6.9) directly and by transforming the problem into an equivalent second-order system. The approach to be followed in this chapter will enable eigenvalue problems with odd-order derivatives to be solved, too. Let $\mathbf{V} = [p,\ q,\ r,\ s,\ t,\ u,\ v,\ \lambda,\ \eta,\ \xi]^T$, and let $D \equiv \text{diag}\{\frac{d}{dx}\}$ be a matrix of order 10. Then the equivalent first-order system is

$$\begin{bmatrix} p' \\ q' \\ r' \\ s' \\ t' \\ u' \\ v' \\ \lambda' \\ \eta' \\ \xi' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ A^{10} & 0 & -k_4 & 0 & k_3 & 0 & -k_2 & 0 & k_1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \\ t \\ u \\ v \\ \lambda \\ \eta \\ \xi \end{bmatrix}$$

$$+ RA^2 \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -A^4 & 0 & k_5 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \\ t \\ u \\ v \\ \lambda \\ \eta \\ \xi \end{bmatrix}, \tag{6.11}$$

which is of the form

$$\mathbf{D}\mathbf{V} = \mathbf{B}\mathbf{V} + \mathrm{R}\mathrm{A}^2\mathrm{C}\mathbf{V} \tag{6.12}$$

6.5 NUMERICAL METHOD BASED ON THE (1,1) PADÉ APPROXIMANT

Equation (6.12) may be solved using the recurrence relation

$$\mathbf{V}(\mathrm{x}+\mathrm{h}) = [\exp(\mathrm{hD})]\mathbf{V}(\mathrm{x}) \tag{6.13}$$

Suppose that exp(hD) in (6.13) is replaced by the (1,1) Padé approximant $(\mathrm{I}-\frac{1}{2}\mathrm{hD})^{-1}(\mathrm{I}+\frac{1}{2}\mathrm{hD})$ where I is the identity matrix of order 10. This gives

$$\mathbf{V}(\mathrm{x}+\mathrm{h}) = (\mathrm{I}-\frac{1}{2}\mathrm{hD})^{-1}(\mathrm{I}+\frac{1}{2}\mathrm{hD})\mathbf{V}(x), \tag{6.14}$$

i.e.

$$(\mathrm{I}-\frac{1}{2}\mathrm{hD})\mathbf{V}(\mathrm{x}+\mathrm{h}) = (\mathrm{I}+\frac{1}{2}\mathrm{hD})\mathbf{V}(\mathrm{x}). \tag{6.15}$$

Since

$$\mathrm{D}\mathbf{V}(\mathrm{x}) = \mathrm{B}\mathbf{V}(\mathrm{x}) + \mathrm{RA}^2\mathrm{C}\mathbf{V}(\mathrm{x}), \tag{6.16}$$

it follows that

$$\mathrm{D}\mathbf{V}(\mathrm{x}+\mathrm{h}) = \mathrm{B}\mathbf{V}(\mathrm{x}+\mathrm{h}) + \mathrm{RA}^2\mathrm{C}\mathbf{V}(\mathrm{x}+\mathrm{h}). \tag{6.17}$$

Then, from (6.15),

$$\mathbf{V}(\mathrm{x}+\mathrm{h}) - \frac{1}{2}\mathrm{h}[\mathrm{B}\mathbf{V}(\mathrm{x}+\mathrm{h}) + \mathrm{RA}^2\mathrm{C}\mathbf{V}(\mathrm{x}+\mathrm{h})] = \mathbf{V}(\mathrm{x}) + \frac{1}{2}\mathrm{h}[\mathrm{B}\mathbf{V}(\mathrm{x}) + \mathrm{RA}^2\mathrm{C}\mathbf{V}(\mathrm{x})], \tag{6.18}$$

giving

$$\begin{aligned}(\mathrm{I}-\tfrac{1}{2}\mathrm{hB})\mathbf{V}(\mathrm{x}+\mathrm{h}) - (\mathrm{I}+\tfrac{1}{2}\mathrm{hB})\mathbf{V}(\mathrm{x}) \\ = \mathrm{RA}^2[\tfrac{1}{2}\mathrm{hC}\mathbf{V}(\mathrm{x}+\mathrm{h}) + \tfrac{1}{2}\mathrm{hC}\mathbf{V}(\mathrm{x})],\end{aligned} \tag{6.19}$$

This is of the form

$$\mathrm{P}\mathbf{V}(\mathrm{x}+\mathrm{h}) - \mathrm{Q}\mathbf{V}(\mathrm{x}) = \mathrm{RA}^2[\mathrm{S}\mathbf{V}(\mathrm{x}+\mathrm{h}) + \mathrm{S}\mathbf{V}(\mathrm{x})]; \tag{6.20}$$

with x=0, h, 2h, 3h, ..., Nh, that is $x = x_0, x_1, x_2, x_3, \ldots, x_N$.

In (6.20), the matrices P, Q and S are given by

$$P = \begin{bmatrix} 1 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}h \\ -\frac{1}{2}hA^{10} & 0 & \frac{1}{2}hk_4 & 0 & -\frac{1}{2}hk_3 & 0 & \frac{1}{2}hk_2 & 0 & -\frac{1}{2}hk_1 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2}h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2}h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}h \\ \frac{1}{2}hA^{10} & 0 & -\frac{1}{2}hk_4 & 0 & \frac{1}{2}hk_3 & 0 & -\frac{1}{2}hk_2 & 0 & \frac{1}{2}hk_1 & 1 \end{bmatrix}$$

and

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{2}hA^4 & 0 & \frac{1}{2}hk_5 & 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

6.6 NOTATION AND DETAILS AT THE MESH POINTS

Let

$$\mathbf{V}_0 = [0,\ q_0,\ 0,\ s_0,\ 0,\ u_0,\ 0,\ \lambda_0,\ 0,\ \xi_0]^T,$$

$$\mathbf{V}_m = [p_m,\ q_m,\ r_m,\ s_m,\ t_m,\ u_m,\ v_m,\ \lambda_m,\ \eta_m, \xi_m]^T,\quad m = 1, 2, 3, \ldots, N,$$

$$\mathbf{V}_{N+1} = [0,\ q_{N+1},\ 0,\ s_{N+1},\ 0,\ u_{N+1},\ 0,\ \lambda_{N+1},\ 0,\ \xi_{N+1}]^T,$$

and

$$\mathbf{W} = [q_0,\ s_0,\ u_0,\ \lambda_0,\ \xi_0,\ q_{N+1},\ s_{N+1},\ u_{N+1},\ \lambda_{N+1},\ \xi_{N+1}]^T.$$

Then (6.20), for the general point x_m, becomes

$$-Q\mathbf{V}_m + P\mathbf{V}_{m+1} = RA^2(S\mathbf{V}_m + S\mathbf{V}_{m+1})\ ;\quad m = 1, 2,\ \ldots\ , N-1. \tag{6.21}$$

Recall the mesh points are $x_m = a + mh$ ($m = 0, 1, 2,\ \ldots,\ N,\ N+1$).

For $x = x_0$ equation (6.20) may be written

$$-Q\mathbf{V}_0 + P\mathbf{V}_1 = RA^2(S\mathbf{V}_0 + S\mathbf{V}_1). \tag{6.22}$$

This gives

$$0 - \frac{1}{2}hq_0 + p_1 - \frac{1}{2}q_1 = RA^2(0 + 0), \tag{6.23}$$

$$-q_0 + 0 + q_1 - \frac{1}{2}r_1 = RA^2(0+0), \tag{6.24}$$

$$0 - \frac{1}{2}hs_0 + r_1 - \frac{1}{2}s_1 = RA^2(0+0), \tag{6.25}$$

$$-s_0 + 0 + s_1 - \frac{1}{2}t_1 = RA^2(0+0), \tag{6.26}$$

$$0 - \frac{1}{2}hu_0 + t_1 - \frac{1}{2}u_1 = RA^2(0+0), \tag{6.27}$$

$$-u_0 + 0 + u_1 - \frac{1}{2}v_1 = RA^2(0+0), \tag{6.28}$$

$$0 - \frac{1}{2}h\lambda_0 + t_1 - \frac{1}{2}\lambda_1 = RA^2(0+0), \tag{6.29}$$

$$-\lambda_0 + 0 + \lambda_1 - \frac{1}{2}\eta_1 = RA^2(0+0), \tag{6.30}$$

$$0 - \frac{1}{2}h\xi_0 + \eta_1 - \frac{1}{2}\xi_1 = RA^2(0+0), \tag{6.31}$$

$$\begin{aligned} &-\xi_0 - \tfrac{1}{2}hA^{10}p_1 + \tfrac{1}{2}hk_4r_1 - \tfrac{1}{2}hk_3t_1 + \tfrac{1}{2}hk_2v_1 - \tfrac{1}{2}hk_1\eta_1 + \xi_1 \\ &= RA^2(0 - \tfrac{1}{2}A^4p_1 + \tfrac{1}{2}hk_5r_1 - \tfrac{1}{2}ht_1). \end{aligned} \tag{6.32}$$

i.e.

$$\begin{bmatrix} -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ s_0 \\ u_0 \\ \lambda_0 \\ \xi_0 \\ q_{N+1} \\ s_{N+1} \\ u_{N+1} \\ \lambda_{N+1} \\ \xi_{N+1} \end{bmatrix} + P\mathbf{V}_1 = RA^2S\mathbf{V}_1,$$

i.e.

$$P\mathbf{V}_1 + S_1\mathbf{W} = RA^2S\mathbf{V}_1, \tag{6.33}$$

For $x = x_N$ (6.21) then becomes

$$-Q\mathbf{V}_N + P\mathbf{V}_{N+1} = RA^2(S\mathbf{V}_N + S\mathbf{V}_{N+1}). \tag{6.34}$$

This gives

$$-p_N - \frac{1}{2}hq_N + 0 - \frac{1}{2}q_{N+1} = RA^2(0+0), \tag{6.35}$$

$$-q_N - \frac{1}{2}hr_N + q_{N+1} - 0 = RA^2(0+0), \tag{6.36}$$

$$-r_N - \frac{1}{2}hs_N + 0 - \frac{1}{2}s_{N+1} = RA^2(0+0), \tag{6.37}$$

$$-s_N - \frac{1}{2}ht_N + s_{N+1} - 0 = RA^2(0+0), \tag{6.38}$$

$$-t_N - \frac{1}{2}hu_N + 0 - \frac{1}{2}u_{N+1} = RA^2(0+0), \tag{6.39}$$

$$-u_N - \frac{1}{2}hv_N + u_{N+1} - 0 = RA^2(0+0), \tag{6.40}$$

$$-v_N - \frac{1}{2}h\lambda_N + 0 - \frac{1}{2}\lambda_{N+1} = RA^2(0+0), \tag{6.41}$$

$$-\lambda_N - \frac{1}{2}h\eta_N + \lambda_{N+1} - 0 = RA^2(0+0), \tag{6.42}$$

$$-\eta_N - \frac{1}{2}h\xi_N + 0 - \frac{1}{2}\xi_{N+1} = RA^2(0+0), \tag{6.43}$$

$$\begin{aligned} &-\tfrac{1}{2}A^{10}p_N + \tfrac{1}{2}hk_4r_N - \tfrac{1}{2}k_3t_N + \tfrac{1}{2}k_2v_N - \tfrac{1}{2}k_1\eta_N - \xi_N + \xi_{N+1} \\ &= RA^2[-\tfrac{1}{2}hA^4p_N + \tfrac{1}{2}hk_5r_N - \tfrac{1}{2}ht_N] \end{aligned} \tag{6.44}$$

Equations (6.34)—(6.44) may be written in the form

$$-\mathbf{Q}\mathbf{V}_N + \begin{bmatrix} 0 - \frac{1}{2}hq_{N+1} \\ q_{N+1} - 0 \\ 0 - \frac{1}{2}hs_{N+1} \\ s_{N+1} - 0 \\ 0 - \frac{1}{2}hu_{N+1} \\ u_{N+1} - 0 \\ 0 - \frac{1}{2}h\lambda_{N+1} \\ \lambda_{N+1} - 0 \\ 0 - \frac{1}{2}h\xi_{N+1} \\ \xi_{N+1} - 0 \end{bmatrix} = RA^2\mathbf{S}\mathbf{V}_N, \tag{6.45}$$

i.e.

$$-Q\mathbf{V}_N + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_0 \\ s_0 \\ u_0 \\ \lambda_0 \\ \xi_0 \\ q_{N+1} \\ s_{N+1} \\ u_{N+1} \\ \lambda_{N+1} \\ \xi_{N+1} \end{bmatrix} = RA^2S\mathbf{V}_N,$$

i.e.

$$-Q\mathbf{V}_N + S_{N+1}\mathbf{W} = RA^2S\mathbf{V}_N. \tag{6.46}$$

6.7 IMPLEMENTATION

The next main aim is to form the block matrix

$$\begin{bmatrix} P & & & & & & & S_1 \\ -Q & P & & & & & & \\ & -Q & P & & & & & \\ & & \ddots & \ddots & & & & \\ & & & -Q & P & & & \\ & & & & \ddots & \ddots & & \\ & & & & & -Q & P & \\ & & & & & & -Q & S_{N+1} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \\ \vdots \\ \mathbf{V}_m \\ \vdots \\ \mathbf{V}_N \\ \mathbf{W} \end{bmatrix}$$

$$= RA^2 \begin{bmatrix} S & & & & & & & 0 \\ S & S & & & & & & \\ & S & S & & & & & \\ & & \ddots & \ddots & & & & \\ & & & S & S & & & \\ & & & & \ddots & \ddots & & \\ & & & & & S & S & \\ & & & & & & S & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \\ \vdots \\ \mathbf{V}_m \\ \vdots \\ \mathbf{V}_N \\ \mathbf{W} \end{bmatrix}. \tag{6.47}$$

This is of the form of the generalized eigenvalue problem

$$L\mathbf{v} = \Lambda Z\mathbf{v}, \tag{6.48}$$

where $\Lambda = RA^2$ is "the eigenvalue". This may be written in

$$(L - \Lambda Z)\mathbf{v} = \mathbf{0}, \tag{6.49}$$

and the NAG routine F02BJF may be used to obtain the eigenvalues. Of course, only the smallest eigenvalue (they should all be real and positive) is of interest. (See Chandrasekhar (1961).)

6.8 NUMERICAL RESULTS

The eigenvalue problem (6.7)—(6.9) was solved using the block-formulation (6.47) first of all. The number of interior points of the descretization of the interval $0 \leq x \leq 1$ was taken to be $N = 50$. To compare the computed results with those in the literature, it is covenient to define Q_1 and T_1 by the relationships

$$Q_1 = \frac{Q}{\pi_2} \quad \& \quad T_1 = \frac{T}{\pi_2}. \tag{6.50}$$

Numerical results were obtained for $T_1 = 1000$ and $T_1 = 10000$ with a range of values of Q_1. The numerical values of R_1 for the values of A used in Twizell et al. (1994), were obtained using the NAG library package F02BJF and are given in Tables 6.1 and 6.2 together with the corresponding results, R_C and R_{TBD} of Chandrasekhar (1961) and Twizell et al. (1994) respectively. The values of A reported by Chandrasekhar (1961) are almost identical to those used in Twizell et al. (1994).

Table 6.1: Values of R_1 and A for $T_1 = 1000$ using the method (6.47)

Q_1	A	R_1	R_C	R_{TBD}
10	7.90	2.017×10^4	2.016×10^4	2.016×10^4
50	4.50	1.605×10^4	1.605×10^4	1.604×10^4
100	5.22	1.953×10^4	1.952×10^4	1.951×10^4
500	7.46	6.384×10^4	6.380×10^4	6.377×10^4
1000	8.52	1.192×10^5	1.192×10^5	1.192×10^5

Table 6.2: Values of R_1 and A for $T_1 = 10000$ using the method (6.47)

Q_1	A	R_1	R_C	R_{TBD}
10	12.59	8.983×10^4	8.979×10^4	8.977×10^4
50	3.68	8.556×10^4	8.118×10^4	8.116×10^4
50	3.68	8.148×10^4	8.118×10^4	8.116×10^4
100	3.91	5.547×10^4	5.544×10^4	5.542×10^4
500	6.51	7.550×10^4	7.545×10^4	7.543×10^4
1000	7.98	1.267×10^5	1.267×10^5	1.267×10^5

Tables 6.1 and 6.2 show that, for the values of Q_1 tested, the results ob-

tained by transforming the tenth-order eigenvalue problem into a first-order system are very similar to those reported by Chandrasekhar (1961) and Twizell et al. (1994). This shows that the technique developed in this thesis is reliable and may be adapted for other linear tenth-order eigenvalue problems.

Appendix A

TWELFTH-ORDER EIGENVALUE PROBLEMS

A.1 SUMMARY

Numerical methods are developed for the solution of the twelfth-order eigenvalue problems arising in the modelling of instabilities associated with a rotating fluid heated from below which may also be subject to a uniform magnetic field in the same direction as gravity (Chandrasekhar, 1961).

A.2 ROTATION AND A MAGNETIC FIELD: INSTABILITY AS ORDINARY CONVECTION

Numerical methods for solving high-order eigenvalue problems directly may suffer word-length problems due to the high condition numbers involved. This was experienced in earlier chapters of this thesis. One way of minimizing such difficulties is to transform the given differential equation into a system of lower-order equations and then to use appropriate techniques for solving low-order boundary-value problems.

A.3 TWELFTH-ORDER EIGENVALUE PROBLEMS

Consider again the general twelfth-order differential equation

$$(D^{12} - A^2 - p_1\sigma)[(D^2 - A^2)\{(D^2 - A^2 - \sigma)(D^2 - A^2 - p_2\sigma) - QD^2\}^2$$
$$+TD^2(D^2 - A^2 - p_2\sigma)^2]w(x) + RA^2[(D^2 - A^2 - \sigma)(D^2 - A^2 - p_2\sigma)$$
$$-QD^2](D^2 - A^2 - p_2\sigma)w(x) = 0, \quad 0 < x < 1. \tag{A.1}$$

This is clearly a linear ODE which may be written in the form

$$\begin{aligned}&(D^{12} - k_1D^{10} + k_2D^8 - k_3D^6 + k_4D^4 - k_5D^2 + k_6)w(x)\\&+i(-k_7D^{10} + k_8D^8 - k_9D^6 + k_{10}D^4 - k_{11}D^2 + k_{12})w(x)\\&+RA^2(D^6 - k_{13}D^4 + k_{14}D^2 - k_{15})w(x) = 0\\&+iRA^2(-k_{16}D^4 + k_{17}D^2 - k_{18})w(x) = 0, \quad 0 < x < 1.\end{aligned} \tag{A.2}$$

and with it will be associated the free-free boundary conditions

$$w^{(2i)}(0) = w^{(2i)}(1) = 0; \qquad i = 0, 1, 2, 3, 4, 5. \tag{A.3}$$

In (A.2), the coefficients k_i $(i = 1, 2, \ldots, 18)$ are given by

$$k_1 = 6A^2 + 2Q,$$

$$k_2 = 15A^4 + 8A^2Q + Q^2 - [2p_2 + 2p_1(1 + p_2) + (1 + p_2)^2]\mu^2 + T,$$

$$k_3 = 20A^6 + 8A^4Q - 4[2p_2 + 2p_1(1 + p_2) + (1 + p_2)^2]\mu^2A^2 + 3A^2T$$
$$+2A^2Q^2 - 2(p_1 + p_2 + p_1p_2)\mu^2,$$

$$k_4 = 15A^8 + 8A^6Q - 6[2p_2 + 2p_1(1 + p_2) + (1 + p_2)^2]\mu^2A^4 + 3A^2T$$
$$+A^4Q^2 - 4(p_1 + p_2 + p_1p_2)\mu^2A^2 + [p_2^2 + 2p_1p_2(1 + p_2)]\mu^4$$
$$-p_2(2p_1 + p_2)T,$$

$$k_5 = 6A^{10} + 2A^8Q - 4[2p_2 + 2p_1(1 + p_2) + (1 + p_2)^2]\mu^2A^6 + A^6T$$
$$-2(p_1 + p_2 + p_1p_2)\mu^2A^4 + 2[p_2^2 + 2p_1p_2(1 + p_2)]\mu^4A^2$$
$$-p_2(2p_1 + p_2)T\mu^2A^2,$$

$$k_6 = A^{12} - [2p_2 + 2p_1(1+p_2) + (1+p_2)^2]\mu^2 A^8 + [p_2^2 + 2p_1p_2(1+p_2)]\mu^4 A^4,$$

$$k_7 = (2 + p_1 + 2p_2)\mu,$$

$$k_8 = 2(1 + p_1 + 2p_2)\mu Q + 5(2 + p_1 + 2p_2)\mu A^2,$$

$$k_9 = 10(2 + p_1 + 2p_2)\mu A^4 + 6(1 + p_1 + p_2)\mu Q A^2$$
$$-[2p_1p_2 + 2p_2(1+p_2) + p_1(1+p_2)^2]\mu^3 + 3(p_1 + 2P_2)T\mu + p_1\mu Q^2,$$

$$k_{10} = 10(2 + p_1 + 2p_2)\mu A^6 + 6(1 + p_1 + p_2)\mu Q A^4 + 2(p_1 + 2p_2)T\mu A^2$$
$$-3[2p_1p_2 + 2p_2(1+p_2) + p_1(1+p_2)^2]\mu^3 A^2 + p_1\mu Q^2 A^2,$$

$$k_{11} = 5(2 + p_1 + 2p_2)\mu A^8 + 2(1 + p_1 + p_2)\mu Q A^6$$
$$-3[(2p_1p_2 + 2p_2(1+p_2) + p_1(1+p_2)^2]\mu^3 A^4$$
$$+(p_1 + 2p_2)T\mu A^4 + p_1p_2^2\mu^5 - p_1p_2^2\mu^3 T,$$

$$k_{12} = (2 + p_1 + 2p_2)\mu A^{10} - [2p_1p_2 + 2p_2(1+p_2) + p_1(1+p_2)^2]\mu^3 A^6$$
$$+p_1{p_2}^2\mu^5 A^2,$$

$$k_{13} = 3A^2 + Q,$$

$$k_{14} = 3A^4 + QA^2 - p_2(2 + p_2)\mu^2,$$

$$k_{15} = A^6 - p_2(2 + p_2)\mu^2 A^2,$$

$$k_{16} = (1 + 2p_2)\mu,$$

$$k_{17} = 2(1 + 2p_2)\mu A^2 + p_2\mu Q,$$

$$k_{18} = (1 + 2p_2)\mu A^4 + p_2^2\mu^3.$$

A first-order numerical method will be used to estimate the critical values of R associated with the eigenvalue problem (A.2), (A.3) for different values of A, T and Q together with the corresponding value of $\sigma = i\mu$. The variables, parameters and constants in (A.1), and the other terms on which they depend are described in Chapter 6.

A.4 FIRST-ORDER SYSTEM

Consider the linear differential equation (A.2) written in the form

$$\begin{aligned}
D^{12}w &= k_1D^{10}w - k_2D^8w + k_3D^6w - k_4D^4w + k_5D^2w - k_6w \\
&+ik_7D^{10}w - ik_8D^8w + ik_9D^6w - ik_{10}D^4w + ik_{11}D^2w - ik_{12}w \\
&-RA^2D^6w + RA^2k_{13}D^4w - RA^2k_{14}D^2w + RA^2k_{15}w \\
&+iRA^2k_{16}D^4w - iRA^2k_{17}D^2w + iRA^2k_{18}w, \quad 0 < x < 1.
\end{aligned} \tag{A.4}$$

Let

$$\begin{aligned}
w &= p, \\
w'(x) &= q, \\
w''(x) &= r, \\
w'''(x) &= s, \\
w^{(iv)}(x) &= t, \\
w^{(v)}(x) &= u, \\
w^{(vi)}(x) &= v, \\
w^{(vii)}(x) &= \lambda, \\
w^{(viii)}(x) &= \eta, \\
w^{(ix)}(x) &= \xi, \\
w^{(x)}(x) &= \alpha, \\
w^{(xi)}(x) &= \beta,
\end{aligned}$$

so that

$$\begin{aligned}
p' &= w' = q, \\
q' &= w'' = r, \\
r' &= w''' = s,
\end{aligned}$$

$$s' = w^{(iv)} = t,$$

$$t' = w^{(v)} = u,$$

$$u' = w^{(vi)} = v,$$

$$v' = w^{(vii)} = \lambda,$$

$$\lambda' = w^{(viii)} = \eta,$$

$$\eta' = w^{(ix)} = \xi$$

$$\xi' = w^{(x)} = \alpha$$

$$\alpha' = w^{(xi)} = \beta$$

and

$$\begin{aligned}\beta' = w^{(xii)} &= k_1\alpha - k_2\eta + k_3v - k_4t + k_5r - k_6p + ik_7\alpha - ik_8\eta + ik_9v - ik_{10}t \\ &+ik_{11}r - ik_{12}p - RA^2v + RA^2k_{13}t - RA^2k_{14}r + RA^2k_{15}p \\ &+iRA^2k_{16}t - iRA^2k_{17}r + iRA^2k_{18}p.\end{aligned}$$

Twizell et al. (1994) solve (A.2), (A.3) directly and by transforming the problem into an equivalent second-order system. The approach to be followed in this appendix will enable eigenvalue problems with odd-order derivatives to be solved, too. Let $\mathbf{V} = [p,\ q,\ r,\ s,\ t,\ u,\ v,\ \lambda,\ \eta,\ \xi,\ \alpha,\ \beta]^T$, and let, now, $D \equiv \mathrm{diag}\{\frac{d}{dx}\}$ be a matrix of order 12. Then the equivalent first-order system is

$$\begin{bmatrix} p' \\ q' \\ r' \\ s' \\ t' \\ u' \\ v' \\ \lambda' \\ \eta' \\ \xi' \\ \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ k_6 & 0 & -k_5 & 0 & k_4 & 0 & -k_3 & 0 & -k_2 & 0 & k_1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \\ t \\ u \\ v \\ \lambda \\ \eta \\ \xi \\ \alpha \\ \beta \end{bmatrix}$$

$$+i \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -k_{12} & 0 & k_{11} & 0 & -k_{10} & 0 & k_9 & 0 & -k_8 & 0 & k_7 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \\ t \\ u \\ v \\ \lambda \\ \eta \\ \xi \\ \alpha \\ \beta \end{bmatrix}$$

$$
+\mathrm{RA}^2 \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\mathrm{k}_{15} & 0 & -\mathrm{k}_{14} & 0 & \mathrm{k}_{13} & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \mathrm{p} \\ \mathrm{q} \\ \mathrm{r} \\ \mathrm{s} \\ \mathrm{t} \\ \mathrm{u} \\ \mathrm{v} \\ \lambda \\ \eta \\ \xi \\ \alpha \\ \beta \end{bmatrix}
$$

$$
+\,\mathrm{iRA}^2 \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\mathrm{k}_{18} & 0 & -\mathrm{k}_{17} & 0 & \mathrm{k}_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \mathrm{p} \\ \mathrm{q} \\ \mathrm{r} \\ \mathrm{s} \\ \mathrm{t} \\ \mathrm{u} \\ \mathrm{v} \\ \lambda \\ \eta \\ \xi \\ \alpha \\ \beta \end{bmatrix}
\qquad \text{(A.5)}
$$

which is of the form

$$
\mathrm{D}\mathbf{V} = \mathrm{C}\mathbf{V} + \mathrm{iE}\mathbf{V} + \mathrm{RA}^2\mathrm{F}\mathbf{V} + \mathrm{iRA}^2\mathrm{G}\mathbf{V} \qquad \text{(A.6)}
$$

A.5 NUMERICAL METHOD BASED ON THE (1,1) PADÉ APPROXIMANT

Equation (A.6) may be solved using the recurrence relation

$$\mathbf{V}(x+h) = [\exp(hD)]\mathbf{V}(x) \tag{A.7}$$

Suppose that exp(hD) in (A.7) is replaced by the (1,1) Padé approximant $(I - \frac{1}{2}hD)^{-1}(I + \frac{1}{2}hD)$ where I is the identity matrix of order 12. This gives

$$\mathbf{V}(x+h) = (I - \frac{1}{2}hD)^{-1}(I + \frac{1}{2}hD)\mathbf{V}(x), \tag{A.8}$$

thus

$$(I - \frac{1}{2}hD)\mathbf{V}(x+h) = (I + \frac{1}{2}hD)\mathbf{V}(x). \tag{A.9}$$

Since

$$D\mathbf{V}(x) = C\mathbf{V}(x) + iE\mathbf{V}(x) + RA^2F\mathbf{V}(x) + iRA^2G\mathbf{V}(x), \tag{A.10}$$

it follows that

$$D\mathbf{V}(x+h) = C\mathbf{V}(x+h) + iE\mathbf{V}(x+h) + RA^2F\mathbf{V}(x+h) + iRA^2G\mathbf{V}(x+h). \tag{A.11}$$

Then, from (A.9),

$$\begin{aligned} &\mathbf{V}(x+h) - \tfrac{1}{2}h[C\mathbf{V}(x+h) + iE\mathbf{V}(x+h) + RA^2F\mathbf{V}(x+h) + iRA^2G\mathbf{V}(x+h)] \\ &\quad = \mathbf{V}(x) + \tfrac{1}{2}h[C\mathbf{V}(x) + iE\mathbf{V}(x) + RA^2F\mathbf{V}(x) + iRA^2G\mathbf{V}(x)], \end{aligned} \tag{A.12}$$

giving

$$\begin{aligned} &(I - \tfrac{1}{2}hC - i\tfrac{1}{2}hE - \tfrac{1}{2}hRA^2F - i\tfrac{1}{2}hRA^2G)\mathbf{V}(x+h) \\ &\quad = (I + \tfrac{1}{2}hC + i\tfrac{1}{2}hE + \tfrac{1}{2}hRA^2F + i\tfrac{1}{2}hRA^2G)\mathbf{V}(x+h), \end{aligned} \tag{A.13}$$

implies that

$$\begin{aligned} &(I - \tfrac{1}{2}hC)\mathbf{V}(x+h) - (I + \tfrac{1}{2}hC)\mathbf{V}(x) \\ &\quad = i[\tfrac{1}{2}hE\mathbf{V}(x+h) + \tfrac{1}{2}hE\mathbf{V}(x)] \\ &\quad\quad +RA^2[\tfrac{1}{2}hF\mathbf{V}(x+h) + \tfrac{1}{2}hF\mathbf{V}(x)] \\ &\quad\quad\quad +iRA^2[\tfrac{1}{2}hG\mathbf{V}(x+h) + \tfrac{1}{2}hG\mathbf{V}(x)], \end{aligned} \tag{A.14}$$

This is of the form

$$\mathbf{P V}(x+h) - \mathbf{Q V}(x) = i[S_1\mathbf{V}(x+h) + S_1\mathbf{V}(x)] + RA^2[S_2\mathbf{V}(x+h) + S_2\mathbf{V}(x)] \\ +iRA^2[S_3\mathbf{V}(x+h) + S_3\mathbf{V}(x)] \tag{A.15}$$

with x=0, h, 2h, 3h, ..., Nh, that is $x = x_0,\ x_1,\ x_2,\ x_3,\ \dots,x_N$.

In (A.15), the matrices P, Q, S_1, S_2 and S_3 are given by

$$P = \begin{bmatrix} 1 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 \\ a_0 & 0 & a_1 & 0 & a_2 & 0 & a_3 & 0 & a_4 & 0 & a_5 & 1 \end{bmatrix}$$

with

$$a = -\frac{1}{2}h,\ a_0 = \frac{1}{2}k_6,\ a_1 = -\frac{1}{2}k_5,\ a_2 = \frac{1}{2}k_4, \\ a_3 = -\frac{1}{2}k_3,\ a_4 = \frac{1}{2}k_2,\ a_5 = -\frac{1}{2}k_1,$$

$$Q = \begin{bmatrix} 1 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & 0 \\ b_0 & 0 & b_1 & 0 & b_2 & 0 & b_3 & 0 & b_4 & 0 & b_5 & 1 \end{bmatrix}$$

with

$$b = \frac{1}{2}h, \ b_0 = -\frac{1}{2}k_6, \ b_1 = \frac{1}{2}k_5, \ b_2 = -\frac{1}{2}k_4,$$

$$b_3 = \frac{1}{2}k_3, \ b_4 = -\frac{1}{2}k_2, \ b_5 = \frac{1}{2}k_1,$$

$$S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_0 & 0 & c_1 & 0 & c_2 & 0 & c_3 & 0 & c_4 & 0 & c_5 & 0 \end{bmatrix}$$

with

$$c_0 = -\frac{1}{2}k_{12},\ c_1 = \frac{1}{2}k_{11},\ c_2 = -\frac{1}{2}k_{10},\ c_3 = \frac{1}{2}k_9,$$
$$c_4 = -\frac{1}{2}k_8,\ c_5 = \frac{1}{2}k_7,$$

$$S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_0 & 0 & d_1 & 0 & d_2 & 0 & d_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with

$$d_0 = \frac{1}{2}k_{15},\ d_1 = -\frac{1}{2}k_{14},\ d_2 = \frac{1}{2}k_{13},\ d_3 = -\frac{1}{2}h,$$

and

$$S_3 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ e_0&0&e_1&0&e_2&0&0&0&0&0&0&0 \end{bmatrix}$$

with

$$e_0 = \frac{1}{2}k_{18},\ e_1 = -\frac{1}{2}k_{17},\ e_2 = \frac{1}{2}k_{16}.$$

A.6 NOTATION AND DETAILS AT THE MESH POINTS

Let

$$\mathbf{V}_0 = [0,\ q_0,\ 0,\ s_0,\ 0,\ u_0,\ 0,\ \lambda_0,\ 0,\ \xi_0,\ 0,\ \beta_0]^T,$$

$$\mathbf{V}_m = [p_m,\ q_m,\ r_m,\ s_m,\ t_m,\ u_m,\ v_m,\ \lambda_m,\ \eta_m,\ \xi_m,\ \alpha_m,\ \beta_m]^T,$$

$m = 1, 2, 3, \ldots, N$,

$$\mathbf{V}_{N+1} = [0,\ q_{N+1},\ 0,\ s_{N+1},\ 0,\ u_{N+1},\ 0,\ \lambda_{N+1},\ 0,\ \xi_{N+1},\ 0,\ \beta_{N+1}]^T,$$

and

$$\mathbf{W} = [q_0,\ s_0,\ u_0,\ \lambda_0,\ \xi_0,\ \beta_0,\ q_{N+1},\ s_{N+1},\ u_{N+1},\ \lambda_{N+1},\ \xi_{N+1},\ \beta_{N+1}]^T.$$

Then (A.15), for the general point x_m, becomes

$$-Q\mathbf{V}_m + P\mathbf{V}_{m+1} = i(S_1\mathbf{V}_m + S_1\mathbf{V}_{m+1}) + RA^2(S_2\mathbf{V}_m + S_2\mathbf{V}_{m+1}) + iRA^2(S_3\mathbf{V}_m + S_3\mathbf{V}_{m+1}) \;;\quad m = 1, 2, \ldots, N-1. \tag{A.16}$$

Recall the mesh points are $x_m = a + mh \quad (m = 0, 1, 2, \ldots, N, N+1)$.

For $x = x_0$ equation (A.16) may be written

$$-Q\mathbf{V}_0 + P\mathbf{V}_1 = i(S_1\mathbf{V}_0 + S_1\mathbf{V}_1) + RA^2(S_2\mathbf{V}_0 + S_2\mathbf{V}_1) + iRA^2(S_3\mathbf{V}_0 + S_3\mathbf{V}_1). \tag{A.17}$$

This gives

$$0 - \frac{1}{2}hq_0 + p_1 - \frac{1}{2}q_1 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.18}$$

$$-q_0 + 0 + q_1 - \frac{1}{2}r_1 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.19}$$

$$0 - \frac{1}{2}hs_0 + r_1 - \frac{1}{2}s_1 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.20}$$

$$-s_0 + 0 + s_1 - \frac{1}{2}t_1 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.21}$$

$$0 - \frac{1}{2}hu_0 + t_1 - \frac{1}{2}u_1 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.22}$$

$$-u_0 + 0 + u_1 - \frac{1}{2}v_1 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.23}$$

$$0 - \frac{1}{2}h\lambda_0 + t_1 - \frac{1}{2}\lambda_1 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.24}$$

$$-\lambda_0 + 0 + \lambda_1 - \frac{1}{2}\eta_1 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.25}$$

$$0 - \frac{1}{2}h\xi_0 + \eta_1 - \frac{1}{2}\xi_1 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.26}$$

$$-\xi_0 + 0 + \xi_1 - \frac{1}{2}\alpha_1 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.27}$$

$$0 - \frac{1}{2}h\beta_0 + \alpha_1 - \frac{1}{2}\beta_1 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.28}$$

$$\begin{aligned}
&-\beta_0 + \tfrac{1}{2}hk_6p_1 - \tfrac{1}{2}hk_5r_1 + \tfrac{1}{2}hk_4t_1 - \tfrac{1}{2}hk_3v_1 + \tfrac{1}{2}hk_2\eta_1 - \tfrac{1}{2}hk_1\alpha_1 + \beta_1 \\
&\quad = i(0 - \tfrac{1}{2}hk_{12}p_1 + \tfrac{1}{2}hk_{11}r_1 - \tfrac{1}{2}hk_{10}t_1) + \tfrac{1}{2}k_9v_1 - \tfrac{1}{2}hk_8\eta_1 + \tfrac{1}{2}hk_7\alpha_1) \\
&\qquad + RA^2(0 + \tfrac{1}{2}hk_{15}p_1 - \tfrac{1}{2}hk_{14}r_1 + \tfrac{1}{2}hk_{13}t_1 - \tfrac{1}{2}v_1) \\
&\qquad\quad + iRA^2(0 + \tfrac{1}{2}hk_{18}p_1 - \tfrac{1}{2}hk_{17}r_1 + \tfrac{1}{2}hk_{16}t_1)
\end{aligned} \tag{A.29}$$

i.e.

$$\begin{bmatrix} -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ s_0 \\ u_0 \\ \lambda_0 \\ \xi_0 \\ \alpha_0 \\ q_{N+1} \\ s_{N+1} \\ u_{N+1} \\ \lambda_{N+1} \\ \xi_{N+1} \\ \alpha_{N+1} \end{bmatrix} + P\mathbf{V}_1$$

$$= iS_1\mathbf{V}_1 + RA^2S_2\mathbf{V}_1 + iRA^2S_3\mathbf{V}_1,$$

that is

$$P\mathbf{V}_1 + S_0\mathbf{W} = iS_1\mathbf{V}_1 + RA^2S_2\mathbf{V}_1 + iRA^2S_3\mathbf{V}_1, \tag{A.30}$$

For $x = x_N$ (A.16) then becomes

$$\begin{aligned} -Q\mathbf{V}_N + P\mathbf{V}_{N+1} &= i(S_1\mathbf{V}_N + S_1\mathbf{V}_{N+1}) + RA^2(S_2\mathbf{V}_N + S_2\mathbf{V}_{N+1}) \\ &+iRA^2(S_3\mathbf{V}_N + S_3\mathbf{V}_{N+1}). \end{aligned} \tag{A.31}$$

This gives

$$-p_N - \frac{1}{2}hq_N + 0 - \frac{1}{2}q_{N+1} = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.32}$$

$$-q_N - \frac{1}{2}hr_N + q_{N+1} - 0 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.33}$$

$$-r_N - \frac{1}{2}hs_N + 0 - \frac{1}{2}s_{N+1} = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.34}$$

$$-s_N - \frac{1}{2}ht_N + s_{N+1} - 0 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.35}$$

$$-t_N - \frac{1}{2}hu_N + 0 - \frac{1}{2}u_{N+1} = i(0+0) + RA^2(0+0) + iRA^2(0+0), \tag{A.36}$$

$$-u_N - \frac{1}{2}hv_N + u_{N+1} - 0 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \quad (A.37)$$

$$-v_N - \frac{1}{2}h\lambda_N + 0 - \frac{1}{2}\lambda_{N+1} = i(0+0) + RA^2(0+0) + iRA^2(0+0), \quad (A.38)$$

$$-\lambda_N - \frac{1}{2}h\eta_N + \lambda_{N+1} - 0 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \quad (A.39)$$

$$-\eta_N - \frac{1}{2}h\xi_N + 0 - \frac{1}{2}\xi_{N+1} = i(0+0) + RA^2(0+0) + iRA^2(0+0), \quad (A.40)$$

$$-\xi_N - \frac{1}{2}h\alpha_N + \frac{1}{2}\xi_{N+1} - 0 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \quad (A.41)$$

$$-\alpha_N - \frac{1}{2}h\beta_N + 0 - \frac{1}{2}\beta_{N+1} - 0 = i(0+0) + RA^2(0+0) + iRA^2(0+0), \quad (A.42)$$

$$\begin{aligned} &\tfrac{1}{2}k_6p_N - \tfrac{1}{2}hk_5r_N + \tfrac{1}{2}k_4t_N - \tfrac{1}{2}k_3v_N + \tfrac{1}{2}k_2\eta_N - \tfrac{1}{2}k_1\alpha_N - \beta_N + \beta_{N+1} \\ &\quad = i[-\tfrac{1}{2}hk_{12}p_N + \tfrac{1}{2}hk_{11}r_N - \tfrac{1}{2}hk_{10}t_N + \tfrac{1}{2}hk_9v_N - \tfrac{1}{2}hk_8\eta_N + \tfrac{1}{2}hk_7\alpha_N] \\ &\qquad + RA^2[\tfrac{1}{2}hk_{15}p_N - \tfrac{1}{2}hk_{14}r_N + \tfrac{1}{2}hk_{13}t_N - \tfrac{1}{2}hv_N] \\ &\qquad + iRA^2[\tfrac{1}{2}hk_{18}p_N - \tfrac{1}{2}hk_{17}r_N + \tfrac{1}{2}hk_{16}t_N] \end{aligned} \quad (A.43)$$

Equations (A.32)—(A.43) may be written in the form

$$-\mathbf{Q}\mathbf{V}_N + \begin{bmatrix} 0 - \frac{1}{2}hq_{N+1} \\ q_{N+1} - 0 \\ 0 - \frac{1}{2}hs_{N+1} \\ s_{N+1} - 0 \\ 0 - \frac{1}{2}hu_{N+1} \\ u_{N+1} - 0 \\ 0 - \frac{1}{2}h\lambda_{N+1} \\ \lambda_{N+1} - 0 \\ 0 - \frac{1}{2}h\xi_{N+1} \\ \xi_{N+1} - 0 \\ 0 - \frac{1}{2}h\beta_{N+1} \\ \beta_{N+1} - 0 \end{bmatrix} = iS_1\mathbf{V}_N + RA^2S_2\mathbf{V}_N + iRA^2S_3\mathbf{V}_N, \quad (A.44)$$

that is

$$
-\mathbf{QV}_N + \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} q_0 \\ s_0 \\ u_0 \\ \lambda_0 \\ \xi_0 \\ \beta_0 \\ q_{N+1} \\ s_{N+1} \\ u_{N+1} \\ \lambda_{N+1} \\ \xi_{N+1} \\ \beta_{N+1} \end{bmatrix}
$$

$$
= iS_1\mathbf{V}_N + RA^2S_2\mathbf{V}_N + iRA^2S_3\mathbf{V}_N,
$$

that is

$$
-\mathbf{QV}_N + S_{N+1}\mathbf{W} = iS_1\mathbf{V}_N + RA^2S_2\mathbf{V}_N + iRA^2S_3\mathbf{V}_N. \tag{A.45}
$$

A.7 IMPLEMENTATION

The next main aim is to form the block matrix

$$
\begin{bmatrix}
P & & & & & & & S_0 \\
-Q & P & & & & & & \\
& -Q & P & & & & & \\
& & \ddots & \ddots & & & & \\
& & & -Q & P & & & \\
& & & & \ddots & \ddots & & \\
& & & & & -Q & P & \\
& & & & & & -Q & S_{N+1}
\end{bmatrix}
\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \\ \vdots \\ \mathbf{V}_m \\ \vdots \\ \mathbf{V}_N \\ \mathbf{W} \end{bmatrix}
$$

$$
= \mathrm{i}\begin{bmatrix}
S_1 & & & & & & & 0 \\
S_1 & S_1 & & & & & & \\
 & S_1 & S_1 & & & & & \\
 & & \ddots & \ddots & & & & \\
 & & & S_1 & S-1 & & & \\
 & & & & \ddots & \ddots & & \\
 & & & & & S-1 & S_1 & \\
 & & & & & & S_1 & 0
\end{bmatrix}
\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \\ \vdots \\ \mathbf{V}_m \\ \vdots \\ \mathbf{V}_N \\ \mathbf{W} \end{bmatrix}
$$

$$
+ RA^2\begin{bmatrix}
S_2 & & & & & & & 0 \\
S_2 & S_2 & & & & & & \\
 & S_2 & S_2 & & & & & \\
 & & \ddots & \ddots & & & & \\
 & & & S_2 & S_2 & & & \\
 & & & & \ddots & \ddots & & \\
 & & & & & S_2 & S_2 & \\
 & & & & & & S_2 & 0
\end{bmatrix}
\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \\ \vdots \\ \mathbf{V}_m \\ \vdots \\ \mathbf{V}_N \\ \mathbf{W} \end{bmatrix}
$$

$$
= \mathrm{i}RA^2\begin{bmatrix}
S_3 & & & & & & & 0 \\
S_3 & S & & & & & & \\
 & S_3 & S_3 & & & & & \\
 & & \ddots & \ddots & & & & \\
 & & & S_3 & S_3 & & & \\
 & & & & \ddots & \ddots & & \\
 & & & & & S_3 & S_3 & \\
 & & & & & & S_3 & 0
\end{bmatrix}
\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{V}_3 \\ \vdots \\ \mathbf{V}_m \\ \vdots \\ \mathbf{V}_N \\ \mathbf{W} \end{bmatrix}. \tag{A.46}
$$

That is

$$
J\mathbf{V} = \mathrm{i}C_1\mathbf{V} + RA^2C_2\mathbf{V} + \mathrm{i}RA^2C_3\mathbf{V}, \tag{A.47}
$$

which may be written as

$$
(J - \mathrm{i}C_1)\mathbf{V} = RA^2(C_2 + \mathrm{i}C_3)\mathbf{V}. \tag{A.48}
$$

This is of the form of the generalized eigenvalue problem

$$\mathbf{L}\mathbf{v} = \Lambda \mathbf{Z}\mathbf{v}, \tag{A.49}$$

where $\Lambda = RA^2$ is "the eigenvalue". It may be written in the form

$$(\mathbf{L} - \Lambda \mathbf{Z})\mathbf{v} = \mathbf{0}, \tag{A.50}$$

and the NAG routine F02BJF may be used to obtain the eigenvalues. Of course, only the smallest eigenvalue (they should all be real and positive) is of interest. (See Chandrasekhar (1961).)

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