High frequency diffraction of a electromagnetic plane wave by an imperfectly conducting rectangular cylinder.

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6 February 2010,Revised, February 3, 2012

Abstract. We shall consider the problem of determining the scattered far wave field produced when a plane E-polarized wave is incident on an imperfectly conducting rectangular cylinder. By using the uniform asymptotic solution for the problem of the diffraction of a plane wave by a right-angled impedance wedge, in conjunction with Keller's method, and multiple diffraction then a high frequency far field solution to the problem is given for two edge diffractions.

Keywords: High frequency, Diffraction,Impedance rectangle,Wedge asymptotics,Absorbing rectangular cylinder. LaTeX

1. Introduction

The electromagnetic wave propagation of radio, television, and mobile phone signals in cities depends on the effect of building corners and their surface cladding. In particular the effect of these building corners is of paramount importance for the signal strength of the phones, see references in Nechayev[1] and Bertoni[2]. Modern buildings can be clad with solar cells and other esoteric Metamaterials that have a degree of conductivity in there makeup Driessen[3]. Skyscrapers have metal reinforcement structures imbedded into concrete, and to produce even greater height and strength it is proposed to build them with carbon fibre materials, this again introduces an amount of conductivity. Increasingly important today is the need to provide electric shielding for electronic equipment from stray electromagnetic waves; a metal lined room surrounded by a dielectric building material is one way to provide such protection Ando[4]. In stealth technology some buildings and structures are designed to absorb as much incident radiation as possible to avoid detection; such surfaces should ideally be perfect absorbers Driessen[3],Chen[5],Rawlins[6],Tretyakov[7]. The radiation signature of such dielectric buildings with a degree of conductivity, and hence a complex refractive index, is of practical importance for many purposes. Experimental determination of such complex refractive indices have been carried out by a number of researchers for some common building materials by Sato[8], Li[9], and see references in Bertoni[2]; and for thin film suitable for cladding by Siqueiros[10] and Driessen and de Dood[3]. A building of rectangular cross-section can be modelled by four of these corners. There has been some work on modelling such absorbent corners and it has been shown that with appropriate polarization buildings can be effectively modelled for diffraction effects by a rectangular impedance wedge in two dimensions see Zhao[11],El-Sallabi[12],Demetrescu[13][14]. To obtain quantitative and qualitative results for the signal strength far from the building when there are multiple diffraction from such corners an effective approach is to use the Keller method of the geometrical theory of diffraction(GTD). This method requires information about the "diffraction coefficient" which are obtained from the solution of canonical impedance wedge problems. These coefficients need to be uniformly valid in the angular variables in order that the method can be used successfully when considering multiple diffractions at different corners. In a previous work Rawlins[15] this aspect was addressed by using a simple, analytically convenient, exact solution to the specific problem of the diffraction of a plane wave by a right-angled impedance wedge obtained by Rawlins[16]. In this work, Rawlins[15], useful asymptotic results were obtained for the far-field across singular ray directions where the usual diffraction coefficient used in high frequency methods breaks down. In this work we shall use these results to apply to the practical situation of the scattering by an absorbing rectangular building. The determination of the far field when a high-frequency E-polarized electromagnetic plane-wave is obliquely incident on an imperfectly conducting rectangular cylinder is obtained by applying Keller's method of geometrical diffraction. Oblique incidence corresponds to the situation where incident plane wave ray is not running parallel along any of the faces of the rectangular cylinder. To achieve this the uniform results of Rawlins[15] for the diffraction coefficient for a right-angled impedance wedge is used in conjunction with the multiple diffraction that arises from waves travelling from corner to corner of the rectangle.

There are a number of works on the diffraction of electromagnetic waves by perfectly conducting rectangular cylinders, in particular Morse[17], van Bladel[18],Mei and van Bladel[19],Mei[20],Hinata[21],Cheung[22]. The only theoretical work, known to the author, to have dealt with the diffraction of electromagnetic waves by imperfectly conducting rectangular cylinders is by Topsakal[23].This work uses the modified Wiener-Hopf technique to produce integral equations that are asymptotically approximated for high frequency when the impedance parameter is purely imaginary. Thus this solution represents high frequency diffraction by an ideal rectangular cylinder whose surfaces support surface waves without the cylinder being absorbent.

In section 2 we shall formulate the mathematical boundary value problem that describes the physical problem of the diffraction of an E-polarized plane wave by an absorbing rectangular cylinder. In section 3 we shall introduce the Keller method of geometric theory of diffraction(GTD) and its extensions to deal with multiple diffraction.We shall derive results for the diffraction coefficient for the canonical problem of diffraction by an impedance corner.We shall introduce appropriate coordinate systems. These results will be used in combination with Keller's method of GTD to cope with the effect of multiple diffraction by the corners of the rectangle. In section 4 we shall derive the diffracted far field for when the incident plane wave is obliquely incident upon the rectangular cylinder and the observation point is not on the specular or shadow boundaries created by the plane wave incident upon the rectangular cylinder. In section 5 the special situations where the observation point is on these specular and shadow boundaries is calculated. The results of sections 4 and 5 are combined to produce the complete far-field expressions. These are used together with Mathematica to produce graphical plots of the modulus of the scattered far field in section 6. In the next section 7 the scattering cross section for a highly conducting but absorbing rectangular cylinder for oblique incidence is derived. This important quantity, is a measure of the electromagnetic shadow cast by the diffracting object, and is used in scattering applications and is derived for oblique incidence by carrying out some asymptotic approximations on the complicated expression for the forward far field. Finally the work ends with conclusions which discuss certain aspects of the work and further work and generalizations.

2. Formulation of the boundary value problem

An electrically-polarized incident plane wave $\mathbf{E}_i = u_i(r,\theta)\mathrm{e}^{-i\omega t}\hat{\mathbf{z}}$

$$u_i(r,\theta) = \mathrm{e}^{-ikr\sin(\theta+\theta_0)}, \tag{1}$$

is incident on an imperfectly conducting rectangular cylinder: $|x| \leq a$, and $|y| \leq b, -\infty < z < \infty$; see Figure 1, where the polar coordinates (r,θ) are defined by $x = r\cos\theta, y = r\sin\theta$, and $\hat{\mathbf{z}}$ is the usual unit vector in the positive z-direction. The permeability, permittivity, and conductivity of the cylinder are μ, ϵ, and σ respectively; and the complex refractive index of the cylinder material is given by

$$N = \sqrt{\frac{\mu}{\mu_0}\left(\frac{\epsilon}{\epsilon_0} + \frac{i\sigma}{\omega\epsilon_0}\right)}.$$

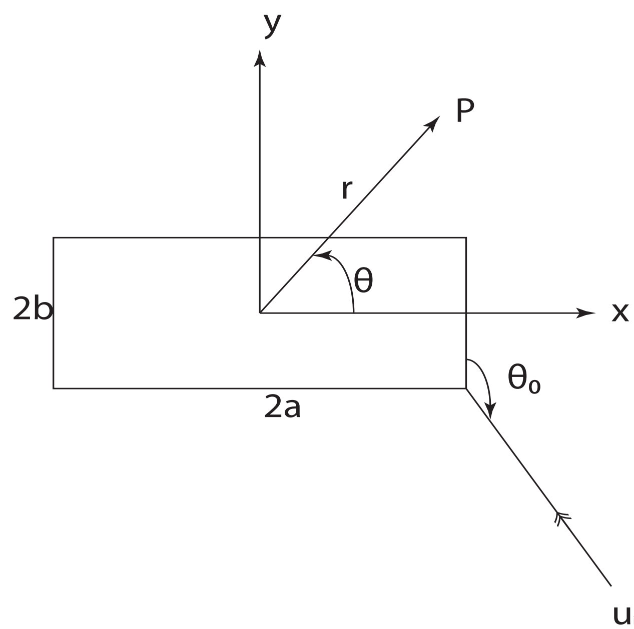

Figure 1. Diffraction of a plane wave by an absorbing rectangular cylinder.

We can formulate the problem mathematically as follows:

If the total field is denoted by $\mathbf{E} = \mathbf{E}_i + \mathbf{E}_s$ where $\mathbf{E}_s$ is the scattered field then if $\mathbf{E} = u(r,\theta)\mathrm{e}^{-i\omega t}\hat{\mathbf{z}}$ denotes the total electric field parallel to the z-axis then Maxwell's equations give,

$$(\Delta + k^2)u = 0, \quad |x| \geq a, |y| \geq b, -\infty < z < \infty,$$

where

$$\Delta \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2},$$

with $k^2 = \epsilon_0 \mu_0 \omega^2$. For a unique solution u, must satisfy edge conditions at the corners and a radiation condition at infinity. The boundary conditions appropriate to the present problem are given by

$$\frac{\partial u}{\partial y} - ik\cos\vartheta u = 0, \quad (|x| \leq a, \quad y = b),$$

$$\frac{\partial u}{\partial x} + ik\cos\vartheta u = 0, \quad (|y| \leq b, \quad x = -a),$$
$$\frac{\partial u}{\partial y} + ik\cos\vartheta u = 0, \quad (|x| \leq a, \quad y = -b),$$

$$\frac{\partial u}{\partial x} - ik\cos\vartheta u = 0, \quad (|y| \leq b, \quad x = a).$$

where $\cos\vartheta = -\frac{\mu_0}{\mu}N$, and for absorbing surfaces $\pi < \Re\vartheta \leq 3\pi/2, \Im\vartheta < 0$, so that $\vartheta = 2\pi - \arccos[-\frac{\mu_0}{\mu}N]$. The details of the derivation and the validity of the impedance boundary condition are given in Senior[1][24],van Bladel[18], Tretyakov[7]. An exact closed form solution of such a boundary problem is not so far possible. However for practical purposes we may make some realistic practical assumptions that can lead to useful computational results. The sides of the cylinder are assumed to be large compared to the incident wavelength (i.e. $kb \gg 1$). The problem under consideration is that of finding the field $u(P)$ at the point $P(r,\theta)$, $0 \leq \theta \leq 2\pi$, where r is large compared to the dimensions of the cylinder, $r \gg \sqrt{a^2+b^2}$, and hence the scattered far field $u_s(P)$. Clearly from the symmetry of the problem we need only consider an angle of incidence within the range $\pi/2 \leq \theta_0 \leq \pi$. To achieve this objective we shall need to use the results for the uniform, and nonuniform, asymptotics of the solution to the problem of the diffraction of an electrically-polarized plane wave by a right-angled impedance wedge given in Rawlins[15]; in conjunction with Keller's theory of geometrical diffraction (GTD) Keller[25]. Since the scatterer has more than one corner the effect of multiple diffractions will have to be considered, and thus an outline of Keller's theory will be given with relevance to the problem under consideration.

3. Keller's geometrical theory of diffraction and multiple diffraction

According to Keller's GTD the diffracted field $u_d(P)$ at a point P is equal to the sum of the fields on all rays through P:

$$u_d(P) = \sum_{rays} u_j(P). \tag{2}$$

Here $u_j(P)$ is the diffracted field on the j^{th} such ray, and if this is an m-fold diffracted ray then

$$u_d(P) = \frac{\mathrm{e}^{iks_j}}{k^{\frac{m}{2}}} \sum_{n=0}^{\infty} \frac{A_{jn}(P)}{(ik)^n}, \tag{3}$$

where $k(=2\pi/\lambda)$ is the propagation constant, s_j the arc length along the ray, and the function A_{jn} depends on the geometry and material

[1] The harmonic time variation in this work is incorrect, it should be $\mathrm{e}^{-i\omega t}$.

of the diffracting object. For a rectangular cylinder all the diffracted rays are produced by wedges of 90^0 angle. Hence the inclusion of higher $A_{jn}(n = 1, 2, ...)$ in the expansion (3) involves the use of more terms in the asymptotic solution of the wedge diffraction problem. The calculation of the diffraction coefficient corresponding to these higher-order terms necessitates the solution of the wedge diffraction problem for non-plane-wave incidence. However, as is shown in the works of Zitron[26],and Martin[27], the relevant non-plane waves are expressible in terms of linear combinations of plane waves and their derivatives. Thus the diffraction coefficients are easily found; and therefore it is possible to calculate the off shadow far fields corresponding to wedge excitations which are not shadow boundary fields. In order to calculate diffraction coefficients corresponding to shadow boundary fields we show that these fields too are expressible in terms of plane waves and their derivatives. As far as the solution to the canonical problem of the diffraction of a plane wave by an impedance wedge is concerned the complete solution to this problem has already been derived in detail by Rawlins[15]. In particular the far field expression is given by

$$u_d(r,\theta,\theta_0) = D(\theta,\theta_0)\frac{\mathrm{e}^{ikr}}{\sqrt{r}} + O((kr)^{-\frac{3}{2}}), \tag{4}$$

where the "diffraction coefficient" $D(\theta, \theta_0)$ is given by Rawlins[15] as

$$\begin{aligned} D(\theta,\theta_0) \; = \; & \frac{2\mathrm{e}^{i\frac{\pi}{4}}(\cos\theta-\cos\vartheta)(\sin\theta+\cos\vartheta)(\cos\frac{4\theta_0}{3}-\cos\frac{4(\pi+\vartheta)}{3})}{\sqrt{6\pi k}(\cos\theta_0+\cos\vartheta)(\sin\theta_0-\cos\vartheta)(\cos\frac{4(\theta-\pi-\vartheta)}{3}+\frac{1}{2})} \\ & \frac{(2\cos\frac{2\theta_0}{3}\cos\frac{2\theta}{3}+\frac{1}{2}-\cos\frac{4(\pi+\vartheta)}{3})\sin\frac{2\theta}{3}\sin\frac{2\theta_0}{3}}{(\cos\frac{4(\theta+\pi+\vartheta)}{3}+\frac{1}{2})(\cos\frac{2(\theta-\theta_0)}{3}+\frac{1}{2})(\cos\frac{2(\theta+\theta_0)}{3}+\frac{1}{2})}. \end{aligned} \tag{5}$$

An important property of the diffraction coefficient (5) is that

$$D(\theta,\theta_0) = D(\frac{3\pi}{2}-\theta, \frac{3\pi}{2}-\theta_0), \tag{6}$$

which means that the angle of incidence θ_0 and the angle of observation θ can be measured from either face of the corner wedge provided they are both measured from the same datum face. Other useful properties of $D(\theta, \theta_0)$ which we will require later in an application of the Keller method is the Karp-Karal lemma:

$$D(0,\theta_0) = D(\theta, 0) = 0; \tag{7}$$

and if we use the notation

$$\lim_{\theta\to 0}\frac{\partial D(\theta,\theta_0)}{\partial\theta} = D_\theta(0,\theta_0), \quad \lim_{\theta_0\to 0}\frac{\partial D(\theta,\theta_0)}{\partial\theta_0} = D_{\theta_0}(\theta,0),$$

then

$$D_\theta(0,\theta_0) = \frac{4e^{i\frac{\pi}{4}}(1-\cos\vartheta)(\cos\frac{4\theta_0}{3}-\cos\frac{4(\pi+\vartheta)}{3})}{3\sqrt{6\pi k}(\cos\theta_0+\cos\vartheta)(\sin\theta_0-\cos\vartheta)} \frac{(2\cos\frac{2\theta_0}{3}+\frac{1}{2}-\cos\frac{4(\pi+\vartheta)}{3})\sin\frac{2\theta_0}{3}\cos\vartheta}{(\cos\frac{4(\pi+\vartheta)}{3}+\frac{1}{2})^2(\cos\frac{2\theta_0}{3}+\frac{1}{2})^2}, \tag{8}$$

$$D_{\theta_0}(\theta,0) = \frac{-4e^{i\frac{\pi}{4}}(\cos\theta-\cos\vartheta)(\sin\theta+\cos\vartheta)(1-\cos\frac{4(\pi+\vartheta)}{3})}{3\sqrt{6\pi k}(1+\cos\vartheta)\cos\vartheta(\cos\frac{4(\theta-\pi-\vartheta)}{3}+\frac{1}{2})} \frac{(2\cos\frac{2\theta}{3}+\frac{1}{2}-\cos\frac{4(\pi+\vartheta)}{3})\sin\frac{2\theta}{3}}{(\cos\frac{4(\theta+\pi+\vartheta)}{3}+\frac{1}{2})(\cos\frac{2\theta}{3}+\frac{1}{2})^2}. \tag{9}$$

By applying a formula due to Zitron[26], to the expression (4) we obtain the field on and near to the ray determined by the point $P_1(r_1,\theta_1)$. More precisely, we obtain the asymptotic expansion of the diffracted field, $u_d(B_1;E_1)$, at any point $B_1 = B_1(\xi,\eta)$ in the neighbourhood of the point P_1, see Figure 2.

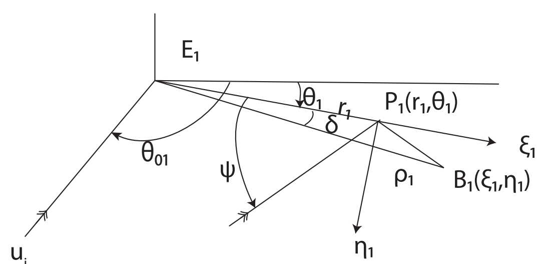

Figure 2. The coordinate system set up at an edge.

$$
\begin{aligned}
u_d(B_1;E_1) \;=\;& \frac{\mathrm{e}^{ikr_1}}{\sqrt{r_1}}[D(\theta_1,\theta_{01})p(0) \\
& +[\frac{D(\theta_1,\theta_{01})p''(0)}{2ik} - \frac{D_\theta(\theta_1,\theta_{01})p'(0)}{ik} \\
+\;& D^{(1)}(\theta_1,\theta_{01})p(0)]\frac{1}{r_1} + O((kr_1)^{-2})],
\end{aligned} \tag{10}
$$

where

$$
D^{(1)}(\theta_1,\theta_{01}) = (2ik)^{-1}[\frac{1}{4}D(\theta_1,\theta_{01}) + D_{\theta\theta}(\theta_1,\theta_{01})], \tag{11}
$$

and

$$
p(\psi) = \mathrm{e}^{ik(\xi\cos\psi - \eta\sin\psi)}.
$$

It will be noticed that (10) is expressed in terms of $p(0)$ and derivatives of $p(0)$, where $p(\psi)$ is a plane wave expression. Since (10), which may be continued asymptotically to any desired order of $(r_1k)^{-1}$, is linear in $p(\psi)$ and its derivatives the fields resulting from successive interactions are readily derived. Thus the expression for the diffracted field when an incident plane wave first illuminates E_1 and the resulting diffracted field, given by (10), is then incident on the second wedge E_2, see Figure 3, is given by

$$
\begin{aligned}
u_d(B_1;E_1) \;=\;& \frac{\mathrm{e}^{ikr_1}}{\sqrt{r_1}}[D(\theta_1,\theta_{01})u_d(B_2;E_2) \\
& +[\frac{D(\theta_1,\theta_{01})u_d''(B_2;E_2)}{2ik} - \frac{D_\theta(\theta_1,\theta_{01})u_d'(B_2;E_2)}{ik} \\
+\;& D^{(1)}(\theta_1,\theta_{01})u_d(B_2;E_2)]\frac{1}{r_1} + O((kr_1)^{-2})].
\end{aligned} \tag{12}
$$

In the expression (12), (r_{12},θ_{12}) are the polar coordinates of E_2 with respect to E_1, see Figure 3; $u_d(B_2;E_2)$ is given by (10) with the subscript 2 in place of 1, and the primes on this function indicates derivatives with respect to $\pm\theta_{02}$ depending respectively on whether or not ψ and θ_{02} have the same orientation.

Equation (10) is obtained assuming oblique incidence. whenever the incident rays are parallel to a side (i.e. when $\theta_{01} = \frac{\pi}{2}$) the value of the field given by (10) must be divided by two. This normalization is due to the coalescing of the incident and reflected field at this angle of incidence. The expressions (10) and (13) have been obtained under the assumption that the poles of the integrand do not reside near to

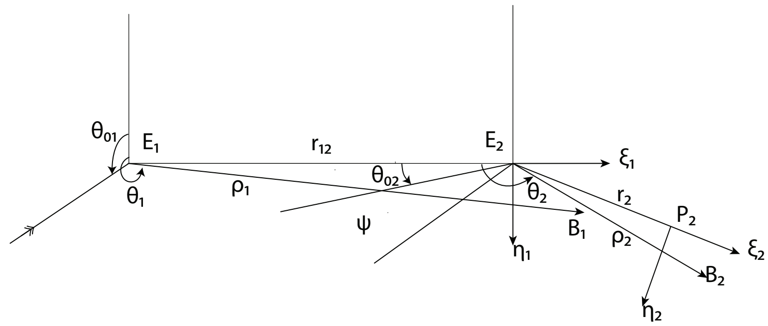

Figure 3. The angles ψ and θ_{02} are measured from the same face $\overline{E_1E_2}$ and have therefore the same orientation.

the saddle point in the integral (9) of Rawlins[15]. As a result these expressions are not valid near to the shadow or specular boundaries, see Figure 4, indeed the expressions (10) and (12) become infinitely large near these boundaries.

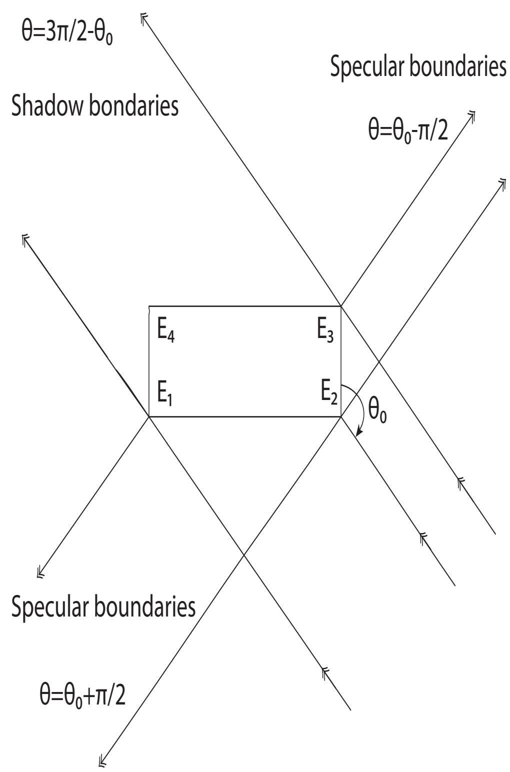

Figure 4. The ray geometry of the shadow and specular boundaries.

To determine the field near to the above mentioned boundaries it is necessary to use the asymptotic expansions (49) to $(52)^2$ of Rawlins[15]; and in order to account for the effects of multiple diffraction these results must be expressed in terms of the plane wave $p(\psi) = e^{ik(\xi\cos\psi - \eta\sin\psi)}$, and its derivatives. Thus introducing the coordinate system

$$\delta = \arctan\frac{\eta}{r+\eta}, \qquad \varrho = [(r+\xi)^2 + \eta^2]^{\frac{1}{2}}, \tag{13}$$

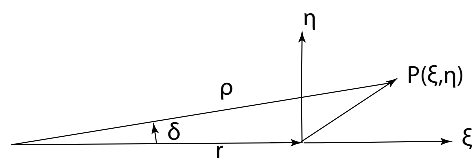

Figure 5. The (ξ, η) coordinate system.

where δ is defined as being positive for the situation shown in Figure 5, the asymptotic expressions (49) to(52) of Rawlins[15] become

$$I(\delta) = \mathbf{D}(\theta,\theta_0)\left[\mathrm{sgn}\delta \mathrm{e}^{\mathrm{ik(r+\xi)}}\left\{\frac{1}{2} - \frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}{\sqrt{\pi}}\left(\frac{\mathrm{k}|\eta|}{\sqrt{2\mathrm{kr}}} + \frac{-3\sqrt{2}\mathrm{k}^2\xi|\eta| + \mathrm{i}\sqrt{2}\mathrm{k}^3|\eta|^3}{12(\mathrm{kr})^{\frac{3}{2}}}\right)\right\}\right]$$
$$-\frac{17\mathbf{D}(\theta,\theta_0)k\eta \mathrm{e}^{ik(r+\xi)+i\frac{\pi}{4}}}{216\sqrt{2\pi}(kr)^{\frac{3}{2}}} + \frac{B_2\mathrm{e}^{ik(r+\xi)+i\frac{\pi}{4}}k\eta}{4\sqrt{2\pi}(kr)^{\frac{3}{2}}} + O\left[(kr)^{-\frac{5}{2}}\right], \tag{14}$$

$$J(\delta) = \frac{B_0'\mathrm{e}^{ik(r+\xi)}}{\sqrt{\pi}}\left[\frac{3\mathrm{e}^{i\frac{\pi}{4}}}{4\sqrt{2kr}} - \frac{3\mathrm{e}^{-i\frac{\pi}{4}}}{2}\left\{\frac{\sqrt{\pi}\mathrm{e}^{\frac{i\pi}{4}}|\eta|}{4r} - \frac{k^2|\eta|^2}{(2kr)^{\frac{3}{2}}}\right\} - \frac{3\mathrm{e}^{i\frac{\pi}{4}}i(-k^2\eta^2 - ik\eta\xi)}{8\sqrt{2}(kr)^{\frac{3}{2}}}\right]$$
$$-\frac{\mathrm{e}^{-i\frac{\pi}{4}}}{2\sqrt{2\pi}(kr)^{\frac{3}{2}}}\left(\frac{3B_2'}{8} - \frac{17B_0'}{144}\right)\mathrm{e}^{ik(r+\xi)} + O\left[(kr)^{-\frac{5}{2}}\right], \tag{15}$$

[2] In section 8 of Rawlins[15] $\delta = \psi \pm \delta'(\delta' > 0)$ so that the right hand side of equation (52) should be $J(\psi \pm \delta')$.

$$I(\psi \pm \delta) = \frac{\mathbf{D}(\theta,\theta_0)\mathrm{e}^{ik(r+\xi)+i\frac{\pi}{4}}}{3\sqrt{2\pi kr}}\left[\cot\frac{\psi}{3} \mp \frac{\eta}{3r(\sin\frac{\psi}{3})^2} + \frac{(ik\eta^2-\xi)}{2r}\cot\frac{\psi}{3}\right]$$
$$-\frac{i\mathbf{D}(\theta,\theta_0)\mathrm{e}^{ik(r+\xi)+i\frac{\pi}{4}}}{24\sqrt{2\pi kr}kr}\cot\frac{\psi}{3}\left[1+\frac{8}{9}(\csc\frac{\psi}{3})^2\right] + \frac{B_2 i\mathrm{e}^{ik(r+\xi)+i\frac{\pi}{4}}}{12\sqrt{2\pi}(kr)^{\frac{3}{2}}}\cot\frac{\psi}{3} + O\left[(kr)^{-\frac{5}{2}}\right]. \tag{16}$$

$$J(\psi \pm \delta) = -\frac{B_0'\mathrm{e}^{ik(r+\xi)-i\frac{\pi}{4}}}{12\sqrt{2\pi}(kr)^{\frac{3}{2}}}(\csc\frac{\psi}{3})^2 + O\left[(kr)^{-\frac{5}{2}}\right]. \tag{17}$$

The various quantities that appear in the above expressions are defined in the work of Rawlins [15], in particular

$$\begin{aligned}\mathbf{D}(\gamma,\theta_0) &= \frac{(\cos\gamma-\cos\vartheta)(\sin\gamma+\cos\vartheta)}{(\cos\theta_0+\cos\vartheta)(\sin\theta_0-\cos\vartheta)} \\ &\times \frac{(\cos\frac{4\theta_0}{3}-\cos\frac{4(\pi+\vartheta)}{3})(2\cos\frac{2\theta_0}{3}\cos\frac{2\gamma}{3}+\frac{1}{2}-\cos\frac{4(\pi+\vartheta)}{3})}{(\cos\frac{4(\gamma-\pi-\vartheta)}{3}+\frac{1}{2})(\cos\frac{4(\gamma+\pi+\vartheta)}{3}+\frac{1}{2})}.\end{aligned} \tag{18}$$

Expressing the above formulae in terms of the plane wave function $p(\psi)$ gives for $\eta > 0$

$$I(\delta) = \mathbf{D}(\theta,\theta_0)\left[\frac{\mathrm{e}^{ikr}}{2}p(0) - \frac{\mathrm{e}^{ikr-i\frac{\pi}{4}}}{\sqrt{\pi}}\left\{\frac{ip'(0)}{\sqrt{2kr}} + \frac{\sqrt{2}[p'(0)+p'''(0)]}{12(kr)^{\frac{3}{2}}}\right\}\right]$$
$$-\frac{17\mathbf{D}(\theta,\theta_0)\mathrm{e}^{ikr+i\frac{\pi}{4}}}{216\sqrt{2\pi}(kr)^{\frac{3}{2}}}ip'(0) + \frac{B_2\mathrm{e}^{ikr+i\frac{\pi}{4}}}{4\sqrt{2\pi}(kr)^{\frac{3}{2}}}ip'(0) + O\left[(kr)^{-\frac{5}{2}}\right], \tag{19}$$

$$J(\delta) = \frac{B_0'\mathrm{e}^{ikr}}{\sqrt{\pi}}\left\{\frac{3\mathrm{e}^{\frac{\pi}{4}}}{4\sqrt{2kr}}p(0) - \frac{3\sqrt{\pi}}{8kr}ip'(0) + \frac{[17p(0)-108\overline{p''(\pi)}]\mathrm{e}^{-i\frac{\pi}{4}}}{288\sqrt{2}(kr)^{\frac{3}{2}}}\right\}$$
$$-\frac{54B_2'\mathrm{e}^{ikr-i\frac{\pi}{4}}}{288\sqrt{2}(kr)^{\frac{3}{2}}}p(0) + O\left[(kr)^{-\frac{5}{2}}\right], \tag{20}$$

$$J(\psi \pm \delta) = -\frac{B_0' \mathrm{e}^{ikr - i\frac{\pi}{4}}}{12\sqrt{2\pi}(kr)^{\frac{3}{2}}} (\csc\frac{\psi}{3})^2 + O\left[(kr)^{-\frac{5}{2}}\right], \tag{21}$$

$$\begin{aligned} I(\psi \pm \delta) &= \frac{\mathbf{D}(\theta, \theta_0)\mathrm{e}^{ikr + i\frac{\pi}{4}}}{3\sqrt{2\pi kr}} \\ &\times \left[\cot\frac{\psi}{3} \mp \frac{8ip'(0)}{24kr(\sin\frac{\psi}{3})^2} - \frac{\cot\frac{\psi}{3}(1 + \frac{8}{9}(\csc\frac{\psi}{3})^2)ip(0) + 4\cot\frac{\psi}{3}ip''(0)}{8kr}\right] \\ &+ \frac{B_2 i \mathrm{e}^{ikr + i\frac{\pi}{4}}}{12\sqrt{2\pi}(kr)^{\frac{3}{2}}}\cot\frac{\psi}{3} + O\left[(kr)^{-\frac{5}{2}}\right], \end{aligned} \tag{22}$$

where $p(0) = \mathrm{e}^{ik\xi}, p'(0) = -ik\eta\mathrm{e}^{ik\xi}, p''(0) = -(k^2\eta^2 + ik\eta)\mathrm{e}^{ik\xi}$, $p''(\pi) = (-k^2\eta^2 + ik\eta)\mathrm{e}^{ik\xi}, p''(0) = (-3k^2\eta\xi + ik\eta + ik^3\eta^3)\mathrm{e}^{ik\xi}$. The expressions (19) to (22) will be required in the next chapter and the dominant terms of (14) to (17) will be required when dealing with the field near the shadow or specular boundaries.

4. Diffracted field for oblique incidence

We now derive the far field expression for the diffracted field in all regions around the cylinder for oblique incidence except in specified directions. To obtain the result in terms of the coordinates (r, θ) of the observation point P we use asymptotic approximations to the length and the angles of various rays from the edges of the rectangle to the observation point $P(r, \theta)$. From Figure 6 on applying the cosine rule we get for large r

$$r_i = r - l\cos\theta\cos\beta - l\sin\theta\sin\beta + O\left[r^{-2}\right], \tag{23}$$

but $x_i = l\cos\beta$ and $y_i = l\sin\beta$, where x_i and y_i are the coordinates of the point on the cylinder. Hence

$$r_i = r - x_i\cos\theta - y_i\sin\theta.$$

The diffracted rays being dealt with are those from the corners of the rectangle $E_1(= r_1), E_2(= r_2), E_3(= r_3)$, and $E_4(= r_4)$ so that

$$r_1 = r + a\cos\theta + b\sin\theta + O\left[r^{-2}\right],$$