State Estimation for Discrete-Time Neural Networks with Markov-Mode-Dependent Lower and Upper Bounds on the Distributed Delays

Yurong Liu*, Zidong Wang and Xiaohui Liu

Abstract

This paper is concerned with the state estimation problem for a new class of discrete-time neural networks with Markovian jumping parameters and mixed time-delays. The parameters of the neural networks under consideration switch over time subject to a Markov chain. The networks involve both the discrete time-varying delay and the mode-dependent distributed time-delay characterized by the upper and lower boundaries dependent on the Markov chain. By constructing novel Lyapunov-Krasovskii functionals, sufficient conditions are firstly established to guarantee the exponential stability in mean square for the addressed discrete-time neural networks with Markovian jumping parameters and mixed time-delays. Then, the state estimation problem is coped with for the same neural network where the goal is to design a desired state estimator such that the estimation error approaches zero exponentially in mean square. The derived conditions for both the stability and the existence of desired estimators are expressed in the form of matrix inequalities that can be solved by the semi-definite programme method. A numerical simulation example is exploited to demonstrate the usefulness of the main results obtained.

Keywords

Discrete-time neural networks; Mixed time delays; Markovian jumping parameters; Exponential stability; State estimate; Linear matrix inequality.

I. Introduction

In the last few decades, recurrent neural networks (RNNs) have found successful applications in a variety of areas including pattern recognition, associative memory and combinational optimization [2,6,12,13,18,19]. An increasing research interest has been devoted to the study of dynamical behaviors of various kinds of neural networks and, accordingly, a great number of important research results have been published. Among others, the stability analysis and state estimation problems serve as two of the most investigated ones that have received considerable research attention. On one hand, if a neural network is employed to solve some optimization problems, it is highly desirable for the neural network to have a unique globally stable equilibrium, and it is not surprising that the stability analysis of neural networks has been an ever hot research topic resulting in enormous stability conditions reported in the literature. On the other hand, in many practical applications, the states of neural networks are crucial for some specific design objectives. However, a common situation is that the states of neural networks are not completely accessible and only partial information can be obtained via the output of neural networks. Consequently, it becomes necessary to estimate the neuron state from the given output. The state estimation has recently drawn particular research attention, see, e.g., [4,10,14,24,32] and the references therein.

As is well known, the time delay is often encountered as a characteristic of signal transmission between neurons, which is usually one of the major sources of instability and poor performance. For the dynamical

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behavior analysis of delayed neural networks, various types of time delays (such as constant delays, time-varying delays and distributed delays) have been taken into account by using a variety of techniques that include the linear matrix inequality (LMI) approach, the Lyapunov functional method, the M-matrix theory, the topological degree theory, and the inequality analysis techniques. For example, in [5, 7–9, 20, 22, 33], the global asymptotic stability analysis problem has been dealt with for a class of neural networks with time-delays by using an effective LMI approach. Recently, the global stability analysis problem for general RNNs with *mixed time-delays* (also called *discrete and distributed delays* [23]) has received an increasing research attention and many relevant results have been available in the literature, see e.g. [3, 11, 34] and the references therein.

Very recently, the RNNs with Markovian jumping parameters have gained particular attention from many researchers, where the RNNs may experience abrupt change in network structure and parameters, and the switches from a mode to another are usually subject to a Markov chain. In the context of Markovian jumping RNNs, the exponential stability has been studied in many papers, see e.g. [1, 3, 15, 17, 21, 25, 35]. In particular, the asymptotical stability has been investigated in [25] for continuous-time RNNs with Markovian jumping parameters, and stability analysis and synchronization problems have been dealt with in [15] for a class of discrete-time Markovian jumping RNNs with mixed time-delays. In [3], the problem of the global exponential stability has been investigated for neutral-type impulsive neural networks with mixed delays and Markovian jumping parameters. In [35], the delay-dependent mean square exponential stability has been established for a class of delayed stochastic Hopfield neural networks with Markovian jump parameters. It should be pointed out that, most of the available stability analysis results have been concerned with the continuous-time Markovian jumping RNNs with or without mixed time-delays, and the corresponding results in the discrete-time case have been scattered, see e.g. [1, 17] where the distributed time-delays have not been taken into consideration.

Although the state estimation problem has stirred a great deal of research attention for neural networks, a literature search reveals that there have been very few results on *discrete-time neural networks with Markovian jumping parameters and mode-dependant distributed delays*. Some recent papers [4, 16, 17, 32] are worth mentioning here. In [4], the state estimation problem has been investigated for *continuous-time* Markovian jumping neural networks with *mode-independent* discrete-type time-delays. In [17], a mode-dependent stability criterion has been established for discrete-time neural networks with stochastic disturbances, Markovian jumping parameters and mode-dependent discrete time delay, but the distributed time-delay has not been included. In [32], the state estimation problem has been studied for discrete-time neural networks with Markovian jumping parameters and time-varying discrete time-delays under the assumption that only partial entries in the transition probability matrix are known, but the distributed time-delay has not been taken into account. In [16], the state estimation problem has been addressed for a class of discrete-time Markovian jumping neural networks with mixed mode-dependent time-delays, where the discrete time-delay is time-invariant and the distributed time-delay includes *mode-dependent lower bound only*. So far, to the best of the authors' knowledge, the state estimation problem has not yet been investigated for discrete-time Markovian jumping neural networks with Markov-mode-dependent lower and upper bounds on the distributed delays. Two possible reasons for such a gap are that, 1) when the *lower and upper bounds* of the distributed time-delays are both subject to Markovian switching (i.e., mode dependent), the corresponding stability analysis becomes more complicated since a new Lyapunov functional is required to reflect the Markovian jumps of the delay bounds; and 2) the structure of the state estimator has to be chosen with great care so that the exponential decay rate of the estimation error can be guaranteed. To this end, the purpose of this paper is to shorten such a gap.

Summarizing the above discussion, in this paper, we study the state estimation problem for a new class of discrete-time neural networks with Markovian jumping parameters as well as mixed time-delays. The mixed time-delays comprise both the time-varying discrete delay and distributed delay with mode-dependent upper and lower bounds. The purpose of the addressed state estimation problem is to design a desired state estimator such that the state estimates converges exponentially to the actual states of the original RNNs. *The main contribution of this paper is twofold: 1) a novel yet more comprehensive type of distributed delays (in discrete-time) is put forward whose upper and lower bounds are changeable subject to a given Markov chain; and 2) a novel Lyapunov functional is constructed to reflect the Markov jumping nature of the distributed delays in the exponential estimator design procedure.* A numerical simulation example is employed to illustrate the effectiveness of the proposed estimator design scheme.

Notations: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite); I is the identity matrix with compatible dimension. $|\cdot|$ refers to the Euclidean vector norm. If A is a matrix, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalue, respectively. In symmetric block matrices, we use an asterisk "$*$" to represent a term that is induced by symmetry and diag$\{\cdots\}$ stands for a block-diagonal matrix. $\mathbb{E}[x]$ and $\mathbb{E}[x|y]$ will, respectively, mean the expectation of x and the expectation of x conditional on y. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem formulation

Let $r(k)$ $(k \geq 0)$ be a Markov chain taking values in a finite state space $\mathcal{S} = \{1, 2, ..., N\}$ with probability transition matrix $\Pi = (\pi_{ij})_{N\times N}$ given by

$$\Pr\Big\{r(k+1) = j \mid r(k) = i\Big\} = \pi_{ij}, \quad \forall i, j \in \mathcal{S}$$

where $\pi_{ij} \geq 0$ $(i, j \in \mathcal{S})$ is the transition rate from i to j and $\sum_{j=1}^{N} \pi_{ij} = 1,\ \forall i \in \mathcal{S}$.

Consider a discrete-time n-neuron neural network with N modes described by the following dynamical system:

$$x(k+1) = D(r(k))x(k) + A(r(k))F(x(k)) + B(r(k))G(x(k-\tau_0(k))) + C(r(k))\sum_{v=\tau_{1,r(k)}}^{\tau_{2,r(k)}} H(x(k-v)) \tag{1a}$$

$$x(s) = \phi(s),\ s = -\tau, -\tau+1, \ldots, -1, 0 \tag{1b}$$

where $x(k) = (x_1(k), x_2(k), \ldots, x_n(k))^T$ is the neural state vector; the constant matrices $D(r(k)) = \mathrm{diag}\{d_1(r(k)), d_2(r(k)), \ldots, d_n(r(k))\}$ describe the rate with which the each neuron will reset its potential to the resting state in isolation when disconnected from the networks and external inputs; $A(r(k)) = [a_{ij}(r(k))]_{n\times n},\ B(r(k)) = [b_{ij}(r(k))]_{n\times n}$ and $C(r(k)) = [c_{ij}(r(k))]_{n\times n}$ are, respectively, the connection weight matrix, the delayed connection weight matrix and the distributively delayed connection weight matrix; $\tau_0(k)$ denotes the time-varying delay while $\tau_{1,r(t)}$ and $\tau_{2,r(t)}$ $(0 \leq \tau_{1,r(t)} \leq \tau_{2,r(t)})$ represent the mode-dependent upper and lower bounds of distributed time delay, respectively; $F(x(k)) = (f_1(x_1(k)), f_2(x_2(k)), \ldots, f_n(x_n(k)))^T$, $G(x(k-\tau_0(k))) = (g_1(x_1(k-\tau_0(k))), g_2(x_2(k-\tau_0(k))), \ldots, g_n(x_n(k-\tau_0(k))))^T$ and $H(x(k)) = (h_1(x_1(k)), h_2(x_2(k)), \ldots, h_n(x_n(k)))^T$ are the nonlinear activation functions; and $\phi(s)$ describes the initial condition.

In system (1), $\tau = \max\{\overline{\tau}_0,\ \overline{\tau}_2\}$ with $\overline{\tau}_0 = \max\{\tau_0(k) \mid k \geq 0\}$ and $\overline{\tau}_2 = \max\{\tau_{2,i} \mid i \in \mathcal{S}\}$. In addition, we also denote $\underline{\tau}_0 = \min\{\tau_0(k) \mid k \geq 0\}$, $\underline{\tau}_1 = \min\{\tau_{1,i} \mid i \in \mathcal{S}\}$, $\overline{\tau}_1 = \max\{\tau_{1,i} \mid i \in \mathcal{S}\}$, $\underline{\tau}_2 = \min\{\tau_{2,i} \mid i \in \mathcal{S}\}$, and $\underline{\pi} = \min\{\pi_{ii} \mid i \in \mathcal{S}\}$.

For neuron activation functions, we make the following assumptions.

Assumption 1: [14] For the activation functions $F(\cdot), G(\cdot)$ and $H(\cdot)$, there exist constants $\lambda_i^-, \lambda_i^+, \sigma_i^-, \sigma_i^+$,

υ_i^- and υ_i^+ such that

$$\lambda_i^- \leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq \lambda_i^+, \tag{2}$$

$$\sigma_i^- \leq \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \leq \sigma_i^+, \tag{3}$$

$$\upsilon_i^- \leq \frac{h_i(s_1) - h_i(s_2)}{s_1 - s_2} \leq \upsilon_i^+. \tag{4}$$

Assumption 2: $F(\cdot), G(\cdot)$ and $H(\cdot)$ are bounded functions and satisfy $F(0) = G(0) = H(0) = 0$.

Remark 1: As pointed out in [14], the constants λ_i^-, λ_i^+, σ_i^-, σ_i^+, υ_i^- and υ_i^+ in Assumption 1 are allowed to be positive, negative or zero. Hence, the resulting activation functions could be non-monotonic, and are more general than the usual sigmoid functions. Also, under Assumption 2, it is obvious that the origin $x = 0$ is the equilibrium point of system (1). Note that the existence of equilibrium points of a neural network can be guaranteed by boundedness of the activation functions (see [14]), and there is a standard way to shift the system equilibrium point to the origin. Therefore, Assumption 2 is made here without loss of generality.

Definition 1: Neural network (1) is said to be asymptotically stable in mean square if, for any solution $x(k)$ of (1), the following holds:

$$\lim_{k\to\infty} \mathbb{E}[|x(k)|^2] = 0.$$

Furthermore, neural network (1) is said to be exponentially stable in mean square if there exist constants $\mu > 1$ and $\beta > 0$ such that, for any solution $x(k)$ of (1),

$$\mathbb{E}[|x(k)|^2] \leq \beta\mu^{-k} \max_{-\tau\leq i\leq 0} \mathbb{E}[|x(i)|^2],\ \forall k > 0.$$

In this paper, we shall firstly deal with the exponential stability problem for the system (1). By constructing novel Lyapunov-Krasovskii functional, sufficient conditions are established to guarantee the exponential stability in mean square, and then we turn to the dynamics analysis problem of the state estimation errors and the design algorithm of the *exponential estimator* for system (1). The criteria, either for stability analysis or for state estimator design, are expressed in the form of matrix inequalities that can be solved by the semi-definite programme method.

III. Exponential Stability

Before proceeding to the stability analysis for system (1), we introduce two lemmas that will be useful in deriving our results.

Lemma 1: [15] Let $M \in \mathbb{R}^{n\times n}$ be a positive semi-definite matrix, $\mathbf{x}_i \in \mathbb{R}^n$ be a vector and $a_i \geq 0$ $(i = 1, 2, ...)$ be scalars. If the series concerned are convergent, then the following inequality holds:

$$\left(\sum_{i=1}^{+\infty} a_i\mathbf{x}_i\right)^T M \left(\sum_{i=1}^{+\infty} a_i\mathbf{x}_i\right) \leq \left(\sum_{i=1}^{+\infty} a_i\right) \sum_{i=1}^{+\infty} a_i\mathbf{x}_i^T M\mathbf{x}_i \tag{5}$$

Lemma 2: (Schur Complement) [25] Given constant matrices $\Omega_1, \Omega_2, \Omega_3$ where $\Omega_1 = \Omega_1^T$ and $\Omega_2 > 0$, then

$$\Omega_1 + \Omega_3^T\Omega_2^{-1}\Omega_3 < 0$$

if only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0.$$

Hereafter, we denote

$$\Lambda_1 = \mathrm{diag}\big\{\lambda_1^+\lambda_1^-, \lambda_2^+\lambda_2^-, ..., \lambda_n^+\lambda_n^-\big\},\ \Lambda_2 = \mathrm{diag}\Big\{\frac{\lambda_1^+ + \lambda_1^-}{2}, \frac{\lambda_2^+ + \lambda_2^-}{2}, ..., \frac{\lambda_n^+ + \lambda_n^-}{2}\Big\},$$
$$\Sigma_1 = \mathrm{diag}\big\{\sigma_1^+\sigma_1^-, \sigma_2^+\sigma_2^-, ..., \sigma_n^+\sigma_n^-\big\},\ \Sigma_2 = \mathrm{diag}\Big\{\frac{\sigma_1^+ + \sigma_1^-}{2}, \frac{\sigma_2^+ + \sigma_2^-}{2}, ..., \frac{\sigma_n^+ + \sigma_n^-}{2}\Big\},$$
$$\Upsilon_1 = \mathrm{diag}\big\{\upsilon_1^+\upsilon_1^-, \upsilon_2^+\upsilon_2^-, ..., \upsilon_n^+\upsilon_n^-\big\},\ \Upsilon_2 = \mathrm{diag}\Big\{\frac{\upsilon_1^+ + \upsilon_1^-}{2}, \frac{\upsilon_2^+ + \upsilon_2^-}{2}, ..., \frac{\upsilon_n^+ + \upsilon_n^-}{2}\Big\}.$$

For the stability of network (1), we have the following results.

Theorem 1: Under assumptions 1 and 2, the delayed neural network (1) is exponentially stable in mean square if there exist a set of matrices $P_i > 0$, two matrices $Q > 0$ and $R > 0$, and three sets of diagonal matrices $\Omega_i > 0, \Theta_i > 0$ and $\Delta_i > 0$ such that the following LMIs hold:

$$\Phi_i := \begin{pmatrix} \Xi_i & \Omega_i\Lambda_2 & \Theta_i\Sigma_2 & 0 & \Delta_i\Upsilon_2 & 0 & D(i)\overline{P}_i \\ * & -\Omega_i & 0 & 0 & 0 & 0 & A^T(i)\overline{P}_i \\ * & * & \Pi_i & 0 & 0 & 0 & 0 \\ * & * & * & -Q & 0 & 0 & B^T(i)\overline{P}_i \\ * & * & * & * & \kappa_i R - \Delta_i & 0 & 0 \\ * & * & * & * & * & -\frac{1}{\tau_{2,i}-\tau_{1,i}}R & C^T(i)\overline{P}_i \\ * & * & * & * & * & * & -\overline{P}_i \end{pmatrix} < 0, \quad (i \in \mathcal{S}) \tag{6}$$

where

$$\overline{P}_i = \sum_{j=1}^{N} \pi_{ij}P_j, \quad \Xi_i = -P_i - \Omega_i\Lambda_1 - \Theta_i\Sigma_1 - \Delta_i\Upsilon_1, \quad \Pi_i = (\overline{\tau}_0 - \underline{\tau}_0 + 1)Q - \Theta_i, \tag{7}$$

$$\kappa_i = \sum_{j=1}^{N} \pi_{ij}(\tau_{2,j} - \tau_{1,j}) + \frac{1}{2}(1-\underline{\pi})(\overline{\tau}_1 - \underline{\tau}_1)(\overline{\tau}_1 + \underline{\tau}_1 - 3) + \frac{1}{2}(1-\underline{\pi})(\overline{\tau}_2 - \underline{\tau}_2)(\overline{\tau}_2 + \underline{\tau}_2 - 1). \tag{8}$$

Proof: To simplify the notation, let us denote

$$\begin{aligned} \mathbf{x}_k &= \big[x^T(k), x^T(k-1), \cdots, x^T(k-\tau)\big]^T, \\ \chi(i) &= \big[D(i) \quad A(i) \quad 0 \quad B(i) \quad 0 \quad C(i)\big], \\ \xi(k,i) &= \Big[x^T(k) \quad F^T(x(k)) \quad G^T(x(k)) \quad G^T(x(k-\tau_0(k))) \quad H^T(x(k)) \quad \sum_{v=\tau_{1,i}}^{\tau_{2,i}} H^T(x(k-v))\Big]^T. \end{aligned}$$

By Lemma 2, inequality (6) is equivalent to

$$\hat{\Phi}_i + \chi^T(i)\overline{P}_i\chi(i) < 0,\ i \in \mathcal{S} \tag{9}$$

where

$$\hat{\Phi}_i = \begin{pmatrix} \Xi_i & \Omega_i\Lambda_2 & \Theta_i\Sigma_2 & 0 & \Delta_i\Upsilon_2 & 0 \\ * & -\Omega_i & 0 & 0 & 0 & 0 \\ * & * & \Pi_i & 0 & 0 & 0 \\ * & * & * & -Q & 0 & 0 \\ * & * & * & * & \kappa_i R - \Delta_i & 0 \\ * & * & * & * & * & -\frac{1}{\tau_{2,i}-\tau_{1,i}}R \end{pmatrix}.$$

To start with the stability analysis, consider the stochastic Lyapunov-Krasovskii functional $V(\mathbf{x}_k, k, r(k))$ as follows:

$$\begin{aligned} V(\mathbf{x}_k, k, r(k)) =& V_1(\mathbf{x}_k, k, r(k)) + V_2(\mathbf{x}_k, k, r(k)) + V_3(\mathbf{x}_k, k, r(k)) \\ &+ V_4(\mathbf{x}_k, k, r(k)) + V_5(\mathbf{x}_k, k, r(k)) + V_6(\mathbf{x}_k, k, r(k)) \end{aligned} \tag{10}$$

where

$$\begin{aligned}
V_1(\mathbf{x}_k,k,r(k)) &= x^T(k)P_{r(k)}x(k), && (11)\\
V_2(\mathbf{x}_k,k,r(k)) &= \sum_{v=k-\tau_0(k)}^{k-1} G^T(x(v))QG(x(v)), && (12)\\
V_3(\mathbf{x}_k,k,r(k)) &= \sum_{\iota=k-\overline{\tau}_0+1}^{k-\underline{\tau}_0} \sum_{v=\iota}^{k-1} G^T(x(v))QG(x(v)), && (13)\\
V_4(\mathbf{x}_k,k,r(k)) &= \sum_{\iota=\tau_{1,r(k)}}^{\tau_{2,r(k)}} \sum_{v=k-\iota}^{k-1} H^T(x(v))RH(x(v)), && (14)\\
V_5(\mathbf{x}_k,k,r(k)) &= (1-\underline{\pi})\sum_{s=\underline{\tau}_1}^{\overline{\tau}_1-1}\sum_{\iota=1}^{s-1}\sum_{v=k-\iota}^{k-1} H^T(x(v))RH(x(v)). && (15)\\
V_6(\mathbf{x}_k,k,r(k)) &= (1-\underline{\pi})\sum_{s=\underline{\tau}_2+1}^{\overline{\tau}_2}\sum_{\iota=1}^{s-1}\sum_{v=k-\iota}^{k-1} H^T(x(v))RH(x(v)). && (16)\\
& && (17)
\end{aligned}$$

For $i \in \mathcal{S}$, it can be calculated that

$$\begin{aligned}
&\mathbb{E}[V_1(\mathbf{x}_{k+1},k+1,r(k+1)) \mid \mathbf{x}_k, r(k)=i] - V_1(\mathbf{x}_k,k,i)\\
={}& \Big[D(i)x(k) + A(i)F(x(k)) + B(i)G(x(k-\tau_1(k))) + C(i)\sum_{v=1}^{\tau_{2,i}} H(x(k-v))\Big]^T \overline{P}_i \Big[D(i)x(k)\\
&+ A(i)F(x(k)) + B(i)G(x(k-\tau_1(k))) + C(i)\sum_{v=1}^{\tau_{2,i}} H(x(k-v))\Big] - x^T(k)P_ix(k)\\
={}& \xi^T(k,i)\chi^T(i)\overline{P}_i\chi(i)\xi(k,i) - x^T(k)P_ix(k), \qquad (18)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}[V_2(\mathbf{x}_{k+1},k+1,r(k+1)) \mid \mathbf{x}_k, r(k)=i] - V_2(\mathbf{x}_k,k,i)\\
={}& \sum_{v=k-\tau_0(k+1)}^{k} G^T(x(v))QG(x(v)) - \sum_{v=k-\tau_0(k)}^{k-1} G^T(x(v))QG(x(v))\\
={}& G^T(x(k))QG(x(k)) - G^T(x(k-\tau_0(k)))QG(x(k-\tau_0(k)))\\
&+ \sum_{v=k-\tau_1(k+1)+1}^{k-1} G^T(x(v))QG(x(v)) - \sum_{v=k-\tau_1(k)+1}^{k-1} G^T(x(v))QG(x(v))\\
\leq{}& G^T(x(k))QG(x(k)) - G^T(x(k-\tau_0(k)))QG(x(k-\tau_0(k))) + \sum_{v=k-\overline{\tau}_0+1}^{k-\underline{\tau}_0} G^T(x(v))QG(x(v)), \qquad (19)
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}[V_3(\mathbf{x}_{k+1},k+1,r(k+1)) \mid \mathbf{x}_k, r(k)=i] - V_3(\mathbf{x}_k,k,i)\\
=\;& \sum_{\iota=k-\overline{\tau}_0+2}^{k-\underline{\tau}_1+1} \sum_{v=\iota}^{k} G^T(x(v))QG(x(v)) - \sum_{j=k-\overline{\tau}_0+1}^{k-\underline{\tau}_0} \sum_{v=\iota}^{k-1} G^T(x(v))QG(x(v))\\
=\;& \sum_{\iota=k-\overline{\tau}_0+1}^{k-\underline{\tau}_0} \sum_{v=\iota+1}^{k} G^T(x(v))QG(x(v)) - \sum_{\iota=k-\overline{\tau}_0+1}^{k-\underline{\tau}_0} \sum_{v=\iota}^{k-1} G^T(x(v))QG(x(v))\\
=\;& \sum_{\iota=k-\overline{\tau}_0+1}^{k-\underline{\tau}_0} \big(G^T(x(k))QG(x(k)) - G^T(x(\iota))QG(x(\iota))\big)\\
=\;& (\overline{\tau}_0-\underline{\tau}_0)G^T(x(k))QG(x(k)) - \sum_{v=k-\overline{\tau}_0+1}^{k-\underline{\tau}_0} G^T(x(v))QG(x(v)),
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\mathbb{E}[V_4(\mathbf{x}_{k+1},k+1,r(k+1)) \mid \mathbf{x}_k, r(k)=i] - V_4(\mathbf{x}_k,k,i)\\
=\;& \sum_{j=1}^{N}\pi_{ij}\sum_{\iota=\tau_{1,j}}^{\tau_{2,j}}\sum_{v=k-\iota+1}^{k} H^T(x(v))RH(x(v)) - \sum_{\iota=\tau_{1,i}}^{\tau_{2,i}}\sum_{v=k-\iota}^{k-1} H^T(x(v))RH(x(v))\\
=\;& \sum_{j=1}^{N}\pi_{ij}(\tau_{2,j}-\tau_{1,j})H^T(x(k))RH(x(k)) + \sum_{j=1}^{N}\pi_{ij}\Bigg[\sum_{\iota=\tau_{1,j}}^{\tau_{2,j}}\sum_{v=k-\iota+1}^{k-1} H^T(x(v))RH(x(v))\\
&- \sum_{\iota=\tau_{1,i}}^{\tau_{2,i}}\sum_{v=k-\iota+1}^{k-1} H^T(x(v))RH(x(v))\Bigg] + \sum_{\iota=\tau_{1,i}}^{\tau_{2,i}}\sum_{v=k-\iota+1}^{k-1} H^T(x(v))RH(x(v))\\
&- \sum_{\iota=\tau_{1,i}}^{\tau_{2,i}}\sum_{v=k-\iota}^{k-1} H^T(x(v))RH(x(v))\\
=\;& \hat{\tau}_i H^T(x(k))RH(x(k)) + \sum_{j\neq i}\pi_{ij}\Bigg[\sum_{\iota=\tau_{1,j}}^{\tau_{2,j}}\sum_{v=k-\iota+1}^{k-1} H^T(x(v))RH(x(v))\\
&- \sum_{\iota=\tau_{1,i}}^{\tau_{2,i}}\sum_{v=k-\iota+1}^{k-1} H^T(x(v))RH(x(v))\Bigg] - \sum_{\iota=\tau_{1,i}}^{\tau_{2,i}} H^T(x(k-\iota))RH(x(k-\iota))\\
\leq\;& \hat{\tau}_i H^T(x(k))RH(x(k)) + \sum_{j\neq 1}\pi_{ij}\Bigg[\sum_{\iota=\underline{\tau}_1}^{\overline{\tau}_2}\sum_{v=k-\iota+1}^{k-1} H^T(x(v))RH(x(v))\\
&- \sum_{\iota=\overline{\tau}_1}^{\underline{\tau}_2}\sum_{v=k-\iota+1}^{k-1} H^T(x(v))RH(x(v))\Bigg] - \sum_{\iota=\tau_{1,i}}^{\tau_{2,i}} H^T(x(k-\iota))RH(x(k-\iota))\\
\leq\;& \hat{\tau}_i H^T(x(k))RH(x(k)) + (1-\underline{\pi})\sum_{\iota=\underline{\tau}_2+1}^{\overline{\tau}_2}\sum_{v=k-\iota+1}^{k-1} H^T(x(v))RH(x(v))\\
&+ (1-\underline{\pi})\sum_{\iota=\underline{\tau}_1}^{\overline{\tau}_1-1}\sum_{v=k-\iota+1}^{k-1} H^T(x(v))RH(x(v)) - \sum_{\iota=\tau_{1,i}}^{\tau_{2,i}} H^T(x(k-\iota))RH(x(k-\iota))
\end{aligned}
\tag{21}
$$

where $\hat{\tau}_i = \sum_{j=1}^{N} \pi_{ij}(\tau_{2,j} - \tau_{1,j})$ and

$$\begin{aligned}
&\mathbb{E}[V_5(\mathbf{x}_{k+1}, k+1, r(k+1)) \mid \mathbf{x}_k, r(k) = i] - V_5(\mathbf{x}_k, k, i) \\
= \;& (1-\underline{\pi})\Big[\sum_{s=\underline{\tau}_1}^{\overline{\tau}_1 - 1}\sum_{\iota=1}^{s-1}\sum_{v=k-\iota+1}^{k} H^T(x(v))RH(x(v)) - \sum_{s=\underline{\tau}_1}^{\overline{\tau}_1 - 1}\sum_{\iota=1}^{s-1}\sum_{v=k-\iota}^{k-1} H^T(x(v))RH(x(v))\Big] \\
= \;& (1-\underline{\pi})\Big[\sum_{s=\underline{\tau}_1}^{\overline{\tau}_1 - 1}\sum_{\iota=1}^{s-1}\big(H^T(x(k))RH(x(k)) - H^T(x(k-\iota))RH(x(k-\iota))\big)\Big] \\
= \;& (1-\underline{\pi})\Big[\frac{1}{2}(\overline{\tau}_1 - \underline{\tau}_1)(\overline{\tau}_1 + \underline{\tau}_1 - 3)H^T(x(k))RH(x(k)) - \sum_{\iota=\underline{\tau}_1}^{\overline{\tau}_1 - 1}\sum_{v=k-\iota+1}^{k-1} H^T(x(v))RH(x(v))\Big].
\end{aligned} \tag{22}$$

Similarly, we have

$$\begin{aligned}
&\mathbb{E}[V_6(\mathbf{x}_{k+1}, k+1, r(k+1)) \mid \mathbf{x}_k, r(k) = i] - V_6(\mathbf{x}_k, k, i) \\
= \;& (1-\underline{\pi})\Big[\frac{1}{2}(\overline{\tau}_2 - \underline{\tau}_2)(\overline{\tau}_2 + \underline{\tau}_2 - 1)H^T(x(k))RH(x(k)) - \sum_{\iota=\underline{\tau}_2+1}^{\overline{\tau}_2}\sum_{v=k-\iota+1}^{k-1} H^T(x(v))RH(x(v))\Big].
\end{aligned} \tag{23}$$

From (18)-(23), it follows that

$$\begin{aligned}
&\mathbb{E}[V(\mathbf{x}_{k+1}, k+1, r(k+1)) \mid \mathbf{x}_k, r(k) = i] - V(\mathbf{x}_k, k, i) \\
\leq \;& \xi^T(k,i)\chi^T(i)\overline{P}_i\chi(i)\xi(k,i) - x^T(k)P_ix(k) + (\overline{\tau}_0 - \underline{\tau}_0 + 1)G^T(x(k))QG(x(k)) \\
& - G^T(k-\tau_0(k))QG(k-\tau_0(k)) + \kappa_i H^T(x(k-v))RH(x(k-v)) \\
& - \sum_{v=1}^{\tau_{2,i}} H^T(x(k-v))RH(x(k-v)),
\end{aligned} \tag{24}$$

where κ_i is defined in (8).

From Lemma 1, one has

$$-\sum_{v=\tau_{2,i}}^{\tau_{2,i}} H^T(x(k-v))RH(x(k-v)) \leq -\frac{1}{\tau_{2,i} - \tau_{1,i}}\Big(\sum_{v=\tau_{2,i}}^{\tau_{2,i}} H(x(k-v))\Big)^T R \sum_{v=\tau_{1,i}}^{\tau_{2,i}} H(x(k-v)). \tag{25}$$

Also, notice that condition (2) is equivalent to

$$(f_i(x_i) - l_i^+ x_i)(f_i(x_i) - l_i^- x_i) \leq 0,$$

or

$$\begin{bmatrix} x \\ F(x) \end{bmatrix}^T \begin{bmatrix} l_i^+ l_i^- e_i e_i^T & -\frac{l_i^+ + l_i^-}{2} e_i e_i^T \\ -\frac{l_i^+ + l_i^-}{2} e_i e_i^T & e_i e_i^T \end{bmatrix} \begin{bmatrix} x \\ F(x) \end{bmatrix} \leq 0, \quad i = 1, ..., n,$$

where e_k denotes the unit column vector having "1" element on its kth row and zeros elsewhere.

Since $\Omega_i = \mathrm{diag}\{\omega_{i1}, \omega_{i2}, ..., \omega_{in}\} \geq 0$, it follows readily that

$$\sum_{i=1}^{n} \omega_i \begin{bmatrix} x \\ F(x) \end{bmatrix}^T \begin{bmatrix} l_i^+ l_i^- e_i e_i^T & -\frac{l_i^+ + l_i^-}{2} e_i e_i^T \\ -\frac{l_i^+ + l_i^-}{2} e_i e_i^T & e_i e_i^T \end{bmatrix} \begin{bmatrix} x \\ F(x) \end{bmatrix} \leq 0,$$

or

$$\begin{bmatrix} x \\ F(x) \end{bmatrix}^T \begin{bmatrix} \Omega_i \Lambda_1 & -\Omega_i \Lambda_2 \\ -\Omega_i \Lambda_2 & \Omega_i \end{bmatrix} \begin{bmatrix} x \\ F(x) \end{bmatrix} \leq 0,$$

which implies that

$$x^T(k)\Omega_i\Lambda_1x(k) - 2x^T(k)\Omega_i\Lambda_2F(x(k)) + F^T(x(k))\Omega_iF(x(k)) \leq 0. \tag{26}$$

Similarly, we have from (3) and (4) that

$$x^T(k)\Theta_i\Sigma_1x(k) - 2x^T(k)\Theta_i\Sigma_2G(x(k)) + G^T(x(k))\Theta_iG(x(k)) \leq 0, \tag{27}$$
$$x^T(k)\Delta_i\Upsilon_1x(k) - 2x^T(k)\Delta_i\Upsilon_2H(x(k)) + H^T(x(k))\Delta_iH(x(k)) \leq 0. \tag{28}$$

Considering (25)-(28), it follows from (24) that

$$\begin{aligned}
&\mathbb{E}[V(\mathbf{x}_{k+1}, k+1, r(k+1)) \mid \mathbf{x}_k, r(k) = i] - V(\mathbf{x}_k, k, i) \\
\leq\ & \xi^T(k,i)\chi^T(i)\overline{P}_i\chi(i)\xi(k,i) - x^T(k)P_ix(k) + (\overline{\tau}_0 - \underline{\tau}_0 + 1)G^T(x(k))QG^T(x(k)) \\
& - G^T(k-\tau_0(k))QG(k-\tau_0(k)) + \kappa_iH^T(x(k-v))RH(x(k-v)) \\
& - \frac{1}{\tau_{2,i}-\tau_{1,i}}\Big(\sum_{v=\tau_{1,i}}^{\tau_{2,i}} H(x(k-v))\Big)^T R \sum_{v=\tau_{1,i}}^{\tau_{2,i}} H(x(k-v)) - \Big(x^T(k)\Omega_i\Lambda_1x(k) - 2x^T(k)\Omega_i\Lambda_2F(x(k)) \\
& + F^T(x(k))\Omega_iF(x(k))\Big) - \Big(x^T(k)\Theta_i\Sigma_1x(k) - 2x^T(k)\Theta_i\Sigma_2G(x(k)) + G^T(x(k))\Theta_iG(x(k))\Big) \\
& - \Big(x^T(k)\Delta_i\Upsilon_1x(k) - 2x^T(k)\Delta_i\Upsilon_2H(x(k)) + H^T(x(k))\Delta_iH(x(k))\Big) \\
& = \xi^T(k,i)\Big(\chi^T(i)\overline{P}_i\chi(i) + \hat{\Phi}_i\Big)\xi(k,i).
\end{aligned} \tag{29}$$

Setting $\alpha_0 = \max_{i\in\mathcal{S}}\Big\{\lambda_{\max}\big(\chi^T(i)\overline{P}_i\chi(i) + \hat{\Phi}_i\big)\Big\}$, we obtain from (9) that $\alpha_0 < 0$. Furthermore, it follows from (29) that

$$\mathbb{E}[V(\mathbf{x}_{k+1}, k+1, r(k+1)) \mid \mathbf{x}_k, r(k) = i] - V(\mathbf{x}_k, k, i) \leq \alpha_0|x(k)|^2,$$

which implies

$$\Delta\mathbb{E}[V(\mathbf{x}_k, k, r(k))] = \mathbb{E}[V(\mathbf{x}_{k+1}, k+1, r(k+1))] - \mathbb{E}[V(\mathbf{x}_k, k, r(k))] \leq \alpha_0\mathbb{E}[|x(k)|^2]. \tag{30}$$

Next, we proceed to deal with the exponential stability of system (1). Firstly, by the definition of $V(\mathbf{x}_k, k, r(k))$, it is obvious that there exist two constants $c_1 > 0$ and $c_2 > 0$ satisfying

$$c_1\mathbb{E}[|x(k)|^2] \leq \mathbb{E}[V(\mathbf{x}_k, k, r(k))] \leq c_2\sum_{i=k-\tau}^{k}\mathbb{E}[|x(i)|^2]. \tag{31}$$

Then, for arbitrary positive $\mu > 1$ and nonnegative integer j, we have

$$\begin{aligned}
&\mu^{j+1}\mathbb{E}[V(\mathbf{x}_{j+1}, j+1, r(j+1))] - \mu^j\mathbb{E}[V(\mathbf{x}_j, j, r(j))] \\
&= \mu^{j+1}\Delta\mathbb{E}[V(\mathbf{x}_j, j, r(j))] + \mu^j(\mu-1)\mathbb{E}[V(\mathbf{x}_j, j, r(j))] \\
&\leq \vartheta_1(\mu)\mu^j\mathbb{E}[|x(j)|^2] + \vartheta_2(\mu)\mu^j\sum_{i=j-\tau}^{j-1}\mathbb{E}[|x(i)|^2]
\end{aligned} \tag{32}$$

with

$$\vartheta_1(\mu) = \mu\alpha_0 + (\mu-1)c_2, \quad \vartheta_2(\mu) = (\mu-1)c_2.$$

Let k be any positive integer. Then, summing up both sides of (32) from 0 to k with respect to j yields

$$\mu^k\mathbb{E}[V(\mathbf{x}_k, k, r(k))] - \mathbb{E}[V(\mathbf{x}_0, 0, r(0))] \leq \vartheta_1(\mu)\sum_{j=0}^{k-1}\mu^j\mathbb{E}[|x(j)|^2] + \vartheta_2(\mu)\sum_{j=0}^{k-1}\mu^j\sum_{i=j-\tau}^{j-1}\mathbb{E}[|x(i)|^2]. \tag{33}$$

It can be verified that

$$\begin{aligned}
&\sum_{j=0}^{k-1}\mu^j\sum_{i=j-\tau}^{j-1}\mathbb{E}[|x(i)|^2]\\
&=\sum_{i=-\tau}^{-1}\sum_{j=0}^{i+\tau}\mu^j\mathbb{E}[|x(i)|^2]+\sum_{i=0}^{k-\tau-1}\sum_{j=i+1}^{i+\tau}\mu^j\mathbb{E}[|x(i)|^2]+\sum_{i=k-\tau}^{k-2}\sum_{j=i+1}^{k-1}\mu^j\mathbb{E}[|x(i)|^2]\\
&\le\tau\sum_{i=-\tau}^{-1}\mu^{i+\tau}\mathbb{E}[|x(i)|^2]+\tau\sum_{i=0}^{k-\tau-1}\mu^{i+\tau}\mathbb{E}[|x(i)|^2]+\tau\sum_{i=k-\tau}^{k-2}\mu^{i+\tau}\mathbb{E}[|x(i)|^2]\\
&\le\tau\mu^\tau\sum_{i=-\tau}^{-1}\mathbb{E}[|x(i)|^2]+\tau\mu^\tau\sum_{i=0}^{k-1}\mu^i\mathbb{E}[|x(i)|^2]. \qquad (34)
\end{aligned}$$

From (33), together with (31) and (34), it follows that

$$\mu^k\mathbb{E}[V(\mathbf{x}_k,k,r(k))]\le\mathbb{E}[V(\mathbf{x}_0,0,r(0))]+\tau\vartheta_2(\mu)\mu^\tau\sum_{i=-\tau}^{-1}\mathbb{E}[|x(i)|^2]+\big(\vartheta_1(\mu)+\tau\vartheta_2(\mu)\mu^\tau\big)\sum_{j=0}^{k-1}\mu^j\mathbb{E}[|x(j)|^2] \quad (35)$$

Picking a $\mu_0>1$ such that $\vartheta_1(\mu_0)+\tau\vartheta_2(\mu_0)\mu_0^\tau=0$, it follows from (35) that

$$\mu_0^k\mathbb{E}[V(\mathbf{x}_k,k,r(k))]\le\mathbb{E}[V(\mathbf{x}_0,0,r(0))]+\tau\vartheta_2(\mu_0)\mu_0^\tau\sum_{i=-\tau}^{-1}\mathbb{E}[|x(i)|^2] \quad (36)$$

which, together with (31), implies that

$$c_1\mu_0^k\mathbb{E}[|x(k)|^2]\le\big(c_2+\tau\vartheta_2(\mu_0)\mu_0^\tau\big)\sum_{i=-\tau}^{0}\mathbb{E}[|x(i)|^2].$$

There, we obtain

$$\mathbb{E}[|x(k)|^2]\le\frac{c_2+\tau\vartheta_2(\mu_0)\mu_0^\tau}{c_1}(1+\tau)\mu_0^{-k}\max_{i=-\tau}\mathbb{E}[|x(i)|^2],$$

which shows that the system (1) is exponentially stable in mean square. The proof of this theorem is complete. ■

IV. State estimation

In biological or artificial neural networks, it is usually the case that the states of neural networks are not completely accessible and only partial information can be obtained from the output of neural networks. Consequently, it is important to estimate the neuron state from the given output for some specific design objectives in many practical applications, and there is a need to construct an estimator to approach the state of the neural network (1) in an asymptotical or exponential way.

Assume that the measurement from the output from the network (1) is given by

$$y(k)=M(r(k))x(k). \quad (37)$$

Here, $y(k)$ is the measurement of (1) and $M(r(k))\in\mathbb{R}^{m\times n}$ $(m<n)$ are mode-dependent output matrices. Note that the assumption of $m<n$ means that only partial information about the system states can be accessible by the measurement outputs.

In order to estimate the state of system (1), we construct the following state estimator:

$$\begin{aligned}\hat{x}(k+1) =& D(r(k))\hat{x}(k) + A(r(k))F(\hat{x}(k)) + B(r(k))G(\hat{x}(k-\tau_0(k))) \\ &+ C(r(k))\sum_{v=\tau_{1,r(k)}}^{\tau_{2,r(k)}} H(\hat{x}(k-v)) + K(r(k))[y(k) - M(r(k))\hat{x}(k)]\end{aligned} \tag{38}$$

where $\hat{x}(k)$ is the state estimate of (1) and $K(i) \in \mathbb{R}^{n\times m}$ $(i \in \mathcal{S})$ are the state estimate gain matrices to be designed. For the activation functions, we still assume that (2)-(4) hold. Let $\boldsymbol{e}(k) = (e_1(k), e_2(k), ..., e_n(k))^T \triangleq \hat{x}(k) - x(k)$ be the state estimation error. Also, denote $\tilde{F}(\boldsymbol{e}(k)) = \big(\tilde{f}_1(e_1(k)), \tilde{f}_2(e_2(k)), ..., \tilde{f}_n(e_n(k))\big)^T \triangleq F(\hat{x}(k)) - F(x(k))$, $\tilde{G}(\boldsymbol{e}(k)) = \big(\tilde{g}_1(e_1(k)), \tilde{g}_2(e_2(k)), ..., \tilde{g}_n(e_n(k))\big)^T \triangleq G(\hat{x}(k)) - G(x(k))$, and $\tilde{H}(\boldsymbol{e}(k)) = \big(\tilde{h}_1(e_1(k)), \tilde{h}_2(e_2(k)), ..., \tilde{h}_n(e_n(k))\big)^T \triangleq H(\hat{x}(k)) - H(x(k))$. Then, the estimation error is subject to the dynamics governed by

$$\begin{aligned}\boldsymbol{e}(k+1) =& D_M(r(k))\boldsymbol{e}(k) + A(r(k))\tilde{F}(\boldsymbol{e}(k)) + B(r(k))\tilde{G}(\boldsymbol{e}(k-\tau_0(k))) \\ &+ C(r(k))\sum_{v=\tau_{1,r(k)}}^{\tau_{2,r(k)}} \tilde{H}(\boldsymbol{e}(k-v))\end{aligned} \tag{39}$$

where $D_M(r(k)) = D(r(k)) - K(r(k))M(r(k))$.

Remark 2: Under Assumption 1, it is easy to verify that (2)-(4) still hold for functions $\tilde{F}, \tilde{G}$ and $\tilde{H}$ with f_i, g_i and h_i replaced by $\tilde{f}_i, \tilde{g}_i$ and $\tilde{h}_i$, respectively.

Definition 2: System (38) is said to be an *asymptotic state estimator* of neural network (1) if the estimation error satisfies

$$\lim_{k\to+\infty} \mathbb{E}[|\boldsymbol{e}(k)|^2] = 0;$$

and system (38) is said to be an *exponential state estimator* of neural network (1) if there exist constants $\mu > 1$ and $\beta > 0$ such that the estimation error satisfies

$$\mathbb{E}[|\boldsymbol{e}(k)|^2] \le \beta\mu^{-k} \max_{-\tau\le i\le 0} \mathbb{E}[|\boldsymbol{e}(i)|^2], \ \forall k > 0.$$

For the dynamics analysis of estimation error, we have the following result.

Theorem 2: Let $K(i)$ $(i \in \mathcal{S})$ be known constant matrices. Then, under Assumption 1, state estimator (38) becomes an exponential state estimator of the delayed neural network (1) if there exist a set of matrices $P_i > 0$, two matrices $Q > 0$ and $R > 0$, and three sets of diagonal matrices $\Omega_i > 0, \Theta_i > 0$ and $\Delta_i > 0$ such that the following LMIs hold:

$$\Psi_i := \begin{pmatrix} \Xi_i & \Omega_i\Lambda_2 & \Theta_i\Sigma_2 & 0 & \Delta_i\Upsilon_2 & 0 & D_M(i)\overline{P}_i \\ * & -\Omega_i & 0 & 0 & 0 & 0 & A^T(i)\overline{P}_i \\ * & * & \Pi_i & 0 & 0 & 0 & 0 \\ * & * & * & -Q & 0 & 0 & B^T(i)\overline{P}_i \\ * & * & * & * & \kappa_i R - \Delta_i & 0 & 0 \\ * & * & * & * & * & -\frac{1}{\tau_{2,i}-\tau_{1,i}}R & C^T(i)\overline{P}_i \\ * & * & * & * & * & * & -\overline{P}_i \end{pmatrix} < 0, \quad (i \in \mathcal{S}) \tag{40}$$

where $\overline{P}_i, \Xi_i, \Pi_i$, and κ_i are defined as in Theorem 1.

Proof: Clearly, for system (38) to be an exponential state estimator of the delayed neural network (1), we need to guarantee the exponential stability of the error dynamics (39) in mean square. It should be pointed out that, the exponential stability of (39) can be established by following the derivation similar to Theorem 1. To avoid duplication, the detailed proof of Theorem 2 is omitted here. ■

In Theorem 2, $K(i)$ $(i \in \mathcal{S})$ are assumed to be known matrices. However, what is important in practice is how to determine these matrices, in other words, how to actually design the state estimator. To answer this question, we have the following results.

Theorem 3: Under Assumption 1, state estimator (38) becomes an exponential state estimator of the delayed neural network (1) if there exist a set of matrices X_i $(i \in \mathcal{S})$ a set of matrices $P_i > 0$, two matrices $Q > 0$ and $R > 0$, and three sets of diagonal matrices $\Omega_i > 0, \Theta_i > 0$ and $\Delta_i > 0$ such that the following LMIs hold:

$$\Psi_i := \begin{pmatrix} \Xi_i & \Omega_i\Lambda_2 & \Theta_i\Sigma_2 & 0 & \Delta_i\Upsilon_2 & 0 & \overline{P}_i D(i) - M^T(i)X_i^T \\ * & -\Omega_i & 0 & 0 & 0 & 0 & A^T(i)\overline{P}_i \\ * & * & \Pi_i & 0 & 0 & 0 & 0 \\ * & * & * & -Q & 0 & 0 & B^T(i)\overline{P}_i \\ * & * & * & * & \kappa_i R - \Delta_i & 0 & 0 \\ * & * & * & * & * & -\frac{1}{\tau_{2,i}-\tau_{1,i}}R & C^T(i)\overline{P}_i \\ * & * & * & * & * & * & -\overline{P}_i \end{pmatrix} < 0, \quad (i \in \mathcal{S}) \tag{41}$$

where $\overline{P}_i, \Xi_i, \Pi_i$, and κ_i are defined as in Theorem 1. Accordingly, the estimate gain matrices can be chosen as $K(i) = \overline{P}_i^{-1} X_i$ $(i \in \mathcal{S})$.

Proof: It is an immediate result from Theorem 2. ■

Remark 3: In Theorems 1-3, an LMI-based approach has been developed for the stability analysis and estimator design for neural network (1). We mention here that such a treatment would be easily extended to deal with the networks with white noises. The criteria derived here are in the form of linear matrix inequalities that can be effectively solved and checked by the algorithms such as the interior-point method.

V. A numerical examples

In this section, an example is presented here to demonstrate the effectiveness of our main results.

Example 1: Consider a three-neuron neural network (1) with the following parameters:

$$D(1) = \begin{bmatrix} 1.3 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}, \; A(1) = \begin{bmatrix} 0.3 & -0.4 & 0.2 \\ 0.1 & -0.4 & 0 \\ 0 & -0.1 & 0.3 \end{bmatrix}, \; B(1) = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.1 & -0.2 & 0 \\ 0.2 & -0.1 & -0.2 \end{bmatrix},$$

$$C(1) = \begin{bmatrix} 0.2 & 0.1 & -0.1 \\ 0 & 0.3 & 0.3 \\ -0.3 & 0 & 0.2 \end{bmatrix}, \; D(2) = \begin{bmatrix} -1.4 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}, \; A(2) = \begin{bmatrix} 0.4 & -0.2 & 0.1 \\ 0.1 & -0.1 & 0.2 \\ 0.1 & 0 & 0.2 \end{bmatrix},$$

$$B(2) = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.2 & -0.2 & 0 \\ 0.3 & -0.1 & -0.1 \end{bmatrix}, \; C(2) = \begin{bmatrix} 0.2 & -0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 \\ 0.7 & 0 & 0.2 \end{bmatrix}, \; \Pi = \begin{bmatrix} 0.4 & 0.6 \\ 0.55 & 0.45 \end{bmatrix},$$

$$\tau_0(k) = 8 + (-1)^k, \; \tau_{1,1} = 1, \; \tau_{1,2} = 2, \; \tau_{2,1} = 2, \; \tau_{2,2} = 3.$$

Take the activation functions as follows:

$$f_1(s) = g_1(s) = h_1(s) = \tanh(-0.4s), \quad f_2(s) = g_2(s) = h_2(s) = \tanh(0.4s),$$
$$f_3(s) = g_3(s) = h_3(s) = 0.8\tanh(0.2s).$$

It is easy to verify that $\Lambda_1 = \Sigma_1 = \Upsilon_1 = 0$, $\Lambda_2 = \Sigma_2 = \Upsilon_2 = \text{diag}\{-0.2, 0.2, 0.1\}$. The state evolution of network (1) with the above parameters is shown in Fig. 1

It can be seen from Fig. 1 that the original network (1) is unstable, and our aim is to estimate the network states through available measurement output exponentially. Now, in the output (37), we take $M(1) = M(2) =$

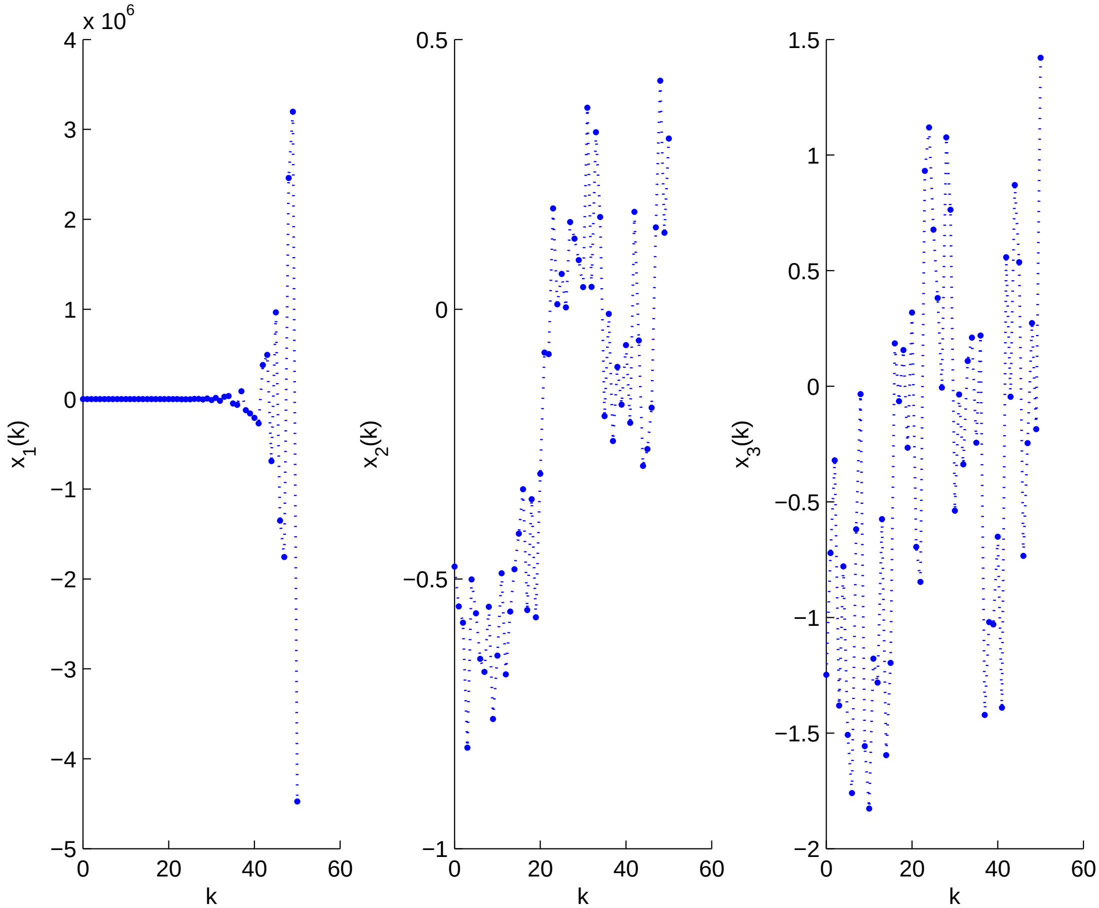

Fig. 1. The State Evolution of Original System

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. To design the exponential state estimator, by using the Matlab LMI Toolbox, we solve the LMIs (41) and obtain the feasible solution as follows:

$$P_1 = \begin{bmatrix} 110.8222 & -3.9401 & -13.5736 \\ -3.9401 & 120.7580 & -24.6933 \\ -13.5736 & -24.6933 & 56.6580 \end{bmatrix}, \ P_2 = \begin{bmatrix} 104.6359 & -6.2682 & -9.3142 \\ -6.2682 & 116.4385 & -23.9323 \\ -9.3142 & -23.9323 & 67.1470 \end{bmatrix},$$

$$Q = \begin{bmatrix} 54.2485 & 0.2484 & -0.0796 \\ 0.2484 & 48.5650 & 0.3910 \\ -0.0796 & 0.3910 & 42.3273 \end{bmatrix}, \ R = \begin{bmatrix} 37.1554 & 0.5641 & 0.4685 \\ 0.5641 & 30.3276 & -0.6087 \\ 0.4685 & -0.6087 & 26.9403 \end{bmatrix},$$

$$\Omega_1 = \mathrm{diag}\{121.5379, 137.0317, 121.2769\}, \ \Omega_2 = \mathrm{diag}\{123.3261, 113.9034, 117.8766\}$$

$$\Theta_1 = \mathrm{diag}\{247.4227, 229.0166, 214.5096\}, \ \Theta_2 = \mathrm{diag}\{247.8891, 228.9265, 212.7504\},$$

$$\Delta_1 = \mathrm{diag}\{164.1961, 148.8719, 143.5709\}, \ \Delta_2 = \mathrm{diag}\{192.9705, 146.9838, 144.7939\}.$$

From Theorem 3, it follows that the state estimator (38) is indeed an exponential state estimator of the delayed neural network (1). That is, the estimation errors $e_i(k)$ tend to zero as $k \to \infty$, which is further confirmed by numerical simulation as demonstrated in Fig. 2.

Remark 4: Notice that the design of exponential estimator would be trivial if the original system (1) is stable itself. However, in Example 1, the system matrices $A(1)$ and $A(2)$ have been chosen to be unstable since the spectral radii of $A(1)$ and $A(2)$ are both larger than one. The simulation shows that, although the

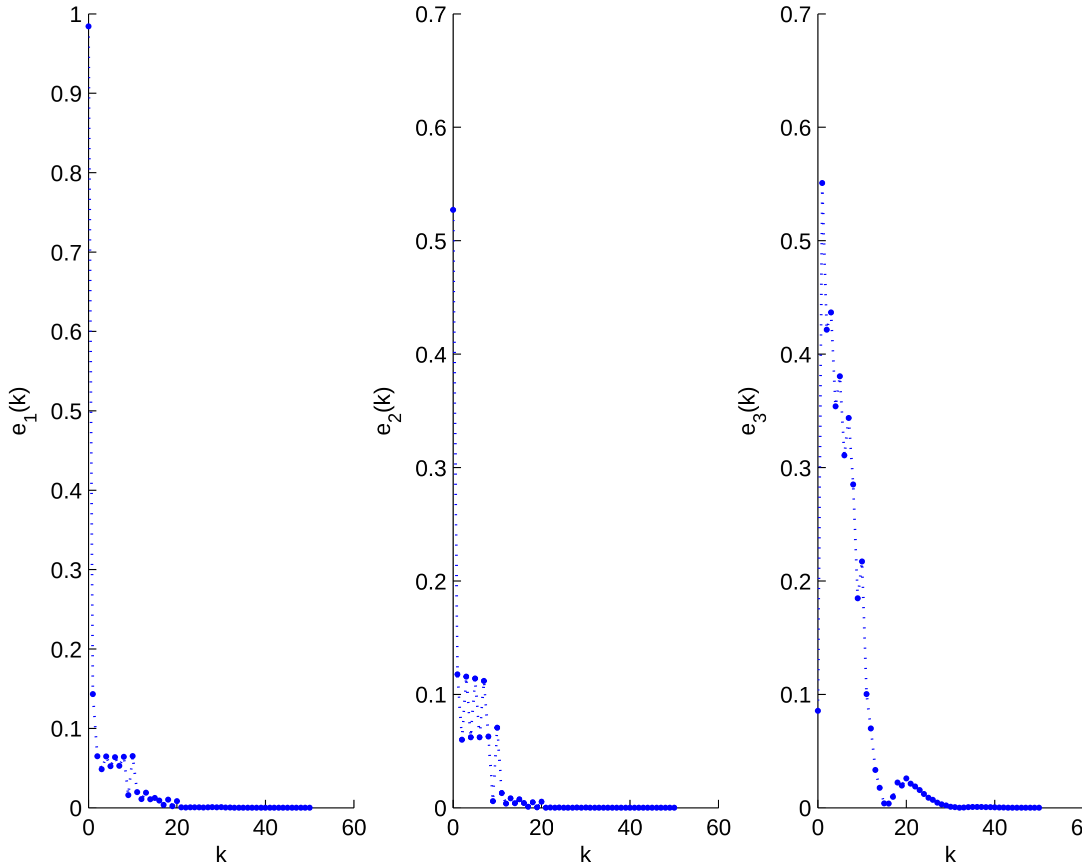

Fig. 2. The Evolution of Estimation Errors

system (1) with given parameters is unstable (see Fig. 1), the resulting estimation errors approach zero (see Fig. 2) exponentially. This illustrates the effectiveness of the proposed methods.

VI. Conclusions

In this paper, we have introduced a new class of discrete-time neural networks with Markovian jumping parameters as well as mixed time-delays. The networks involved include both time-varying discrete time-delay and distributed time-delay whose lower and upper bounds are mode-dependent. By employing new Lyapunov-Krasovskii functionals, LMI-based conditions are established for the networks under study to be exponentially stable in mean square. Moreover, the design of desired exponential state estimator has also been reduced to the feasibility problem of LMIs, and the estimate gain matrices are explicitly given. Numerical simulation has further demonstrated the effectiveness of the main results obtained. The future research topics would include the extension of the main results obtained in this paper to more general networks with missing measurements, degraded measurements, sensor saturations, mode-dependent mixed time-delays, nonlinear disturbances, and randomly occurring nonlinearities [26–31].

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