Robust $\mathcal{H}_\infty$ Sliding Mode Control for Nonlinear Stochastic Systems with Multiple Data Packet Losses

Lifeng Ma*, Zidong Wang, Yuming Bo and Zhi Guo

Abstract

In this paper, an $\mathcal{H}_\infty$ sliding mode control (SMC) problem is studied for a class of discrete-time nonlinear stochastic systems with multiple data packet losses. The phenomenon of data packet losses, which is assumed to occur in a random way, is taken into consideration in the process of data transmission through both the state feedback loop and the measurement output. The probability for the data packet loss for each individual state variable is governed by a corresponding individual random variable satisfying a certain probabilistic distribution over the interval $[0 \quad 1]$. The discrete-time system considered is also subject to norm-bounded parameter uncertainties and external nonlinear disturbances, which enter the system state equation in both matched and unmatched ways. A novel stochastic discrete-time switching function is proposed to facilitate the sliding mode controller design. Sufficient conditions are derived by means of the linear matrix inequality (LMI) approach. It is shown that the system dynamics in the specified sliding surface is exponentially stable in the mean square with a prescribed $\mathcal{H}_\infty$ noise attenuation level if an LMI with an equality constraint is feasible. A discrete-time SMC controller is designed capable of guaranteeing the discrete-time sliding mode reaching condition of the specified sliding surface with probability 1. Finally, a simulation example is given to show the effectiveness of the proposed method.

Keywords

Discrete-time systems; Nonlinear systems; Multiple data packet losses; Discrete-time sliding mode control; Robust control.

I. Introduction

In the past few decades, the sliding mode control (also known as variable structure control) problem originated in [33] has been extensively studied because of its advantage of strong robustness against model uncertainties, parameter variations and external disturbances, see [6, 16, 26–28] and the references therein. In the sliding mode control, trajectories are forced to reach a sliding manifold in finite time and then stay on the manifold for all future time. It is worth mentioning that, in the existing literature concerning SMC problem for nonlinear systems, the nonlinearities and uncertainties taken into consideration are mainly under the matching conditions, that is to say, the nonlinear and uncertain terms enter the state equation at the same point as the control input and consequently the motion on the sliding manifold is independent of those matched terms, see [23, 24, 34]. However, in engineering practice, a large part of external nonlinear disturbances and parameter uncertainties cannot be treated as matched type of nonlinearities.

In recent years, since most control strategies are implemented in a discrete-time setting (e.g., networked control systems), the sliding mode control (SMC) problem for discrete-time systems has gained considerable

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research interests and many results have been reported in the literature, see [1, 3, 5, 7, 8, 14, 18, 20, 21, 40, 43]. To be specific, an integral type SMC schemes was proposed in [1, 3] for sampled-data systems and a class of nonlinear discrete-time systems, respectively. By applying adaptive laws, in [5, 7], the authors synthesized sliding mode controllers for discrete-time systems with stochastic as well as deterministic disturbances. A methodology for designing sliding mode controllers was put forward in [8] for a class of linear multi-input discrete-time systems with matching perturbations. In [18], using dead-beat control technique, the authors presented a discrete variable structure control method with a finite-time step to reach the switching surface. In [20, 21], the discrete-time SMC problems were solved via output feedback. It is worth mentioning that in [14], a novel reaching law approach was developed, which was conveniently and widely applied in literature to handle robust SMC control problems for discrete-time systems, see [40, 43] for some latest publications.

On another research front, in most practical systems nowadays such as a target tracking system, there may be certain observations that consist of noise only when the target is absent due to its high maneuverability. In other words, the measurements are not consecutive but usually subject to partial or complete information missing. Such a phenomenon is referred to as measurement missing, information dropout or data packet losses, which occurs frequently for a variety of reasons such as sensor temporal failure, network congestion, accidental loss of some collected data or network-induced delay, and might leads to system performance degradation and sometimes even instability [9, 32, 37, 38]. Therefore, in the past few years, a great deal of research efforts has been made to solve the control and filtering problems in the presence of data packet losses. Such a data packet loss phenomenon is usually characterized in a probabilistic way. As a result, in [22,30,31,35], the packet losses phenomenon has been modeled by different kinds of stochastic variables, among which the binary random variable sequence taking on values of 0 and 1, also known as Bernoulli distributed model, have been widely applied because of its simplicity [35, 44]. It is worth mentioning that, in a recent paper [39], it has been assumed that the missing probability for each sensor is governed by an individual random variable satisfying a certain probabilistic distribution over the interval $[0 \;\; 1]$, which is more general than most existing literature and includes the Bernoulli distribution as a special case.

Up to now, to the best of the authors' knowledge, the $\mathcal{H}_\infty$ sliding mode control problem has not been studied for discrete-time uncertain nonlinear stochastic system with multiple data packet losses. By using the discrete-time sliding motion concept, this paper aims to design a state feedback controller such that 1) the system state trajectories are globally driven onto the pre-specified sliding surface with probability 1, then results in a non-increasing zigzag motion on the sliding surface; 2) the exponentially mean-square stability and the $\mathcal{H}_\infty$ noise attenuation level of the system are simultaneously achieved on the pre-specified sliding surface. The main contribution of this paper lies in (a) a new description of data packet losses, which is much more general than the existing literature, is proposed and considered in both the state-feedback loop and the measurement output channel; (b) in the light of the presented data packet losses model, a novel stochastic switching function is proposed for the discrete-time SMC problem and then a control law is designed to drive the state trajectories onto a pre-specified stochastic sliding surface with probability 1; (c) an algorithm is proposed which is capable of handling both matched and unmatched external nonlinear disturbances as well as internal parameter variations.

The rest of this paper is arranged as follows. Section II formulates an uncertain nonlinear stochastic system with multiple data packet losses. In Section III, a novel switching function is first put forward and then two LMI-based sufficient conditions are given to obtain the parameters in the proposed switching function for simultaneously ensuring the exponentially mean-square stability and $\mathcal{H}_\infty$ performance in the sliding surface. Secondly, an SMC law is synthesized to drive the state trajectories onto the specified surface with probability 1. In Section IV, an illustrative numerical example is provided to show the effectiveness and usefulness of the proposed approach. Section V gives our conclusions.

Notation The following notation will be used in this paper. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the

n-dimensional Euclidean space and the set of all $n \times m$ matrices, and $\mathbb{I}^+$ denotes the set of nonnegative integers. The notation $X \geq Y$ (respectively $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively positive definite). $\mathbb{E}\{x\}$ stands for the expectation of stochastic variable x and $\mathbb{E}\{x|y\}$ for the expectation of x conditional on y. The superscript "T" denotes the transpose. $\mathrm{diag}\{F_1, F_2, \ldots\}$ denotes a block diagonal matrix whose diagonal blocks are given by F_1, F_2 .... The symbol "$*$" in a matrix means that the corresponding term of the matrix can be obtained by symmetric property.

II. Problem Formulation

Consider an Itô-type nonlinear stochastic system governed by the following state-space equation:

$$x(k+1) = (A + \Delta A)x(k) + B(u(k) + f(x(k))) + E_1 g(x(k)) + E_2 x(k)\omega(k), \tag{1}$$

and the output equation:

$$y(k) = Cx(k) + E_3 h(x(k)) + D\nu(k), \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $y(k) \in \mathbb{R}^p$ is the output signal, $u(k) \in \mathbb{R}^m$ is the control input, $\nu(k) \in l_2$ is a stochastic external disturbance. A, B, C, D, E_1, E_2 and E_3 are known constant real-valued matrices with appropriate dimensions. The nonlinear function $f(x(k))$ represents the matched bounded disturbance. $\omega(k)$ is a scalar Wiener process (Brownian Motion) on $(\Omega, \mathcal{F}, \mathcal{P})$ with

$$\mathbb{E}\{\omega(k)\} = 0, \qquad \mathbb{E}\left\{\omega^2(k)\right\} = 1, \qquad \mathbb{E}\{\omega(k)\omega(j)\} = 0 \ \ (k \neq j). \tag{3}$$

The matrix ΔA is the real-valued norm-bounded parameter uncertainty

$$\Delta A = MFN, \tag{4}$$

where M and N are known real constant matrices which characterize how the deterministic uncertain parameter in F enters the nominal matrix A with

$$F^{\mathrm{T}} F \leqslant I. \tag{5}$$

The parameter uncertainty ΔA is said to be admissible if both (4) and (5) are satisfied.

The vector-valued nonlinear functions $g(x(k))$ and $h(x(k))$ stand for the unmatched external nonlinearities, satisfying:

$$\begin{aligned} &[g(x) - g(z) - U_1(x-z)]^{\mathrm{T}} [g(x) - g(z) - U_2(x-z)] \leqslant 0, \quad g(0) = 0, \quad \forall x, z \in \mathbb{R}^n, \\ &[h(x) - h(z) - V_1(x-z)]^{\mathrm{T}} [h(x) - h(z) - V_2(x-z)] \leqslant 0, \quad h(0) = 0, \quad \forall x, z \in \mathbb{R}^n, \end{aligned} \tag{6}$$

where U_1, U_2, V_1 and $V_2 \in \mathbb{R}^{n \times n}$ are known real constant matrices, with $U = U_1 - U_2$ and $V = V_1 - V_2$ being positive definite matrices.

In this paper, the phenomenon of multiple data packet losses, which frequently occur in a networked environment, is also taken into consideration. We use the following formula to describe such a multiple data packet losses situation:

$$\bar{x}(k) = \Theta x(k), \tag{7}$$

where $\bar{x}(k)$ is the actual signal obtained from the process of sampling the feedback or output signal. The matrix of Θ is defined as

$$\Theta = \mathrm{diag}\{\theta_1(k), \theta_2(k), \cdots, \theta_n(k)\}$$

with $\theta_i(k)$ $(i = 1, 2, \cdots, n)$ being n unrelated random variables which are also unrelated with $\omega(k)$. It is assumed that $\theta_i(k)$ has the probabilistic density function $\varrho_i(s)$ $(i = 1, 2, \cdots, n)$ on the interval $[0 \quad 1]$ with mathematical expectation μ_i and variance σ_i^2. Note that $\theta_i(k)$ could satisfy any discrete probabilistic distributions on the interval [0 1]. Due to multiple data packet losses (7), the output equation (2) should be amended as

$$\begin{aligned} y(k) &= C\bar{x}(k) + E_3 h(x(k)) + D\nu(k) \\ &= C\Theta x(k) + E_3 h(x(k)) + D\nu(k) \\ &= \sum_{i=1}^{n} C_i \theta_i(k) x(k) + E_3 h(x(k)) + D\nu(k), \end{aligned} \tag{8}$$

where

$$C_i := C \cdot \mathrm{diag}\{\underbrace{0, \cdots, 0}_{i-1}, 1, \underbrace{0, \cdots, 0}_{n-i}\}. \tag{9}$$

In the sequel, we denote $\bar{\Theta} = \mathbb{E}\{\Theta\}$.

Remark 1: In the formulation of the multiple data packet losses (7), $\theta_i(k)$ could take value on the interval [0 1], hence it includes the widely used Bernoulli distribution as a special case. To be specific, when $\theta_i(k) = 0$ (respectively, $0 < \theta_i(k) < 1$), the ith state variable $x_i(k)$ is completely (respectively, partially) lost at the sampling instant k. The main difference between the model for output missing proposed in [39, 44] and this paper is that, the former focus on the data missing phenomenon caused by sensors' failures while, in this paper, we are interested in the situation that the data packet, due to complex circumstances such as network congestion and transmission lines aging, is completely or partially lost before it reaches the sensors.

III. Design of Sliding Model Controllers

In this section, we first propose a switching function in a *stochastic* form for the uncertain nonlinear system (1) with data packet losses. Then, two theorems will be given in order to design the switching function parameters capable of simultaneously ensuring the exponentially mean square stability and the $\mathcal{H}_\infty$ performance in the sliding motion. It is shown that the controller design problem in the sliding motion can be solved if an LMI with an equality constraint is feasible. Finally, a controller is synthesized to satisfy the improved discrete-time sliding motion reaching condition to drive the trajectories of system (1) onto the pre-specified sliding surface with probability 1.

A. Sliding Surface

In this paper, considering the existence of random data packet losses in the feedback loop, we choose the switching function as follows:

$$s(k) = G\bar{x}(k) = G\Theta x(k), \tag{10}$$

where G is designed such that $G\bar{\Theta}B$ is nonsingular and $G\bar{\Theta}E = 0$, where $E := \begin{bmatrix} E_1 & E_2 \end{bmatrix}$. In this paper, we select $G\bar{\Theta} = B^{\mathrm{T}}P$ with $P > 0$ being a positive definite matrix to confirm the non-singularity of $G\bar{\Theta}B$.

It can be seen that the switching function (10) serves as a stochastic difference equation due to the existence of the random variable matrix Θ. Therefore, the traditional necessary condition for discrete-time quasi-sliding motion, stated as $s(k+1) = s(k) = 0$, should be re-formulated on $(\Omega, \mathcal{F}, \mathcal{P})$ as follows:

$$\mathcal{P}\{s(k+1) = s(k) = 0\} = 1. \tag{11}$$

In order to obtain the equivalent control law of the sliding motion, we take

$$\mathbb{E}\{s(k+1)\} = \mathbb{E}\{s(k)\} = 0. \tag{12}$$

Solving the above for $u(k)$, the equivalent control law of the sliding motion is given by

$$u_{eq}(k) = -(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P(A + \Delta A)x(k) - f(x(k)). \tag{13}$$

Substituting (13) as $u(k)$ into (1) yields

$$x(k+1) = A_K x(k) + E_1 g(x(k), k) + E_2 x(k)\omega(k) \tag{14}$$

where $A_K := A + \Delta A - B(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P(A + \Delta A)$. The expression (14) is the sliding mode dynamics of system (1) in the specified switching surface $\mathbb{E}\{s(k+1)\} = \mathbb{E}\{s(k)\} = 0$.

Remark 2: It is the first time in the literature that a stochastic switching function (10) is introduced to deal with the discrete-time SMC problem for stochastic systems with multiple random data packet losses. The reason why we use (12) as the necessary condition for a discrete-time sliding motion is that it is meaningless to solve a stochastic difference equation $s(k+1) = s(k) = 0$ for an equivalent control law in sliding motion followed by the deterministic SMC controller parameters. As a result, both the equivalent control law (13) and the sliding mode dynamics (14) exist in a probabilistic sense. Moreover, throughout this paper, the SMC controller is designed to satisfy the discrete-time sliding motion reaching condition as well as the necessary condition on sliding surface also in a probabilistic sense.

Remark 3: In the output equation (8) and stochastic switching function (10), the matrix Θ is employed to describe the random data packet losses in the output channel as well as in the state feedback loop. Generally, in most real-world engineering practices, the probabilities of data packet losses through feedback channel and output channel might not be identical to each other since they are always transmitted by different ways. Nevertheless, the packet loss probabilities are assumed to be the same in this paper purely for avoiding unnecessarily complicated notations. It should be pointed out that our main results can be easily extended to more general cases where different data transmitting channels have different packet loss probabilities.

Remark 4: The condition $G\bar{\Theta}E = 0$ is applied to eliminate the unmatched nonlinearity $g(x(k))$ and the Brownian motion on the sliding surface so as to obtain the deterministic form of switching parameters that will be used to synthesize the SMC controller. For continuous-time stochastic systems, such a methodology has been used in [23, 24], where the unmatched external nonlinearity is not taken into consideration. It is worth mentioning that, by the proposed technique in this paper, we could deal with a wide range of nonlinearities, either stochastic or deterministic.

Before stating the designing goal, we introduce the following stability concept for system (14).

Definition 1: The system (14) is said to be robustly mean square stable if, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\mathbb{E}\{\|x(k)\|^2\} < \varepsilon$ $(k > 0)$ when $\mathbb{E}\{\|x(0)\|^2\} < \delta(\varepsilon)$. And if $\lim_{k\to\infty} \mathbb{E}\{\|x(k)\|^2\} = 0$ for any $x(0) \in \mathbb{R}^n$, then the system (14) is said to be asymptotically mean square stable. Moreover, if there exist constants $\beta \geqslant 1$ and $0 < \tau < 1$ such that $\mathbb{E}\{\|x(k)\|^2\} \leqslant \beta\tau^k \mathbb{E}\{\|x(0)\|^2\}$, then the system (14) is said to be exponentially mean square stable.

In this paper, we aim to synthesize an SMC law such that, for all admissible parameter uncertainties and multiple data packet losses, the following two requirements are achieved simultaneously:

(Q1) The state trajectory of system (1) is globally driven onto the pre-specified sliding surface (10) with probability 1 and, subsequently, the sliding motion is exponentially mean square stable.

(Q2) For a given scalar $\gamma > 0$, with $\nu(k) \neq 0$, the controlled output $y(k)$ satisfies

$$\sum_{k=0}^{\infty} \mathbb{E}\left\{\|y(k)\|^2\right\} \leqslant \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\left\{\|\nu(k)\|^2\right\}, \tag{15}$$

under the zero initial condition.

The problem addressed above is referred to as the robust $\mathcal{H}_\infty$ sliding mode control for nonlinear stochastic systems with multiple packet losses.

B. Stability and $\mathcal{H}_\infty$ Performance on Sliding Surface

In this subsection, we present two theorems to determine the parameters appeared in switching function (10). These parameters are necessary for designing the SMC controller to fulfil the control tasks (*Q1*) and

(*Q2*).

To begin with, we introduce the following lemmas which will be used later.

Lemma 1: [36] Let $\mathcal{W}(k) = x^{\mathrm{T}}(k)Px(k)$ be a Lyapunov functional where $P > 0$. If there exist real scalars ζ, μ, υ and $0 < \psi < 1$ such that both

$$\mu\|x(k)\|^2 \leqslant \mathcal{W}(k) \leqslant \upsilon\|x(k)\|^2 \tag{16}$$

and

$$\mathbb{E}\{\mathcal{W}(k+1)|x(k)\} - \mathcal{W}(k) \leqslant \zeta - \psi\mathcal{W}(k) \tag{17}$$

hold, then the process $x(k)$ satisfies

$$\mathbb{E}\{\|x(k)\|^2\} \leqslant \frac{\upsilon}{\mu}\|x(0)\|^2(1-\psi)^k + \frac{\lambda}{\mu\psi}. \tag{18}$$

Lemma 2: For any real vectors a, b and matrix $P > 0$ of compatible dimensions,

$$a^{\mathrm{T}}b + b^{\mathrm{T}}a \leqslant a^{\mathrm{T}}Pa + b^{\mathrm{T}}P^{-1}b. \tag{19}$$

Lemma 3: (Schur Complement) Given constant matrices $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ where $\mathcal{S}_1 = \mathcal{S}_1^{\mathrm{T}}$ and $0 < \mathcal{S}_2 = \mathcal{S}_2^{\mathrm{T}}$, then $\mathcal{S}_1 + \mathcal{S}_3^{\mathrm{T}}\mathcal{S}_2^{-1}\mathcal{S}_3 < 0$ if and only if

$$\begin{bmatrix} \mathcal{S}_1 & \mathcal{S}_3^{\mathrm{T}} \\ \mathcal{S}_3 & -\mathcal{S}_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\mathcal{S}_2 & \mathcal{S}_3 \\ \mathcal{S}_3^{\mathrm{T}} & \mathcal{S}_1 \end{bmatrix} < 0. \tag{20}$$

Lemma 4: (S-procedure) Let $J = J^{\mathrm{T}}$, M and N be real matrices of appropriate dimensions, and F satisfy (5). Then $J + MFN + N^{\mathrm{T}}F^{\mathrm{T}}M^{\mathrm{T}} < 0$ if and only if there exists a positive scalar ε such that $J + \varepsilon MM^{\mathrm{T}} + \varepsilon^{-1}N^{\mathrm{T}}N < 0$ or, equivalently,

$$\begin{bmatrix} J & \varepsilon M & N^{\mathrm{T}} \\ \varepsilon M^{\mathrm{T}} & -\varepsilon I & 0 \\ N & 0 & -\varepsilon I \end{bmatrix} < 0. \tag{21}$$

Denote

$$\widetilde{U} = \frac{U_1^{\mathrm{T}}U_2 + U_2^{\mathrm{T}}U_1}{2}, \quad \bar{U} = \frac{-U_1^{\mathrm{T}} - U_2^{\mathrm{T}}}{2}, \quad \widetilde{V} = \frac{V_1^{\mathrm{T}}V_2 + V_2^{\mathrm{T}}V_1}{2}, \quad \bar{V} = \frac{-V_1^{\mathrm{T}} - V_2^{\mathrm{T}}}{2}.$$

The following theorem presents a sufficient condition for the exponentially mean square stability of the sliding motion dynamics (14).

Theorem 1: The system (14) is exponentially stable in the mean square if there exist a positive definite matrix $P > 0$, positive scalars $\epsilon > 0$ and $\varphi_1 > 0$, such that

$$\begin{bmatrix} -P-\varphi_1\widetilde{U} & -\varphi_1\bar{U} & 2A^{\mathrm{T}}P & 2A^{\mathrm{T}}PB & E_2^{\mathrm{T}}P & 0 & \epsilon N^{\mathrm{T}} \\ -\varphi_1\bar{U}^{\mathrm{T}} & 2E_1^{\mathrm{T}}PE_1-\varphi_1 I & 0 & 0 & 0 & 0 & 0 \\ 2PA & 0 & -P & 0 & 0 & 2PM & 0 \\ 2B^{\mathrm{T}}PA & 0 & 0 & -B^{\mathrm{T}}PB & 0 & 2B^{\mathrm{T}}PM & 0 \\ PE_2 & 0 & 0 & 0 & -P & 0 & 0 \\ 0 & 0 & 2M^{\mathrm{T}}P & 2M^{\mathrm{T}}PB & 0 & -\epsilon I & 0 \\ \epsilon N & 0 & 0 & 0 & 0 & 0 & -\epsilon I \end{bmatrix} < 0, \tag{22}$$

$$B^{\mathrm{T}}PE = 0. \tag{23}$$

Proof: For system (14), we choose the Lyapunov functional by $\mathcal{W}(k) = x^{\mathrm{T}}(k)Px(k)$. Then, along the trajectory, we have

$$\begin{aligned}\mathbb{E}\{\Delta\mathcal{W}|x(k)\} =&\mathbb{E}\{\mathcal{W}(k+1)|x(k)\}-\mathcal{W}(k)\\ =&\mathbb{E}\{x^{\mathrm{T}}(k+1)Px(k+1)|x(k)\}-x^{\mathrm{T}}(k)Px(k)\\ =&\mathbb{E}\{(A_Kx(k)+E_1g(x(k))+E_2x(k)\omega(k))^{\mathrm{T}}\\ &\times P(A_Kx(k)+E_1g(x(k))+E_2x(k)\omega(k))|x(k)\}-x^{\mathrm{T}}(k)Px(k)\\ =&x^{\mathrm{T}}(k)(A_K^{\mathrm{T}}PA_K+E_2^{\mathrm{T}}PE_2-P)x(k)+g^{\mathrm{T}}(x(k))E_1^{\mathrm{T}}PE_1g(x(k))+2x^{\mathrm{T}}(k)A_K^{\mathrm{T}}PE_1g(x(k)).\end{aligned} \tag{24}$$

By Lemma 2, it is easy to obtain

$$2x^{\mathrm{T}}(k)A_K^{\mathrm{T}}PE_1g(x(k))\leqslant x^{\mathrm{T}}(k)A_K^{\mathrm{T}}PA_Kx(k)+g^{\mathrm{T}}(x(k))E_1^{\mathrm{T}}PE_1g(x(k)), \tag{25}$$

and

$$\begin{aligned}A_K^{\mathrm{T}}PA_K&=(A+\Delta A-B(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P(A+\Delta A))^{\mathrm{T}}P(A+\Delta A-B(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P(A+\Delta A))\\ &\leqslant 2(A+\Delta A)^{\mathrm{T}}P(A+\Delta A)+2\bar{A}^{\mathrm{T}}P\bar{A},\end{aligned} \tag{26}$$

where $\bar{A}$ is defined as $\bar{A}\triangleq -B(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P(A+\Delta A)$. To this end, we have

$$A_K^{\mathrm{T}}PA_K\leqslant 2(A+\Delta A)^{\mathrm{T}}P(A+\Delta A)+2(A+\Delta A)^{\mathrm{T}}PB(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P(A+\Delta A). \tag{27}$$

Notice that, when $z=0$, inequality (6) is equivalent to

$$\begin{aligned}\begin{bmatrix} x(k)\\ g(x(k))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\widetilde{U} & \bar{U}\\ \bar{U}^{\mathrm{T}} & I\end{bmatrix}\begin{bmatrix} x(k)\\ g(x(k))\end{bmatrix}\leqslant 0,\\ \begin{bmatrix} x(k)\\ h(x(k))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\widetilde{V} & \bar{V}\\ \bar{V}^{\mathrm{T}} & I\end{bmatrix}\begin{bmatrix} x(k)\\ h(x(k))\end{bmatrix}\leqslant 0.\end{aligned} \tag{28}$$

Therefore, for some $\varphi_1>0$,

$$\begin{aligned}\mathbb{E}\{\Delta\mathcal{W}|x(k)\}&\leqslant\mathbb{E}\{\Delta\mathcal{W}|x(k)\}-\varphi_1\begin{bmatrix} x(k)\\ g(x(k))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\widetilde{U} & \bar{U}\\ \bar{U}^{\mathrm{T}} & I\end{bmatrix}\begin{bmatrix} x(k)\\ g(x(k))\end{bmatrix}\\ &=\xi^{\mathrm{T}}(k)\begin{bmatrix}2A_K^{\mathrm{T}}PA_K+E_2^{\mathrm{T}}PE_2-P-\varphi_1\widetilde{U} & -\varphi_1\bar{U}\\ -\varphi_1\bar{U}^{\mathrm{T}} & 2E_1^{\mathrm{T}}PE_1-\varphi_1 I\end{bmatrix}\xi(k),\end{aligned} \tag{29}$$

where $\xi(k)\triangleq\begin{bmatrix} x^{\mathrm{T}}(k) & g^{\mathrm{T}}(x(k))\end{bmatrix}^{\mathrm{T}}$.

By Schur Complement,

$$\begin{aligned}&\begin{bmatrix}2A_K^{\mathrm{T}}PA_K+E_2^{\mathrm{T}}PE_2-P-\varphi_1\widetilde{U} & -\varphi_1\bar{U}\\ -\varphi_1\bar{U}^{\mathrm{T}} & 2E_1^{\mathrm{T}}PE_1-\varphi_1 I\end{bmatrix}<0\\ \iff&\begin{bmatrix}-P-\varphi_1\widetilde{U} & -\varphi_1\bar{U} & \sqrt{2}A_K^{\mathrm{T}}P & E_2^{\mathrm{T}}P\\ -\varphi_1\bar{U}^{\mathrm{T}} & 2E_1^{\mathrm{T}}PE_1-\varphi_1 I & 0 & 0\\ \sqrt{2}PA_K & 0 & -P & 0\\ PE_2 & 0 & 0 & -P\end{bmatrix}<0.\end{aligned} \tag{30}$$

It is easy to see that (30) is implied by

$$\begin{bmatrix}-P-\varphi_1\widetilde{U} & -\varphi_1\bar{U} & 2(A+\Delta A)^{\mathrm{T}}P & 2(A+\Delta A)^{\mathrm{T}}PB & E_2^{\mathrm{T}}P\\ -\varphi_1\bar{U}^{\mathrm{T}} & 2E_1^{\mathrm{T}}PE_1-\varphi_1 I & 0 & 0 & 0\\ 2P(A+\Delta A) & 0 & -P & 0 & 0\\ 2B^{\mathrm{T}}P(A+\Delta A) & 0 & 0 & -B^{\mathrm{T}}PB & 0\\ PE_2 & 0 & 0 & 0 & -P\end{bmatrix}<0. \tag{31}$$

Now, rewrite matrix inequality (31) into the following form:

$$\begin{bmatrix} -P-\varphi_1\widetilde{U} & -\varphi_1\bar{U} & 2A^{\mathrm{T}}P & 2A^{\mathrm{T}}PB & E_2^{\mathrm{T}}P \\ -\varphi_1\bar{U}^{\mathrm{T}} & 2E_1^{\mathrm{T}}PE_1-\varphi_1 I & 0 & 0 & 0 \\ 2PA & 0 & -P & 0 & 0 \\ 2B^{\mathrm{T}}PA & 0 & 0 & -B^{\mathrm{T}}PB & 0 \\ PE_2 & 0 & 0 & 0 & -P \end{bmatrix} + \bar{M}F\bar{N} + \bar{N}^{\mathrm{T}}F^{\mathrm{T}}\bar{M}^{\mathrm{T}} < 0, \tag{32}$$

where

$$\bar{M} = \begin{bmatrix} 0 & 0 & 2M^{\mathrm{T}}P & 2M^{\mathrm{T}}PB & 0 \end{bmatrix}^{\mathrm{T}},$$
$$\bar{N} = \begin{bmatrix} N & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Lemma 4, we can see that (32) is true if and only if there exists a $\epsilon > 0$ such that:

$$\begin{bmatrix} -P-\varphi_1\widetilde{U} & -\varphi_1\bar{U} & 2A^{\mathrm{T}}P & 2A^{\mathrm{T}}PB & E_2^{\mathrm{T}}P \\ -\varphi_1\bar{U}^{\mathrm{T}} & 2E_1^{\mathrm{T}}PE_1-\varphi_1 I & 0 & 0 & 0 \\ 2PA & 0 & -P & 0 & 0 \\ 2B^{\mathrm{T}}PA & 0 & 0 & -B^{\mathrm{T}}PB & 0 \\ PE_2 & 0 & 0 & 0 & -P \end{bmatrix} + \epsilon^{-1}\bar{M}\bar{M}^{\mathrm{T}} + \epsilon N^{\mathrm{T}}\bar{N} < 0. \tag{33}$$

It follows again from Lemma 4 that that (33) is equivalent to (22). Then, we have from (29) that $\mathbb{E}\{\Delta\mathcal{W}|x(k)\} < 0$ which indicates the sliding motion dynamics (14) is asymptotically mean square stable. Moreover, from (22), it is seen that

$$\Omega \triangleq \begin{bmatrix} 2A_K^{\mathrm{T}}PA_K + E_2^{\mathrm{T}}PE_2 - P - \varphi_1\widetilde{U} & -\varphi_1\bar{U} \\ -\varphi_1\bar{U}^{\mathrm{T}} & 2E_1^{\mathrm{T}}PE_1 - \varphi_1 I \end{bmatrix} < 0, \tag{34}$$

from which we know that there must exist a sufficiently small scalar α satisfying $0 < \alpha < \lambda_{\max}(P)$ such that $\Omega < -\alpha I$. Therefore, it follows that

$$\mathbb{E}\{\Delta\mathcal{W}|x(k)\} = \mathbb{E}\{\mathcal{W}(k+1)|x(k)\} - \mathcal{W}(k) \leqslant -\alpha\xi^{\mathrm{T}}(k)\xi(k) \leqslant -\alpha x^{\mathrm{T}}(k)x(k) \leqslant -\frac{\alpha}{\lambda_{\max}(P)}\mathcal{W}(k). \tag{35}$$

Then, the exponentially mean square stable of system (14) can be verified immediately from Lemma 2 and Definition 1. The proof is complete. ■

By means of LMI, the following theorem establishes a unified framework within which the exponentially mean square stability can be guaranteed together with the pre-specified $\mathcal{H}_\infty$ noise attenuation level.

Theorem 2: Consider the system (14). For the pre-specified $\mathcal{H}_\infty$ noise attenuation level $\gamma > 0$, if there exist a positive definite matrix $P > 0$, positive scalars $\epsilon > 0$, $\varphi_1 > 0$ and $\varphi_2 > 0$ satisfying

$$\begin{bmatrix} \Upsilon_{11} & -\varphi_1\bar{U} & -\varphi_2\bar{V}+\bar{\Theta}C^{\mathrm{T}}E_3 & \bar{\Theta}C^{\mathrm{T}}D & 2A^{\mathrm{T}}P & 2A^{\mathrm{T}}PB & 0 & \epsilon N^{\mathrm{T}} \\ * & -\varphi_1 I + 2E_1^{\mathrm{T}}PE_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\varphi_2 I + E_3^{\mathrm{T}}E_3 & E_3^{\mathrm{T}}D & 0 & 0 & 0 & 0 \\ * & * & * & D^{\mathrm{T}}D-\gamma^2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & -P & 0 & 2PM & 0 \\ * & * & * & * & * & -B^{\mathrm{T}}PB & 2B^{\mathrm{T}}PM & 0 \\ * & * & * & * & * & * & -\epsilon I & 0 \\ * & * & * & * & * & * & * & -\epsilon I \end{bmatrix} < 0, \tag{36}$$

$$B^{\mathrm{T}}PE = 0, \tag{37}$$

where

$$\Upsilon_{11} = E_2^{\mathrm{T}} P E_2 + \bar{\Theta} C^{\mathrm{T}} C \bar{\Theta} + \sum_{i=1}^{n} \sigma_i^2 C_i^{\mathrm{T}} C_i - P - \varphi_1 \widetilde{U} - \varphi_2 \widetilde{V},$$

then the system (14) is exponentially mean square stable and, meanwhile, the $\mathcal{H}_\infty$ performance is achieved.

Proof: It is obvious that Theorem 2 implies Theorem 1, and therefore the system (14) is exponentially mean square stable.

Next, for any $\nu(k) \neq 0$, we have

$$\begin{aligned}
&\mathbb{E}\{y^{\mathrm{T}}(k)y(k)\} \\
=&\mathbb{E}\{(C\Theta x(k) + E_3 h(x(k)) + D\nu(k))^{\mathrm{T}}(C\Theta x(k) + E_3 h(x(k)) + D\nu(k))\} \\
=&\mathbb{E}\{x^{\mathrm{T}}(k)\Theta C^{\mathrm{T}} C\Theta x(k) + 2x^{\mathrm{T}}(k)\Theta C^{\mathrm{T}} E_3 h(x(k)) + 2x^{\mathrm{T}}(k)\Theta C^{\mathrm{T}} D\nu(k)\} \\
&+ h^{\mathrm{T}}(x(k))E_3^{\mathrm{T}} E_3 h(x(k)) + \nu^{\mathrm{T}}(k)D^{\mathrm{T}} D\nu(k) + 2\nu^{\mathrm{T}}(k)D^{\mathrm{T}} E_3 h(x(k)).
\end{aligned} \tag{38}$$

Defining $\widetilde{\Theta} \triangleq \Theta - \bar{\Theta}$, we obtain

$$\begin{aligned}
\mathbb{E}\{x^{\mathrm{T}}(k)\Theta C^{\mathrm{T}} C\Theta x(k)\} &= \mathbb{E}\{x^{\mathrm{T}}(k)(\bar{\Theta} + \widetilde{\Theta})C^{\mathrm{T}} C(\bar{\Theta} + \widetilde{\Theta})x(k)\} \\
&= x^{\mathrm{T}}(k)\bar{\Theta} C^{\mathrm{T}} C\bar{\Theta} x(k) + \mathbb{E}\{2x^{\mathrm{T}}(k)\bar{\Theta} C^{\mathrm{T}} C\widetilde{\Theta} x(k)\} + \mathbb{E}\{x^{\mathrm{T}}(k)\widetilde{\Theta} C^{\mathrm{T}} C\widetilde{\Theta} x(k)\} \\
&= x^{\mathrm{T}}(k)\bar{\Theta} C^{\mathrm{T}} C\bar{\Theta} x(k) + x^{\mathrm{T}}(k)(\sum_{i=1}^{n} \sigma_i^2 C_i^{\mathrm{T}} C_i)x(k).
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
&\mathbb{E}\{\mathcal{W}(k+1)|x(k)\} - \mathcal{W}(k) + \mathbb{E}\{y^{\mathrm{T}}(k)y(k)\} - \gamma^2 \mathbb{E}\{\nu^{\mathrm{T}}(k)\nu(k)\} \\
=&\mathbb{E}\{x^{\mathrm{T}}(k)(A_K^{\mathrm{T}} P A_K + E_2^{\mathrm{T}} P E_2 + \bar{\Theta} C^{\mathrm{T}} C\bar{\Theta} + \sum_{i=1}^{n} \sigma_i^2 C_i^{\mathrm{T}} C_i - P)x(k) \\
&+ 2x^{\mathrm{T}}(k)A_K^{\mathrm{T}} P E_1 g(x(k)) + g^{\mathrm{T}}(x(k))E_1^{\mathrm{T}} P E_1 g(x(k)) + 2x^{\mathrm{T}}(k)\bar{\Theta} C^{\mathrm{T}} E_3 h(x(k)) + 2x^{\mathrm{T}}(k)\bar{\Theta} C^{\mathrm{T}} D\nu(k) \\
&+ h^{\mathrm{T}}(x(k))E_3^{\mathrm{T}} E_3 h(x(k)) + \nu^{\mathrm{T}}(k)D^{\mathrm{T}} D\nu(k) + 2\nu^{\mathrm{T}}(k)D^{\mathrm{T}} E_3 h(x(k))\} - \gamma^2 \mathbb{E}\{\nu^{\mathrm{T}}(k)\nu(k)\}
\end{aligned} \tag{40}$$

Taking (25) and (28) into consideration, for some $\varphi_1 > 0$ and $\varphi_2 > 0$, we have

$$\begin{aligned}
&\mathbb{E}\{\mathcal{W}(k+1)|x(k)\} - \mathcal{W}(k) + \mathbb{E}\{y^{\mathrm{T}}(k)y(k)\} - \gamma^2 \mathbb{E}\{\nu^{\mathrm{T}}(k)\nu(k)\} \\
\leqslant&\mathbb{E}\{\mathcal{W}(k+1)|x(k)\} - \mathcal{W}(k) + \mathbb{E}\{y^{\mathrm{T}}(k)y(k)\} - \gamma^2 \mathbb{E}\{\nu^{\mathrm{T}}(k)\nu(k)\} \\
&- \varphi_1 \begin{bmatrix} x(k) \\ g(x(k)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \widetilde{U} & \bar{U} \\ \bar{U}^{\mathrm{T}} & I \end{bmatrix} \begin{bmatrix} x(k) \\ g(x(k)) \end{bmatrix} \\
&- \varphi_2 \begin{bmatrix} x(k) \\ h(x(k)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \widetilde{V} & \bar{V} \\ \bar{V}^{\mathrm{T}} & I \end{bmatrix} \begin{bmatrix} x(k) \\ h(x(k)) \end{bmatrix} \triangleq \mathbb{E}\{\eta(k)^{\mathrm{T}} \Upsilon \eta(k)\}.
\end{aligned} \tag{41}$$

Here,

$$\Upsilon \triangleq \begin{bmatrix} \Upsilon_{11} + 2A_K^{\mathrm{T}} P A_K & -\varphi_1 \bar{U} & -\varphi_2 \bar{V} + \bar{\Theta} C^{\mathrm{T}} E_3 & \bar{\Theta} C^{\mathrm{T}} D \\ * & -\varphi_1 I + 2E_1^{\mathrm{T}} P E_1 & 0 & 0 \\ * & * & -\varphi_2 I + E_3^{\mathrm{T}} E_3 & E_3^{\mathrm{T}} D \\ * & * & * & D^{\mathrm{T}} D - \gamma^2 I \end{bmatrix},$$

$$\eta(k) \triangleq \begin{bmatrix} x^{\mathrm{T}}(k) & g^{\mathrm{T}}(x(k)) & h^{\mathrm{T}}(x(k)) & \nu^{\mathrm{T}}(k) \end{bmatrix}^{\mathrm{T}}.$$

Using Schur Complement, $\Upsilon < 0$ is true if

$$\begin{bmatrix} \Upsilon_{11} & -\varphi_1\bar{U} & -\varphi_2\bar{V}+\bar{\Theta}C^{\mathrm{T}}E_3 & \bar{\Theta}C^{\mathrm{T}}D & 2(A+\Delta A)^{\mathrm{T}}P & 2(A+\Delta A)^{\mathrm{T}}PB \\ * & -\varphi_1 I+E_1^{\mathrm{T}}PE_1 & 0 & 0 & 0 & 0 \\ * & * & -\varphi_2 I+E_3^{\mathrm{T}}E_3 & E_3^{\mathrm{T}}D & 0 & 0 \\ * & * & * & D^{\mathrm{T}}D-\gamma^2 I & 0 & 0 \\ * & * & * & * & -P & 0 \\ * & * & * & * & * & -B^{\mathrm{T}}PB \end{bmatrix} < 0 \qquad (42)$$

which, by Lemma 4, is equivalent to (36), and therefore

$$\mathbb{E}\{\mathcal{W}(k+1)|x(k)\} - \mathcal{W}(k) + \mathbb{E}\{y^{\mathrm{T}}(k)y(k)\} - \gamma^2\mathbb{E}\{\nu^{\mathrm{T}}(k)\nu(k)\} < 0. \qquad (43)$$

Next, taking the sum on both sides of (43) from 0 to ∞ with respect to k leads to

$$\sum_{k=0}^{\infty}\left[\mathbb{E}\{\mathcal{W}(k+1)|x(k)\} - \mathcal{W}(k) + \mathbb{E}\{y^{\mathrm{T}}(k)y(k)\} - \gamma^2\mathbb{E}\{\nu^{\mathrm{T}}(k)\nu(k)\}\right] < 0, \qquad (44)$$

or

$$\sum_{k=0}^{\infty}\mathbb{E}\{\|y(k)\|^2\} < \gamma^2\sum_{k=0}^{\infty}\mathbb{E}\{\|\nu(k)\|^2\} + \mathcal{W}(0) - \mathcal{W}(\infty). \qquad (45)$$

Since $x(0) = 0$ and the system (14) is exponentially mean square stable, we can easily obtain

$$\sum_{k=0}^{\infty}\mathbb{E}\{\|y(k)\|^2\} < \gamma^2\sum_{k=0}^{\infty}\mathbb{E}\{\|\nu(k)\|^2\}, \qquad (46)$$

which completes the proof. ■

C. Computational Algorithm

Notice that the condition in Theorem 2 is presented as the feasibility problem of an LMI with an equality constraint. By means of the proposed method in [23], as the condition $B^{\mathrm{T}}PE = 0$ is equivalent to $\mathrm{tr}[(B^{\mathrm{T}}PE)^{\mathrm{T}}B^{\mathrm{T}}PE] = 0$, we first introduce the condition $(B^{\mathrm{T}}PE)^{\mathrm{T}}B^{\mathrm{T}}PE \leqslant \phi I$. By Schur Complement, the condition can be expressed as

$$\begin{bmatrix} -\phi I & E^{\mathrm{T}}PB \\ B^{\mathrm{T}}PE & -I \end{bmatrix} < 0. \qquad (47)$$

Hence, the original nonconvex feasibility problem can be converted into the following minimization problem:

$$\min\phi \qquad \text{subject to} \quad (36) \ \text{and} \ (47). \qquad (48)$$

If this infinum equals zero, the solutions will satisfy the LMI (36) with the equality $B^{\mathrm{T}}PE = 0$. Thus, the exponentially mean square stability and $\mathcal{H}_\infty$ performance of system (14) are simultaneously achieved.

D. Reaching Condition Analysis

In this subsection, we will synthesize a sliding mode controller, with the pre-specified stochastic switching function (10) and sliding surface (12), to meet the discrete-time sliding mode reaching condition. That is to say, the trajectory of (1) starting from any initial state is globally driven onto the sliding surface (12) in finite time with probability 1, and then results in a sliding motion within a band called quasi-sliding mode band (QSMB) [14], along the sliding surface in the subsequent time.

To begin with, since the system parameter uncertainty ΔA and external disturbance $f(x(k))$ are both assumed to be bounded, $D_a \triangleq B^{\mathrm{T}}P\Delta Ax(k)$ and $D_f \triangleq B^{\mathrm{T}}PBf(x(k))$ will also be bounded. Denote d_a^i and

d_f^i as the ith element in D_a and D_f, respectively. Suppose the lower and upper bounds on D_a and D_f are known and given as follows:

$$\begin{aligned} &d_{aL}^i \leqslant d_a^i \leqslant d_{aU}^i, \\ &d_{fL}^i \leqslant d_f^i \leqslant d_{fU}^i, \qquad i = 1, 2, \cdots, m. \end{aligned} \tag{49}$$

where d_{aL}^i d_{aU}^i d_{fL}^i and d_{fU}^i are all known constants. Furthermore, we denote

$$\begin{aligned} &\bar{D}_a = \begin{bmatrix} \bar{d}_a^1 & \bar{d}_a^1 & \cdots & \bar{d}_a^m \end{bmatrix}^{\mathrm{T}}, && \bar{d}_a^i = \frac{d_{aU}^i + d_{aL}^i}{2}, \\ &\widetilde{D}_a = \mathrm{diag}\left\{\widetilde{d}_a^1, \widetilde{d}_a^2, \cdots, \widetilde{d}_a^m\right\}, && \widetilde{d}_a^i = \frac{d_{aU}^i - d_{aL}^i}{2}, \\ &\bar{D}_f = \begin{bmatrix} \bar{d}_f^1 & \bar{d}_f^1 & \cdots & \bar{d}_f^m \end{bmatrix}^{\mathrm{T}}, && \bar{d}_f^i = \frac{d_{fU}^i + d_{fL}^i}{2}, \\ &\widetilde{D}_f = \mathrm{diag}\left\{\widetilde{d}_f^1, \widetilde{d}_f^2, \cdots, \widetilde{d}_f^m\right\}, && \widetilde{d}_f^i = \frac{d_{fU}^i - d_{fL}^i}{2}, \qquad i = 1, 2, \cdots, m. \end{aligned} \tag{50}$$

Remark 5: We should point out that the assumption on the upper and lower bounds of D_a and D_f are standard for discrete-time SMC, see [14] and the references therein. Besides, the bounds of both D_a and D_f might be time-varying or dependent on state $x(k)$, which we will show in Section IV.

Next, we aim to improve the reaching condition proposed in [14] by proposing the following form for system (1) with the sliding surface (12):

$$\mathbb{E}\{\Delta s_i(k)\} = \mathbb{E}\{s_i(k+1) - s_i(k)\} \begin{cases} \leqslant -\rho\lambda_i \cdot \mathrm{sgn}[\mathbb{E}\{s_i(k)\}] - \rho q_i \mathbb{E}\{s_i(k)\} & \text{if } \mathbb{E}\{s_i(k)\} > 0 \\ \geqslant -\rho\lambda_i \cdot \mathrm{sgn}[\mathbb{E}\{s_i(k)\}] - \rho q_i \mathbb{E}\{s_i(k)\} & \text{if } \mathbb{E}\{s_i(k)\} < 0 \end{cases} \tag{51}$$

where ρ represents the sampling period, $\lambda_i > 0$ and $q_i > 0$ $(i = 1, 2, \cdots, m)$ are properly chosen scalars satisfying $0 < 1 - \rho q_i < 1$, $\forall i \in \{1, 2, \cdots, m\}$. We also can rewrite (51) into a compact form as follows:

$$\mathbb{E}\{\Delta s(k)\} = \mathbb{E}\{s(k+1) - s(k)\} \begin{cases} \leqslant -\rho\Lambda \cdot \mathrm{sgn}[\mathbb{E}\{s(k)\}] - \rho Q \mathbb{E}\{s(k)\} & \text{if } \mathbb{E}\{s(k)\} > 0 \\ \geqslant -\rho\Lambda \cdot \mathrm{sgn}[\mathbb{E}\{s(k)\}] - \rho Q \mathbb{E}\{s(k)\} & \text{if } \mathbb{E}\{s(k)\} < 0 \end{cases} \tag{52}$$

where

$$\begin{aligned} \Lambda &= \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_m\} \in \mathbb{R}^{m\times m}, \\ Q &= \mathrm{diag}\{q_1, q_2, \cdots, q_m\} \in \mathbb{R}^{m\times m}. \end{aligned}$$

Now we are ready to give the design technique of the robust SMC controller.

Theorem 3: Consider the uncertain nonlinear stochastic system (1) with the stochastic sliding surface (12) where P is the solution to (36)-(37). If the SMC law is given as

$$\begin{aligned} u(k) = &-(G\bar{\Theta}B)^{-1}(\rho\Lambda \cdot \mathrm{sgn}[G\bar{\Theta}x(k)] + (\rho Q - I)G\bar{\Theta}x(k) + G\bar{\Theta}Ax(k) \\ &+ (\bar{D}_a + \bar{D}_f) + (\widetilde{D}_a + \widetilde{D}_f)\mathrm{sgn}[G\bar{\Theta}x(k)]), \end{aligned} \tag{53}$$

then the state trajectories of the system (1) are driven onto the pre-specified sliding surface (10) with probability 1.

Proof: By (53), with the switching function defined in (10), we can easily obtain

$$\begin{aligned}\mathbb{E}\{\Delta s(k)\} =&\mathbb{E}\{s(k+1)-s(k)\}\\ =&\mathbb{E}\{G\Theta x(k+1)-G\Theta x(k))\}\\ =&G\bar{\Theta}((A+\Delta A)x(k)+Bu(k)+Bf(x(k))-x(k))\\ =&G\bar{\Theta}((A+\Delta A)x(k)-x(k)+Bf(x(k)))\\ &-(\rho\Lambda\cdot\mathrm{sgn}[G\bar{\Theta}x(k)]+(\rho Q-I)G\bar{\Theta}x(k)+G\bar{\Theta}Ax(k)\\ &+(\bar{D}_a+\bar{D}_f)+(\widetilde{D}_a+\widetilde{D}_f)\mathrm{sgn}[G\bar{\Theta}x(k)])\\ =&B^{\mathrm{T}}P\Delta Ax(k)+B^{\mathrm{T}}PBf(x(k))\\ &-\rho\Lambda\cdot\mathrm{sgn}[B^{\mathrm{T}}Px(k)]-\rho QB^{\mathrm{T}}Px(k)\\ &-(\bar{D}_a+\bar{D}_f)-(\widetilde{D}_a+\widetilde{D}_f)\mathrm{sgn}[B^{\mathrm{T}}Px(k)]\\ =&-\rho\Lambda\cdot\mathrm{sgn}[\mathbb{E}\{s(k)\}]-\rho Q\mathbb{E}\{s(k)\}\\ &+B^{\mathrm{T}}P\Delta Ax(k)-(\bar{D}_a+\widetilde{D}_a\mathrm{sgn}[\mathbb{E}\{s(k)\}])\\ &+B^{\mathrm{T}}PBf(x(k))-(\bar{D}_f+\widetilde{D}_f\mathrm{sgn}[\mathbb{E}\{s(k)\}])\end{aligned} \tag{54}$$

and

$$\begin{aligned}\mathbb{E}\{s(k)\}<0\Longrightarrow&\begin{cases}B^{\mathrm{T}}P\Delta Ax(k)\geqslant\bar{D}_a+\widetilde{D}_a\mathrm{sgn}[\mathbb{E}\{s(k)\}]\\ B^{\mathrm{T}}PBf(x(k))\geqslant\bar{D}_f+\widetilde{D}_f\mathrm{sgn}[\mathbb{E}\{s(k)\}]\end{cases}\\ &\Longrightarrow\mathbb{E}\{\Delta s(k)\}\geqslant-\rho\Lambda\cdot\mathrm{sgn}[\mathbb{E}\{s(k)\}]-\rho Q\mathbb{E}\{s(k)\}.\end{aligned} \tag{55}$$

Similarly, we can obtain

$$\mathbb{E}\{s(k)\}>0\Longrightarrow\mathbb{E}\{\Delta s(k)\}\leqslant-\rho\Lambda\cdot\mathrm{sgn}[\mathbb{E}\{s(k)\}]-\rho Q\mathbb{E}\{s(k)\}. \tag{56}$$

Therefore, the reaching condition (52) for discrete-time sliding mode is satisfied. In other words, the trajectory of system (1) will be, with probability 1, globally driven on the pre-specified sliding surface in finite time and result in a non-increasing sliding motion within the quasi-sliding mode band afterwards. The proof ends. ■

Remark 6: We point out that it is not difficult to extend the present results to more general systems that include polytopic parameter uncertainties, stochastic disturbances and constant or time-varying time delays by using the approach proposed and the LMI framework developed. The reason why we discuss the simplified system (1)-(2) is to make our theory more understandable and also to avoid unnecessarily complicated notations.

E. Some Discussions

First, let us discuss the issue of worst-case analysis of the robustness. In this paper, we have considered three kinds of "perturbations", i.e., the parameter uncertainties ΔA, the external disturbances $\nu(k)$ and the packet losses. For ΔA, it has been shown that, as long as the norm-bounded condition (4) holds, our main results are true no matter how ΔA varies within the bounded set. In this sense, we have actually dealt with the worst-case analysis with respect to ΔA. For $\nu(k)$, we have introduced the requirement (Q2), H_∞ performance constraint, to account for the disturbance rejection attenuation level. H_∞ performance, as is well known, can be understood as the worst-case property ("best out of the worst") as long as the disturbance $\nu(k)$ has bounded energy. Therefore, our main results are true, i.e., the disturbance rejection attenuation level is guaranteed no matter how $\nu(k)$ varies within an energy-bounded set.

Second, let us discuss the practical stability/boundedness issue. In this paper, the proposed switching function is actually a *stochastic difference equation*. In case there is a stochastic disturbance, it would be more reasonable to deal with the stability *in a probabilistic way* rather than the absolute stability. In fact, one of the novelties of this paper lies in the stochastic analysis of the sliding mode behavior. Although the stability in probability 1 (considered in this paper) is weaker than the absolute stability, it has been widely used in stochastic control area, see e.g. [2, 25, 45]. For example, in [45], the stability of the stochastic system is introduced in a probabilistic way, and the designed controller is also said to be able to stabilize the system in a probabilistic sense. Accordingly, in this paper, we propose to discuss the sliding mode control problem for the stochastic nonlinear system in a probabilistic way (with probability 1).

In order to show how the stability performances are influenced by the stochastic factors, we have added two figures in the simulation part to illustrate the worst-case (that is, all the data packets are lost during the sampling hence no valid signal can be used for feedback) response. It can be seen that the system is not stable when the states are completely lost.

IV. An Illustrative Example

In this section, we present an illustrative example to demonstrate the effectiveness of the proposed algorithm. The nominal system matrix A is taken from the model of an F-404 aircraft engine system in [10]. Note that this example is actually a special case of the physical models studied in [6, 10, 29, 42]. Moreover, in order to make the model more realistic and more close to the real-world engineering practices, we add the stochastic noises, the data packet loss and the external disturbances caused by the complex and time-varying working conditions in the system. After discretization, the system is as follows:

$$\begin{cases} x(k+1) = \left(\begin{bmatrix} 0.0307 & 0 & 0.0557 \\ 0.0333 & 0.2466 & -0.0091 \\ 0.0071 & 0 & 0.0130 \end{bmatrix} + \begin{bmatrix} 0.01 \\ 0.02 \\ 0 \end{bmatrix} \sin(0.6k) \begin{bmatrix} 0 & 0.01 & 0 \end{bmatrix} \right) x(k) \\ \qquad + \begin{bmatrix} 0.1817 & 0.4286 \\ 0.1597 & 0.0793 \\ 0.1138 & 0.0581 \end{bmatrix} \big(u(k) + f(x(k))\big) + \begin{bmatrix} 0.03 & 0 & -0.01 \\ 0.02 & 0.03 & 0 \\ 0.04 & 0.05 & -0.01 \end{bmatrix} g(x(k)) \\ \qquad + \begin{bmatrix} 0.015 & 0 & -0.01 \\ 0.01 & 0.015 & 0 \\ 0.02 & 0.025 & -0.01 \end{bmatrix} x(k)\omega(k), \\ y(k) = \begin{bmatrix} 0.2 & 0 & -0.1 \\ 0.1 & 0.15 & 0 \end{bmatrix} \Theta x(k) + \begin{bmatrix} -0.01 & 0 & 0.03 \\ 0.01 & 0.02 & 0 \end{bmatrix} h(x(k)) + \begin{bmatrix} 0.015 \\ 0.02 \end{bmatrix} \nu(k). \end{cases} \tag{57}$$

Let

$$\begin{aligned} f(x(k)) &= \begin{bmatrix} 0.5\sin(x_1(k)) \\ 0.6\cos(x_3(k)) \end{bmatrix}, \\ g(x(k)) &= 0.5(U_1+U_2)x(k) + 0.5(U_2-U_1)\sin(x(k))x(k), \\ h(x(k)) &= 0.5(V_1+V_2)x(k) + 0.5(V_2-V_1)\cos(x(k))x(k), \end{aligned} \tag{58}$$

where

$$\begin{aligned} &\sin(x(k)) \triangleq \mathrm{diag}\{\sin(x_1(k)), \sin(x_2(k)), \sin(x_3(k))\}, \\ &\cos(x(k)) \triangleq \mathrm{diag}\{\cos(x_1(k)), \cos(x_2(k)), \cos(x_3(k))\}, \\ &U_1 = \mathrm{diag}\{0.1, 0.2, 0.5\}, \quad U_2 = \mathrm{diag}\{0.1, 0.6, 0.7\}, \\ &V_1 = \mathrm{diag}\{0.3, 0.2, 0.8\}, \quad V_2 = \mathrm{diag}\{0.4, 0.5, 0.6\}. \end{aligned} \tag{59}$$

In addition, we assume the probabilistic density functions of θ_1, θ_2 and θ_3 in $[0\ \ 1]$ are described by

$$\varrho_1(s_1)=\begin{cases}0.8 & s_1=0\\ 0.1 & s_1=0.5\\ 0.1 & s_1=1,\end{cases}\qquad \varrho_2(s_2)=\begin{cases}0.7 & s_2=0\\ 0.2 & s_2=0.5\\ 0.1 & s_2=1,\end{cases}\qquad \varrho_3(s_3)=\begin{cases}0 & s_3=0\\ 0.1 & s_3=0.5\\ 0.9 & s_3=1,\end{cases} \tag{60}$$

from which the expectations and variances can be easily calculated as $\mu_1 = 0.15$, $\sigma_1^2 = 0.1025$, $\mu_2 = 0.2$, $\sigma_2^2 = 0.11$, $\mu_3 = 0.95$ and $\sigma_3^2 = 0.0225$.

Choosing $\gamma = 0.8$ and using Matlab LMI Toolbox to solve problem (48), we have

$$P=\begin{bmatrix} 0.075173 & 0.020922 & -0.060289\\ 0.020922 & 0.18556 & -0.091766\\ -0.060289 & -0.091766 & 0.19063 \end{bmatrix}$$

and $\phi = 9.000011 \times 10^{-7}$ (hence the constraint $B^{\mathrm{T}}PE = 0$ is satisfied). Choose $\rho = 0.05$, $\lambda_j = 1$ and $q_j = 1$ $(j = 1, 2)$. Moreover, In order to design the explicit SMC controller, we suppose $B^{\mathrm{T}}P\Delta Ax(k)$ and $B^{\mathrm{T}}PBf(x(k))$ are bounded by the following conditions:

$$\begin{aligned} &d_{aL}^i = -\|B^{\mathrm{T}}PM\|\|Nx(k)\|, \quad d_{aU}^i = \|B^{\mathrm{T}}PM\|\|Nx(k)\|,\\ &d_{fL}^i = -0.5\|B^{\mathrm{T}}PB\sin(x(k))\|, \quad d_{fU}^i = 0.5\|B^{\mathrm{T}}PB\sin(x(k))\|. \end{aligned} \tag{61}$$

Then, it follows from Theorem 3 that the desired SMC law can be set up with all known parameters. The simulation results are shown in Fig. 1 to Fig. 4, which confirm that the desired requirements are well achieved.

V. Conclusions

A robust SMC design problem for a class of uncertain nonlinear stochastic systems with multiple data packet losses has been studied. Both matched and unmatched nonlinearities have been taken into consideration. The multiple data packet losses are assumed to happen in a random way, and the loss probability of each individual state variable is governed by a corresponding individual stochastic variable obeying a certain probabilistic distribution in the interval $[0\ \ 1]$. We also have introduced, for the first time, a stochastic switching function for the SMC problem of discrete-time stochastic systems. By means of LMI, a sufficient condition for the exponentially mean square stability as well as pre-specified $\mathcal{H}_\infty$ performance index of the system dynamics on the specified sliding surface has been derived. By the reaching condition proposed in this paper, an SMC controller has been designed to globally drive the state trajectory onto the specified surface with probability 1, which gives rise to a non-increasing zigzag motion along the surface. An illustrative numerical example has been given to show the applicability and effectiveness of the proposed method in this paper.

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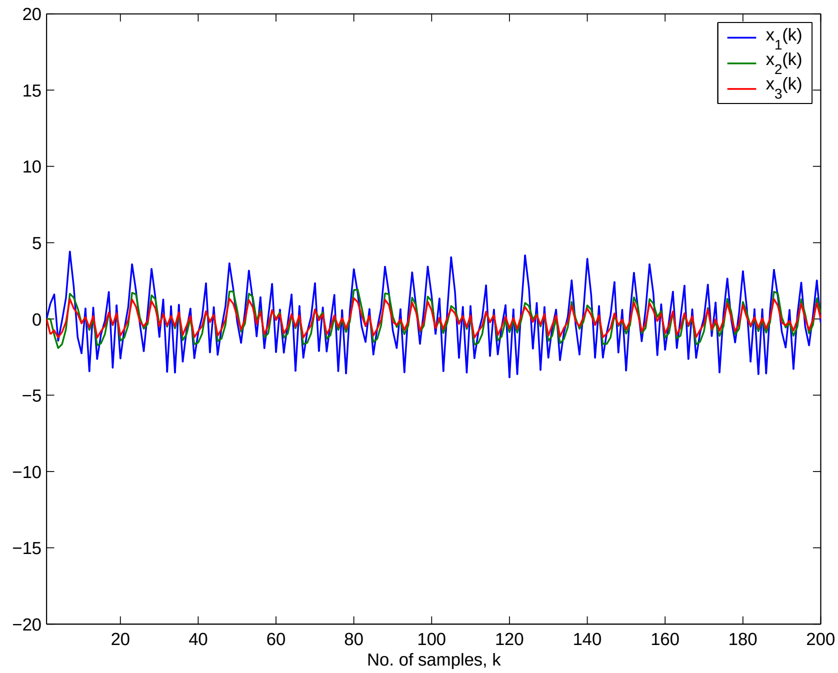

Fig. 1. The state trajectories $x(k)$.

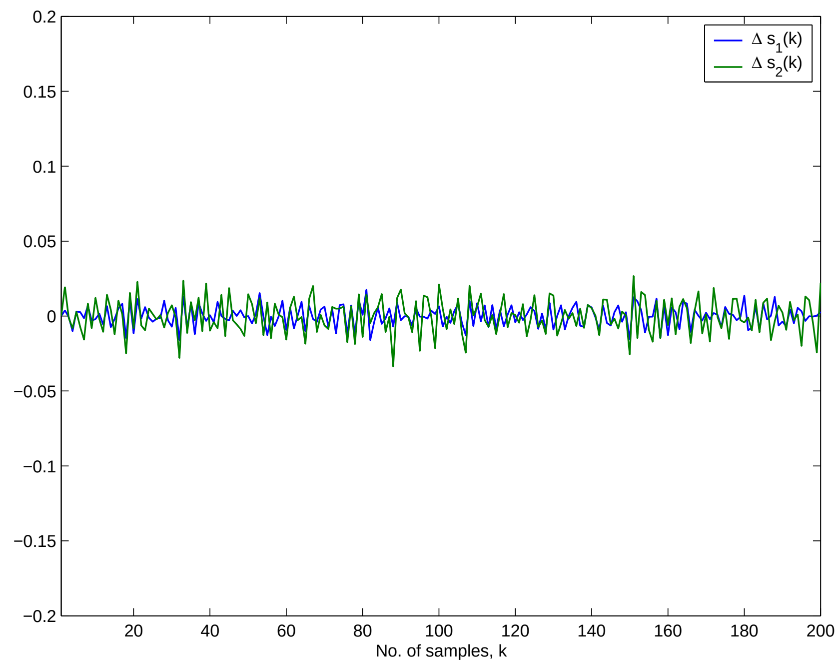

Fig. 2. The signal $\Delta s(k)$.

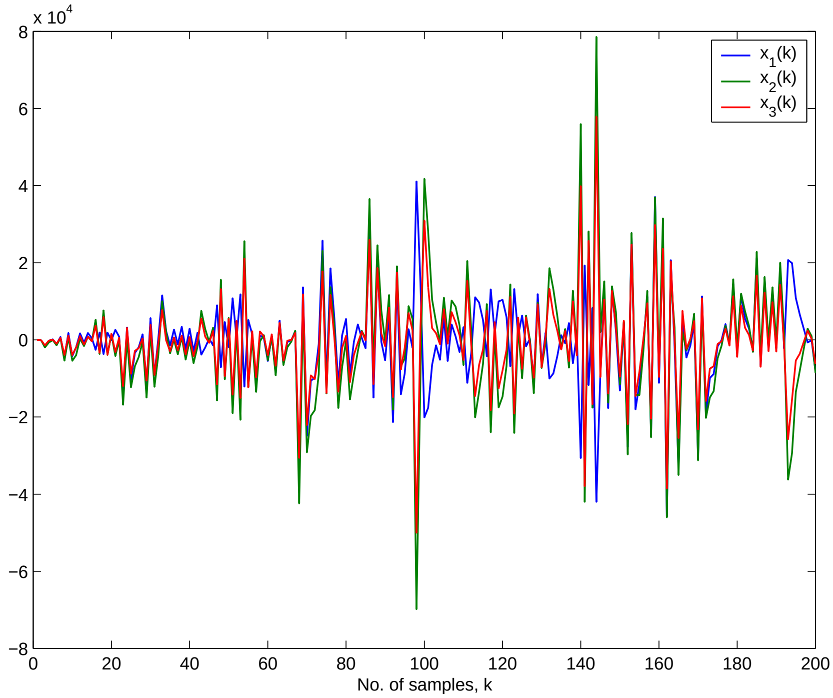

Fig. 3. The state trajectories when the states are completely lost.

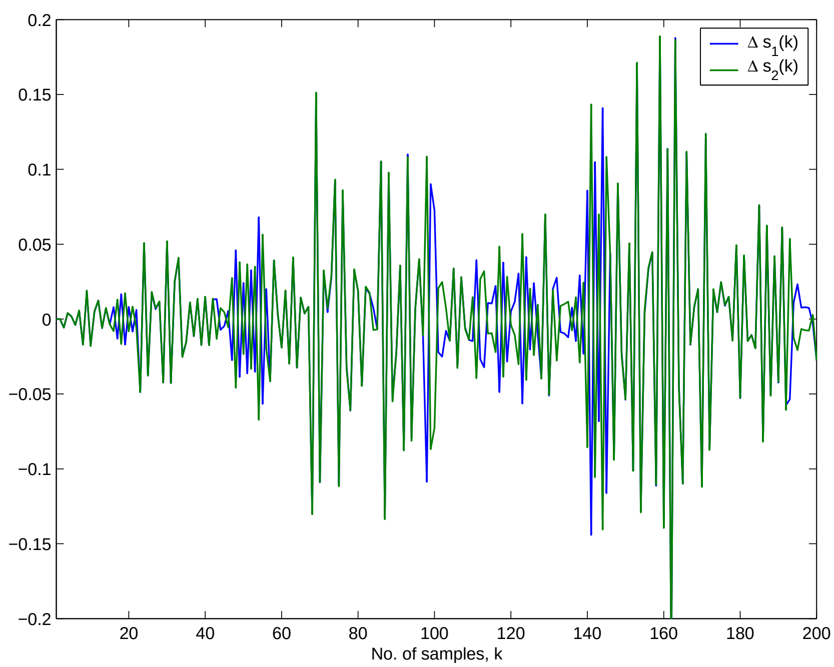

Fig. 4. The signal $\Delta s(k)$ when the states are completely lost.

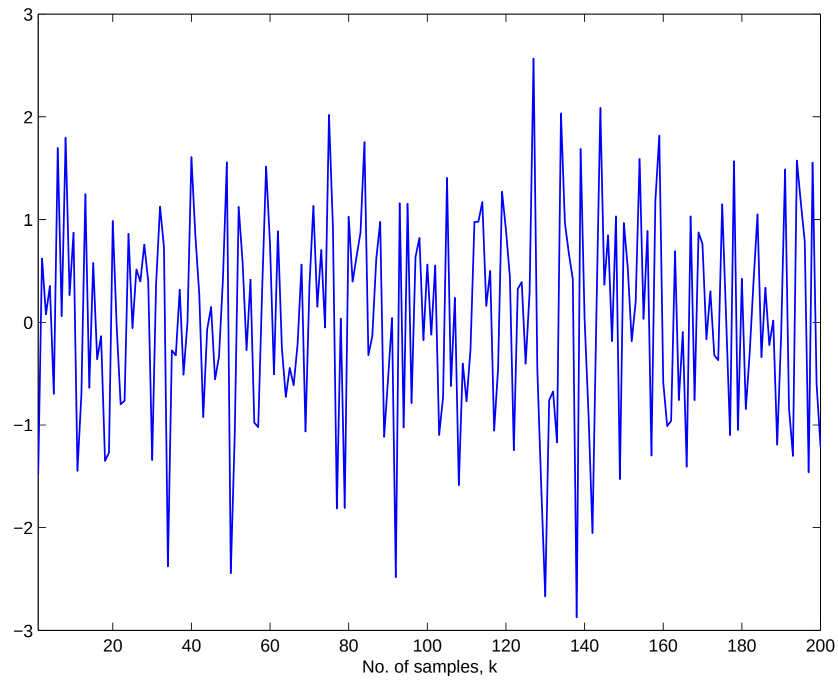

Fig. 5. The noise $\omega(k)$.

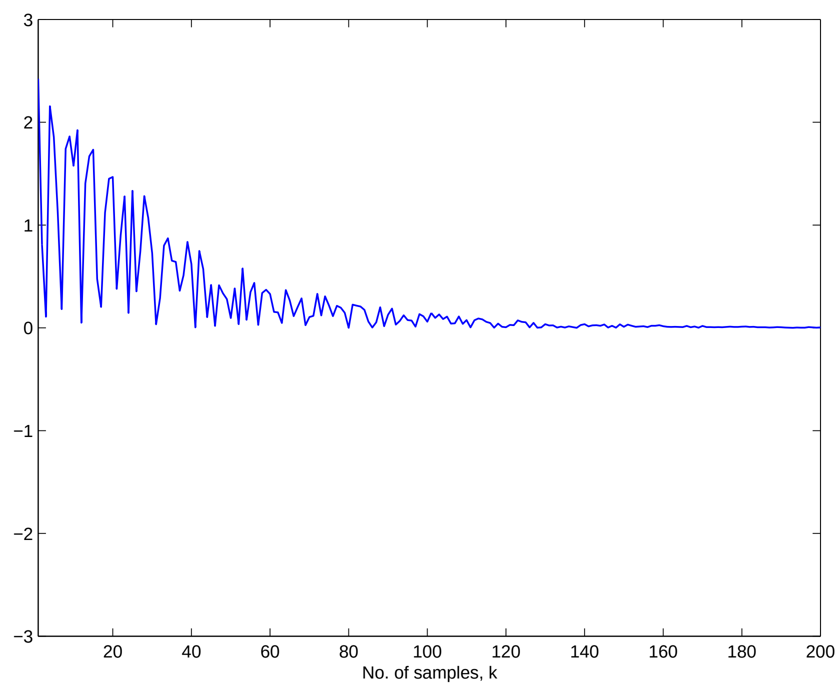

Fig. 6. The disturbance $\nu(k)$.