The Complexity of Two Graph Orientation Problems

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Abstract

We consider two orientation problems in a graph, namely the minimization of the sum of all the shortest path lengths and the minimization of the diameter. Our main result is that for each positive integer k, there is a linear-time algorithm that decides for a planar graph G whether there is an orientation for which the diameter is at most k. We also extend this result from planar graphs to any minor-closed family $\mathcal{F}$ not containing all apex graphs. In contrast, it is known to be NP-complete to decide whether a graph has an orientation such that the sum of all the shortest path lengths is at most an integer specified in the input. We give a simpler proof of this result.

Keywords: graph orientation, diameter, planar graph, graph minors, apex graph

1 Introduction

We consider two problems concerned with orienting the edges of an undirected graph in order to minimize two global measures of distance in the resulting directed graph. Our work is motivated by an application involving the design of urban light rail networks of the sort described in [25]. In such an application, a number of stations are to be linked with unidirectional track in order to minimize some function of the travel times between stations and subject to constraints on cost, engineering and planning. In practice these constraints mean that the choice of which stations to link may be forced upon us and the only control we have is over the choice of direction of each piece of track. Since the stations that are linked tend to be those that are close to each other, we make the simplifying assumption that the travel time along each single piece of track or link is the same. Consequently the network can be viewed as an (unweighted) graph in

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which the vertices represent stations and the edges represent track that is to be built. Furthermore, planning constraints tend to rule out the possibility of tracks crossing, so the graph is usually planar. The aim is to orient the resulting graph to minimize travel time. We assume that each journey in the oriented network progresses along a shortest directed path from the vertex representing the starting station to the vertex representing the destination.

All our graphs are simple, that is they have no loops or parallel edges. When the underlying graph is obvious, we use n and m to denote its numbers of vertices and edges, respectively. We use $\overrightarrow{G}$ to denote a directed graph obtained by orienting the edges of G. We let $d(x, y)$ denote the distance from vertex x to vertex y in a directed graph. The two measures of the quality of an orientation are its *diameter* $\mathrm{diam}(\overrightarrow{G})$, given by $\mathrm{diam}(\overrightarrow{G}) = \max_{x \neq y} d(x, y)$ and the *Wiener Index* $Z(\overrightarrow{G})$, given by $Z(\overrightarrow{G}) = \sum_{x \neq y} d(x, y)$. The name Wiener Index is perhaps not widely used, but is more common in applications in chemistry.

The networks arising in the application tend to be planar and have small degree. Our original aim was to determine the complexity of minimizing $\mathrm{diam}(\overrightarrow{G})$ and $Z(\overrightarrow{G})$ for planar graphs of bounded degree. We have two partial results in this direction.

An important complexity result on graph orientations was established by Chvátal and Thomassen [8], who showed that determining whether a graph may be oriented so that its diameter is at most two is NP-complete. In the next section we consider the problem where we are given a graph G and an integer k and must determine whether G can be oriented so that its Wiener Index is at most k. This was shown to be NP-complete by Hassin and Megiddo in [22]. Using the result from [8], we give a simpler proof of this result. Recently we discovered that the same proof has also been found independently by Caprara, Traversi and Schweizer [7]. We believe that it should be possible to sharpen this result so that the input graph is restricted to having degree at most three but our idea for a proof became extremely complicated, so we have not pursued this. We do not know whether the input may be restricted to planar graphs.

In contrast, a result of Bollobás and Scott [6] shows that an oriented graph with diameter two and n vertices must have at least $(1 + o(1))n \log_2 n$ edges. Since a planar graph with n vertices has at most $3n - 6$ edges, this implies that there is a constant upper bound on the number of vertices in a planar oriented graph with diameter two. So there is a constant time algorithm to determine whether a planar graph can be oriented so that its diameter is at most two. However there are arbitrarily large planar graphs that can be oriented so that their diameter is three, for example, a set of triangles sharing a common edge.

The bulk of this paper is devoted to showing that for any fixed integer l, there is a polynomial time algorithm that will take a planar graph and determine whether it may be oriented so that its diameter is at most l. In fact we establish rather more than this. An *apex* graph is a graph G having a vertex v such that $G - v$ is planar. Using a result of [19], we can extend our main result to apply not just to planar graphs but to any minor closed family of graphs not containing all apex graphs.

In Section 3 we discuss necessary concepts from tree-width. In Section 4 we describe an algorithm that attempts to find a suitable orientation when the input graph has bounded tree-width. Section 5 contains our main result. The main reason that our algorithm works is that the minimum diameter of an orientation of a graph does not increase when an edge is contracted and is at least $\Omega(l)$ for an $l \times l$-grid. Such a parameter is essentially what is called contraction-bidimensional in [14, 15, 16], where a general framework is described for when the corresponding decision problems for these parameters are tractable. Perhaps the most notable example is finding a k-dominating set in a planar graph [1, 21].

2 Complexity of the Wiener Index

Imagine we are given a graph and an integer k and we would like to know whether the graph can be oriented in such a way that the Wiener Index is less than k. This problem has previously been shown to be NP-complete by Hassin and Megiddo [22]. In this section we give a simpler proof of this result. The same proof has been found independently by Caprara, Traversi and Schweizer [7].

Chvátal and Thomassen [8] showed that the problem below is NP-complete:

Problem 1.
Instance: A graph G.
Question: Is it possible to orient G to ensure that $\mathrm{diam}(\overrightarrow{G}) \leq 2$?

From this result we can easily conclude that the following problem concerning the Wiener Index is NP-complete:

Problem 2.
Instance: A graph G, integer k.
Question: Is it possible to orient G to ensure that the Wiener Index of $\overrightarrow{G}$ is at most k?

Theorem 3. *Problem 2 is NP-complete.*

Proof. The problem is clearly in NP. Suppose that G has m edges. Let $k = 2(n^2 - n) - m$. If $\mathrm{diam}(\overrightarrow{G}) \leq 2$ then all pairs of vertices are either joined by an edge or by a path of length two. So $Z(\overrightarrow{G}) = 2(n^2 - n) - m = k$. Conversely if $\mathrm{diam}(\overrightarrow{G}) > 2$, there are $n^2 - n - m$ pairs of vertices joined by paths of length at least two including at least one path of length at least three, so $Z(\overrightarrow{G}) > 2(n^2 - n) - m = k$. Consequently there is an orientation of G with $\mathrm{diam}(\overrightarrow{G}) \leq 2$ if and only if there is an orientation of G with $Z(\overrightarrow{G}) \leq k$. □

We have been unable to determine the complexity of the following problem:

Problem 4.
Instance: Planar graph G and integer k.
Question: Can we orient the edges of G so that $Z(\overrightarrow{G}) \leq k$?

The rest of the paper is dedicated to the investigation of the complexity of the following problem for any fixed integer l and any minor-closed family $\mathcal{F}$ of graphs not containing all apex graphs:

Problem 5.
Instance: Graph G belonging to $\mathcal{F}$.
Question: Can we orient the edges of G so that $\mathrm{diam}(\overrightarrow{G}) \leq l$?

There is a considerable amount of work devoted to orienting graphs to minimize the diameter, see for instance the survey [23] or the book [3]. Much of the focus has been on very specific classes of graphs. One algorithmic result is that for $l \geq 4$, it is NP-complete to determine whether a chordal graph has an orientation of diameter at most l [20].

3 Tree-decompositions

The notion of a tree-decomposition was developed by Robertson and Seymour in [26]. Good introductions to the theory of tree-decompositions can be found, for example, in [4, 24, 28]. The definition of a tree-decomposition is as follows.

Definition 6. A tree-decomposition $\mathcal{T}$ of a graph G is a pair $(T, \mathcal{W})$ where T is a tree and $\mathcal{W} = (W_t : t \in V(T))$ is a family of subsets of $V(G)$ such that:

- $\bigcup_{t \in V(T)} W_t = V(G)$ and every edge in G has both endpoints in W_t for some t;
- if $t, t', t'' \in V(T)$ and t' lies on the path from t to t'' in T then $W_t \cap W_{t''} \subseteq W_{t'}$.

The width of $(T, \mathcal{W})$ is defined to be

$$\max\{|W_t| - 1 : t \in V(T)\}.$$

The tree-width of G is the minimum width among all possible tree-decompositions of G.

One reason for the importance of tree-width is that many NP-hard problems can be solved in polynomial or even linear time when restricted to graphs of bounded tree-width [4, 28].

Bodlaender [5] gave a linear-time algorithm for finding tree decompositions of small width.

Theorem 7. *For any positive integer k, there is an algorithm running in time $O\big(2^{\theta(k^3)} n\big)$ that inputs a graph G and determines whether the tree-width of G is at most k, and if so finds a tree-decomposition of G with width at most k.*

4 Minimizing diameter: the bounded tree-width case

In this section we show that there is an algorithm which for fixed integers k, l, takes as input a graph G and a tree-decomposition of G with width k, and determines whether there is an orientation of G with diameter at most l. The algorithm runs in time $O(cn)$, where c is a constant depending on k and l.

It is possible to construct such an algorithm explicitly. For a full description see [17] or for a brief outline see [18]. There, it is shown that c may be taken to be

$$(l+1)^{2(k+1)^2} 2^{[4(l+1)^{k+1}+2(l+1)^{2k+2}]} k^2.$$

Given that a full description of the algorithm is extremely lengthy and that the constant is so large, we do not describe the explicit algorithm here but instead prove the existence of a linear time algorithm by using the theory of monadic second-order logic of graphs (MSOL) introduced by Courcelle in [10].

We briefly give some background on MSOL here, but for more information see [9], [10] or the forthcoming book [13]. A directed graph $\vec{G}$ is represented by a 4-tuple $< V, E, R_H, R_T >$ where V and E are just the vertex and edge sets respectively of G, and $R_H \subset V \times E$ and $R_T \subset V \times E$ are relations with $(v, e) \in R_H$ if v is the head of e and $(v, e) \in R_T$ if v is the tail of e. Most discussions of MSOL mainly consider undirected graphs where one only has a single relation determining the end-vertices of an edge. An MSOL formula on $< V, E, R_H, R_T >$ may contain *member variables*, denoting members of either V or E, and *set variables*, denoting subsets of either V or E. The atomic formulae of an MSOL formula on $< V, E, R_H, R_T >$ are $v \in U$, $e \in A$, $v = w$, $e = f$, $(v, e) \in R_H$ and $(v, e) \in R_T$ where v, w are variables denoting vertices, e, f are variables denoting edges, U denotes a set of vertices and A denotes a set of edges. Standard logical connectives are permitted and both existential and universal quantification is allowed over both types of variable.

Courcelle [11] showed that for any graph property $\mathcal{P}$, that may be expressed by an MSOL formula, for each k, the problem of deciding whether a graph satisfies $\mathcal{P}$ is solvable in time $O(|E||V|)$ when the input is restricted to graphs having tree-width at most k. Arnborg, Lagergren and Seese [2] gave another proof of this result, reducing the time bound to $O(|V|)$. Originally these results were established for the case of undirected graphs, but the proofs may be extended to directed graphs [13].

Theorem 8. *For any k and l, there exists an algorithm that takes as input a graph G with tree-width at most k and a tree decomposition of G with width at most k, and determines whether G can be oriented so that its diameter is at most l. The algorithm runs in time $O(c_{k,l}n)$ where $c_{k,l}$ depends only on k and l.*

Proof. If we give an undirected graph an arbitrary or base orientation, then quantifying over all subsets of the edges is effectively the same as quantifying over all orientations of the graph, because one may obtain any orientation of a graph by starting with one arbitrary or base orientation and reversing the direction of some of the edges. Given an input graph, we first give it an arbitrary orientation. Now the proof just consists of showing that having a subset of edges which may be reversed to give an orientation with diameter at most l is a property expressible in MSOL.

(The reason for starting with a directed graph and changing the orientation rather than starting with an undirected graph is that it is not easy to specify an arbitrary orientation in MSOL, because an orientation is essentially a relation on the ordered pairs of endpoints of each edge and quantification over functions and relations on the vertices and edges is not allowed in MSOL. See [12].)

Clearly it is necessary for G to be connected in order to have an orientation with finite diameter. Since connectivity is easily expressed in MSOL we shall assume that G is indeed connected. Let

$$\begin{aligned} g(u,v,A) = {} & (u \in V) \wedge (v \in V) \\ & \wedge \big[u = v \vee (\exists e : e \in E \wedge uR_He \wedge vR_Te \wedge e \in A) \\ & \vee (\exists e : e \in E \wedge vR_He \wedge uR_Te \wedge e \notin A)\big]. \end{aligned}$$

So g determines if either $u = v$ or there is an edge from u to v in the orientation formed by reversing the edges of A. We can now express the property of having an orientation of diameter at most l by

$$\begin{aligned} \exists A : {} & \forall x \forall v_0 \forall v_l [(x \in A \to x \in E) \\ & \wedge ((v_0 \in V) \wedge (v_l \in V) \to \exists v_1, \ldots, v_{l-1} : \\ & (g(v_0, v_1, A) \wedge g(v_1, v_2, A) \wedge \cdots \wedge g(v_{l-1}, v_l, A)))]. \end{aligned}$$

□

In the next section we shall see how to extend this result to certain classes of graphs containing members with arbitrarily large tree-width.

5 Minimizing the diameter when an apex graph is excluded

It is now straightforward to establish our main result using the following restatement of a theorem of Eppstein [19].

Theorem 9. *If $\mathcal{F}$ is a minor-closed family of graphs that does not include all apex graphs, then there is a function f such that any graph $G \in \mathcal{F}$ with diameter at most d has tree-width at most $O(f(d))$.*

Theorem 10. *For any minor-closed family $\mathcal{F}$ of graphs that does not include all apex graphs and for every l, Problem 5 is solvable in time $O(cn)$ where c depends only on l and $\mathcal{F}$.*

Proof. Fix $\mathcal{F}$ and l. Then, by Theorem 9, there exists k (depending only on $\mathcal{F}$ and l) such that any graph in $\mathcal{F}$ with diameter at most l has tree-width at most k. So given an input graph G, using Bodlaender's algorithm we can determine whether G has tree-width at most k and if so find a tree decomposition with width at most k in time $O(c'n)$ where c' depends only on k. If G has tree-width at most k then Theorem 8 implies that there is an algorithm to determine whether G may oriented to have diameter at most l running in time $O(c''n)$ where c'' depends only on k and l. On the other hand if the tree-width of G exceeds k then its diameter exceeds l and so it cannot be oriented to have diameter at most l. $\square$

A consequence of this result is that for fixed $\mathcal{F}$ not containing all apex graphs, Problem 5 is fixed parameter tractable with respect to l.

Originally our main aim was to establish this result for the special case where $\mathcal{F}$ is the class of planar graphs. Because of this and the fact that we can obtain an expression for the constant in the running time bound, we give a sketch proof of this special case that does not use Theorem 9.

Lemma 11. *Any planar graph having a $(2l+1) \times (2l+1)$-grid-minor has diameter at least l.*

Proof. Suppose that G is a planar graph having a $(2l+1) \times (2l+1)$-grid-minor. We may assume that G is connected because otherwise $\operatorname{diam}(G) = \infty$. So a $(2l+1) \times (2l+1)$-grid may be obtained from G by a sequence of contractions followed by a series of deletions of edges. If an edge of a graph is contracted then its diameter cannot increase. Let K be the graph obtained from G after all the contractions of edges, so $\operatorname{diam}(G) \geq \operatorname{diam}(K)$.

It follows from Whitney's Theorem [29] that a $(2l+1) \times (2l+1)$-grid has a unique embedding on a sphere. K is a simple planar graph of which the $(2l+1) \times (2l+1)$-grid is a spanning subgraph. The only edges of K that are not present in the grid must have both endpoints in the same face of the grid. Consequently $\operatorname{diam}(G) \geq \operatorname{diam}(K) \geq l$ and so the diameter of G is at least l. $\square$

We need the following result from [27].

Theorem 12. *Any planar graph with no $g \times g$-grid-minor has tree-width at most $6g - 5$.*

We can now establish the result for planar graphs.

Theorem 13. *For every l, Problem 5 is solvable in time*

$$O\big(nl^2(l+1)^{2(12l+14)^2} 2^{[4(l+1)^{12l+14}+2(l+1)^{24l+28}]}\big)$$

when restricted to planar graphs.

Proof. Let G be a planar graph. Using Bodlaender's algorithm [5] we can determine in time $O\big(2^{\theta(l^3)}n\big)$ if G has tree-width at most $12l + 13$. If so then the algorithm will also find a corresponding tree-decomposition if one exists and then the algorithm from [17, 18] discussed at the beginning of Section 4 may be used to determine whether G can be oriented so that its diameter is at most l.

On the other hand if G has tree-width at least $12l+14$, then by Theorem 12, it has a $(2l + 3) \times (2l + 3)$-grid-minor and therefore by Lemma 11 its diameter is at least $l + 1$ and hence the diameter of any orientation is at least $l + 1$. $\square$

6 Conclusion

We have shown that Problem 1 which is NP-complete for arbitrary graphs becomes decidable in polynomial time for graphs belonging to any minor-closed family that does not contain all apex graphs and in particular planar graphs, even for bounds on the diameter larger than two. It would be interesting to try to find a more efficient algorithm for this problem, not depending on graph minor theory, and also to determine the complexity when l is part of the input. Furthermore it also remains to determine the complexity of minimizing the Wiener index for planar graphs.

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