# Positivity-Preserving $H_\infty$ Model Reduction for Positive Systems

Ping Li James Lam Zidong Wang Paresh Date

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**Abstract**

This paper is concerned with the model reduction of positive systems. For a given stable positive system, our attention is focused on the construction of a reduced-order model in such a way that the positivity of the original system is preserved and the error system is stable with a prescribed $H_\infty$ performance. Based upon a system augmentation approach, a novel characterization on the stability with $H_\infty$ performance of the error system is first obtained in terms of linear matrix inequality (LMI). Then, a necessary and sufficient condition for the existence of a desired reduced-order model is derived accordingly. A significance of the proposed approach is that the reduced-order system matrices can be parametrized by a positive definite matrix with flexible structure, which is fully independent of the Lyapunov matrix; thus, the positivity constraint on the reduced-order system can be readily embedded in the model reduction problem. Furthermore, iterative LMI approaches with primal and dual forms are developed to solve the positivity-preserving $H_\infty$ model reduction problem. Finally, a compartmental network is provided to show the effectiveness of the proposed techniques.

**Keywords:** $H_\infty$ performance; Iterative algorithm; Linear matrix inequality; Model reduction; Positive systems.

## 1 Introduction

In many practical systems, there is such a kind of systems whose state variables are confined to be positive. Such systems are frequently encountered in various fields, for instance, biomedicine [1], pharmacokinetics [2], chemical reactions [3] and internet congestion control [4]. These systems belong to the class of positive systems, whose state variable and output are always positive (at least nonnegative) whenever the initial state and the input are positive [5]. *Positivity* of the system state for all times will bring about many new issues, which cannot be solved in general by using well-established methods for general linear systems, mainly due to the fact that positive systems are defined on cones rather than linear spaces. Therefore, the study on this kind of systems has drawn the attention of many researchers in recent years [6] [7] [8] [9].

On the other hand, mathematical modeling of positive systems often results in complex high-order models, which will bring serious difficulties to analysis and synthesis of positive systems, irrespective of the computational resources available [10]. Therefore, it is necessary to replace high-order models by reduced

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Ping Li (pingli@hkusua.hku.hk) and James Lam (james.lam@hku.hk) are with the Department of Mechanical Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong.

Zidong Wang (Zidong.Wang@brunel.ac.uk) is with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex, UB8 3PH, UK.

Paresh Date (paresh.date@brunel.ac.uk) is with the Department of Mathematical Sciences, Brunel University, Uxbridge, Middlesex, UB8 3PH, UK.

ones with respect to some given criterion. In fact, such a topic is actually a model reduction problem in control area, and has received considerable attention in the past decades [11] [12] [13] [14]. Amongst the many optimality criteria for approximation, one is based on the $H_\infty$ norm of the associated error system. The characterization of $H_\infty$ model reduction solution was first proposed in [15], and many important results have been reported for various kinds of systems, such as stochastic systems [16] and switched systems [17]. Very recently, based on the methods of balanced truncation and matrices inequalities, the model reduction problem for positive systems has been investigated in [18] and [19], respectively. It should be pointed out that traditional approaches developed for general linear systems, including the widely adopted projection approach and similarity transformation [16] [20], are no longer applicable for positive systems in general, since they cannot guarantee the positivity of the reduced-order system. This indicates that conventional approaches, if used to construct a reduced-order system, may generate a meaningless approximation for the actual system whose state is always positive all the time. Therefore, it is necessary to develop new approaches to the $H_\infty$ model reduction problem for positive systems with positivity preserved. However, such a problem has not been well studied in the literature, and still remains as a challenging open issue.

In this paper, we are concerned with the $H_\infty$ model reduction problem for positive systems. More specifically, for a given positive linear continuous-time system, the aim is to construct a *positive* lower-order system such that the $H_\infty$ norm of the difference between the original system and the desired lower-order one satisfies a prescribed $H_\infty$ norm bound constraint. The main body is divided into two parts. In the first part, by virtue of a system augmentation approach, the associated error system is represented as a singular system form, and a novel characterization on the stability of the error system under the $H_\infty$ performance is derived in the form of LMI. An important feature of the results reported here is that the reduced-order system matrices can be parametrized by a positive definite matrix with flexible structure, which is fully independent of the Lyapunov matrix. Such a characterization will greatly facilitate the parametrization with positivity constraints. In the second part, a necessary and sufficient condition for the existence of a desired reduced-order system is firstly proposed, then iterative LMI approaches are developed to compute the reduced-order system matrices and optimize the initial values, respectively. Moreover, a dual approach, together with the primal one, is further incorporated to improve the solvability of the positive-preserving $H_\infty$ model reduction problem.

## 2 System Description and Preliminaries

*Notation:* Let $\mathbb{R}$ be the set of real numbers; $\mathbb{R}^n$ denotes the $n$-column real vectors; $\mathbb{R}^{n\times m}$ is the set of all real matrices of dimension $n \times m$. $\mathbb{R}_+^{n\times m}$ represents the $n \times m$ dimensional matrices with positive components and $\mathbb{R}_+^n \triangleq \mathbb{R}_+^{n\times 1}$. A matrix is said to be *positive*, if all its elements are positive. For a matrix $A \in \mathbb{R}^{n\times n}$, it is called *Metzler*, if all its off-diagonal elements are positive. $I$ represents the identity matrix with appropriate dimension; For any real symmetric matrices $P$, $Q$, the notation $P \geq Q$ (respectively, $P > Q$) means that the matrix $P-Q$ is positive semi-definite (respectively, positive definite). The notation $L_2[0,\infty)$ represents the space of square Lebesgue integrable functions over $[0,\infty)$ with the usual norm $\|\cdot\|_2$. For a transfer function matrix $\mathcal{G}$, its $H_\infty$ norm is denoted as $\|\mathcal{G}\|_\infty$. For a real matrix $A$, $A^\perp$ denotes the orthonormal complement of $A$, that is, $AA^\perp = 0$ and $\left(A^\perp\right)^T A^\perp = I$. In addition, $\mathrm{Her}(M) \triangleq \frac{1}{2}\left(M^T + M\right)$ is defined for any matrix $M \in \mathbb{R}^{n\times n}$; $\mathrm{diag}(A_1, A_2, \ldots, A_N)$ denotes the block diagonal matrix with $A_1$, $A_2$, $\ldots$, $A_N$ on the diagonal. The superscript "$T$" denotes the transpose for vectors or matrices. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

Consider the following linear asymptotically stable system:

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & Ax(t)+Bu(t),\\ y(t) & = & Cx(t)+Du(t),\\ x(0) & = & x_0,\end{array}\right. \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^q$ is the output or measurement vector. In this paper, we assume $u \in L_2[0,\infty)$, and $A$, $B$, $C$ and $D$ are real constant matrices with appropriate dimensions.

We give the following definition on positive linear systems.

**Definition 1 ([5])** *System (1) is said to be a positive linear system if for all $x_0 \in \mathbb{R}^n_+$ and all input $u(t) \in \mathbb{R}^m_+$, we have $x(t) \in \mathbb{R}^n_+$ and $y(t) \in \mathbb{R}^q_+$ for $t > 0$.*

The following lemma provides a well-known characterization of positive linear systems.

**Lemma 1 ([5])** *The system in (1) is positive if and only if $A$ is Metzler, $B$, $C$ and $D$ are positive.*

In this paper, we aim at approximating system (1) by a reduced-order stable system described by

$$\left\{\begin{array}{rcl}\dot{x}_r(t) & = & A_r x_r(t)+B_r u(t),\\ y_r(t) & = & C_r x_r(t)+D_r u(t),\\ x_r(0) & = & x_{r0},\end{array}\right. \tag{2}$$

where $x_r(t) \in \mathbb{R}^{n_r}$ is the state vector of the reduced-order system (2) with $0 < n_r < n$, and $y_r(t) \in \mathbb{R}^q$. $A_r$, $B_r$, $C_r$ and $D_r$ are matrices to be determined later.

For the stable system in (1), the transfer matrix from input $u(t)$ to output $y(t)$ is given by

$$\mathcal{G}_{uy}(s) = C\left(sI - A\right)^{-1} B + D. \tag{3}$$

Traditionally, the $H_\infty$ model reduction problem was formulated by finding a reduced-order system (2), such that

$$\left\|\mathcal{G}_{uy} - \mathcal{G}_{uy_r}\right\|_\infty < \gamma, \tag{4}$$

where

$$\mathcal{G}_{uy_r}(s) = C_r\left(sI - A_r\right)^{-1} B_r + D_r \tag{5}$$

is the transfer matrix of system (2) from $u(t)$ to $y_r(t)$, and $\gamma > 0$ is a prescribed scalar.

However, such a specification is not sufficient for positive systems, since as an approximation of system (1), it is naturally desirable that system (2) should also be positive, like system (1) itself. That is, in addition to the $H_\infty$ performance in (4), the positivity should also be preserved when considering the model reduction problem for the positive system in (1). To ensure the positivity of system (2), if follows from Lemma 1 that $A_r$ should be Metzler, $B_r$, $C_r$ and $D_r$ should be positive.

For convenience, denote set $\mathbb{S} \triangleq \{(A_r, B_r, C_r, D_r) : A_r \text{ is Metzler, } B_r, C_r \text{ and } D_r \text{ are positive}\}$.

Let $\hat{x}(t) = \left[x^T(t), x_r^T(t)\right]^T$ and $e(t) = y(t) - y_r(t)$. Then, from (1) and (2), we obtain the associated error system as

$$\left\{\begin{array}{rcl}\dot{\hat{x}}(t) & = & \hat{A}\hat{x}(t)+\hat{B}u(t),\\ e(t) & = & \hat{C}\hat{x}(t)+\hat{D}u(t),\end{array}\right. \tag{6}$$

where

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}, \ \hat{B} = \begin{bmatrix} B \\ B_r \end{bmatrix},$$
$$\hat{C} = \begin{bmatrix} C & -C_r \end{bmatrix}, \ \hat{D} = D - D_r.$$

Obviously, condition in (4) is equivalent to

$$\|\mathcal{G}_{ue}\|_\infty < \gamma, \tag{7}$$

where

$$\mathcal{G}_{ue}(s) = \hat{C}\left(sI - \hat{A}\right)^{-1}\hat{B} + \hat{D} \tag{8}$$

is the transfer matrix from $u(t)$ to $e(t)$. In addition, the stability of system (1) and (2) is naturally equivalent to that of system (6).

Based on the above discussion, the problem of positivity-preserving $H_\infty$ model reduction for positive systems in (1) to be addressed in this paper is formulated as follows.

*Problem PP-$H_\infty$-MR (Positivity-Preserving $H_\infty$ Model Reduction):* Given a disturbance attenuation level $\gamma > 0$, construct a system (2) such that the following two requirements are fulfilled simultaneously.

(1) $(A_r, B_r, C_r, D_r) \in \mathbb{S}$.

(2) The error system in (6) is asymptotically stable and satisfies the $H_\infty$ performance $\|\mathcal{G}_{ue}\|_\infty < \gamma$.

The following result gives a fundamental characterization on the stability of (6) with $H_\infty$ performance, which will be used later.

**Lemma 2 ([20])** *The error system in (6) is asymptotically stable and satisfies $\|\mathcal{G}_{ue}\|_\infty < \gamma$, if and only if there exists a matrix $\hat{P} > 0$, such that*

$$\begin{bmatrix} \hat{A}^T\hat{P} + \hat{P}\hat{A} & \hat{P}\hat{B} & \hat{C}^T \\ \hat{B}^T\hat{P} & -\gamma I & \hat{D}^T \\ \hat{C} & \hat{D} & -\gamma I \end{bmatrix} < 0, \tag{9}$$

*where $\hat{P}$ is usually referred to as the Lyapunov matrix.*

## 3 Analysis of Associated Error System

This section focuses on the stability analysis of the error system in (6) under the $H_\infty$ performance. To achieve this, we first represent system (6) by means of a system augmentation approach, which will facilitate the parametrization on the positivity constraint. Then, a novel characterization on the stability and the $H_\infty$ performance of (6) is developed in terms of linear matrix inequality, which will play a key role for the computation of the reduced-order system matrices.

### 3.1 System Augmentation Approach

To facilitate the construction of system (2), we define

$$G_r = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}, \tag{10}$$

which collects the representation for the system matrices in (2) into one matrix. We further make the following definitions:

$$\begin{aligned}
\bar{A} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \ \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \ \bar{D} = D, \\
\bar{F} &= \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \ \bar{M} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \\
\bar{N} &= \begin{bmatrix} 0 \\ I \end{bmatrix}, \ \bar{H} = \begin{bmatrix} 0 & -I \end{bmatrix},
\end{aligned}$$

which are entirely in terms of the state space matrices for system (1), then we have

$$\hat{A} = \bar{A} + \bar{F}G_r\bar{M}, \ \hat{B} = \bar{B} + \bar{F}G_r\bar{N}, \tag{11}$$

$$\hat{C} = \bar{C} + \bar{H}G_r\bar{M}, \ \hat{D} = \bar{D} + \bar{H}G_r\bar{N}. \tag{12}$$

Although the system matrices in (2) are encapsulated into $G_r$, one can see that it is still embedded with two other matrices. In addition, when applying Lemma 2, we have that $G_r$ is still coupled with the Lyapunov matrix $\hat{P}$, which makes them hard to solve. More significantly, such a problem will become more difficult and arduous, in particular when additional constraints on $G_r$ are taken into account.

For convenience, denote set $\tilde{\mathbb{S}} \triangleq \{G_r : G_r \text{ is defined in (10) with } (A_r, B_r, C_r, D_r) \in \mathbb{S}\}$.

To overcome these difficulties, we introduce an auxiliary variable $\hat{\vartheta}(t) = G_r\bar{M}\hat{x}(t) + G_r\bar{N}u(t)$, and define $\boldsymbol{x}(t) = \begin{bmatrix} \hat{x}^T(t) & \hat{\vartheta}^T(t) \end{bmatrix}^T$ accordingly. Then the error system in (6) can be equivalently described by the following descriptor (or semi-state) system:

$$\begin{cases} \boldsymbol{E}\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t), \\ e(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}u(t), \end{cases} \tag{13}$$

where

$$\begin{aligned}
\boldsymbol{E} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{A} = \begin{bmatrix} \bar{A} & \bar{F} \\ G_r\bar{M} & -I \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} \bar{B} \\ G_r\bar{N} \end{bmatrix}, \\
\boldsymbol{C} &= \begin{bmatrix} \bar{C} & \bar{H} \end{bmatrix}, \ \boldsymbol{D} = \bar{D}.
\end{aligned}$$

**Remark 1** *A major obstacle for the construction of the reduced-order system in (2) is that it should be positive, which results in the additional constraints on the system matrices $A_r$, $B_r$, $C_r$, and $D_r$. Focusing on this, one can see that the advantage of the above manipulations lies in the following aspects. First, these system matrices are assembled to a single matrix $G_r$, which will be convenient for the synthesis consideration. Second, by means of system augmentation approach in (13), $G_r$ is successfully extracted from the middle of two matrices, and can be further parametrized by a free positive definite matrix, which will be shown later. Such an approach will introduce the flexibility to the construction of $G_r$, in particular when $G_r$ has some certain constraints.*

### 3.2 Novel Stability and $H_\infty$ Performance Characterization

**Theorem 1** *Given the system matrices $A_r$, $B_r$, $C_r$ and $D_r$. Then the following statements are equivalent:*

*(i) The error system in (6) is asymptotically stable, and satisfies $\|\mathcal{G}_{ue}\|_\infty < \gamma$.*

*(ii) There exist matrices $\hat{P} > 0$ and diagonal $X > 0$ such that*

$$\Xi \triangleq \begin{bmatrix} \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}^T\boldsymbol{A} & \boldsymbol{P}^T(\boldsymbol{I}+\boldsymbol{J})\boldsymbol{B} & \boldsymbol{C}^T \\ \boldsymbol{B}^T(\boldsymbol{I}+\boldsymbol{J})^T\boldsymbol{P} & -\boldsymbol{B}^T\boldsymbol{J}^T\left(\boldsymbol{P}+\boldsymbol{P}^T\right)\boldsymbol{J}\boldsymbol{B} - \gamma I & \boldsymbol{D}^T \\ \boldsymbol{C} & \boldsymbol{D} & -\gamma I \end{bmatrix} < 0, \tag{14}$$

*where*

$$\boldsymbol{P} = \begin{bmatrix} \hat{P} & 0 \\ -\frac{1}{2}XG_r\bar{M} & \frac{1}{2}X \end{bmatrix}, \quad \boldsymbol{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \boldsymbol{J} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

**Proof:** *(ii)⇒(i).* Suppose there exist matrices $\hat{P} > 0$ and diagonal $X > 0$ such that (14) holds. Define a nonsingular matrix

$$T \triangleq \begin{bmatrix} I & 0 & 0 & 0 \\ G_r\bar{M} & G_r\bar{N} & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}.$$

Pre- and post-multiplying (14) by $T^T$ and $T$, respectively, we have

$$\Theta \triangleq T^T\Xi T = \begin{bmatrix} \hat{A}^T\hat{P} + \hat{P}\hat{A} & \hat{P}\hat{B} & \hat{C}^T & \hat{P}\bar{F} \\ \hat{B}^T\hat{P} & -\gamma I & \hat{D}^T & 0 \\ \hat{C} & \hat{D} & -\gamma I & \bar{H} \\ \bar{F}^T\hat{P} & 0 & \bar{H}^T & -X \end{bmatrix} < 0. \tag{15}$$

Based on Lemma 2, the third leading principal submatrix of $\Theta$ indicates that the error system in (6) is asymptotically stable, and satisfies $\|\mathcal{G}_{ue}\|_\infty < \gamma$, which completes this part of the proof.

*(i)⇒(ii).* If the error system in (6) is asymptotically stable, and satisfies $\|\mathcal{G}_{ue}\|_\infty < \gamma$, then it follows from Lemma 2 that there exists a matrix $\hat{P} > 0$, such that

$$\begin{bmatrix} \hat{A}^T\hat{P} + \hat{P}\hat{A} & \hat{P}\hat{B} & \hat{C}^T \\ \hat{B}^T\hat{P} & -\gamma I & \hat{D}^T \\ \hat{C} & \hat{D} & -\gamma I \end{bmatrix} < 0.$$

Then, for any diagonal $S > 0$, there exists a sufficiently large scalar $\alpha > 0$ such that

$$-\alpha S - \begin{bmatrix} \hat{P}\bar{F} \\ 0 \\ \bar{H} \end{bmatrix}^T \begin{bmatrix} \hat{A}^T\hat{P} + \hat{P}\hat{A} & \hat{P}\hat{B} & \hat{C}^T \\ \hat{B}^T\hat{P} & -\gamma I & \hat{D}^T \\ \hat{C} & \hat{D} & -\gamma I \end{bmatrix}^{-1} \begin{bmatrix} \hat{P}\bar{F} \\ 0 \\ \bar{H} \end{bmatrix} < 0. \tag{16}$$

By choosing $X = \alpha S$ and applying Schur complement equivalence to (16), we have

$$\Xi = T^{-T}\Theta T^{-1} < 0,$$

which completes the whole proof. □

**Remark 2** *Although the conditions in (9) and (14) are equivalent, it should be pointed out that the LMI formulation in (14) has some advantages over the one in (9). First, with the LMI characterization in (14), the reduced-order system matrices, or $G_r$ equivalently, are not coupled with the Lyapunov matrix $\hat{P}$ any more, but can be parametrized by a positive definite matrix $X$, which is fully independent of $\hat{P}$. Second, it follows from (16) that, if the error system in (6) is asymptotically stable and satisfies $\|\mathcal{G}_{ue}\|_\infty < \gamma$, the existence of $X$ will be naturally guaranteed. Finally, it is worth pointing out that the structure of $X$ is rather flexible. In fact, from the proof of ((ii)⇒(i)), $X$ can be chosen as $X = \alpha S$, where $S$ is required only to be* a positive definite matrix *with $\alpha$ being sufficiently large. The freedom on the structure of $X$ will greatly facilitate the synthesis considered in this paper when additional constraints on $G_r$ are imposed, which will be shown subsequently.*

# 4 Synthesis Condition and Algorithm

This section is devoted to the synthesis of the reduced-order system in (2). Based on the analysis in Section 3, a necessary and sufficient condition for the existence of a solution to *Problem PP-$H_\infty$-MR* is obtained. Then, iterative LMI approaches are developed to compute the reduced-order system matrices and optimize the initial values, respectively. A dual approach to solving *Problem PP-$H_\infty$-MR* is further addressed subsequently, and the combination of the primal and the dual approach is proposed in the last subsection.

## 4.1 Existence of Positive Reduced-Order System

**Theorem 2** Problem PP-$H_\infty$-MR *is solvable, if and only if there exists a matrix $\hat{P} > 0$, a diagonal matrix $X > 0$, matrices $U$, $V$, $L_1$, $L_2$, $L_3$ and $L_4$ such that*

$$
\begin{aligned}
L &= \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \in \tilde{\mathbb{S}}, && (17)\\
\Xi(U,V) &\triangleq \begin{bmatrix} \Xi_{11} & \hat{P}\bar{F} + \bar{M}^T L^T & \Xi_{13} & \bar{C}^T \\ \bar{F}^T\hat{P} + L\bar{M} & -X & L\bar{N} & \bar{H}^T \\ \Xi_{13}^T & \bar{N}^T L^T & \Xi_{33} & \bar{D}^T \\ \bar{C} & \bar{H} & \bar{D} & -\gamma I \end{bmatrix} < 0, && (18)
\end{aligned}
$$

*where*

$$
\begin{aligned}
\Xi_{11} &= 2\mathrm{Her}\left(\bar{A}^T\hat{P} - U^T L\bar{M}\right) + U^T X U,\\
\Xi_{13} &= \hat{P}\bar{B} - \bar{M}^T L^T V - U^T L\bar{N} + U^T X V,\\
\Xi_{33} &= -2\mathrm{Her}\left(V^T L\bar{N}\right) + V^T X V - \gamma I.
\end{aligned}
$$

*In this case, the system matrices of (2) can be given as*

$$
G_r = X^{-1} L. \tag{19}
$$

**Proof:** By expanding (14), we have

$$
\begin{bmatrix} \bar{A}^T\hat{P} + \hat{P}\bar{A} - \bar{M}^T G_r^T X G_r \bar{M} & \hat{P}\bar{F} + \bar{M}^T G_r^T X & \hat{P}\bar{B} - \bar{M}^T G_r^T X G_r \bar{N} & \bar{C}^T \\ \bar{F}^T\hat{P} + X G_r \bar{M} & -X & X G_r \bar{N} & \bar{H}^T \\ \bar{B}^T\hat{P} - \bar{N}^T G_r^T X G_r \bar{M} & \bar{N}^T G_r^T X & -\bar{N}^T G_r^T X G_r \bar{N} - \gamma I & \bar{D}^T \\ \bar{C} & \bar{H} & \bar{D} & -\gamma I \end{bmatrix} < 0. \tag{20}
$$

*Sufficiency:* It follows from (17) and $X > 0$ diagonal, we have that $A_r$ Metzler, $B_r$, $C_r$ and $D_r$ positive. From (19), we have $L = XG_r$. Substituting this into (18), and observing that, for any $U$ and $V$,

$$\begin{aligned} -\Phi^T G_r^T X G_r \Phi &\leq -\Phi^T G_r^T X G_r \Phi + (\Psi - G_r \Phi)^T X (\Psi - G_r \Phi) \\ &= -2\mathrm{Her}\left(\Psi^T X G_r \Phi\right) + \Psi^T X \Psi, \end{aligned}$$

where

$$\Phi = \left[ \begin{array}{ccc} \bar{M} & 0 & \bar{N} \end{array} \right], \ \Psi = \left[ \begin{array}{ccc} U & 0 & V \end{array} \right], \tag{21}$$

we obtain that (20) holds, which further indicates that (14) holds. According to Theorem 1, this completes the sufficiency proof.

*Necessity:* If *Problem PP-$H_\infty$-MR* is solvable, then for the given error system in (6), it follows from Theorem 1 that there exists a matrix $\hat{P} > 0$, and a diagonal matrix $X > 0$ such that (14) holds, or equivalently, (20) holds. Then, by choosing $U = G_r \bar{M}$ and $V = G_r \bar{N}$, we have that

$$-\Phi^T G_r^T X G_r \Phi = -2\mathrm{Her}\left(\Psi^T X G_r \Phi\right) + \Psi^T X \Psi,$$

where $\Phi$ and $\Psi$ are defined in (21). Substituting this into (20), and letting $L = XG_r$, one has that (18) holds. This completes the whole proof. □

**Remark 3** *From the proof in Theorem 2, one can see that the construction matrix $G_r$ is not coupled with $\hat{P}$, but can be parametrized by $X$, which makes the construction specification for $G_r \in \tilde{\mathbb{S}}$ possible. More specifically, due to the fact that the structure of $X$ is rather flexible, we can designate $X$ to be a positive diagonal matrix. As a matter of fact, $X$ can be chosen as a positive diagonal matrix, or even a positive scalar matrix, whereas no conservatism will be introduced consequently.*

## 4.2 Iterative Approaches to Reduced-Order System Matrices Computation

Let us explain the conditions in Theorem 2 from a computational perspective. Obviously, the condition in (17) can be viewed as a set of LMIs, which can be readily verified by standard software. Now, we turn to inequality (18), which is generally not a linear matrix inequality with respect to the parameters $\hat{P}$, $X$, $U$, $V$ and $L$. However, it can be easily observed that if $U$ and $V$ are held fixed, then it becomes an LMI problem with respect to the remaining parameters. Note that the LMI problem is convex and can be efficiently solved if a feasible solution exists [21]. This leaves a natural problem about how to choose $U$ and $V$ properly. Define a scalar $\alpha$ satisfying

$$\Xi(U, V) < \alpha \Pi, \tag{22}$$

where

$$\Pi = \mathrm{diag}\left( \begin{array}{cccc} I & 0 & I & 0 \end{array} \right) \tag{23}$$

and $\Xi(U, V)$ is defined in (18). Inspired by [22], it follows from the proof of Theorem 2 that $\alpha$ will achieve its minimum when $U = X^{-1} L \bar{M}$ and $V = X^{-1} L \bar{N}$, which leads to an iterative approach to solve inequality (18).

Now, we are in a position to develop the following iterative LMI algorithm:

*Algorithm 1 (ILMI Approach):*

1. **START**: Set $j = 1$. For a given $H_\infty$ performance level $\gamma$, compute the initial matrices $U_1$ and $V_1$ such that the following auxiliary system,

$$\left\{ \begin{array}{rcl} \dot{\bar{x}}(t) & = & \bar{A}\bar{x}(t) + \bar{F}\bar{\vartheta}(t) + \bar{B}u(t), \\ e(t) & = & \bar{C}\bar{x}(t) + \bar{H}\bar{\vartheta}(t) + \bar{D}u(t), \end{array} \right. \tag{24}$$

with $\bar{\vartheta}(t) = U_1\bar{x}(t) + V_1u(t)$ is asymptotically stable and the transfer function $\mathcal{T}_{ue}(s)$ from $u(t)$ to $e(t)$ satisfies $\|\mathcal{T}_{ue}\|_\infty < \gamma$.

2. For fixed $U_j$ and $V_j$, solve the following convex optimization problem for the parameters in $\Omega \triangleq \left\{\hat{P} > 0,\ X > 0 \text{ is diagonal, and } L_1,\ L_2,\ L_3 \text{ and } L_4\right\}$ :

$$\alpha_j^* := \min_{\Omega} \alpha_j \ \ \text{s.t.} \ \left\{ \begin{array}{l} L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \in \tilde{\mathbb{S}} \\ \Xi\left(U_j, V_j\right) < \alpha_j \Pi \\ \alpha_j \geq \mu \end{array} \right. ,$$

where $\mu \leq 0$ is an arbitrary scalar. Denote the corresponding value of $X$ and $L$ as $X_j$ and $L_j$, respectively.

3. If $\alpha_j^* \leq 0$, then a desired parametric matrix $G_r$ is obtained as $G_r := X_j^{-1}L_j$. **STOP**. If not, then go to next step.

4. If $\left|\left(\alpha_j^* - \alpha_{j-1}^*\right)/\alpha_j^*\right| < \delta_1$, where $\delta_1$ is a prescribed tolerance, then go to next step. If not, update $U_{j+1}$ and $V_{j+1}$ as

$$U_{j+1} := X_j^{-1}L_j\bar{M},\ V_{j+1} := X_j^{-1}L_j\bar{N}.$$

Set $j := j + 1$, then go to Step 2.

5. A solution to *Problem PP-$H_\infty$-MR* may not exist. **STOP**.

**Remark 4** *Note that the constraint $\alpha_j \geq \mu$ is only added to make $\alpha_j^*$ bounded from below by a negative scalar, and will not affect the search of $\alpha_j^*$, since we are only interested in the case $\alpha_j^* \leq 0$. Meanwhile, it can be seen that the sequence $\alpha_j^*$ is monotonically decreasing with respect to $j$, that is, $\alpha_{j+1}^* \leq \alpha_j^*$. Therefore, the convergence of the iterative process is naturally guaranteed.*

The problem in Step 1 is convex, which can be regarded as a state-feedback $H_\infty$ control problem. Furthermore, if there are no matrices $U_1$ and $V_1$ such that system (24) is stable and satisfies $\|\mathcal{T}_{ue}\|_\infty < \gamma$, then we can conclude immediately that there does not exist a solution to *Problem PP-$H_\infty$-MR*. In addition, it follows from Lemma 2 that finding $U_1$ and $V_1$ is equivalent to finding $\bar{Q} > 0$, $W_1$ and $V_1$ such that

$$\begin{bmatrix} 2\mathrm{Her}\left(\bar{A}\bar{Q} + \bar{F}W_1\right) & \bar{B} + \bar{F}V_1 & \bar{Q}\bar{C}^T + W_1^T\bar{H}^T \\ \bar{B}^T + V_1^T\bar{F}^T & -\gamma I & \bar{D}^T + V_1^T\bar{H}^T \\ \bar{C}\bar{Q} + \bar{H}W_1 & \bar{D} + \bar{H}V_1 & -\gamma I \end{bmatrix} < 0 \tag{25}$$

holds, then $U_1$ can be obtained as $U_1 = W_1\bar{Q}^{-1}$, and $V_1$ can be given directly from (25).

In addition, it is well known that the performance of an iterative algorithm usually depends on the initial starting points, and a poor selection of the initial value often results in the iterates being trapped at local minima and the iterative process becomes sluggish, whereas good starting points lead to fast solution. This leaves the problem for the optimization of $U_1$ and $V_1$.

In what follows of this subsection, we will show how to further improve the solvability of *Algorithm 1* by optimizing the initial values $U_1$ and $V_1$. It can be seen that if, for some $G_r^*$, $\hat{P}^* > 0$, diagonal $X^* > 0$, (20) holds, then (18) will also be feasible, that is, $\Xi(U,V) < 0$, provided that $(\Psi - G_r^*\Phi)^T X^* (\Psi - G_r^*\Phi)$ is "small" enough. Thus, the solvability of the algorithm can be further improved by choosing the initial values $U_1$ and $V_1$ such that $\left\|(\Psi_1 - G_r^*\Phi)^T X^* (\Psi_1 - G_r^*\Phi)\right\|$ is small enough, where

$$\Psi_1 = \begin{bmatrix} U_1 & 0 & V_1 \end{bmatrix} \tag{26}$$

and $\Phi$ is defined in (21). As we indicated previously that $X^*$ should be large, an alternative way is then to make $\|\Psi_1 - G_r^*\Phi\|$ sufficiently small. The following proposition provides an equivalent characterization of how $\|\Psi_1 - G_r^*\Phi\|$ can be made as small as possible.

**Theorem 3** *Given $\Phi$ and $\Psi_1$ defined in (21) and (26), respectively, for a sufficiently small scalar $\varepsilon > 0$, the following statements are equivalent:*

*(i) There exists $G_r^*$ such that system (6) is asymptotically stable with $\|\mathcal{G}_{ue}\|_\infty < \gamma$, and $\|\Psi_1 - G_r^*\Phi\| \leq \varepsilon$.*

*(ii) System (24) is asymptotically stable with $\|\mathcal{T}_{ue}\|_\infty < \gamma$, and $\left\|\Psi_1\Phi^\perp\right\| \leq \varepsilon$.*

**Proof:** *(i)⇒(ii).* First, we have

$$\left\|\Psi_1\Phi^\perp\right\| = \left\|\Psi_1\Phi^\perp - G_r^*\Phi\Phi^\perp\right\| \leq \|\Psi_1 - G_r^*\Phi\| \left\|\Phi^\perp\right\| \leq \varepsilon.$$

In the following, we shall prove that system (24) is asymptotically stable with $\|\mathcal{T}_{ue}\|_\infty < \gamma$. By defining

$$\mathcal{A} = \begin{bmatrix} \bar{A} & \bar{B} & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ \bar{C} & \bar{D} & -\frac{\gamma}{2}I \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} \bar{F} \\ 0 \\ \bar{H} \end{bmatrix}, \ \boldsymbol{K} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}, \tag{27}$$

then according to Lemma 2, if system (6) is asymptotically stable with $\|\mathcal{G}_{ue}\|_\infty < \gamma$, then there exists $\bar{P} > 0$ such that

$$\mathrm{Her}\left((\mathcal{A} + \mathcal{B}G_r^*\mathcal{K})^T \mathcal{P}\right) < 0 \tag{28}$$

holds, where

$$\mathcal{K} = \Phi\boldsymbol{K}, \ \mathcal{P} = \mathrm{diag}\begin{pmatrix} \bar{P} & I & I \end{pmatrix}.$$

In addition,

$$\begin{aligned} & \mathrm{Her}\left((\mathcal{A} + \mathcal{B}\Psi_1\boldsymbol{K})^T \mathcal{P}\right) \\ = \ & \mathrm{Her}\left((\mathcal{A} + \mathcal{B}G_r^*\mathcal{K})^T \mathcal{P}\right) + \mathrm{Her}\left((\mathcal{B}(\Psi_1 - G_r^*\Phi)\boldsymbol{K})^T \mathcal{P}\right). \end{aligned} \tag{29}$$

Thus, if $\|\Psi_1 - G_r^*\Phi\|$ is sufficiently small, then based on (28) and (29), we have that

$$\mathrm{Her}\left((\mathcal{A} + \mathcal{B}\Psi_1\boldsymbol{K})^T \mathcal{P}\right) < 0, \tag{30}$$

which is equivalent to inequality (25) with $\bar{Q} = \bar{P}^{-1}$. Hence, system (24) is asymptotically stable with $\|\mathcal{T}_{ue}\|_\infty < \gamma$, which completes the first part of the proof.

*(ii)⇒(i).* For $\Phi$ defined in (21), we have that $\Phi\Phi^T = I$, and $\Phi^\perp$ can be explicitly given as

$$\Phi^\perp = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}.$$

If we select $G_r^* = \Psi_1\Phi^T$, then

$$(\Psi_1 - G_r^*\Phi)\begin{bmatrix} \Phi^T & \Phi^\perp \end{bmatrix} = \begin{bmatrix} 0 & \Psi_1\Phi^\perp \end{bmatrix}.$$

Note that $\begin{bmatrix} \Phi^T & \Phi^\perp \end{bmatrix}$ is a permutation matrix, and consider the fact that

$$\Psi_1 - G_r^*\Phi = \begin{bmatrix} 0 & \Psi_1\Phi^\perp \end{bmatrix}\begin{bmatrix} \Phi^T & \Phi^\perp \end{bmatrix}^{-1},$$

we have

$$\begin{aligned} \|\Psi_1 - G_r^*\Phi\| &\leq \left\|\begin{bmatrix} 0 & \Psi_1\Phi^\perp \end{bmatrix}\right\|\left\|\begin{bmatrix} \Phi^T & \Phi^\perp \end{bmatrix}^{-1}\right\| \\ &\leq \varepsilon. \end{aligned} \tag{31}$$

In addition, if system (24) is asymptotically stable with $\|\mathcal{T}_{ue}\|_\infty < \gamma$, then $\mathrm{Her}\left((\mathcal{A} + \mathcal{B}\Psi_1\boldsymbol{K})^T\mathcal{P}\right) < 0$. With the similar proof line of (*(i)⇒(ii)*), we have that (28) holds, which further indicates there exists $G_r^*$ such that system (6) is asymptotically stable with $\|\mathcal{G}_{ue}\|_\infty < \gamma$. This, together with (31), gives that *(i)* holds, which completes the whole proof. □

Clearly, Theorem 3 shows that if $U_1$ and $V_1$ can be chosen in such a way that (30) holds and $\left\|\Psi_1\Phi^\perp\right\|$ is small enough, then it will be more likely for (18) to have a feasible solution. Typically, if $\varepsilon = 0$, that is, $\left\|\Psi_1\Phi^\perp\right\| = 0$, then (18) will also be feasible. Furthermore, in view of (17) and the proof of (*(ii)⇒(i)*) in Theorem 3, we have that $\Psi_1\Phi^T \in \tilde{\mathbb{S}}$. Summarizing the above discussion, we propose the following algorithm to find $U_1$ and $V_1$ such that the solvability of *Algorithm 1* can be further improved.

*Algorithm 2 (Initial Optimization):*

1. **START**: Set $j = 1$. Solve the LMI in (25) to obtain $U_j$ and $V_j$, respectively.

2. For fixed $\Psi_j = \begin{bmatrix} U_j & 0 & V_j \end{bmatrix}$, solve the following LMI with respect to $\bar{P}_j$ :

$$\mathrm{Her}\left((\mathcal{A} + \mathcal{B}\Psi_j\boldsymbol{K})^T\mathcal{P}_j\right) < 0. \tag{32}$$

   where $\mathcal{A}$, $\mathcal{B}$, and $\boldsymbol{K}$ are defined in (27), and $\mathcal{P}_j = \mathrm{diag}\begin{pmatrix} \bar{P}_j & I & I \end{pmatrix}$.

3. For fixed $\bar{P}_j$, solve the following convex optimization problem with respect to $\Psi_j$ and $\varepsilon_j$ :

$$\varepsilon_j^* := \min_{\Psi_j} \varepsilon_j \quad \text{s.t.} \begin{cases} (32) \text{ holds} \\ \Psi_j\Phi^T \in \tilde{\mathbb{S}} \\ \begin{bmatrix} -\varepsilon_j I & (\Psi_j\Phi^\perp)^T \\ \Psi_j\Phi^\perp & -I \end{bmatrix} < 0 \end{cases}.$$

   Correspondingly, denote $\Psi_j^*$ as the optimized value of $\Psi_j$.

4. If $\left|\left(\varepsilon_j^* - \varepsilon_{j-1}^*\right)/\varepsilon_j^*\right| < \delta_2$, where $\delta_2$ is a prescribed tolerance, then go to next step. If not, set $\Psi_{j+1} := \Psi_j^*$ and $j := j+1$, then go to Step 2.

5. An initial choice of $U_1$ and $V_1$ is obtained as $\begin{bmatrix} U_1 & 0 & V_1 \end{bmatrix} := \Psi_j^*$. **STOP**.

**Remark 5** *Note that the stopping criterion suggested in Step 4 is heuristic, and the convergence of Algorithm 2 is naturally guaranteed, since $\varepsilon_j^*$ is monotonically decreasing with respect to $j$ with a lower bound of zero.*

### 4.3 Further Results via Dual Approach

This subsection further studies the positive model reduction problem via dual approach. To be specific, we introduce a new auxiliary variable $\breve{\vartheta}(t) = \bar{M}\hat{x}(t) + \bar{N}u(t)$, and define $\breve{\boldsymbol{x}}(t) = \begin{bmatrix} \hat{x}^T(t) & \breve{\vartheta}^T(t) \end{bmatrix}^T$ accordingly. Thus, the error system in (6) can be equivalently described by the dual of system (13), shown as follows:

$$\left\{ \begin{array}{rcl} \breve{\boldsymbol{E}}\dot{\breve{\boldsymbol{x}}}(t) & = & \breve{\boldsymbol{A}}\breve{\boldsymbol{x}}(t) + \breve{\boldsymbol{B}}u(t), \\ e(t) & = & \breve{\boldsymbol{C}}\breve{\boldsymbol{x}}(t) + \breve{\boldsymbol{D}}u(t), \end{array} \right. \tag{33}$$

where

$$\begin{aligned} \breve{\boldsymbol{E}} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \ \breve{\boldsymbol{A}} = \begin{bmatrix} \bar{A} & \bar{F}G_r \\ \bar{M} & -I \end{bmatrix}, \ \breve{\boldsymbol{B}} = \begin{bmatrix} \bar{B} \\ \bar{N} \end{bmatrix}, \\ \breve{\boldsymbol{C}} &= \begin{bmatrix} \bar{C} & \bar{H}G_r \end{bmatrix}, \ \breve{\boldsymbol{D}} = \bar{D}. \end{aligned}$$

The motivation for introducing (33) is that optimality in the primal direction does not imply optimality in the dual direction, therefore, it is possible that the primal problem which cannot be solved directly may have solutions in their dual form.

In the following, we shall propose an equivalent characterization on the stability with $H_\infty$ performance of (6) as follows, which can be viewed as a dual form of Theorem 1.

**Theorem 4** *Given the system matrices $A_r$, $B_r$, $C_r$ and $D_r$. Then the following statements are equivalent:*

*(i) The error system in (6) is asymptotically stable, and satisfies $\|\mathcal{G}_{ue}\|_\infty < \gamma$.*

*(ii) There exist matrices $\hat{Q} > 0$ and diagonal $Z > 0$ such that*

$$\breve{\Xi} \triangleq \begin{bmatrix} \breve{\boldsymbol{A}}\boldsymbol{Q} + \boldsymbol{Q}^T\breve{\boldsymbol{A}}^T & \boldsymbol{Q}^T(\boldsymbol{I}+\boldsymbol{J})\breve{\boldsymbol{C}}^T & \breve{\boldsymbol{B}} \\ \breve{\boldsymbol{C}}(\boldsymbol{I}+\boldsymbol{J})^T\boldsymbol{Q} & -\breve{\boldsymbol{C}}\boldsymbol{J}^T\left(\boldsymbol{Q}+\boldsymbol{Q}^T\right)\boldsymbol{J}\breve{\boldsymbol{C}}^T - \gamma I & \breve{\boldsymbol{D}} \\ \breve{\boldsymbol{B}}^T & \breve{\boldsymbol{D}}^T & -\gamma I \end{bmatrix} < 0, \tag{34}$$

*where*

$$\boldsymbol{Q} = \begin{bmatrix} \hat{Q} & 0 \\ -\frac{1}{2}ZG_r^T\bar{F}^T & \frac{1}{2}Z \end{bmatrix}, \ \boldsymbol{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \ \boldsymbol{J} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

**Sketch of Proof:** Define the nonsingular matrix $\breve{T}$ as

$$\breve{T} \triangleq \begin{bmatrix} I & 0 & 0 & 0 \\ (\bar{F}G_r)^T & (\bar{H}G_r)^T & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix},$$

then, by following a similar line of the proof in Theorem 1, the results can be readily obtained. □

Consequently, the dual condition for the existence of a positive reduced-order system (2) can be established as follows.

**Theorem 5** Problem PP-$H_\infty$-MR *is solvable, if and only if there exists a matrix* $\hat{Q} > 0$, *a diagonal matrix* $Z > 0$, *matrices* $U$, $V$, $L_1$, $L_2$, $L_3$ *and* $L_4$ *such that*

$$L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \in \tilde{\mathbb{S}}, \tag{35}$$

$$\breve{\Xi}(U,V) \triangleq \begin{bmatrix} \breve{\Xi}_{11} & \hat{Q}\bar{M}^T + \bar{F}L & \breve{\Xi}_{13} & \bar{B} \\ \bar{M}\hat{Q} + L^T\bar{F}^T & -Z & L^T\bar{H}^T & \bar{N} \\ \breve{\Xi}_{13}^T & \bar{H}L & \breve{\Xi}_{33} & \bar{D} \\ \bar{B}^T & \bar{N}^T & \bar{D}^T & -\gamma I \end{bmatrix} < 0, \tag{36}$$

*where*

$$\begin{aligned} \breve{\Xi}_{11} &= 2\mathrm{Her}\left(\bar{A}\hat{Q} - UL^T\bar{F}^T\right) + UZU^T, \\ \breve{\Xi}_{13} &= \hat{Q}\bar{C}^T - \bar{F}LV^T - UL^T\bar{H}^T + UZV^T, \\ \breve{\Xi}_{33} &= -2\mathrm{Her}\left(VL^T\bar{H}^T\right) + VZV^T - \gamma I. \end{aligned}$$

*In this case, the system matrices of (2) can be given as*

$$G_r = LZ^{-1}. \tag{37}$$

Based on Theorem 5, we present the following algorithm, which can be viewed as a dual of *Algorithm 1*.

*Algorithm 3 (Dual ILMI Approach):*

1. **START**: Set $j = 1$. For a given $H_\infty$ performance level $\gamma$, compute the initial matrices $U_1$ and $V_1$ such that the following auxiliary system,

$$\left\{ \begin{aligned} \dot{\bar{x}}(t) &= \bar{A}^T\bar{x}(t) + \bar{M}^T\bar{\nu}(t) + \bar{C}^Tw(t), \\ e(t) &= \bar{B}^T\bar{x}(t) + \bar{N}^T\bar{\nu}(t) + \bar{D}^Tw(t), \end{aligned} \right. \tag{38}$$

   with $\bar{\nu}(t) = U_1^T\bar{x}(t) + V_1^Tw(t)$ is asymptotically stable and the transfer function $\mathcal{T}_{we}(s)$ from $w(t)$ to $e(t)$ satisfies $\|\mathcal{T}_{we}\|_\infty < \gamma$.

2. For fixed $U_j$ and $V_j$, solve the following convex optimization problem for the parameters in $\digamma \triangleq \left\{\hat{Q} > 0,\ Z > 0 \text{ is diagonal},\ L_1,\ L_2,\ L_3 \text{ and } L_4\right\}$ :

$$\beta_j^* := \min_{\digamma} \beta_j \quad \text{s.t.} \quad \left\{ \begin{array}{l} L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \in \tilde{\mathbb{S}} \\ \breve{\Xi}(U_j, V_j) < \beta_j \Pi \\ \beta_j \geq \sigma \end{array} \right. ,$$

   where $\sigma \leq 0$ is an arbitrary scalar. Denote the corresponding value of $Z$ and $L$ as $Z_j$ and $L_j$, respectively.

3. If $\beta_j^* \leq 0$, then a desired parametric matrix $G_r$ is obtained as $G_r := L_jZ_j^{-1}$. **STOP**. If not, then go to next step.

4. If $\left|\left(\beta_j^* - \beta_{j-1}^*\right)/\beta_j^*\right| < \delta_3$, where $\delta_3$ is a prescribed tolerance, then go to next step. If not, update $U_{j+1}$ and $V_{j+1}$ as
$$U_{j+1} := \bar{F} L_j Z_j^{-1},\ V_{j+1} := \bar{H} L_j Z_j^{-1}.$$
Set $j := j+1$, then go to Step 2.

5. A solution to *Problem PP-$H_\infty$-MR* may not exist. **STOP**.

The reason for the selection of $U_1$ and $V_1$ is the same as the one proposed in the last subsection, and optimization of the initial values can be readily performed by using a similar approach in *Algorithm 2*. Note that $U_1$ and $V_1$ are the "state feedback controller matrix" for system (38), or "observer matrices" for system (24), which motivates us to call *Algorithm 3* as the dual iterative LMI approach.

### 4.4 Combination of Primal and Dual Approaches

On one hand, although one may solve the positive model reduction problem by implementing *Algorithm 1* or *Algorithm 3* separately, it is possible that the iterates may get trapped in a local minimum. More specifically, it may happen that the original problem is actually feasible, but a local minimum is achieved and is unable to confirm the feasibility of the problem. On the other hand, as we stated before, the optimality in the primal direction does not imply that in the dual direction, thus, we may combine *Algorithms 1* and *3* together to further improve the solvability of *Problem PP-$H_\infty$-MR*. We summarize it in the following algorithm:

*Algorithm 4 (Primal-Dual ILMI Approach)*

1. **START**: Run *Algorithm 2* to find an initial value of $U_1$ and $V_1$. Set $k = 0$, and $N$ as the maximum number of iterations allowed.

2. While $k < N$ do

   (a) For fixed $U_1$ and $V_1$, run *Algorithm 1*, if $\alpha_j^* < 0$, then $G_r$ can be readily obtained as $G_r := X_j^{-1} L_j$. **STOP**. Otherwise, running *Algorithm 1* until $\alpha_j^*$ converges, which gives a temporary matrix $G_{rp}^*$ as $G_{rp}^* := X_j^{-1} L_j$. Set $U_1 := \bar{F} G_{rp}^*$ and $V_1 := \bar{H} G_{rp}^*$, then go to next step.

   (b) For fixed $U_1$ and $V_1$, run *Algorithm 3*, if $\beta_j^* < 0$, then $G_r$ can be readily obtained as $G_r := L_j Z_j^{-1}$. **STOP**. Otherwise, running *Algorithm 3* until $\beta_j^*$ converges, which gives another temporary matrix $G_{rd}^*$ as $G_{rd}^* := L_j Z_j^{-1}$. Set $U_1 := G_{rd}^* \bar{M}$, $V_1 := G_{rd}^* \bar{N}$, and $k := k+1$.

   End (while)

3. A solution to *Problem PP-$H_\infty$-MR* may not exist. **STOP**.

Although *Algorithm 4* is proposed to improve the solvability of *Problem PP-$H_\infty$-MR*, it still cannot be guaranteed to converge to the global optimum. In this sense, our algorithm does not completely solve the problem. Nevertheless, the effectiveness of the proposed approach will be further illustrated in the following section.

# 5 Illustrative Example

Compartmental networks consist of a finite number of homogeneous, well-mixed subsystems, called compartments, which exchange with each other and the environment [23]. Consider a compartmental network of $n$ compartments shown schematically in Figure 1.

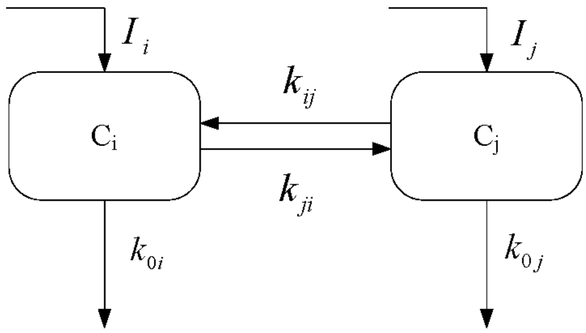


Figure 1: Profile of linear compartmental networks

The quantity (or, concentration) of material involved in compartment $i$ at time $t$ is denoted as $x_i(t)$; $k_{ij}x_j$ is the flow rate from compartment $j$ to compartment $i$ ($k_{0i}x_i$ is the outflow of compartment $i$); the inflow of compartment $i$ is represented by $I_i(t) = \sum_{j=1}^{m} b_{ij}u_j(t)$, where $u_j(t)$ is the $j$th input resource. Thus, the mathematical description for compartment $i$ can be given as follows:

$$\dot{x}_i(t) = \sum_{j\neq i}^{n} [k_{ij}x_j - k_{ji}x_i] - k_{0i}x_i + \sum_{j=1}^{m} b_{ij}u_j(t), \ i = 1, 2, \ldots, n. \tag{39}$$

Then, the system matrix $A$ in (1) can be formulated as $A = [a_{ij}]_{n\times n}$ where

$$a_{ij} = \begin{cases} -\left(\sum_{j\neq i}^{n} k_{ji} + k_{0i}\right), & i = j, \\ k_{ij}, & i \neq j, \end{cases}.$$

For illustration, we have constructed the compartmental model with six states shown in Figure 2.

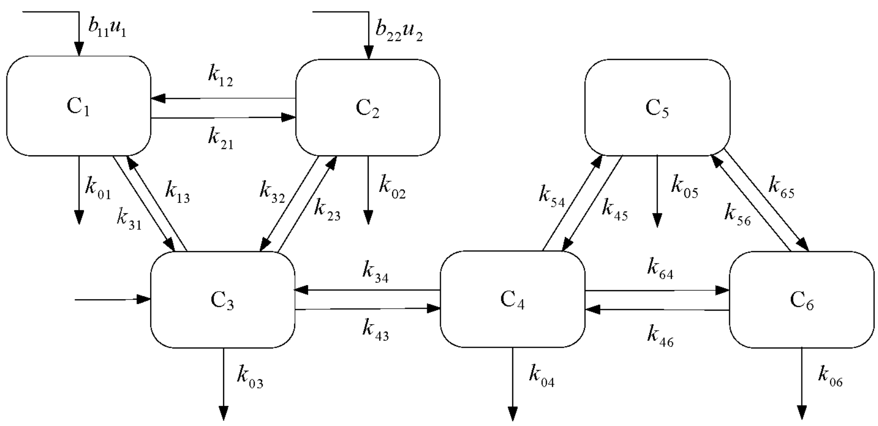


Figure 2: Compartmental network with 6 state components

One can see that there are two completely connected subsystems each with three state components, and they are linked through compartments 3 and 4. Here, we assume

$$\begin{aligned}
k_{01} &= 1.0,\ k_{21} = 0.3,\ k_{31} = 0.2,\ k_{02} = 0.8,\ k_{12} = 0.6,\\
k_{32} &= 0.5,\ k_{03} = 1.0,\ k_{13} = 1.0,\ k_{23} = 0.2,\ k_{43} = 0.5,\\
k_{04} &= 1.0,\ k_{34} = 1.0,\ k_{54} = 0.4,\ k_{64} = 0.6,\ k_{05} = 0.5,\\
k_{45} &= 0.6,\ k_{65} = 0.5,\ k_{06} = 0.8,\ k_{46} = 0.5,\ k_{56} = 0.3,\\
b_{11} &= 1.0,\ b_{22} = 1.0,
\end{aligned}$$

that is,

$$A = \begin{bmatrix} -1.5 & 0.6 & 1.0 & 0 & 0 & 0 \\ 0.3 & -1.9 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0.5 & -2.7 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & -3 & 0.6 & 0.5 \\ 0 & 0 & 0 & 0.4 & -1.6 & 0.3 \\ 0 & 0 & 0 & 0.6 & 0.5 & -1.6 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It can be verified that the system is stable, and we assume that the $H_\infty$ performance level is prescribed as $\gamma = 0.1$. The aim of this example is to construct 2-order positive reduced-order systems in two cases (with different outputs) below in the form of (2) to approximate the original system.

*Case I.* Output is the sum of the quantity of material in the compartmental network, that is,

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

We first try to apply *Algorithm 1* to solve *Problem PP-$H_\infty$-MR*. Using the Matlab LMI Control Toolbox, we obtain an initial value of $U_1$ and $V_1$ as

$$U_1^{int} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix},\ V_1^{int} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It can be checked that no desired reduced-order system in (2) can be found by *Algorithm* 1. However, if *Algorithm 2* is further applied to optimize $U_1$ and $V_1$ with the tolerance level $\delta_2 = 10^{-2}$, then optimized values of $U_1$ and $V_1$ can be obtained as

$$U_1^{opt} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -2.5175 & 1.6949 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.6321 & -2.5014 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.0279 & 1.9659 \end{bmatrix},\ V_1^{opt} = \begin{bmatrix} 0.2121 & 0.2300 \\ 0.2116 & 0.2332 \\ 0.0494 & 0.0376 \end{bmatrix}.$$

With these new initial values, we implement *Algorithm 1* to solve *Problem PP-$H_\infty$-MR* again, and an $\alpha_1^* = -0.0443 < 0$ is found after one iteration. Therefore, the condition in Theorem 2 is feasible with the following solution:

$$X = \begin{bmatrix} 253.5817 & 0 & 0 \\ 0 & 363.4548 & 0 \\ 0 & 0 & 122.0049 \end{bmatrix},\ L = \begin{bmatrix} -455.9701 & 241.7357 & 54.3521 & 59.1674 \\ 160.5375 & -474.4492 & 77.3205 & 84.9729 \\ 219.9069 & 267.2964 & 5.9949 & 4.5524 \end{bmatrix}.$$

Then, according to (19), a desired positive 2nd order model in (2) can be readily obtained with the system matrices given as

$$\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right] = \left[\begin{array}{cc|cc} -1.7981 & 0.9533 & 0.2143 & 0.2333 \\ 0.4417 & -1.3054 & 0.2127 & 0.2338 \\ \hline 1.8024 & 2.1909 & 0.0491 & 0.0373 \end{array}\right].$$

*Case II.* Output is the quantity of material in Compartments 1 and 2, respectively, that is,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It can be easily verified that no solution can be found by running *Algorithm 1*, irrespective of whether *Algorithm 2* is applied or not. However, if *Algorithm 4* is utilized to solve the problem with the tolerance levels $\delta_1$, $\delta_2$, and $\delta_3$ all specified as $10^{-2}$, then after six iterations, a $\beta_j^* = 0$ is found in Step 2.b. Therefore, the condition in Theorem 5 is feasible with the following solution:

$$Z = \begin{bmatrix} 167.5471 & 0 & 0 & 0 \\ 0 & 38.1587 & 0 & 0 \\ 0 & 0 & 753.6758 & 0 \\ 0 & 0 & 0 & 165.3818 \end{bmatrix}, \quad L = \begin{bmatrix} -307.0492 & 15.6431 & 1.5285 & 470.1873 \\ 5.9141 & -35.2889 & 131.2422 & 0.0484 \\ 12.6363 & 142.1916 & 72.9519 & 0.0622 \\ 46.7395 & 42.7307 & 0.3559 & 13.9471 \end{bmatrix}.$$

Then, according to (37), a desired positive 2nd order model in (2) can be readily obtained with the system matrices given as

$$\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right] = \left[\begin{array}{cc|cc} -1.8326 & 0.4099 & 0.0020 & 2.8430 \\ 0.0353 & -0.9248 & 0.1741 & 0.0003 \\ \hline 0.0754 & 3.7263 & 0.0968 & 0.0004 \\ 0.2790 & 1.1198 & 0.0005 & 0.0843 \end{array}\right].$$

It can be easily verified that the $H_\infty$ performance of the associated error system is 0.0977, which is less than the prescribed $H_\infty$ norm bound $\gamma = 0.1$. Figure 3 shows that the maximum singular value is below the $H_\infty$ error bound.

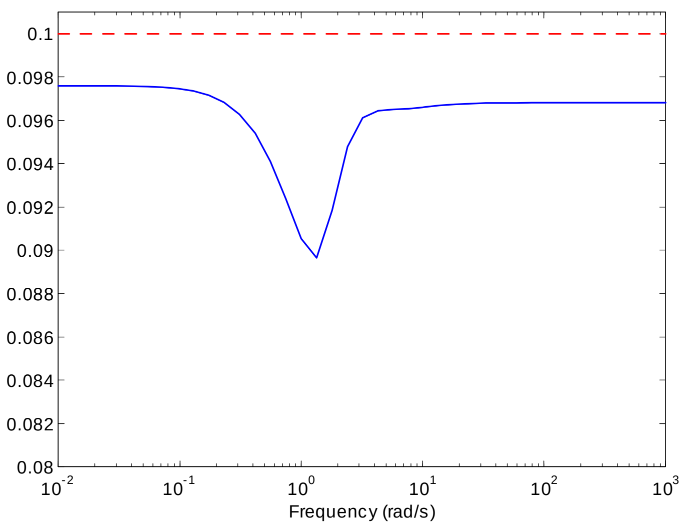


Figure 3: Maximum singular value of associated error system

In addition, under the excitation of $L_2$-input

$$u(t) = \begin{bmatrix} e^{-0.001t} & \frac{1}{0.2+0.005t}\left|\cos\left(\frac{\pi}{10}t\right)\right| \end{bmatrix}^T,$$

and zero initial conditions, Figures 4 and 5 depict the output trajectories of the original positive system and those of the reduced-order positive system, respectively. It can be observed from these simulation results that the obtained reduced model preserves the positivity and approximates the original system very well.

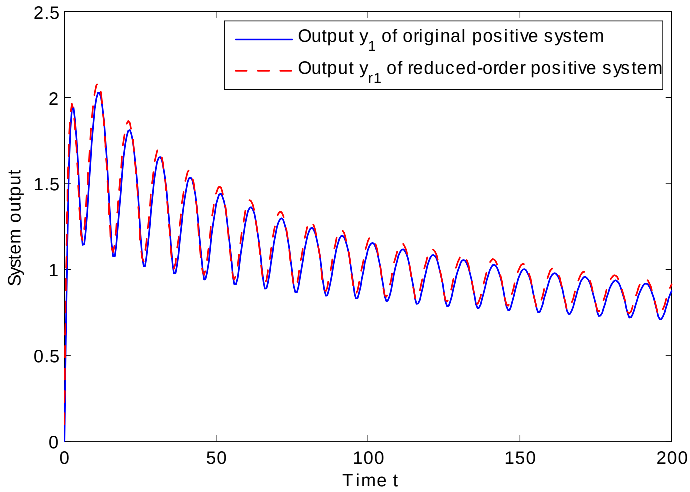


Figure 4: Output $y_1$ of original positive system and $y_{r1}$ of reduced-order positive system

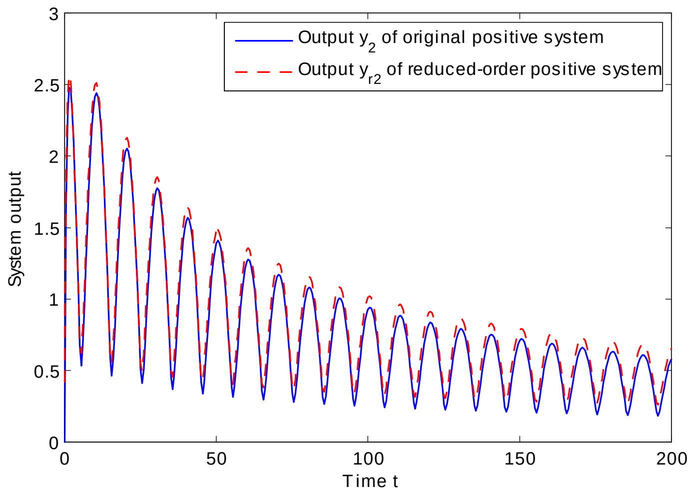


Figure 5: Output $y_2$ of original positive system and $y_{r2}$ of reduced-order positive system

## 6 Conclusion

In this paper, we have presented a model reduction approach that preserves positivity and stability with $H_\infty$ performance of positive systems. In particular, we have proposed a novel characterization on the stability and

$H_\infty$ performance of the associated error system by means of a system augmentation method, which ensures the separation of the reduced-order system matrices to be constructed from the Lyapunov matrix. Based on this new characterization, a necessary and sufficient condition for the existence of a desired reduced-order system has been established in terms of matrix equalities, and a primal iterative LMI approach has been developed to solve the condition. A heuristic algorithm has also been proposed to optimize the initial values. Furthermore, a dual iterative LMI approach, together with the primal one, has been utilized to improve the solvability of the positive-preserving $H_\infty$ model reduction problem. Finally, the effectiveness of the proposed method has been illustrated by a compartmental network. The approach adopted in this paper can be applied to tackle problems involving some constraints on elements of the required system matrices, such as positivity and boundedness.

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