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ANALYSIS OF DIRECT BOUNDARY-DOMAIN INTEGRAL EQUATIONS FOR A MIXED BVP WITH VARIABLE COEFFICIENT, I: EQUIVALENCE AND INVERTIBILITY

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ABSTRACT. A mixed (Dirichlet-Neumann) boundary value problem (BVP) for the "stationary heat transfer" partial differential equation with variable coefficient is reduced to some systems of nonstandard segregated direct parametrix-based boundary-domain integral equations (BDIEs). The BDIE systems contain integral operators defined on the domain under consideration as well as potential-type and pseudodifferential operators defined on open submanifolds of the boundary. It is shown that the BDIE systems are equivalent to the original mixed BVP, and the operators of the BDIE systems are invertible in appropriate Sobolev spaces.

1. Introduction. The boundary integral equation (BIE) method has been intensively developed over recent decades both in theory and in engineering applications. Its popularity is due to the possibility of reducing a boundary value problem (BVP) for a partial differential equation in a domain to an integral equation on the boundary of the domain. This approach diminishes the problem dimensionality by one which is very important for construction of various numerical algorithms requiring small computer resources. The main ingredient necessary for reduction of a BVP to a boundary integral equation (BIE) is a fundamental solution to the original partial differential equation, available in an analytical form and/or cheaply calculated. After the

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fundamental solution is used in the corresponding Green formulae, as a first step, one can reduce the BVP to a BIE. The next essential step is to establish the equivalence of the original BVP and the corresponding BIE, and to show the invertibility of the BIE operator which is a crucial moment for further numerical analysis.

However, such a fundamental solution is generally not available if the coefficients of the original partial differential equation are not constant. One can use, in this case, a parametrix (Levi function), which is usually available, instead of the fundamental solution in the Green formulae. This allows a reduction of a boundary value problem not to *Boundary* Integral Equations (BIEs) but to *Boundary-Domain* Integral Equations (BDIEs) (cf. [**26, 30, 35**] and the references therein).

In spite of the fact that mixed BVPs with variable coefficients, on the one hand, and boundary integral equations for BVPs with constant coefficients on the other hand, are well studied nowadays, this is not the case for the *boundary-domain* integral equations associated with BVPs with *variable coefficients*. For example, the classical works [**16, 19, 30**], formulating the parametrix-based integral identities, deal with only the so-called indirect BDIEs for Dirichlet and Neumann BVPs. The indirect BDIE method has been essentially developed in the reference [**35**]. Particularly, a general algorithm for the construction of Levi functions of arbitrary degree has been obtained for elliptic systems in the sense of Douglis-Nirenberg, which has been applied then to the Dirichlet problem arising in the shell theory.

On the other hand, the direct integral equation method is very popular in computational engineering applications since the unknowns involved in the integral equations have clear physical and mechanical meanings in contrast to the indirect approach (see, e.g., [**1**] and the discussion in [**6**]). However, to the best knowledge of the authors the parametrix-based *direct boundary-domain integral equations* for basic and, especially, mixed BVPs with variable coefficients have not been rigorously analyzed, and the questions about the equivalence of BDIEs to the original BVPs as well as about the invertibility of the corresponding operators in appropriate function spaces remained open.

In this paper, an explicit parametrix and the parametrix-based third Green's identity is used to reduce the mixed (Dirichlet-Neumann) boundary value problem for the "heat transfer" partial differential equation with a variable coefficient (heat conductivity) to four different versions of direct segregated Boundary-Domain Integral Equation Systems (BDIES). The original BVP is relevant to modeling electrostatics, ani-plane elasticity, stationary flow in porous media, and diffusion phenomena, in inhomogeneous media. The BDIES are *segregated* in the sense that the unknown boundary functions are considered as formally unrelated to the unknown function inside the domain (while their connection is revealed when they solve the system). The BDIES represent nonstandard systems of equations containing integral operators defined on the domain under consideration and potential type and pseudo-differential operators defined on open submanifolds of the boundary. The domain integral equation of each of the BDIE system is of the second kind, while the remaining (boundary) integral equations are formally of the first or the second or the third kind. We give in the present part of the paper a rigorous analysis of BDIESs and show that the mixed BVP and the corresponding BDIES are *equivalent* and the boundary domain integral operators obtained are *invertible* in appropriate Sobolev-Slobodetski (Bessel-potential) spaces. Depending on the data of the mixed BVP (geometry of the domain, the division of the boundary into the Dirichlet and Neumann parts, special analytical and structural properties of the variable coefficients and the given boundary functions) it may occur that one of the four different types of BDIES mentioned above proves to be more efficient in comparison with others. Along with the pure theoretical interest, this is a basic motivation for reducing the mixed BVP to the different type BDIES described and analyzed in this paper.

The BDIES solution regularity and asymptotics are analyzed in the second part of the paper, [**4**], using the Bessel-potential and Besov spaces.

2. Formulation of the boundary value problem. Let $\Omega = \Omega^+$ be a bounded open three-dimensional region of $\mathbb{R}^3$, $\Omega^- = \mathbb{R}^3 \backslash \overline{\Omega}^+$. For simplicity, we assume that the boundary $S := \partial\Omega$ is a simply connected, closed, infinitely smooth surface. Moreover, $S = \overline{S}_D \cup \overline{S}_N$ where S_D and S_N are nonempty, nonintersecting ($S_D \cap S_N = \varnothing$), simply connected submanifolds of S with infinitely smooth boundary curve $\partial S_D = \partial S_N \in C^\infty$. Let $a \in C^\infty(\mathbb{R}^3)$, $a(x) > 0$ for $x \in \mathbb{R}^3$. Let also $\partial_j = \partial_{x_j} := \partial/\partial x_j$, $j = 1, 2, 3$, $\partial_x = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.

We consider the following scalar elliptic differential equation

(2.1)
$$Lu(x) := L(x,\partial_x)\,u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\Big(a(x)\frac{\partial u(x)}{\partial x_i}\Big) = f(x), \quad x\in\Omega,$$

where u is an unknown function and f is a given function in Ω. When $a=1$, the operator L becomes the Laplace operator Δ.

In what follows $W_p^r(\Omega^+)$ denotes the Sobolev-Slobodetski spaces; $H^s(S) = H_2^s(S)$ the Bessel potential spaces; $H^s_{\rm loc}(\overline{\Omega^-}) = H^s_{2,\rm loc}(\overline{\Omega^-})$ consist of the distributions that belong to $H_2^s(\omega)$ for every compact $\omega\subset\overline{\Omega^-}$; here, $r\geq 0$ and $s\in\mathbb{R}$ are arbitrary real numbers, see e.g., [**20, 34, 40**]. We recall that $H^r = W_2^r$ for $r\geq 0$.

For the operator L from (2.1), following [**7, 15**], we will use the space

$$H^{1,0}(\Omega;L) := \{g\in H^1(\Omega) : L\,g\in L_2(\Omega)\} \tag{2.2}$$

with the norm $\|g\|^2_{H^{1,0}(\Omega;L)} := \|g\|^2_{H^1(\Omega)} + \|Lg\|^2_{L_2(\Omega)}$.

For $S_1\subset S$, we will use the subspace $\widetilde{H}^s(S_1) = \{g : g\in H^s(S), \operatorname{supp} g\subset\overline{S_1}\}$ of $H^s(S)$, while $H^s(S_1) = \{r_{S_1}g : g\in H^s(S)\}$ denotes the space of restriction on S_1 of functions from $H^s(S)$, where r_{S_1} denotes the restriction operator on S_1.

From the trace theorem, see [**20, 23, 34, 40**], for $u\in H^s(\Omega)$ $(u\in H^s_{\rm loc}(\overline{\Omega^-}))$, $s>1/2$, it follows that $u|_S^{\pm} := \gamma_S^{\pm}\,u\in H^{s-\frac12}(S)$, where $\gamma_S^{\pm}$ is the trace operator on S from $\Omega^{\pm}$. We will use also notations $u^{\pm}$ or $[u]^{\pm}$ for the traces $u|_S^{\pm}$, when this will cause no confusion.

For $u\in H^s(\Omega)$, $s>3/2$, we can denote by $T^{\pm}$ the corresponding conormal derivative operator on S understood in the trace sense,

(2.3)
$$T^{\pm}(x,n(x),\partial_x)\,u(x) := \sum_{i=1}^{3} a(x)\,n_i(x)\Big(\frac{\partial u(x)}{\partial x_i}\Big)^{\pm} = a(x)\Big(\frac{\partial u(x)}{\partial n(x)}\Big)^{\pm},$$

where $n(x)$ is the exterior (to Ω) unit normal vector at the point $x\in S$.

For $u\in H^{1,0}(\Omega;L)$ $(u\in H^{1,0}_{\rm loc}(\overline{\Omega^-};L))$ we can correctly define the (canonical) conormal derivative $T^{\pm}u\in H^{-1/2}(S)$ with the help of

Green's formula, cf. [**7, 23**],

$$\left\langle T^{\pm}u,\, w\right\rangle_S := \pm\int_{\Omega^{\pm}} \left[(\gamma_{-1}w)Lu + E(u,\gamma_{-1}w)\right] dx, \qquad \text{for all } w \in H^{1/2}(S), \tag{2.4}$$

where $\gamma_{-1} : H^{1/2}(S) \to H^1_{\text{comp}}(\mathbb{R}^3)$ is a continuous right inverse to the trace operator,

$$E(u,v) := \sum_{i=1}^{3} a(x)\, \frac{\partial u(x)}{\partial x_i}\, \frac{\partial v(x)}{\partial x_i},$$

and $\langle\cdot,\cdot\rangle_S$ denote the duality brackets between the spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$ which extend the usual $L_2(S)$ scalar product.

We will derive and investigate boundary-domain integral equation systems for the following *mixed boundary value problem*.

Find a function $u \in H^1(\Omega)$ satisfying the conditions

$$L\,u = f \text{ in } \Omega^+, \tag{2.5}$$

$$r_{S_D}\, u^+ = \varphi_0 \text{ on } S_D, \tag{2.6}$$

$$r_{S_N}\, T^+ u = \psi_0 \text{ on } S_N, \tag{2.7}$$

where $\varphi_0 \in H^{1/2}(S_D)$, $\psi_0 \in H^{-1/2}(S_N)$ and $f \in L_2(\Omega)$.

Equation (2.5) is understood in the distributional sense, condition (2.6) is understood in the trace sense, while equality (2.7) is understood in the functional sense, cf. (2.4).

By [**7**, Lemma 3.4], the first Green identity holds for any $u \in H^{1,0}(\Omega^+; L)$, $v \in H^1(\Omega^+)$,

$$\left\langle T^+u, v^+\right\rangle_S = \int_{\Omega^+} [vLu + E(u,v)]\, dx. \tag{2.8}$$

We have the following well-known uniqueness theorem.

Theorem 2.1. *The homogeneous version of* BVP (2.5)–(2.7), *i.e., with* $f = 0$, $\varphi_0 = 0$, $\psi_0 = 0$, *has only the trivial solution.*

Proof. The proof immediately follows from Green's formula (2.8) with $v = u$ as a solution of the homogeneous mixed boundary value problem. □

Clearly, nonhomogeneous problems (2.5)–(2.7) may possess at most one solution due to the problem linearity.

3. Parametrix and potential type operators.

3.1. Parametrix. We will say that a function $P(x,y)$ of two variables $x, y \in \Omega$ is a parametrix for the operator $L(x,\partial_x)$ in $\mathbb{R}^3$ if (cf., e.g., [**30**])

$$L(x,\partial_x)\, P(x,y) = \delta(x-y) + R(x,y), \tag{3.1}$$

where $\delta(\cdot)$ is the Dirac distribution and $R(x,y)$ possesses a weak (integrable) singularity at $x = y$, i.e., $R(x,y) = \mathcal{O}\left(|x-y|^{-\varkappa}\right)$ with $\varkappa < 3$.

It is easy to see that, for the operator $L(x,\partial_x)$ given by (2.1), the function

$$P(x,y) = \frac{-1}{4\pi\, a(y)\, |x-y|}, \quad x, y \in \mathbb{R}^3, \tag{3.2}$$

is a parametrix (Levi function), while the remainder R in (3.1) is

$$R(x,y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi\, a(y)\, |x-y|^3}\, \frac{\partial a(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^3, \tag{3.3}$$

and thus is weakly singular, $\mathcal{O}\left(|x-y|^{-2}\right)$, due to the smoothness of the function $a(x)$.

We evidently have that the parametrix $P(x,y)$ given by (3.2) is a fundamental solution to the operator $L(y,\partial_x) := a(y)\Delta(\partial_x)$ with "frozen" coefficient $a(x) = a(y)$, i.e.,

$$L(y,\partial_x)\, P(x,y) = \delta(x-y).$$

Note that parametrix (3.2) used further in this paper does not belong to the class discussed in [**23**] since the reminder (3.3) does not belong to $C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$.

3.2. Surface potentials. Let us introduce the single and the double layer surface potential operators, corresponding to the parametrix (3.2),

$$Vg(y) := -\int_S P(x,y)\, g(x)\, dS_x, \quad y \notin S, \tag{3.4}$$

$$Wg(y) := -\int_S \big[T(x, n(x), \partial_x)\, P(x,y)\big] g(x)\, dS_x, \quad y \notin S, \tag{3.5}$$

where g is some scalar density function.

Let us introduce also the following boundary integral (pseudo-differential) operators:

$$\mathcal{V}\, g(y) := -\int_S P(x,y)\, g(x)\, dS_x, \tag{3.6}$$

$$\mathcal{W}\, g(y) := -\int_S \big[T(x, n(x), \partial_x)\, P(x,y)\big]\, g(x)\, dS_x, \tag{3.7}$$

$$\mathcal{W}'\, g(y) := -\int_S \big[T(y, n(y), \partial_y)\, P(x,y)\big]\, g(x)\, dS_x, \tag{3.8}$$

$$\mathcal{L}^{\pm} g(y) := T^{\pm} W g(y), \tag{3.9}$$

where $y \in S$.

From definitions (3.2), (3.4)–(3.9), one can obtain representations of the parametrix-based surface potential operators in terms of their counterparts for $a = 1$, i.e., associated with the Laplace operator Δ,

$$Vg = \frac{1}{a} V_\Delta g, \quad Wg = \frac{1}{a} W_\Delta (ag), \tag{3.10}$$

$$\mathcal{V}g = \frac{1}{a} \mathcal{V}_\Delta g, \quad \mathcal{W}g = \frac{1}{a} \mathcal{W}_\Delta (ag), \tag{3.11}$$

$$\mathcal{W}'g = \mathcal{W}'_\Delta g + \left[a \frac{\partial}{\partial n} \left(\frac{1}{a} \right) \right] \mathcal{V}_\Delta g, \tag{3.12}$$

$$\mathcal{L}^{\pm} g = \mathcal{L}_\Delta (ag) + \left[a \frac{\partial}{\partial n} \left(\frac{1}{a} \right) \right] W^{\pm}_\Delta (ag) \tag{3.13}$$

where the subscript Δ means that the corresponding surface potentials are constructed by means of the harmonic fundamental solution $P_\Delta =$

$-(4\pi\,|x-y|)^{-1}$. It is taken into account that a and its derivatives are continuous in $\mathbb{R}^3$ and $\mathcal{L}_{\Delta}(ag) := \mathcal{L}^{+}_{\Delta}(ag) = \mathcal{L}^{-}_{\Delta}(ag)$ by the Liapunov-Tauber theorem.

The mapping and jump properties of the potentials of type (3.4)–(3.9) are well studied nowadays (see, e.g., [**2, 11–14, 17, 18, 21, 23, 30, 33, 34, 36, 41**]; see also [**7, 9, 10, 31, 32, 37**], where the coerciveness properties of the boundary operators and also the case of Lipschitz domains are considered).

We provide below in this section some results on jump and mapping properties and invertibility of the parametrix-based potentials in the Sobolev spaces H^s, while the corresponding results (and references) for the Bessel-potential (H^s_p) and Besov spaces are left to the second part of the paper, [**4**].

Theorems 3.1–3.3 below are well known (see, e.g., the above references) for the case $a=$ const. Taking into account (3.10)–(3.13), one can easily prove they hold true also for the variable positive coefficient $a \in C^\infty(\mathbb{R})$.

Theorem 3.1. *Let $s \in \mathbb{R}$. The following operators are continuous*

$$V : H^s(S) \longrightarrow H^{s+\frac{3}{2}}(\Omega^+), \quad \left[H^s(S) \longrightarrow H^{s+\frac{3}{2}}_{\rm loc}(\overline{\Omega^-})\right],$$
$$W : H^s(S) \longrightarrow H^{s+\frac{1}{2}}(\Omega^+), \quad \left[H^s(S) \longrightarrow H^{s+\frac{1}{2}}_{\rm loc}(\overline{\Omega^-})\right].$$

Theorem 3.2. *Let $s \in \mathbb{R}$. The following pseudo-differential operators are continuous*

$$\mathcal{V} : H^s(S) \longrightarrow H^{s+1}(S),$$
$$\mathcal{W},\ \mathcal{W}' : H^s(S) \longrightarrow H^{s+1}(S),$$
$$\mathcal{L}^{\pm} : H^s(S) \longrightarrow H^{s-1}(S).$$

Theorem 3.3. *Let $g_1 \in H^{-1/2}(S)$, and $g_2 \in H^{1/2}(S)$. Then the following jump relations hold on S*:

$$[Vg_1(y)]^{\pm} = \mathcal{V}g_1(y) \tag{3.14}$$

$$[Wg_2(y)]^{\pm} = \mp\frac{1}{2}\,g_2(y) + \mathcal{W}g_2(y), \tag{3.15}$$

$$T^{\pm}Vg_1(y) = \pm\frac{1}{2}\,g_1(y) + \mathcal{W}'\,g_1(y), \tag{3.16}$$

where $y \in S$.

Theorem 3.4. *Let* $s \in \mathbb{R}$. *Let* S_1 *and* S_2 *with* $\partial S_1, \partial S_2 \in C^{\infty}$ *be nonempty open submanifolds of* S. *The operators*

$$r_{S_2}\mathcal{V} : \widetilde{H}^s(S_1) \longrightarrow H^s(S_2), \tag{3.17}$$

$$r_{S_2}\mathcal{W} : \widetilde{H}^s(S_1) \longrightarrow H^s(S_2), \tag{3.18}$$

$$r_{S_2}\mathcal{W}' : \widetilde{H}^s(S_1) \longrightarrow H^s(S_2) \tag{3.19}$$

are compact.

Proof. Theorem 3.2 implies that the operators $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{W}'$ have the following mapping properties:

$$r_{S_2}\mathcal{V} : \widetilde{H}^s(S_1) \longrightarrow H^{s+1}(S_2),$$
$$r_{S_2}\mathcal{W} : \widetilde{H}^s(S_1) \longrightarrow H^{s+1}(S_2),$$
$$r_{S_2}\mathcal{W}' : \widetilde{H}^s(S_1) \longrightarrow H^{s+1}(S_2).$$

Since the embedding $H^{s+1}(S_2) \subset H^s(S_2)$, is compact, the proof follows. □

Theorem 3.5. *Let* S_1 *be a nonempty, simply connected submanifold of* S *with infinitely smooth boundary curve, and* $0 < s < 1$. *Then the operator*

$$r_{S_1}\,\mathcal{V} : \widetilde{H}^{s-1}(S_1) \to H^s(S_1) \tag{3.20}$$

is invertible.

Proof. The lemma claim for operator $\mathcal{V}_{\Delta}$ is well known, (see [**38**, Theorem 2.7 (i)] for $s = 1/2$; for the other $s \in (0,1)$ it can be proved as in [**8**, Theorem 2.4] for the single layer potential of elasticity). Therefore, by the first equality of (3.11) the operator (3.20) is invertible too. See also [**4**, Theorem 3.5]. □

Theorem 3.6. *Let S_1 and $S\backslash\overline{S}_1$ be nonempty, open, simply connected submanifolds of S with an infinitely smooth boundary curve, and $0 < s < 1$. Then*

$$\mathcal{L}^+ + \frac{\partial a}{\partial n}\Big(-\frac{1}{2}\,I + \mathcal{W}\Big) = \mathcal{L}^- + \frac{\partial a}{\partial n}\Big(\frac{1}{2}\,I + \mathcal{W}\Big) \ \text{ on } S. \tag{3.21}$$

Moreover, the pseudo-differential operator

$$r_{S_1}\widehat{\mathcal{L}} : \widetilde{H}^s(S_1) \longrightarrow H^{s-1}(S_1) \tag{3.22}$$

where

$$\widehat{\mathcal{L}}g := \Big[\mathcal{L}^\pm + \frac{\partial a}{\partial n}\Big(\mp\frac{1}{2}\,I + \mathcal{W}\Big)\Big]\,g = \mathcal{L}^+_\Delta(ag) \quad \text{on } S, \tag{3.23}$$

is invertible, while the operators

$$r_{S_1}(\mathcal{L}^\pm - \widehat{\mathcal{L}}) : \widetilde{H}^s(S_1) \longrightarrow H^s(S_1) \tag{3.24}$$

are bounded and the operators

$$r_{S_1}(\mathcal{L}^\pm - \widehat{\mathcal{L}}) : \widetilde{H}^s(S_1) \longrightarrow H^{s-1}(S_1) \tag{3.25}$$

are compact.

Proof. By (3.13) and (3.15), we have

$$\mathcal{L}^\pm g = \mathcal{L}_\Delta(ag) - \frac{\partial a}{\partial n}\Big(\mp\frac{1}{2}\,g + \mathcal{W}\,g\Big).$$

Thus

$$\mathcal{L}_\Delta(ag) = \mathcal{L}^+ g + \frac{\partial a}{\partial n}\Big(-\frac{1}{2}\,I + \mathcal{W}\Big)\,g = \mathcal{L}^- g + \frac{\partial a}{\partial n}\Big(\frac{1}{2}\,I + \mathcal{W}\Big)\,g. \tag{3.26}$$

The operator $r_{S_1}\mathcal{L}_\Delta : \widetilde{H}^s(S_1) \to H^{s-1}(S_1)$ is invertible, see [**38**, Theorem 2.7 (ii)] for $s = 1/2$; for the other $s \in (0,1)$, it can be proved as in [**8**, Theorem 2.4] for the corresponding potential of elasticity, which proves the invertibility of operator (3.22). See also [**4**, Theorem 3.6].

Since

$$\mathcal{L}^\pm - \widehat{\mathcal{L}} = \frac{\partial a}{\partial n}\Big(\pm\frac{1}{2}\, I - \mathcal{W}\Big),$$

the operator (3.24) is bounded due to Theorem 3.2. To prove the compactness of $\mathcal{L}^\pm - \widehat{\mathcal{L}}$, we remark that the imbedding $H^{1/2}(S_1) \subset H^{-1/2}(S_1)$ is compact, which completes the proof. □

Remark 3.7. The analog of Theorem 3.5 for the whole surface S holds true as well, i.e., $\mathcal{V} : H^{s-1}(S) \to H^s(S)$ are invertible for all $s \in \mathbb{R}$ due to, e.g., [**11**, Chapter XI, Part B, Section 2, Remark 1], and the first equality of (3.11).

Generalizations of invertibility results from Theorems 3.5, 3.6 and Remark 3.7 to some ranges of the Bessel-potential (H^s_p) and Besov spaces are available in the second part of this paper, [**4**].

3.3. Volume potentials. Let us introduce the parametrix-based Newtonian and remainder volume potential operators,

$$\mathcal{P}g(y) := \int_{\Omega^+} P(x,y)\, g(x)\, dx, \tag{3.27}$$

$$\mathcal{R}g(y) := \int_{\Omega^+} R(x,y)\, g(x)\, dx. \tag{3.28}$$

Expressions (3.2) and (3.3) imply that

$$\mathcal{P}\, g = \frac{1}{a}\mathcal{P}_\Delta\, g, \tag{3.29}$$

$$\mathcal{R}\, g = -\frac{1}{a}\sum_{j=1}^{3} \partial_j \Big[\mathcal{P}_\Delta\, \big(g\, \partial_j a\big)\Big], \tag{3.30}$$

where

$$\mathcal{P}_\Delta\, h\,(y) := -\frac{1}{4\pi}\int_{\Omega^+} \frac{1}{|x-y|}\, h(x)\, dx \tag{3.31}$$

is the Newtonian potential corresponding to the fundamental solution of the Laplace operator Δ.

Theorem 3.8. *Let Ω^+ be a bounded, open three-dimensional region of $\mathbb{R}^3$ with a simply connected, closed, infinitely smooth boundary $S = \partial\Omega^+$. The following operators are continuous*

$$\mathcal{P} : \widetilde{H}^s(\Omega^+) \longrightarrow H^{s+2}(\Omega^+), \qquad s \in \mathbb{R}, \tag{3.32}$$

$$: H^s(\Omega^+) \longrightarrow H^{s+2}(\Omega^+), \qquad s > -\frac{1}{2}; \tag{3.33}$$

$$\mathcal{R} : \widetilde{H}^s(\Omega^+) \longrightarrow H^{s+1}(\Omega^+), \qquad s \in \mathbb{R}, \tag{3.34}$$

$$: H^s(\Omega^+) \longrightarrow H^{s+1}(\Omega^+), \qquad s > -\frac{1}{2}; \tag{3.35}$$

$$\mathcal{P}^+ : \widetilde{H}^s(\Omega^+) \longrightarrow H^{s+\frac{3}{2}}(S), \qquad s > -\frac{3}{2}, \tag{3.36}$$

$$: H^s(\Omega^+) \longrightarrow H^{s+\frac{3}{2}}(S), \qquad s > -\frac{1}{2}; \tag{3.37}$$

$$\mathcal{R}^+ : \widetilde{H}^s(\Omega^+) \longrightarrow H^{s+\frac{1}{2}}(S), \qquad s > -\frac{1}{2}, \tag{3.38}$$

$$: H^s(\Omega^+) \longrightarrow H^{s+\frac{1}{2}}(S), \qquad s > -\frac{1}{2}; \tag{3.39}$$

$$T^+\mathcal{P} : \widetilde{H}^s(\Omega^+) \longrightarrow H^{s+\frac{1}{2}}(S), \qquad s > -\frac{1}{2}, \tag{3.40}$$

$$: H^s(\Omega^+) \longrightarrow H^{s+\frac{1}{2}}(S), \qquad s > -\frac{1}{2}; \tag{3.41}$$

$$T^+\mathcal{R} : \widetilde{H}^s(\Omega^+) \longrightarrow H^{s-\frac{1}{2}}(S), \qquad s > \frac{1}{2}, \tag{3.42}$$

$$: H^s(\Omega^+) \longrightarrow H^{s-\frac{1}{2}}(S), \qquad s > \frac{1}{2}. \tag{3.43}$$

Proof. Since $S \in C^\infty$, the Newtonian potential $\mathcal{P}_\Delta$ is a pseudodifferential operator of order -2 from "tilde" spaces. Therefore, the continuity of the operators (3.32), (3.34), (3.36), (3.38), (3.40) and (3.42) follows due to the mapping properties of pseudodifferential operators on $\mathbb{R}^n$ (see, e.g., [**14, 29**]) and the trace theorems (see, e.g., [**40**]) along with the formulae (3.29), (3.30) and (3.31).

To prove the remaining items of the theorem we first assume that $-1/2 < s < 1/2$. In this case $H^s(\Omega^+) = \widetilde{H}^s(\Omega^+)$, and the continuity of operator (3.33) is evident due to (3.32).

Now let $g \in H^s(\Omega^+)$ with $1/2 < s < 3/2$. Clearly, $\partial_j g \in H^{s-1}(\Omega^+)$ and $g^+ \in H^{s-\frac{1}{2}}(S)$, due to the continuity of the operator $\partial_j : H^s(\Omega^+) \to H^{s-1}(\Omega^+)$ and the trace theorem. Then, integrating by parts, we have the representation

$$\partial_j \mathcal{P}_\Delta g(y) = \mathcal{P}_\Delta \big(\partial_j g\big)(y) + V_\Delta \left(n_j g^+\right)(y) \quad \text{for } y \in \Omega^+, \tag{3.44}$$

where n_j, $j = 1, 2, 3$, are the components of the outward unit normal vector to S. Due to (3.44) and the mapping properties of the single layer potential (cf., Theorem 3.1) we conclude that $\partial_j \mathcal{P}_\Delta : H^s(\Omega^+) \to H^{s+1}(\Omega^+)$ is continuous for $j = 1, 2, 3$, which along with formulae (3.29) and (3.31) implies the continuity of operator (3.33) for $1/2 < s < 3/2$.

Further, with the help of these results and the representation (3.44), we can easily verify by induction that the operator (3.33) is continuous for $k - 1/2 < s < k + 1/2$, where k is an arbitrary nonnegative integer. For the values $s = k + 1/2$ (with $k = 0, 1, 2, \dots$) the continuity of the operator (3.33) then follows due to the complex interpolation property of Bessel potential spaces, see e.g., [**40**, Chapter 4].

It is evident that (3.37) and (3.41) are then the direct consequences of the trace theorem.

The word for word arguments show that the claims of the theorem concerning the operator $\mathcal{R}$ hold as well, which completes the proof. □

From Theorem 3.8 and Rellich compact imbedding theorem we have the following assertion.

Corollary 3.9. *The operators*

$$\mathcal{R} : H^s(\Omega^+) \longrightarrow H^s(\Omega^+), \tag{3.45}$$

$$r_{S_1} \mathcal{R}^+ : H^s(\Omega^+) \longrightarrow H^{s-\frac{1}{2}}(S_1), \tag{3.46}$$

$$r_{S_1} T^+ \mathcal{R} : H^s(\Omega^+) \longrightarrow H^{s-\frac{3}{2}}(S_1), \tag{3.47}$$

are compact for any $s > 1/2$ and any nonempty, open submanifold S_1 of S with an infinitely smooth boundary curve.

4. Green identities and integral relations. Subtracting from (2.8) its counterpart with exchanged roles of u and v, we arrive at the second Green identity for the operator $L(x, \partial_x)$,

$$\int_{\Omega^+} \big[v\, L(x, \partial_x)u - u\, L(x, \partial_x)v\big]\, dx = \big\langle T^+u,\, v^+\big\rangle_S - \big\langle T^+v,\, u^+\big\rangle_S\,, \tag{4.1}$$

where $u, v \in H^{1,0}(\Omega^+; L)$ are real functions.

For $v(x) := P(x, y)$ and $u \in H^{1,0}(\Omega^+; L)$, we obtain from (4.1) and (3.1) by the standard limiting procedures (cf., [**30**]) the third Green identity,

$$u + \mathcal{R}u - VT^+u + Wu^+ = \mathcal{P}Lu \quad \text{in } \Omega^+. \tag{4.2}$$

If $u \in H^{1,0}(\Omega^+; L)$ is a solution of equation (2.1), then (4.2) gives

$$u + \mathcal{R}u - VT^+u + Wu^+ = \mathcal{P}f \quad \text{in } \Omega^+, \tag{4.3}$$

$$\mathcal{G}u := \frac{1}{2}u^+ + \mathcal{R}^+u - \mathcal{V}T^+u + \mathcal{W}u^+ = [\mathcal{P}f]^+ \quad \text{on } S, \tag{4.4}$$

$$\mathcal{T}u := \frac{1}{2}T^+u + T^+\mathcal{R}u - \mathcal{W}'T^+u + \mathcal{L}^+u^+ = T^+\mathcal{P}f \quad \text{on } S. \tag{4.5}$$

For some functions f, Ψ and Φ, let us consider a more general "indirect" integral relation, associated with (4.3),

$$u(y) + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}f \quad \text{in } \Omega^+. \tag{4.6}$$

Lemma 4.1. *Let $u \in H^1(\Omega^+)$, $f \in L_2(\Omega^+)$, $\Psi \in H^{-1/2}(S)$, $\Phi \in H^{1/2}(S)$ satisfy* (4.6). *Then u belongs to $H^{1,0}(\Omega^+; L)$ and is a solution of* PDE (2.5) *in Ω^+, and*

$$V(\Psi - T^+u)(y) - W(\Phi - u^+)(y) = 0, \quad y \in \Omega^+. \tag{4.7}$$

Proof. First of all, let us prove that $u \in H^{1,0}(\Omega^+; L)$. Indeed, since

$$Lu = \Delta(au) - \sum \partial_i(u\partial_i a),$$

and the last term belongs to $L_2(\Omega^+)$, we need only to show that $\Delta(au) \in L_2(\Omega^+)$. Further, from (4.6) due to (3.11) and (3.29) we have,

$$au = a\mathcal{P}f - a\mathcal{R}u + aV\Psi - aW\Phi = \mathcal{P}_\Delta f - a\mathcal{R}u + V_\Delta\Psi - W_\Delta(a\Phi).$$

Note that the last two terms in the righthand side are harmonic functions. Also $\mathcal{R}u \in H^2(\Omega)$ for $u \in H^1(\Omega)$ and $\Delta[\mathcal{P}_\Delta(f)] = f \in L_2(\Omega^+)$. Therefore, $Lu \in L_2(\Omega^+)$. So, $u \in H^{1,0}(\Omega^+; L)$ and we can write the third Green identity (4.2) for the function u.

Subtracting (4.6) from identity (4.2), we obtain

$$-V\Psi^* + W\Phi^* = \mathcal{P}[Lu - f] \quad \text{in } \Omega^+, \tag{4.8}$$

where $\Psi^* := T^+u - \Psi$ and $\Phi^* := u^+ - \Phi$. Multiplying equality (4.8) by $a(y)$, we get

$$-V_\Delta\Psi^* + W_\Delta(a\Phi^*) = \mathcal{P}_\Delta[Lu - f], \quad \text{in } \Omega^+. \tag{4.9}$$

Applying the Laplace operator Δ to the last equation and taking into consideration that both functions in the lefthand side are harmonic surface potentials, while the righthand side function is the classical Newtonian volume potential, we arrive at the equation

$$Lu - f = 0 \quad \text{in } \Omega^+. \tag{4.10}$$

This shows that u solves differential equation (2.5).

Substituting (4.10) into (4.8) leads to (4.7). □

Lemma 4.2. (i) *Let* $\Psi^* \in H^{-1/2}(S)$. *If*

$$V\Psi^*(y) = 0, \quad y \in \Omega^+, \tag{4.11}$$

then $\Psi^* = 0$.

(ii) *Let* $\Phi^* \in H^{1/2}(S)$. *If*

$$W\Phi^*(y) = 0, \quad y \in \Omega^+, \tag{4.12}$$

then $\Phi^* = 0$.

(iii) *Let $S = \overline{S}_1 \cup \overline{S}_2$, where S_1 and S_2 are nonintersecting simply connected submanifolds of S with infinitely smooth boundaries and S_1 is nonempty. Let $\Psi^* \in \widetilde{H}^{-1/2}(S_1)$, $\Phi^* \in \widetilde{H}^{1/2}(S_2)$. If*

$$V\Psi^*(y) - W\Phi^*(y) = 0, \quad y \in \Omega^+, \tag{4.13}$$

then $\Psi^* = 0$ and $\Phi^* = 0$ on S.

Proof. To prove (i), let us take the trace of (4.11) on S and use (3.14). Then Remark 3.7 implies point (i).

Due to (3.15), the trace of condition (4.12) gives $-(1/2)\Phi^* + \mathcal{W}\Phi^* = 0$ or, due to the second equation of (3.11), $-(1/2)\widehat{\Phi}^* + \mathcal{W}_\Delta \widehat{\Phi}^* = 0$ on S, where $\widehat{\Phi}^* = a\Phi^*$. Since this equation for $\widehat{\Phi}^*$ is uniquely solvable, see e.g., [**11**, Chapter XI, Part B, § 2, Remark 8] and $a(y) \neq 0$, this implies point (ii).

Let us now prove point (iii). Multiplying equation (4.13) by $a(y)$, we have

$$V_\Delta \Psi^* - W_\Delta(a\Phi^*) = 0, \quad \text{in } \Omega^+.$$

Take the traces of this equation and its normal derivative on S_1 and S_2, respectively, to obtain

$$\begin{cases} r_{S_1}\mathcal{V}_\Delta \Psi^* - r_{S_1}\mathcal{W}_\Delta \widehat{\Phi}^* = 0 & \text{on } S_1, \\ r_{S_2}\mathcal{W}'_\Delta \Psi^* - r_{S_2}\mathcal{L}^+_\Delta \widehat{\Phi}^* = 0 & \text{on } S_2, \end{cases} \tag{4.14}$$

where $\widehat{\Phi}^* = a\Phi^*$. We put

$$\mathcal{A}_\Delta := \begin{bmatrix} r_{S_1}\mathcal{V}_\Delta & -r_{S_1}\mathcal{W}_\Delta \\ r_{S_2}\mathcal{W}'{}_\Delta & -r_{S_2}\mathcal{L}^+_\Delta \end{bmatrix}, \qquad \chi = \begin{bmatrix} \Psi^* \\ \widehat{\Phi}^* \end{bmatrix}.$$

Equation (4.14) then can be written as

$$\mathcal{A}_\Delta \chi = 0. \tag{4.15}$$

It is well known that the operators

$$r_{S_1}\mathcal{V}_\Delta : \widetilde{H}^{-\frac{1}{2}}(S_1) \longrightarrow H^{\frac{1}{2}}(S_1), \ -r_{S_2}\mathcal{L}^+_\Delta : \widetilde{H}^{\frac{1}{2}}(S_2) \longrightarrow H^{-\frac{1}{2}}(S_2)$$

are positive definite in the following sense,

$$(4.16)\qquad \begin{aligned} \langle r_{S_1}\mathcal{V}_\Delta\Psi^*, \Psi^*\rangle_{S_1} &\geq c\,\|\Psi^*\|^2_{H^{-\frac{1}{2}}(S)}, \\ \left\langle -r_{S_2}\mathcal{L}^+_\Delta\widehat{\Phi}^*, \widehat{\Phi}^*\right\rangle_{S_2} &\geq c\,\|\widehat{\Phi}^*\|^2_{H^{\frac{1}{2}}(S)} \end{aligned}$$

for arbitrary $\Psi^* \in \widetilde{H}^{-1/2}(S_1)$ and arbitrary $\widehat{\Phi}^* \in \widetilde{H}^{1/2}(S_2)$. For the first estimate (4.16), see e.g., [**11**, Chapter XI, Part B, § 2, Theorem 3]. The second estimate (4.16) can be proved by modifying the proof of [**11**, Chapter XI, Part B, § 2, Theorem 6] for nonclosed surfaces, see also [**33**].

In addition, the operators

$$\begin{aligned} r_{S_1}\mathcal{W}_\Delta &: \widetilde{H}^{\frac{1}{2}}(S_2) \longrightarrow H^{\frac{1}{2}}(S_1), \\ r_{S_2}\mathcal{W}'_\Delta &: \widetilde{H}^{-\frac{1}{2}}(S_1) \longrightarrow H^{-\frac{1}{2}}(S_2) \end{aligned}$$

are mutually adjoint, i.e., $\langle r_{S_1}\mathcal{W}_\Delta\widehat{\Phi}^*, \Psi^*\rangle_{S_1} = \langle\widehat{\Phi}^*, r_{S_2}\mathcal{W}'_\Delta\Psi^*\rangle_{S_2}$ for arbitrary $\Psi^* \in \widetilde{H}^{-1/2}(S_1)$ and arbitrary $\widehat{\Phi}^* \in \widetilde{H}^{1/2}(S_2)$. Consequently, we derive the inequality

$$\langle\mathcal{A}_\Delta\chi,\, \chi\rangle \geq c\left(\|\Psi^*\|^2_{H^{-\frac{1}{2}}(S)} + \|\widehat{\Phi}^*\|^2_{H^{\frac{1}{2}}(S)}\right).$$

Due to (4.15), this implies $\Psi^* = 0$ and $\widehat{\Phi}^* = 0$. Keeping in mind that $a(y) \neq 0$, we have $\Phi^* = 0$ on S, which completes the proof for point (iii). □

5. Boundary-domain integral equations. Let $\Phi_0 \in H^{1/2}(S)$ be a fixed extension of the given function φ_0 from the submanifold S_D to the whole of S (see Dirichlet condition (2.6)). An arbitrary extension $\Phi \in H^{1/2}(S)$ preserving the function space can then be represented as $\Phi = \Phi_0 + \varphi$ with $\varphi \in \widetilde{H}^{1/2}(S_N)$.

Analogously, let $\Psi_0 \in H^{-1/2}(S)$ be a fixed extension of the given function ψ_0 from the submanifold S_N to the whole of S (see the Neumann condition (2.7)). An arbitrary extension $\Psi \in H^{-1/2}(S)$ preserving the function space can then be represented as $\Psi = \Psi_0 + \psi$ with $\psi \in \widetilde{H}^{-1/2}(S_D)$.

If $\psi_0 \in H^{-1/2}(S_N)$ or $\varphi_0 \in H^{1/2}(S_D)$ admits the canonical extension, i.e., can be extended onto the whole boundary S by *zero* preserving the spaces, then evidently one may choose $\Psi_0 \in \widetilde{H}^{-1/2}(S_N)$ or $\Phi_0 \in \widetilde{H}^{1/2}(S_D)$, respectively.

A way of reducing BVP (2.5)–(2.7) to boundary-domain *integral* equations is to substitute the Neumann and Dirichlet boundary conditions (2.6) and (2.7) into (4.3) and either into its trace (4.4) or into its conormal derivative (4.5) on S_D and S_N. To work with the boundary functions defined on the whole boundary S, we will also replace the boundary trace u^+ and the conormal derivative T^+u by new functions $\Phi = \Phi_0 + \varphi$ and $\Psi = \Psi_0 + \psi$, respectively, formally segregated from u. Here $\Phi_0 \in H^{1/2}(S)$ and $\Psi_0 \in H^{-1/2}(S)$ are some of the above-mentioned extensions of the given functions φ_0 from S_D to S and ψ_0 from S_N to S, respectively, while $\varphi \in \widetilde{H}^{1/2}(S_N)$, $\psi \in \widetilde{H}^{-1/2}(S_D)$.

5.1. Boundary-domain integral equation system ($\mathcal{GT}$). Different possibilities exist of reducing BVP (2.5)–(2.7) to a BDIE system. In this subsection we will use equation (4.3) in Ω^+, the restriction of equation (4.4) on S_D, and the restriction of equation (4.5) on S_N, where $\Phi_0 + \varphi$ is substituted for u^+ and $\Psi_0 + \psi$ for T^+u (cf. [**39, 42**] where boundary integral equations of the mixed BVP with the *constant* coefficient a were considered). Then we arrive at the following system with respect to the unknowns u, ψ, and φ:

$$(5.1)\qquad u + \mathcal{R}u - V\psi + W\varphi = F_0 \qquad \text{in } \Omega^+,$$

$$(5.2)\qquad r_{S_D}\,\mathcal{R}^+u - r_{S_D}\mathcal{V}\psi + r_{S_D}\mathcal{W}\varphi = r_{S_D}\,F_0^+ - \varphi_0 \qquad \text{on } S_D,$$

$$(5.3)\qquad r_{S_N}\,T^+\mathcal{R}u - r_{S_N}\mathcal{W}'\psi + r_{S_N}\mathcal{L}^+\varphi = r_{S_N}T^+F_0 - \psi_0 \qquad \text{on } S_N,$$

where

$$(5.4)\qquad F_0 := \mathcal{P}f + V\Psi_0 - W\Phi_0 \quad \text{in } \Omega^+.$$

Note that, for $f \in L_2(\Omega^+)$, $\Psi_0 \in H^{-1/2}(S)$ and $\Phi_0 \in H^{1/2}(S)$, we have the inclusion $F_0 \in H^1(\Omega^+)$ due to mapping properties of the Newtonian (volume) and layer potentials (cf. Theorems 3.1 and 3.10).

Note also that, due to Lemma 4.1, all terms of equation (5.1) belong to $H^{1,0}(\Omega; L)$ and their conormal derivatives are well defined. This fact will be used in the analysis of the BDIE system (5.1)–(5.3) as well as all the other BDIE systems below.

The second and third equations of the system are associated with operator $\mathcal{G}$ on S_D and with operator $\mathcal{T}$ on S_N, respectively.

Let us denote the righthand side of BDIES (5.1)–(5.3) by

$$\mathcal{F}^{\mathcal{G}\mathcal{T}} := [\, F_0, \quad r_{S_D} F_0^+ - \varphi_0, \quad r_{S_N} T^+ F_0 - \psi_0 \,]^\top, \tag{5.5}$$

where the superscript $^\top$ denotes transposition.

Remark 5.1. $\mathcal{F}^{\mathcal{G}\mathcal{T}} = 0$ if and only if $(f,\ \Phi_0,\ \Psi_0) = 0$. Indeed, the latter equality evidently implies the former. Inversely, let $\mathcal{F}^{\mathcal{G}\mathcal{T}} = 0$. Keeping in mind equation (5.4), Lemma 4.1 with $F_0 = 0$ for u implies $f = 0$ and $V\Psi_0 - W\Phi_0 = 0$ in Ω^+. The equalities $\mathcal{F}_2^{\mathcal{G}\mathcal{T}} = 0$ and $\mathcal{F}_3^{\mathcal{G}\mathcal{T}} = 0$ imply $\varphi_0 = 0$ on S_D and $\psi_0 = 0$ on S_N, that is, $\Phi_0 \in \widetilde{H}^{1/2}(S_N)$, $\Psi_0 \in \widetilde{H}^{-1/2}(S_D)$. Lemma 4.2 (iii) then gives $\Phi_0 = 0$ and $\Psi_0 = 0$ on S.

Let us prove that BVP (2.5)–(2.7) in Ω^+, is equivalent to the system of BDIEs (5.1)–(5.3).

Theorem 5.2. *Let $f \in L_2(\Omega^+)$, and let $\Phi_0 \in H^{1/2}(S)$ and $\Psi_0 \in H^{-1/2}(S)$ be some extensions of $\varphi_0 \in H^{1/2}(S_D)$ and $\psi_0 \in H^{-1/2}(S_N)$, respectively.*

(i) *If some $u \in H^1(\Omega^+)$ solves the mixed* BVP (2.5)–(2.7) *in Ω^+, then the solution is unique and the triple $(u, \psi, \varphi) \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$, where*

$$\psi = T^+ u - \Psi_0, \qquad \varphi = u^+ - \Phi_0 \quad \text{on } S, \tag{5.6}$$

solves BDIE *system* (5.1)–(5.3)*.*

(ii) *If a triple $(u, \psi, \varphi) \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ solves* BDIE *system* (5.1)–(5.3)*, then the solution is unique, u solves* BVP (2.5)–(2.7)*, and equations* (5.6) *hold.*

Proof. Let $u \in H^1(\Omega^+)$ be a solution to BVP (2.5)–(2.7). It is unique due to Theorem 2.1. Set $\psi := T^+ u - \Psi_0$ and $\varphi := u^+ - \Phi_0$. Evidently,

$\psi \in \widetilde{H}^{-1/2}(S_D)$ and $\varphi \in \widetilde{H}^{1/2}(S_N)$. Then it immediately follows from relations (4.3)–(4.5) that the triple (u, ψ, φ) solves system (5.1)–(5.3), which completes the proof of item (i).

Now let a triple $(u, \psi, \varphi) \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ solve BDIE system (5.1)–(5.3). Taking the trace of equation (5.1) on S_D using (3.14) and (3.15), and subtracting equation (5.2) from it, we obtain

$$r_{S_D} u^+ = \varphi_0 \quad \text{on } S_D, \tag{5.7}$$

i.e., u satisfies Dirichlet condition (2.6). Taking the conormal derivative of equation (5.1) on S_N using (3.9), (3.16), and subtracting equation (5.3) from it, we obtain

$$r_{S_N} T^+ u = \psi_0 \quad \text{on } S_N, \tag{5.8}$$

i.e., u satisfies the Neumann condition (2.7). Taking into account $\varphi = 0$, $\Phi_0 = \varphi_0$ on S_D and $\psi = 0$, $\Psi_0 = \psi_0$ on S_N, equations (5.7) and (5.8) imply that the first equation (5.6) is satisfied on S_N and the second equation (5.6) is satisfied on S_D.

Equation (5.1) and Lemma 4.1 with $\Psi = \psi + \Psi_0$ and $\Phi = \varphi + \Phi_0$ imply that u is a solution of PDE (2.5) and

$$V\Psi^* - W\Phi^* = 0 \quad \text{in } \Omega^+,$$

where $\Psi^* = \Psi_0 + \psi - T^+ u$ and $\Phi^* = \Phi_0 + \varphi - u^+$. Due to the first equation (5.6) on S_N and the second equation (5.6) on S_D, already proved, we have $\Psi^* \in \widetilde{H}^{-1/2}(S_D)$, $\Phi^* \in \widetilde{H}^{1/2}(S_N)$. Lemma 4.2 (iii) with $S_1 = S_D$ and $S_2 = S_N$ implies $\Psi^* = \Phi^* = 0$, which completes the proof of conditions (5.6).

Uniqueness of the solution to the BDIE system (5.1)–(5.3) follows from (5.6) along with Remark 5.1 and Theorem 2.1. □

System (5.1)–(5.3) can be rewritten in the form

$$\mathcal{A}^{\mathcal{G}\mathcal{T}} \mathcal{U} = \mathcal{F}^{\mathcal{G}\mathcal{T}}, \tag{5.9}$$

where $\mathcal{U}^\top := (u, \psi, \varphi) \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$,

$$\mathcal{A}^{\mathcal{GT}} := \begin{bmatrix} I+\mathcal{R} & -V & W \\ r_{S_D}\mathcal{R}^+ & -r_{S_D}\mathcal{V} & r_{S_D}\mathcal{W} \\ r_{S_N}T^+\mathcal{R} & -r_{S_N}\mathcal{W}' & r_{S_N}\mathcal{L}^+ \end{bmatrix}, \tag{5.10}$$

and $\mathcal{F}^{\mathcal{GT}}$ is defined by (5.5).

Due to the mapping properties of operators V, $\mathcal{V}$, W, $\mathcal{W}$, $\mathcal{P}$, $\mathcal{R}$, $\mathcal{R}^+$ and $T^+\mathcal{R}$ described in Section 3, we have $\mathcal{F}^{\mathcal{GT}} \in H^1(\Omega^+)\times H^{1/2}(S_D)\times H^{-1/2}(S_N)$.

Theorem 5.3. *The operators*

$$\begin{aligned}\mathcal{A}^{\mathcal{GT}} &: H^1_2(\Omega^+)\times \widetilde{H}^{-\frac12}(S_D)\times \widetilde{H}^{\frac12}(S_N)\\ &\longrightarrow H^1(\Omega^+)\times H^{\frac12}(S_D)\times H^{-\frac12}(S_N),\end{aligned} \tag{5.11}$$

$$\begin{aligned}\mathcal{A}^{\mathcal{GT}} &: H^{1,0}_2(\Omega^+;L)\times \widetilde{H}^{-\frac12}(S_D)\times \widetilde{H}^{\frac12}(S_N)\\ &\longrightarrow H^{1,0}(\Omega^+;L)\times H^{\frac12}(S_D)\times H^{-1/2}(S_N)\end{aligned} \tag{5.12}$$

are invertible.

Proof. Let us consider the following operator

$$\mathcal{A}^{\mathcal{GT}}_0 := \begin{bmatrix} I & -V & W \\ 0 & -r_{S_D}\mathcal{V} & 0 \\ 0 & 0 & r_{S_N}\widehat{\mathcal{L}} \end{bmatrix} \tag{5.13}$$

where $\widehat{\mathcal{L}}$ is given by (3.23). As a result of Corollary 3.9 and Theorems 3.4, 3.6, the operator $\mathcal{A}^{\mathcal{GT}}_0$ is a compact perturbation of operator (5.11).

Due to Theorems 3.5 and 3.6 for $\mathcal{V}$ and $\widehat{\mathcal{L}}$, the operator $\mathcal{A}^{\mathcal{GT}}_0$ is an upper triangular matrix operator with the following scalar diagonal invertible operators

$$\begin{aligned} I &: H^1(\Omega^+)\longrightarrow H^1(\Omega^+),\\ r_{S_D}\mathcal{V} &: \widetilde{H}^{-\frac12}(S_D)\longrightarrow H^{\frac12}(S_D),\\ r_{S_N}\widehat{\mathcal{L}} &: \widetilde{H}^{\frac12}(S_N)\longrightarrow H^{-\frac12}(S_N).\end{aligned}$$

Along with the mapping properties of the operators V and W this implies that

$$\mathcal{A}_0^{\mathcal{GT}} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega^+) \times H^{\frac{1}{2}}(S_D) \times H^{-\frac{1}{2}}(S_N)$$

is an invertible operator. Whence it follows that operator (5.11) possesses the Fredholm property and its index is zero.

Uniqueness of solution to system (5.1)–(5.3), provided by Theorem 5.2 (ii), yields that system (5.9) with the zero righthand side has only the trivial solution, which completes the proof for operator (5.11).

To prove invertibility of operator (5.12), we remark that for any $\mathcal{F}^{\mathcal{GT}} \in H^1(\Omega^+) \times H^{1/2}(S_D) \times H^{-1/2}(S_N)$ a solution of system (5.9) can be written as $\mathcal{U} = [\mathcal{A}^{\mathcal{GT}}]^{-1}\mathcal{F}^{\mathcal{GT}}$, where $[\mathcal{A}^{\mathcal{GT}}]^{-1}$ is the continuous inverse operator to operator (5.11). But due to Lemma 4.11 the first equation of system (5.9) implies that $\mathcal{U} = [\mathcal{A}^{\mathcal{GT}}]^{-1}\mathcal{F}^{\mathcal{GT}} \in H^{1,0}(\Omega^+; L) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ if $\mathcal{F}^{\mathcal{GT}} \in H^{1,0}(\Omega^+; L) \times H^{1/2}(S_D) \times H^{-1/2}(S_N)$, and operator $[\mathcal{A}^{\mathcal{GT}}]^{-1}$ is a continuous inverse to (5.12) as well. □

The invertibility of operator $\mathcal{A}$ and Theorem 5.2 lead to the following

Corollary 5.4. *Under the conditions of Theorem* 5.2, *the mixed boundary value problem* (2.5)–(2.7) *as well as* BDIES (5.1)–(5.3) *are uniquely solvable.*

5.2. Boundary-domain integral equation system $(\mathcal{GG})$**.** To obtain the BDIE system $(\mathcal{GG})$ we will use equation (4.3) in Ω^+ and equation (4.4), associated with operator $\mathcal{G}$ on the whole boundary S. After substitution, $\Phi_0 + \varphi$ for u^+ and $\Psi_0 + \psi$ for T^+u, we arrive at the following system of BDIEs $(\mathcal{GG})$,

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \qquad \text{in } \Omega^+, \tag{5.14}$$

$$\frac{1}{2}\varphi + \mathcal{R}^+u - \mathcal{V}\psi(y) + \mathcal{W}\varphi = F_0^+ - \Phi_0 \quad \text{on } S, \tag{5.15}$$

where F_0 is given by (5.4).

Remark 5.5. It is easy to see that $(F_0, F_0^+ - \Phi_0) = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$. Indeed, the latter equality evidently implies the former. Inversely, let $(F_0, F_0^+ - \Phi_0) = 0$. Keeping in mind equation (5.4), Lemma 4.1 with $F_0 = 0$ for u implies $f = 0$ and $V\Psi_0 - W\Phi_0 = 0$ in Ω^+. The equality $F_0^+ - \Phi_0 = 0$ implies $\Phi_0 = 0$ on S. Lemma 4.2 (i) then gives $\Psi_0 = 0$ on S.

Further, we show that system (5.14)–(5.15) is equivalent to the BVP (2.5)–(2.7).

Theorem 5.6. *Let $f \in L_2(\Omega^+)$, and let $\Phi_0 \in H^{1/2}(S)$ and $\Psi_0 \in H^{-1/2}(S)$ be some fixed extensions of $\varphi_0 \in H^{1/2}(S_D)$ and $\psi_0 \in H^{-1/2}(S_N)$, respectively.*

(i) *If some $u \in H^1(\Omega^+)$ solves* BVP (2.5)–(2.7) *in Ω^+, then the triple $(u, \psi, \varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$, where*

$$\psi = T^+u - \Psi_0 \text{ and } \varphi = u^+ - \Phi_0 \quad \text{on } S, \tag{5.16}$$

solves BDIE *system* (5.14)–(5.15).

(ii) *If a triple $(u, \psi, \varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ solves* BDIE *system* (5.14)–(5.15), *then u solves* BVP (2.5)–(2.7), *and ψ and φ satisfy* (5.16). *Moreover,* BDIE *system* (5.14)–(5.15) *is uniquely solvable.*

Proof. Let $u \in H^1(\Omega^+)$ be a solution to BVP (2.5)–(2.7). Set $\psi := T^+u - \Psi_0$ and $\varphi := u^+ - \Phi_0$. Evidently, $\psi \in \widetilde{H}^{-1/2}(S_D)$, $\varphi \in \widetilde{H}^{1/2}(S_N)$. Then it immediately follows from relations (4.3) and (4.4) that the triple (u, ψ, φ) solves system (5.14)–(5.15), which completes the proof of item (i).

Now let a triple $(u, \psi, \varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ solve BDIE system (5.14)–(5.15). Using the properties of single- and double-layer potentials, take the trace of equation (5.14) on S and subtract it from equation (5.15) to obtain

$$u^+ = \Phi_0 + \varphi \quad \text{on } S. \tag{5.17}$$

This means that the second equation in (5.16) holds. Since $\varphi(y) = 0$ on S_D and $\Phi_0 = \varphi_0$ on S_D, we see that the Dirichlet condition (2.6) is satisfied.

Equation (5.14) and Lemma 4.1 with $\Psi = \psi + \Psi_0$ and $\Phi = \psi + \Psi_0$ imply that u is a solution of equation (2.5) and

$$V(\Psi_0 + \psi - T^+u) - W(\Phi_0 + \varphi - u^+) = 0 \quad \text{in } \Omega^+. \tag{5.18}$$

Due to (5.17), the second term vanishes in (5.18) and, by Lemma 4.2 (i), we obtain

$$\Psi_0 + \psi - T^+u = 0 \quad \text{on } S, \tag{5.19}$$

i.e., the first equation (5.16) is satisfied as well. Since $\psi = 0$ on S_N and $\Psi_0 = \psi_0$ on S_N, equation (5.19) implies that u satisfies the Neumann boundary condition (2.7).

Unique solvability of BDIE system (5.14)–(5.15) then follows from Remark 5.5, the unique solvability of the BVP (see Theorem 2.1) and (5.16). □

System (5.14)–(5.15) can be rewritten in the form

$$\mathcal{A}^{\mathcal{G}\mathcal{G}}\mathcal{U} = \mathcal{F}^{\mathcal{G}\mathcal{G}}, \tag{5.20}$$

where $\mathcal{U}^\top := (u, \psi, \varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$,

$$\mathcal{A}^{\mathcal{G}\mathcal{G}} := \begin{bmatrix} I + \mathcal{R} & -V & W \\ \mathcal{R}^+ & -\mathcal{V} & \frac{1}{2}I + \mathcal{W} \end{bmatrix}, \qquad \mathcal{F}^{\mathcal{G}\mathcal{G}} := \begin{bmatrix} F_0 \\ F_0^+ - \Phi_0 \end{bmatrix}. \tag{5.21}$$

Due to the mapping properties of the operators involved in (5.21), see Section 3, we have $\mathcal{F}^{\mathcal{G}\mathcal{G}} \in H^1(\Omega^+) \times H^{1/2}(S)$; moreover, the operator

$$\mathcal{A}^{\mathcal{G}\mathcal{G}} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega^+) \times H^{\frac{1}{2}}(S)$$

is bounded. Now we prove invertibility of the operator $\mathcal{A}^{\mathcal{G}\mathcal{G}}$.

Theorem 5.7. *The operator*

$$\mathcal{A}^{\mathcal{G}\mathcal{G}} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega^+) \times H^{\frac{1}{2}}(S) \tag{5.22}$$

is invertible.

Proof. Evidently the operator (5.22) is bounded and injective, i.e., $\mathcal{A}^{\mathcal{G}\mathcal{G}}\mathcal{U} = 0$ implies $\mathcal{U} = 0$, due to Theorem 5.6 and Remark 5.5.

Let

$$\mathcal{A}_0^{\mathcal{G}\mathcal{G}} := \begin{bmatrix} I & -V & W \\ 0 & -\mathcal{V} & \frac{1}{2}\,I \end{bmatrix}. \tag{5.23}$$

Clearly,

$$\mathcal{A}_0^{\mathcal{G}\mathcal{G}} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega^+) \times H^{\frac{1}{2}}(S) \tag{5.24}$$

is bounded.

Due to the mapping properties of the operators involved in (5.21) and (5.23), see Section 3, the operator

$$\mathcal{A}^{\mathcal{G}\mathcal{G}} - \mathcal{A}_0^{\mathcal{G}\mathcal{G}} \colon H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega^+) \times H^{\frac{1}{2}}(S) \tag{5.25}$$

is compact.

Consider the equation

$$\mathcal{A}_0^{\mathcal{G}\mathcal{G}}\mathcal{U} = \widetilde{F} \tag{5.26}$$

with an unknown vector $\mathcal{U} = (u, \psi, \varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ and a given vector $\widetilde{F} := (\widetilde{F}_1, \widetilde{F}_2)^\top \in H^1(\Omega^+) \times H^{1/2}(S)$. Rewrite (5.26) componentwise

$$u - V\psi + W\varphi = \widetilde{F}_1 \quad \text{in } \Omega^+, \tag{5.27}$$

$$\frac{1}{2}\,\varphi - \mathcal{V}\psi = \widetilde{F}_2 \quad \text{on } S. \tag{5.28}$$

Restriction of equation (5.28) on S_D gives

$$-r_{S_D}\,\mathcal{V}\psi = r_{S_D}\,\widetilde{F}_2. \tag{5.29}$$

Due to Theorem 3.5, equation (5.29) is uniquely solvable, i.e., for arbitrary $\widetilde{F}_2 \in H_2^{1/2}(S)$ there exists a unique $\psi \in \widetilde{H}_2^{-1/2}(S_D)$ satisfying (5.29).

Note that, in accordance with (5.29),

$$[\mathcal{V}\psi + \widetilde{F}_2] \in \widetilde{H}_2^{\frac{1}{2}}(S_N). \tag{5.30}$$

Then (5.28) along with (5.30) yields that φ is determined also uniquely as

$$\varphi = 2\,[\mathcal{V}\psi + \widetilde{F}_2 \in \widetilde{H}_2^{\frac{1}{2}}(S_N)]. \tag{5.31}$$

Thus, equation (5.28) with arbitrary $\widetilde{F}_2 \in H_2^{1/2}(S)$ defines $\varphi \in \widetilde{H}_2^{1/2}(S_N)$ and $\psi \in \widetilde{H}_2^{-1/2}(S_D)$ uniquely. We remark that we then have $V\psi$, $W\varphi \in H_2^1(\Omega^+)$ and, from equation (5.27), we get

$$u = V\psi - W\varphi + \widetilde{F}_1 \text{ in } \Omega^+, \tag{5.32}$$

i.e., the function $u \in H_2^1(\Omega^+)$ is defined uniquely also. The above arguments show that operator (5.24) is invertible. Therefore, operator (5.22) is Fredholm with zero index due to the compactness of operator (5.25). Then the injectivity of (5.22) implies the invertibility. □

5.3. Boundary-domain integral equation system $(\mathcal{TT})$**.** To obtain one more segregated BDIE system, we will use equation (4.3) in Ω^+ and equation (4.5) on S, where $\Phi_0 + \varphi$ is substituted for u^+ and $\Psi_0 + \psi$ for T^+u with functions Φ_0, φ, Ψ_0 and ψ as introduced in the beginning of Section 5. Then we arrive at the following system $(\mathcal{TT})$

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \quad \text{in } \Omega^+, \tag{5.33}$$

$$\frac{1}{2}\,\psi + T^+\mathcal{R}u - \mathcal{W}'\psi + \mathcal{L}^+\varphi = T^+F_0 - \Psi_0 \quad \text{on } S, \tag{5.34}$$

where F_0 is given by (5.4).

Remark 5.8. Similar to Remark 5.5, it is easy to see that

$$(F_0, T^+F_0 - \Psi_0) = 0$$

if and only if $(f, \Phi_0, \Psi_0) = 0$.

Let us prove that BVP (2.5)-(2.7) is equivalent to system (5.33)-(5.34).

Theorem 5.9. *Let* $f \in L_2(\Omega^+)$, *and let* $\Phi_0 \in H^{1/2}(S)$ *and* $\Psi_0 \in H^{-1/2}(S)$ *be some fixed extensions of* $\varphi_0 \in H^{1/2}(S_D)$ *and* $\psi_0 \in H^{-1/2}(S_N)$, *respectively.*

(i) *If some* $u \in H^1(\Omega^+)$ *solves the mixed* BVP (2.5)–(2.7) *in* Ω^+, *then the triple* $(u,\psi,\varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$, *where*

$$\psi = T^+u - \Psi_0 \text{ and } \varphi = u^+ - \Phi_0 \quad \text{on } S, \tag{5.35}$$

solves BDIE *system* (5.33)–(5.34).

(ii) *If a triple* $(u,\psi,\varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ *solves* BDIE *system* (5.33)–(5.34), *then* u *solves* BVP (2.5)–(2.7), *and* ψ *and* φ *satisfy* (5.35). *Moreover,* BDIES (5.33)–(5.34) *is uniquely solvable.*

Proof. Let $u \in H^1(\Omega^+)$ be a solution to BVP (2.5)–(2.7). Set $\psi := T^+u - \Psi_0$ and $\varphi := u^+ - \Phi_0$. Evidently, $\psi \in \widetilde{H}^{-1/2}(S_D)$, $\varphi \in \widetilde{H}^{1/2}(S_N)$. Then it immediately follows from relations (4.3) and (4.5) that the triple (u,ψ,φ) solves system (5.33)–(5.34), which completes the proof of item (i).

Now let a triple $(u,\psi,\varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ solve BDIE system (5.33)–(5.34).

Take the conormal derivative of equation (5.33) on S and subtract it from equation (5.34) to obtain

$$\psi + \Psi_0 - T^+u = 0 \quad \text{on } S, \tag{5.36}$$

that is, the first equation(5.35) is proved. Taking into account $\psi = 0$ on S_N and $\Psi_0 = \psi_0$ on S_N, this implies u satisfies the Neumann condition (2.7).

Equation (5.33) and Lemma 4.1 with $\Psi = \psi + \Psi_0$ and $\Phi = \varphi + \Phi_0$ imply that u is a solution of equation (2.5), and

$$V(\Psi_0 + \psi - T^+u) - W(\Phi_0 + \varphi - u^+) = 0 \quad \text{in } \Omega^+. \tag{5.37}$$

Due to (5.36), the first term vanishes in (5.18) and, by Lemma 4.2 (i), we obtain

$$\Phi_0 + \varphi - u^+ = 0 \quad \text{on } S,$$

which means that the second condition (5.35) holds as well. Taking into account $\varphi = 0$ on S_D and $\Phi_0 = \varphi_0$ on S_D, we conclude that u satisfies the Dirichlet condition (2.6).

Unique solvability of BDIE system (5.33)–(5.34) then follows from (5.35) along with the unique solvability of BVP (2.5)–(2.7) (see Theorem 2.1), and Remark 5.8. □

System (5.33)–(5.34) can be rewritten in the form

$$\mathcal{A}^{\mathcal{T}\mathcal{T}}\mathcal{U} = \mathcal{F}^{\mathcal{T}\mathcal{T}}, \tag{5.38}$$

where $\mathcal{U}^\top := (u, \psi, \varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$,

(5.39)

$$\mathcal{A}^{\mathcal{T}\mathcal{T}} := \begin{bmatrix} I+\mathcal{R} & -V & W \\ T^+\mathcal{R} & \frac{1}{2}I - \mathcal{W}' & \mathcal{L}^+ \end{bmatrix}, \qquad \mathcal{F}^{\mathcal{T}\mathcal{T}} := \begin{bmatrix} F_0 \\ T^+F_0 - \Psi_0 \end{bmatrix}.$$

Due to the mapping properties of the operators involved in (5.39), we have $\mathcal{F}^{\mathcal{T}\mathcal{T}} \in H^1(\Omega^+) \times H^{-1/2}(S)$, and the operator

$$\mathcal{A}^{\mathcal{T}\mathcal{T}} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \to H^1(\Omega^+) \times H^{-\frac{1}{2}}(S)$$

is continuous.

Theorem 5.10. *The operator*

$$\mathcal{A}^{\mathcal{T}\mathcal{T}} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega^+) \times H^{-\frac{1}{2}}(S)$$

is invertible.

Proof. Let

$$\mathcal{A}_0^{\mathcal{T}\mathcal{T}} := \begin{bmatrix} I & -V & W \\ 0 & \frac{1}{2}I & \widehat{\mathcal{L}} \end{bmatrix}.$$

Then the proof follows from the compactness of the mapping

$$\mathcal{A}^{\mathcal{T}\mathcal{T}} - \mathcal{A}_0^{\mathcal{T}\mathcal{T}} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega^+) \times H^{-\frac{1}{2}}(S),$$

and the invertibility of the operators $\widehat{\mathcal{L}}$ (see Theorem 3.6) and $I : H^1(\Omega^+) \to H^1(\Omega^+)$, by arguments similar to those in the proof of Theorem 5.7. □

5.4. Boundary-domain integral equation system $(\mathcal{TG})$. In subsection 5.4 we reduce BVP (2.5)–(2.7) to a segregated BDIE system of "almost" the second kind. We will use for this equation (4.3) in Ω^+, the restriction of equation (4.5) on S_D, and the restriction of equation (4.4) on S_N, where $\Phi_0+\varphi$ is substituted for u^+ and $\Psi_0+\psi$ for T^+u with functions Φ_0, φ, Ψ_0 and ψ as introduced in the beginning of Section 5. Then we arrive at the following system $(\mathcal{TG})$,

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \quad \text{in } \Omega^+, \tag{5.40}$$

$$\frac{1}{2}\psi + r_{S_D} T^+\mathcal{R}u - r_{S_D}\mathcal{W}'\psi + r_{S_D}\mathcal{L}^+\varphi = r_{S_D}T^+F_0 - r_{S_D}\Psi_0 \quad \text{on } S_D, \tag{5.41}$$

$$\frac{1}{2}\varphi + r_{S_N}\mathcal{R}^+u - r_{S_N}\mathcal{V}\psi + r_{S_N}\mathcal{W}\varphi = r_{S_N}F_0^+ - r_{S_N}\Phi_0 \quad \text{on } S_N, \tag{5.42}$$

where F_0 is given by (5.4). The second and the third equations of the system are associated with operator $\mathcal{T}$ on S_D and with operator $\mathcal{G}$ on S_N, respectively.

Let us denote the righthand side of BDIES (5.40)–(5.42) by

$$\mathcal{F}^{\mathcal{TG}} := \left[F_0,\, r_{S_D}T^+F_0 - r_{S_D}\Psi_0,\, r_{S_N}F_0^+ - r_{S_N}\Phi_0\right]^\top. \tag{5.43}$$

Remark 5.11. $\mathcal{F}^{\mathcal{TG}} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$. Indeed, the latter equality evidently implies the former. Inversely, let $\mathcal{F}^{\mathcal{TG}} = 0$. Keeping in mind equation (5.4), Lemma 4.1 with $F_0 = 0$ for u implies $f = 0$ and $V\Psi_0 - W\Phi_0 = 0$ in Ω^+. The equalities $\mathcal{F}_2^{\mathcal{TG}} = 0$ and $\mathcal{F}_3^{\mathcal{TG}} = 0$ imply $\Psi_0 = 0$ on S_D and $\Phi_0 = 0$ on S_N, that is, $\Psi_0 \in \widetilde{H}^{-1/2}(S_N)$, $\Phi_0 \in \widetilde{H}^{1/2}(S_D)$. Lemma 4.2 (iii) then gives $\Phi_0 = 0$ and $\Psi_0 = 0$ on S.

Let us prove that BVP (2.5)–(2.7) is equivalent to the system of BDIEs (5.40)–(5.42).

Theorem 5.12. *Let $f \in L_2(\Omega^+)$ and let $\Phi_0 \in H^{1/2}(S)$ and $\Psi_0 \in H^{-1/2}(S)$ be some fixed extensions of $\varphi_0 \in H^{1/2}(S_D)$ and $\psi_0 \in H^{-1/2}(S_N)$, respectively.*

(i) *If some $u \in H^1(\Omega^+)$ solves the mixed* BVP (2.5)–(2.7) in Ω^+, then the triple $(u, \psi, \varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$, where

$$(5.44) \qquad \psi = T^+u - \Psi_0 \quad \text{and} \quad \varphi = u^+ - \Phi_0 \quad \text{on } S,$$

solves BDIE system (5.40)–(5.42).

(ii) *If a triple $(u, \psi, \varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ solves* BDIE *system* (5.40)–(5.42)*, then u solves* BVP (2.5)–(2.7)*, and ψ and φ satisfy* (5.44)*. Moreover,* BDIES (5.40)–(5.42) *are uniquely solvable.*

Proof. Let $u \in H^1(\Omega^+)$ be a solution to BVP (2.5)–(2.7). Set $\psi := T^+u - \Psi_0$, $\varphi := u^+ - \Phi_0$. Evidently, $\psi \in \widetilde{H}^{-1/2}(S_D)$, $\varphi \in \widetilde{H}^{1/2}(S_N)$. Then it immediately follows from relations (4.3)–(4.5) that the triple $(u, \psi, \varphi)^\top$ solves system (5.40)–(5.42), which completes the proof of item (i).

Now let a triple $(u, \psi, \varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ solve BDIE system (5.40)–(5.42).

Take the conormal derivative of equation (5.40) on S_D and subtract it from equation (5.41) to obtain

$$(5.45) \qquad \psi = r_{S_D} T^+u - r_{S_D} \Psi_0 \quad \text{on } S_D.$$

Further, take the trace of equation (5.40) on S_N and subtract it from equation (5.42). We get

$$(5.46) \qquad \varphi = r_{S_N} u^+ - r_{S_N} \Phi_0 \quad \text{on } S_N.$$

Equations (5.45) and (5.46) imply that the first equation in (5.44) is satisfied on S_D and the second equation in (5.44) is satisfied on S_N.

Equation (5.40) and Lemma 4.1 with $\Psi = \psi + \Psi_0$ and $\Phi = \phi + \Phi_0$ imply that u is a solution of equation (2.5), and $V\Psi^*(y) - W\Phi^*(y) = 0$,

$y \in \Omega^+$, where $\Psi^* = \Psi_0 + \psi - T^+ u$ and $\Phi^* = \Phi_0 + \varphi - u^+$. Due to (5.45) and (5.46), we have $\Psi^* \in \widetilde{H}^{-1/2}(S_N)$, $\Phi^* \in \widetilde{H}^{1/2}(S_D)$. Lemma 4.2 (iii) with $S_1 = S_N$ and $S_2 = S_D$ implies $\Psi^* = \Phi^* = 0$, which completes the proof of conditions (5.44) on the whole boundary S. Taking into account that $\varphi = 0$ on S_D, $\Phi_0 = \varphi_0$ on S_D, $\psi = 0$ on S_N and $\Psi_0 = \psi_0$ on S_N, equations (5.44) imply the boundary conditions (2.6) and (2.7).

Unique solvability of BDIE system (5.40)–(5.42) then follows from Remark 5.11, the unique solvability of BVP (2.5)–(2.7) and from (5.44). □

System (5.40)–(5.42) can be rewritten in the form

$$\mathcal{A}^{\mathcal{TG}}\mathcal{U} = \mathcal{F}^{\mathcal{TG}}, \tag{5.47}$$

where $\mathcal{U}^\top := (u, \psi, \varphi)^\top \in H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$,

$$\mathcal{A}^{\mathcal{TG}} := \begin{bmatrix} I + \mathcal{R} & -V & W \\ r_{S_D} T^+ \mathcal{R} & r_{S_D} \left(\frac{1}{2} I - \mathcal{W}'\right) & r_{S_D} \mathcal{L}^+ \\ r_{S_N} \mathcal{R}^+ & -r_{S_N} \mathcal{V} & r_{S_N} \left(\frac{1}{2} I + \mathcal{W}\right) \end{bmatrix}. \tag{5.48}$$

Due to the mapping properties of the operators involved in (5.48), we have that the operator

$$\mathcal{A}^{\mathcal{TG}} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega^+) \times H^{-\frac{1}{2}}(S_D) \times H^{\frac{1}{2}}(S_N)$$

is continuous. Due to Theorem 5.12 it is also injective.

To prove the invertibility of operator $\mathcal{A}^{\mathcal{TG}}$ we need some auxiliary assertions.

First of all, let us remark that the operator

$$\mathcal{A}^{\mathcal{TG}} : H^{1,0}(\Omega^+; L) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^{1,0}(\Omega^+; L) \times H^{-\frac{1}{2}}(S_D) \times H^{\frac{1}{2}}(S_N),$$

where $H^{1,0}(\Omega^+; L)$ is defined in (2.2), is continuous as well due to the mapping properties of the operators R, V and W, see Section 3. Further, it is also evident that $\mathcal{F}^{\mathcal{TG}}$ given by (5.43) and (5.4) belongs to $H^{1,0}(\Omega^+;L) \times H^{-1/2}(S_D) \times H^{1/2}(S_N)$ if $(f,\Psi_0,\Phi_0)^\top \in L_2(\Omega^+) \times H^{-1/2}(S) \times H^{1/2}(S)$.

Lemma 5.13. *Let $S = \overline{S}_1 \cup \overline{S}_2$, where S_1 and S_2 are nonintersecting simply connected nonempty submanifolds of S with infinitely smooth boundaries. For any triple*

$$\mathcal{F} = (F,\Psi,\Phi)^\top \in H^{1,0}(\Omega^+;L) \times H^{-\frac{1}{2}}(S_1) \times H^{\frac{1}{2}}(S_2),$$

there exists a unique triple

$$(f_*,\Psi_*,\Phi_*)^\top = \widetilde{\mathcal{C}}_{S_1,S_2}\,\mathcal{F} \in L_2(\Omega^+) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S)$$

such that

$$\mathcal{P}\, f_* + V\,\Psi_* - W\,\Phi_* = F \quad \text{in } \Omega^+, \tag{5.49}$$

$$r_{S_1}\,\Psi_* = \Psi \ \text{ on } S_1, \tag{5.50}$$

$$r_{S_2}\,\Phi_* = \Phi \ \text{ on } S_2. \tag{5.51}$$

Moreover, the operator

$$\begin{aligned}\widetilde{\mathcal{C}}_{S_1,S_2} : H^{1,0}(\Omega^+;L) \times H^{-\frac{1}{2}}(S_1) \times H^{\frac{1}{2}}(S_2)\\ \longrightarrow L_2(\Omega^+) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S)\end{aligned} \tag{5.52}$$

is linear and bounded.

Proof. Let Ψ^0 be some fixed extension of the function Ψ from S_1 onto the whole surface S; similarly, denote by Φ^0 some fixed extension of the function Φ from S_2 onto the whole surface S. We assume that these extensions preserve the spaces, i.e., $\Psi^0 \in H^{-1/2}(S)$, $\Phi^0 \in H^{1/2}(S)$, and moreover,

$$\|\Psi^0\|_{H^{-\frac{1}{2}}(S)} \le C_0\|\Psi\|_{H^{-\frac{1}{2}}(S_1)}, \qquad \|\Phi^0\|_{H^{\frac{1}{2}}(S)} \le C_0\|\Phi\|_{H^{\frac{1}{2}}(S_2)}$$

with some positive constant C_0 independent of Ψ and Φ (see, e.g., [**40**, Chapter 4, subsection 4.2]. Then arbitrary extensions of the functions Ψ and Φ in spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$, respectively, can be represented as

$$\Psi_* = \Psi^0 + \widetilde{\psi}, \qquad \widetilde{\psi} \in \widetilde{H}^{-\frac{1}{2}}(S_2), \tag{5.53}$$

$$\Phi_* = \Phi^0 + \widetilde{\varphi}, \qquad \widetilde{\varphi} \in \widetilde{H}^{\frac{1}{2}}(S_1). \tag{5.54}$$

If we look for the unknown functions Ψ_* and Φ_* in the form (5.53) and (5.54), respectively, we see that conditions (5.50) and (5.51) are automatically satisfied for arbitrary $\widetilde{\psi}$ and $\widetilde{\varphi}$.

Thus, we have to show that functions f_*, $\widetilde{\psi}$ and $\widetilde{\varphi}$ can be chosen in such a way that equation (5.49) is satisfied.

Due to relations (3.10) and (3.29), equation (5.49) can be rewritten in the following equivalent form

$$\mathcal{P}_\Delta\, f_* + V_\Delta\,(\Psi^0 + \widetilde{\psi}) - W_\Delta\,(a\,\Phi^0 + a\,\widetilde{\varphi}) = a\,F \quad \text{in } \Omega^+. \tag{5.55}$$

Apply the Laplace operator Δ to equation (5.55) to obtain

$$f_* = \Delta\,(a\,F) \quad \text{in } \Omega^+, \tag{5.56}$$

which shows that function f_* is uniquely defined and belongs to $L_2(\Omega^+)$ since $F \in H^{1,\,0}(\Omega^+; L)$.

Further, substitute (5.56) into (5.55) and rewrite it in the form

$$V_\Delta\,(\widetilde{\psi}) - W_\Delta\,(a\,\widetilde{\varphi}) = a\,F - \mathcal{P}_\Delta\,\big(\Delta\,(a\,F)\big) - V_\Delta\,(\Psi^0) + W_\Delta\,(a\,\Phi^0) \quad \text{in } \Omega^+. \tag{5.57}$$

Denote the known righthand side expression in (5.57) by Q:

$$Q := a\,F - \mathcal{P}_\Delta\,\big(\Delta\,(a\,F)\big) - V_\Delta\,(\Psi^0) + W_\Delta\,(a\,\Phi^0) \quad \text{in } \Omega^+. \tag{5.58}$$

It is easy to check that Q is a harmonic function in Ω^+, as well as the sum of layer potentials in the lefthand side of (5.57).

Let us choose the yet unknown functions $\widetilde{\psi}$ and $\widetilde{\varphi}$ by the conditions

$$r_{S_2}\Big[V_\Delta\,(\widetilde{\psi}) - W_\Delta\,(a\,\widetilde{\varphi})\Big]^+ = r_{S_2}\,[Q]^+ \qquad \text{on } S_2, \tag{5.59}$$

$$r_{S_1}\Big[\partial_n\,\big[V_\Delta\,(\widetilde{\psi}) - W_\Delta\,(a\,\widetilde{\varphi})\big]\Big]^+ = r_{S_1}\,[\partial_n\,Q]^+ \quad \text{on } S_1, \tag{5.60}$$

where ∂_n denotes the normal derivative. As shown in [**39**, Theorem 3.6], the operator generated by the lefthand side of system (5.59)–(5.60) is an isomorphism from $\widetilde{H}^{-1/2}(S_2)\times\widetilde{H}^{1/2}(S_1)$ onto $H^{1/2}(S_2)\times H^{1/2}(S_1)$. Therefore, the system (5.59)–(5.60) is uniquely solvable with respect to $\widetilde{\psi}$ and $\widetilde{\varphi}$ for the arbitrary righthand side. Denote this solution by $\widetilde{\psi}^0$ and $\widetilde{\varphi}^0$. From conditions (5.59)–(5.60), due to the uniqueness theorem for the mixed boundary value problem for harmonic functions, it then follows that

(5.61)
$$V_\Delta\,(\widetilde{\psi}^0)-W_\Delta\,(a\,\widetilde{\varphi}^0)=a\,F-\mathcal{P}_\Delta\big(\Delta\,(a\,F)\big)-V_\Delta\,(\Psi^0)+W_\Delta\,(a\,\Phi^0)\quad\text{in }\Omega^+,$$

i.e.,

$$\mathcal{P}\big(\Delta\,(a\,F)\big)+V\,(\Psi^0+\widetilde{\psi}^0)-W\,(\Phi^0+\widetilde{\varphi}^0)=F\quad\text{in }\Omega^+. \tag{5.62}$$

This yields the existence of a triple $(f_*,\Psi_*,\Phi_*)^\top$ satisfying condition (5.49).

The uniqueness is a consequence of the fact that f_* is defined uniquely by (5.56). Indeed, if $F=0$, $\Psi=0$ and $\Phi=0$, then we have $f_*=0$ and

$$V\,(\Psi_*)-W\,(\Phi_*)=0\quad\text{in }\Omega^+$$

with $\Psi_*\in\widetilde{H}^{-1/2}(S_2)$, $\Phi_*\in\widetilde{H}^{1/2}(S_1)$, whence we conclude $\Psi_*=0$ and $\Phi_*=0$ by Lemma 4.2 (iii).

From the above arguments, it is evident that the operator $\widetilde{\mathcal{C}}_{S_1,S_2}$, see (5.52), is linear and that the norm of the triple

$$\widetilde{\mathcal{C}}_{S_1,S_2}\,\mathcal{F}=(f_*,\Psi_*,\Phi_*)^\top\in L_2(\Omega^+)\times H^{-\frac12}(S)\times H^{\frac12}(S)$$

can be controlled by the norm of the triple

$$\mathcal{F}=(F,\Psi,\Phi)^\top\in H^{1,0}(\Omega^+;L)\times H^{-\frac12}(S_1)\times H^{\frac12}(S_2)$$

in the corresponding function spaces. □

From Lemma 5.13 immediately follows

Corollary 5.14. *For arbitrary triple*

$$\mathcal{F}=(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)^\top\in H^{1,0}(\Omega^+;L)\times H^{-\frac12}(S_1)\times H^{\frac12}(S_2),$$

there exists a unique triple

$$(f_*, \Psi_*, \Phi_*)^\top = \mathcal{C}_{S_1,S_2}\, \mathcal{F} \in L_2(\Omega^+) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S)$$

such that

$$\mathcal{F}_1 = \mathcal{P}\, f_* + V\, \Psi_* - W\, \Phi_* \quad \text{in } \Omega^+, \tag{5.63}$$

$$\mathcal{F}_2 = r_{S_1}\, T^+\, \mathcal{F}_1 - r_{S_1}\, \Psi_* \quad \text{on } S_1, \tag{5.64}$$

$$\mathcal{F}_3 = r_{S_2}\, \mathcal{F}_1^+ - r_{S_2}\, \Phi_* \quad \text{on } S_2. \tag{5.65}$$

Moreover, the operator

$$\mathcal{C}_{S_1,S_2} : H^{1,\,0}(\Omega^+; L) \times H^{-\frac{1}{2}}(S_1) \times H^{\frac{1}{2}}(S_2) \longrightarrow L_2(\Omega^+) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S) \tag{5.66}$$

is linear and bounded.

Now we are in a position to prove the following invertibility result for the matrix operator $\mathcal{A}^{\mathcal{T}\mathcal{G}}$ given by (5.48).

Theorem 5.15. *The operator*

$$\mathcal{A}^{\mathcal{T}\mathcal{G}} : H^{1,0}(\Omega^+; L) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^{1,0}(\Omega^+; L) \times H^{-\frac{1}{2}}(S_D) \times H^{\frac{1}{2}}(S_N) \tag{5.67}$$

is invertible.

Proof. According to Corollary 5.14, any vector

$$\mathcal{F}^{\mathcal{T}\mathcal{G}} = \left(\mathcal{F}_1^{\mathcal{T}\mathcal{G}}, \mathcal{F}_2^{\mathcal{T}\mathcal{G}}, \mathcal{F}_3^{\mathcal{T}\mathcal{G}}\right)^\top \in H^{1,0}(\Omega^+; L) \times H^{-\frac{1}{2}}(S_D) \times H^{\frac{1}{2}}(S_N)$$

can be represented in the form

$$\mathcal{F}_1^{\mathcal{T}\mathcal{G}} = \mathcal{P} f_* + V\Psi_* - W\Phi_* \quad \text{in } \Omega^+, \tag{5.68}$$

$$\mathcal{F}_2^{\mathcal{T}\mathcal{G}} = r_{S_D} T^+ \mathcal{F}_1^{\mathcal{T}\mathcal{G}} - r_{S_D}\Psi_* \quad \text{on } S_D, \tag{5.69}$$

$$\mathcal{F}_3^{\mathcal{T}\mathcal{G}} = r_{S_N} \mathcal{F}_1^{\mathcal{T}\mathcal{G}+} - r_{S_N}\Phi_* \quad \text{on } S_N, \tag{5.70}$$

where

$$(f_*, \Psi_*, \Phi_*)^\top = \mathcal{C}_{S_D,S_N}\, \mathcal{F}^{\mathcal{T}\mathcal{G}}, \tag{5.71}$$

and the operator

$$\mathcal{C}_{S_D,S_N} : H^{1,0}(\Omega^+; L) \times H^{-\frac{1}{2}}(S_D) \times H^{\frac{1}{2}}(S_N) \longrightarrow L_2(\Omega^+) \times H^{-\frac{1}{2}}(S) \times H^{\frac{1}{2}}(S),$$

is linear and bounded.

For any $\mathcal{F}^{\mathcal{G}\mathcal{T}} \in H^{1,\,0}(\Omega^+; L) \times H^{1/2}(S_D) \times H^{-1/2}(S_N)$, let

$$\mathcal{U} = (u, \psi, \varphi)^\top = \big[\,\mathcal{A}^{\mathcal{G}\mathcal{T}}\,\big]^{-1} \mathcal{F}^{\mathcal{G}\mathcal{T}} \in H^{1,0}(\Omega^+; L) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N),$$

where $[\mathcal{A}^{\mathcal{G}\mathcal{T}}]^{-1} : H^{1,\,0}(\Omega^+; L) \times H^{1/2}(S_D) \times H^{-1/2}(S_N) \to H^{1,\,0}_2(\Omega^+; L) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ is the bounded inverse to operator (5.12) from Theorem 5.3 with matrix operator $\mathcal{A}^{\mathcal{G}\mathcal{T}}$ given by (5.10). Then $\mathcal{U}$ is a solution of the equation

$$\mathcal{A}^{\mathcal{G}\mathcal{T}} \mathcal{U} = \mathcal{F}^{\mathcal{G}\mathcal{T}}. \tag{5.72}$$

Let us take

$$\mathcal{F}^{\mathcal{G}\mathcal{T}} = \mathcal{B}^{\mathcal{T}\mathcal{G}} \mathcal{F}^{\mathcal{T}\mathcal{G}}, \tag{5.73}$$

where

$$\mathcal{B}^{\mathcal{T}\mathcal{G}} := \begin{bmatrix} I & 0 & 0 \\ r_{S_D}\gamma^+ - r_{S_D}\big[\mathcal{C}_{S_D,S_N}\big]_{31} & -r_{S_D}\big[\mathcal{C}_{S_D,S_N}\big]_{32} & -r_{S_D}\big[\mathcal{C}_{S_D,S_N}\big]_{33} \\ r_{S_N}T^+ - r_{S_N}\big[\mathcal{C}_{S_D,S_N}\big]_{21} & -r_{S_N}\big[\mathcal{C}_{S_D,S_N}\big]_{22} & -r_{S_N}\big[\mathcal{C}_{S_D,S_N}\big]_{23} \end{bmatrix}. \tag{5.74}$$

In what follows, we will prove that $\mathcal{U}$ is then also a solution to equation (5.47) with the righthand side $\mathcal{F}^{\mathcal{T}\mathcal{G}}$.

Due to (5.71), we can rewrite (5.73) in the form

$$\mathcal{F}^{\mathcal{G}\mathcal{T}}_1 = \mathcal{F}^{\mathcal{T}\mathcal{G}}_1 \quad \text{in } \Omega^+, \tag{5.75}$$

$$\mathcal{F}^{\mathcal{G}\mathcal{T}}_2 = r_{S_D}\mathcal{F}^{\mathcal{T}\mathcal{G}+}_1 - r_{S_D}\Phi_* \quad \text{on } S_D \tag{5.76}$$

$$\mathcal{F}^{\mathcal{G}\mathcal{T}}_3 = r_{S_N}T^+\mathcal{F}^{\mathcal{T}\mathcal{G}}_1 - r_{S_N}\Psi_* \quad \text{on } S_N. \tag{5.77}$$

Taking the trace of the first equation of system (5.72) on S_D and subtracting from it the second equation of the system, we obtain

$$r_{S_D}\, u^+ = r_{S_D} \mathcal{F}_1^{\mathcal{G}\mathcal{T}+} - \mathcal{F}_2^{\mathcal{G}\mathcal{T}} = r_{S_D} \Phi_* \quad \text{on } S_D. \tag{5.78}$$

Taking the conormal derivative of the first equation of system (5.72) on S_N and subtracting from it the third equation of the system, we obtain

$$r_{S_N}\, T^+ u = r_{S_N} T^+ \mathcal{F}_1^{\mathcal{G}\mathcal{T}} - \mathcal{F}_3^{\mathcal{G}\mathcal{T}} = r_{S_N} \Psi_* \quad \text{on } S_N. \tag{5.79}$$

The first equation of system (5.72), representations (5.68), (5.75) and Lemma 4.1 with $f = f_*$, $\Psi = \psi + \Psi_*$ and $\Phi = \varphi + \Phi_*$ imply $V\Psi^* - W\Phi^* = 0$ in Ω^+, where $\Psi^* = \psi + \Psi_* - T^+ u$ and $\Phi^* = \varphi + \Phi_* - u^+$. Due to (5.78) and (5.79), we have $\Psi^* \in \widetilde{H}^{-1/2}(S_D)$ and $\Phi^* \in \widetilde{H}^{1/2}(S_N)$, since $\psi \in \widetilde{H}^{-1/2}(S_D)$ and $\varphi \in \widetilde{H}^{1/2}(S_N)$. Lemma 4.2 (iii) with $S_1 = S_D$ and $S_2 = S_N$ implies $\Psi^* = \Phi^* = 0$, that is,

$$T^+ u = \psi + \Psi_* \quad \text{on } S, \tag{5.80}$$

$$u^+ = \varphi + \Phi_* \quad \text{on } S. \tag{5.81}$$

Taking the conormal derivative of the first equation of system (5.72) on S_D and substituting there $T^+ u$ from (5.80), we obtain

$$r_{S_D} \Big[\psi + \Psi_* + T^+ \mathcal{R} u - \tfrac{1}{2}\psi - \mathcal{W}'\psi(y) + \mathcal{L}^+ \varphi \Big] = r_{S_D} T^+ \mathcal{F}_1^{\mathcal{T}\mathcal{G}}.$$

Taking into account (5.69), this implies that the second equation of system (5.47) is satisfied.

Similarly, taking the trace of the first equation of system (5.72) on S_N and substituting there u^+ from (5.81), we obtain

$$r_{S_N} \Big[\varphi + \Phi_* + \mathcal{R}^+ u - \mathcal{V}\psi - \tfrac{1}{2}\,\varphi + \mathcal{W}\varphi \Big] = r_{S_N} \mathcal{F}_1^{\mathcal{T}\mathcal{G}+}.$$

Taking into account (5.70), this implies the third equation of system (5.47) is satisfied.

The first equations of system (5.47) coincide with the first equations of system (5.72) and consequently is also satisfied. Thus, $\mathcal{U} = [u, \psi, \varphi]^\top =$

$[\mathcal{A}^{\mathcal{G}\mathcal{T}}]^{-1}\mathcal{B}^{\mathcal{T}\mathcal{G}}\mathcal{F}^{\mathcal{T}\mathcal{G}}$ satisfies the whole system (5.47) and $[\mathcal{A}^{\mathcal{G}\mathcal{T}}]^{-1}\mathcal{B}^{\mathcal{T}\mathcal{G}}$ is a bounded right inverse to the operator

$$\begin{aligned}(5.82)\quad \mathcal{A}^{\mathcal{T}\mathcal{G}} : H^{1,0}(\Omega^+;L) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N)\\ \longrightarrow H^{1,0}(\Omega^+;L) \times H^{-\frac{1}{2}}(S_D) \times H^{\frac{1}{2}}(S_N).\end{aligned}$$

On the other hand, system (5.47) with zero righthand side has only the trivial solution due to Theorem 5.12 (ii). This means that, for any righthand side

$$\mathcal{F}^{\mathcal{T}\mathcal{G}} \in H^{1,0}(\Omega^+;L) \times H^{-\frac{1}{2}}(S_D) \times H^{\frac{1}{2}}(S_N),$$

the solution of (5.47) is unique and is given by the formula

$$\mathcal{U} = [u,\psi,\varphi]^\top = \big[\mathcal{A}^{\mathcal{G}\mathcal{T}}\big]^{-1}\mathcal{B}^{\mathcal{T}\mathcal{G}}\mathcal{F}^{\mathcal{T}\mathcal{G}}.$$

Thus, the operator $[\mathcal{A}^{\mathcal{G}\mathcal{T}}]^{-1}\mathcal{B}^{\mathcal{T}\mathcal{G}}$ is a bounded two-side inverse to the operator (5.82). $\square$

Original BVP (2.5)–(2.7) can be written in the form

$$(5.83)\qquad A^{DN}u = F^{DN},$$

where

$$(5.84)\qquad A^{DN} := \begin{bmatrix} L \\ r_{S_D}\gamma^+ \\ r_{S_N}T^+ \end{bmatrix}, \qquad F^{DN} = \begin{bmatrix} f \\ \varphi_0 \\ \psi_0 \end{bmatrix}.$$

The operator $A^{DN} : H^{1,0}(\Omega^+;L) \to L_2(\Omega^+) \times H^{1/2}(S_D) \times H^{-1/2}(S_N)$ is evidently continuous and due to the uniqueness theorem for the BVP is also injective.

The invertibility of operator $\mathcal{A}^{\mathcal{T}\mathcal{G}}$ from Theorem 5.15 and equivalence Theorem 5.12 lead to the following

Corollary 5.16. *The operator $A^{DN} : H^{1,0}(\Omega^+;L) \to L_2(\Omega^+) \times H^{1/2}(S_D) \times H^{-1/2}(S_N)$ is continuous and continuously invertible.*

In the particular case $a(x) = 1$ at $x \in \Omega^+$, (2.5) becomes the classical Laplace equation, $\mathcal{R} = 0$, and BDIES (5.40)–(5.42) splits into the Boundary Integral Equation System (BIES),

(5.85)
$$r_{S_D}\left(\tfrac{1}{2}\psi - \mathcal{W}'_\Delta\psi + \mathcal{L}^+_\Delta\varphi\right) = r_{S_D}T^+F_0 - r_{S_D}\Psi_0 \quad \text{on } S_D,$$

(5.86)
$$r_{S_N}\left(\tfrac{1}{2}\varphi - \mathcal{V}_\Delta\psi + \mathcal{W}_\Delta\varphi\right) = r_{S_N}F_0^+ - r_{S_N}\Phi_0 \quad \text{on } S_N,$$

and the representation formula for u in terms of φ and ψ,

$$u = F_0 + V_\Delta\psi - W_\Delta\varphi \quad \text{in } \Omega^+. \tag{5.87}$$

Then Theorem 5.12 leads to the following

Corollary 5.17. *Let $a = 1$ in Ω^+, $f \in L_2(\Omega^+)$, and let $\Phi_0 \in H^{1/2}(S)$ and $\Psi_0 \in H^{-1/2}(S)$ be some extensions of $\varphi_0 \in H^{1/2}(S_D)$ and $\psi_0 \in H^{-1/2}(S_N)$, respectively.*

(i) *If some $u \in H^1(\Omega^+)$ solves the mixed* BVP (2.5)–(2.7) *in Ω^+, then the solution is unique, the couple $(\psi, \varphi) \in \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ given by* (5.44) *solves* BIE *system* (5.85)–(5.86), *and u satisfies* (5.87).

(ii) *If a couple $(\psi, \varphi) \in \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$ solves* BIE *system* (5.85)–(5.86), *then u given by* (5.87) *solves* BVP (2.5)–(2.7) *and equations* (5.44) *hold. Moreover,* BIES (5.85)–(5.86) *is uniquely solvable in $\widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$.*

System (5.85)–(5.86) can be rewritten in the form

$$\widehat{\mathcal{A}}^{\mathcal{TG}}_\Delta \widehat{\mathcal{U}}_\Delta = \widehat{\mathcal{F}}^{\mathcal{TG}}_\Delta, \tag{5.88}$$

where $\widehat{\mathcal{U}}^\top_\Delta := (\psi, \varphi) \in \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N)$,

(5.89)
$$\widehat{\mathcal{A}}^{\mathcal{TG}}_\Delta := \begin{bmatrix} r_{S_D}\left(\tfrac{1}{2}I - \mathcal{W}'_\Delta\right) & r_{S_D}\mathcal{L}^+_\Delta \\ -r_{S_N}\mathcal{V}_\Delta & r_{S_N}\left(\tfrac{1}{2}I + \mathcal{W}_\Delta\right) \end{bmatrix},$$
$$\widehat{\mathcal{F}}^{\mathcal{TG}}_\Delta := \begin{bmatrix} r_{S_D}T^+F_0 - r_{S_D}\Psi_0 \\ r_{S_N}F_0^+ - r_{S_N}\Phi_0 \end{bmatrix},$$

$\widehat{\mathcal{F}}_{\Delta}^{\mathcal{TG}} \in H^{-1/2}(S_D) \times H^{1/2}(S_N)$. Moreover, the operator $\widehat{\mathcal{A}}_{\Delta}^{\mathcal{TG}} : \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N) \to H^{-1/2}(S_D) \times H^{1/2}(S_N)$ is bounded and injective.

Theorem 5.18. *The operator* $\widehat{\mathcal{A}}_{\Delta}^{\mathcal{TG}} : \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N) \to H^{-1/2}(S_D) \times H^{1/2}(S_N)$ *is invertible.*

Proof. A solution of system (5.88) with an arbitrary $(\widehat{\mathcal{F}}_{\Delta}^{\mathcal{TG}})^{\top} = (\mathcal{F}_{2\Delta}^{\mathcal{TG}}, \mathcal{F}_{3\Delta}^{\mathcal{TG}}) \in H^{-1/2}(S_D) \times H^{1/2}(S_N)$ is delivered by the couple (ψ, φ) satisfying extended system

$$\mathcal{A}_{\Delta}^{\mathcal{TG}} \mathcal{U} = \mathcal{F}_{\Delta 0}^{\mathcal{TG}}, \tag{5.90}$$

where $\mathcal{U} = (u,\, \psi,\, \varphi)^{\top}$, $\mathcal{F}_{\Delta 0}^{\mathcal{TG}} = (0,\, \mathcal{F}_{2\Delta}^{\mathcal{TG}},\, \mathcal{F}_{3\Delta}^{\mathcal{TG}})^{\top}$ and

$$\mathcal{A}_{\Delta}^{\mathcal{TG}} := \begin{bmatrix} I & -V_{\Delta} & W_{\Delta} \\ 0 & r_{S_D} \left(\frac{1}{2} I - \mathcal{W}'_{\Delta} \right) & r_{S_D} \mathcal{L}_{\Delta}^{+} \\ 0 & -r_{S_N} \mathcal{V}_{\Delta} & r_{S_N} \left(\frac{1}{2} I + \mathcal{W}_{\Delta} \right) \end{bmatrix}. \tag{5.91}$$

Operator $\mathcal{A}_{\Delta}^{\mathcal{TG}}$ has a continuous inverse due to Theorem 5.15 for $a = 1$. Consequently, operator $\widehat{\mathcal{A}}_{\Delta}^{\mathcal{TG}}$ has a right bounded inverse, which is also a two-sided inverse due to injectivity of the operator $(\widehat{\mathcal{A}}_{\Delta}^{\mathcal{TG}})$. $\square$

Note that invertibility of the *boundary* integral operator with a structure similar to $\widehat{\mathcal{A}}_{\Delta}^{\mathcal{TG}}$ but associated with homogeneous elasticity was analyzed in [**22**] in weighted spaces of functions with Hölder-continuous derivatives, using the known properties of the original BVP solutions in corresponding spaces.

Now we prove the counterpart of Theorem 5.15 in wider spaces.

Theorem 5.19. *The operator*

$$\mathcal{A}^{\mathcal{TG}} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega^+) \times H^{-\frac{1}{2}}(S_D) \times H^{\frac{1}{2}}(S_N)$$

is invertible.

Proof. Let us consider the following operator,

$$\mathcal{A}_0^{\mathcal{TG}} := \begin{bmatrix} I & -V & W \\ 0 & r_{S_D}\left(\frac{1}{2}I - \mathcal{W}'_\Delta\right) & r_{S_D}\,\widehat{\mathcal{L}} \\ 0 & -r_{S_N}\,\mathcal{V} & r_{S_N}\left(\frac{1}{2}I + \mathcal{W}_\Delta\right) \end{bmatrix}. \tag{5.92}$$

By Theorems 3.4, 3.6 and Corollary 3.45, operator $\mathcal{A}_0^{\mathcal{TG}}$ is a compact perturbation of the operator $\mathcal{A}^{\mathcal{TG}}$. Taking into account relations (3.10), (3.11) and (3.23), the above operator can be represented as

$$\mathcal{A}_0^{\mathcal{TG}} = \operatorname{diag}\left(\tfrac{1}{a}, 1, \tfrac{1}{a}\right) \mathcal{A}_\Delta^{\mathcal{TG}}[\operatorname{diag}(a, 1, a)g],$$

where

$$\operatorname{diag}\left(\tfrac{1}{a}, 1, \tfrac{1}{a}\right) \text{ and } \operatorname{diag}(a, 1, a)$$

are diagonal 3×3 matrices. The operator

$$\mathcal{A}_\Delta^{\mathcal{TG}} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega^+) \times H^{-\frac{1}{2}}(S_D) \times H^{\frac{1}{2}}(S_N), \tag{5.93}$$

where $\mathcal{A}_\Delta^{\mathcal{TG}}$ is given by (5.91), is an upper block-triangular matrix operator with the following diagonal operators

$$I : H^1(\Omega^+) \longrightarrow H^1(\Omega^+),$$
$$\widehat{\mathcal{A}}_\Delta^{\mathcal{TG}} : \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^{-\frac{1}{2}}(S_D) \times H^{\frac{1}{2}}(S_N).$$

The operator $\widehat{\mathcal{A}}_\Delta^{\mathcal{TG}}$ is invertible due to Theorem 5.18. Consequently, (5.93) is an invertible operator as well. Taking into account that $a > \text{const} > 0$ and is bounded, this implies the diagonal matrices

$$\operatorname{diag}\left(\tfrac{1}{a}, 1, \tfrac{1}{a}\right) \quad \text{and} \quad \operatorname{diag}(a, 1, a)$$

are invertible, and the operator

$$\mathcal{A}_0^{\mathcal{TG}} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \longrightarrow H^1(\Omega^+) \times H^{-\frac{1}{2}}(S_D) \times H^{\frac{1}{2}}(S_N)$$

is invertible. This implies that the operator $\mathcal{A}^{\mathcal{T}\mathcal{G}}$ possesses the Fredholm property and its index is zero.

Then injectivity of the operator $\mathcal{A}^{\mathcal{T}\mathcal{G}} : H^1(\Omega^+) \times \widetilde{H}^{-1/2}(S_D) \times \widetilde{H}^{1/2}(S_N) \to H^1(\Omega^+) \times H^{-1/2}(S_D) \times H^{1/2}(S_N)$ implies its invertibility. □

Concluding remarks. Mapping and jump properties of surface and volume integral potentials based on a generalized parametrix (Levi function) of a "Poisson" PDE with *variable* coefficient were presented in the paper in a scale of Sobolev spaces. Four *segregated* boundary-domain integral equation systems were then formulated and analyzed, and their equivalence to the original mixed variable-coefficient BVP was proved in the case of PDE righthand side function from $L_2(\Omega^+)$, and the Dirichlet and the Neumann data from the spaces $H^{1/2}(S_D)$ and $H^{-1/2}(S_N)$, respectively. Invertibility of the operators of the BDIES was proved in the corresponding Sobolev space.

The BDIES $(\mathcal{T}\mathcal{G})$ looks like an operator equation of the second kind, but the operator domain and the range coincide only "up to the tilde." Although the resolvent theory and Neumann series method (cf. [**24, 37** and references therein]) are then not directly applicable to the equation solution, further analysis is needed to find out whether it might be possible after an appropriate modification of the operator and/or the spaces, cf. [**3, 25**].

By the same approach, the corresponding BDIE system can also be considered with the PDE righthand side from $\widetilde{H}^{-1}(\Omega^+)$ (cf. [**27**]), and for unbounded domains. Smoothness of variable coefficients and the boundary can also be essentially relaxed, taking them sufficient only to provide appropriate mapping properties of parametrix-based potentials and ensure invertibility of corresponding classical surface potentials associated with the Laplace operator.

In the formulations considered, unknown boundary traces and conormal derivatives are replaced by auxiliary boundary functions formally segregated from the solution inside the domain. The so-called *united* formulations not involving such auxiliary functions and leading to boundary domain integro-*differential* equations are analyzed in [**28**]. Results of the present paper can also be extended to BDIEs of more general scalar partial differential elliptic operators, particularly with matrix coefficients, as well as to BDIEs of elliptic systems of partial differential equations.

The analysis presented makes a theoretical basis for justification of numerical methods for the BDIES solution. The approach can be extended to analysis of localized BDIES, cf. [**5**], to serve as a theoretical basis for justification of associated localized numerical methods leading to sparsely populated systems of linear algebraic equations (see [**26**] for details).

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