HIGHER ORDER ENERGY EXPANSIONS FOR SOME SINGULARLY PERTURBED NEUMANN PROBLEMS

JUNCHENG WEI AND MATTHIAS WINTER

Abstract- We consider the following singularly perturbed semilinear elliptic problem:

$$\epsilon^2 \Delta u - u + u^p = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega \quad \text{and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded smooth domain in R^N, $\epsilon > 0$ is a small constant and p is a subcritical exponent. Let $J_\epsilon[u] := \int_\Omega (\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1})dx$ be its energy functional, where $u \in H^1(\Omega)$. Ni and Takagi ([15], [16]) proved that for a single boundary spike solution u_ϵ, the following asymptotic expansion holds

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + o(\epsilon) \right],$$

where $c_1 > 0$ is a generic constant, P_ϵ is the unique local maximum point of u_ϵ and $H(P_\epsilon)$ is the boundary mean curvature function. In this paper, we obtain the following higher order expansion of $J_\epsilon[u_\epsilon]$:

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 [c_2 (H(P_\epsilon))^2 + c_3 R(P_\epsilon)] + o(\epsilon^2) \right],$$

where c_2, c_3 are generic constants and $R(P_\epsilon)$ is the Ricci scalar curvature at P_ϵ. In particular $c_3 > 0$. Applications of this expansion will be given.

L'expansion de l'énergie de les solutions de les problèmes de la perturbation singuliere

Résumé. Nous étudions le problème suivant de la perturbation singuliere:

$$\epsilon^2 \Delta u - u + u^p = 0 \text{ dans } \Omega, \quad u > 0 \text{ dans } \Omega \text{ et } \frac{\partial u}{\partial \nu} = 0 \text{ sur } \partial\Omega,$$

où Ω est un domaine ouvert dans R^N, $\epsilon > 0$ est une constante petite et p est un exposant subcritique. L'énergie s'écrit alors $J_\epsilon[u] := \int_\Omega (\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1})dx$, où $u \in H^1(\Omega)$. Ni et Takagi ([15], [16]) montrent que pour une solution u_ϵ avec une pic sur la frontière du domaine, l'éxiste de la expansion asymptotique suivant:

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + o(\epsilon) \right],$$

où $c_1 > 0$ est une constante générique, P_ϵ est le point unique du maximum local de u_ϵ et $H(P_\epsilon)$ est la fonction de la curvature moyenne sur la frontière. Nous dérivons de la expansion suivant de l'ordre plus élevé de $J_\epsilon[u_\epsilon]$:

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 [c_2 (H(P_\epsilon))^2 + c_3 R(P_\epsilon)] + o(\epsilon^2) \right],$$

où c_2, c_3 sont les constantes génériques et $R(P_\epsilon)$ est la curvature scalare de Ricci dans P_ϵ. En particulier $c_3 > 0$. Nous présentons les applications de la expansion.

Version française abrégée-

Nous étudions le problème suivant de la perturbation singuliere:

$$\epsilon^2 \Delta u - u + u^p = 0 \text{ dans } \Omega, \quad u > 0 \text{ dans } \Omega \text{ et } \frac{\partial u}{\partial \nu} = 0 \text{ sur } \partial\Omega,$$

où Ω est un domaine ouvert dans R^N avec une frontière lisse, $\epsilon > 0$ est une constante petite, Δ est l'opérateur de Laplace dans R^N, ν est le normale extérieur sur $\partial\Omega$ et p satisfait $1 < p < (\frac{N+2}{N-2})_+ (= \frac{N+2}{N-2}$ si $N \geq 3; = +\infty$ si $N = 1, 2)$.

L'énergie s'écrit alors $J_\epsilon[u] := \int_\Omega (\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1})dx$, où $u \in H^1(\Omega)$. Ni et Takagi ([15], [16]) montrent que pour une solution u_ϵ avec une pic sur la frontière du domaine, l'éxiste de la expansion asymptotique suivant:

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + o(\epsilon) \right],$$

où $c_1 > 0$ est une constante générique, $P_\epsilon \in \partial\Omega$ est le point unique du maximum local de u_ϵ, $H(P_\epsilon)$ est la fonction de la curvature moyenne sur la frontière et $I[w]$ est l'énergie de l'état fondamental dans R^N.

Dans ce travail nous dérivons de la expansion suivant de l'ordre plus élevé de $J_\epsilon[u_\epsilon]$:

Théorème 1. *Pour une solution u_ϵ de (I) avec une pic sur la frontière du domaine et avec un point unique du maximum local de u_ϵ nous avons pour ϵ petit suffisant:*

$$J_\epsilon = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 [c_2 (H(P_\epsilon))^2 + c_3 R(P_\epsilon)] + o(\epsilon^2) \right],$$

óu c_1, c_2, c_3 sont les constantes génériques. En plus $c_1 > 0, c_3 > 0$.

Le corollaire suivant donnet un affinage des les résultats de [15] et [16].

Corollaire 2. *Pour une solution u_ϵ de l'énergie minimale de (I) et pour ϵ petit suffisant nous avons*

$$H(P_\epsilon) \to \max_{P \in \partial\Omega} H(P), R(P_\epsilon) \to \min_{Q \in \partial\Omega, H(Q) = \max_{P \in \partial\Omega} H(P)} R(Q).$$

Ils sont deux pasles essentielles dans le preuve du Théorème 1. Dans Pas 1 nous trouvons une fonction approximativemente bonne $w_{\epsilon,P}$ avec $\epsilon^2 \Delta \tilde{w}_{\epsilon,P} - \tilde{w}_{\epsilon,P} + w^p_{\epsilon,P} = O(\epsilon^2)$. Dans Pas 2 nous montrons que $u_\epsilon = \tilde{w}_{\epsilon,P_\epsilon} + O(\epsilon^\tau)$ pour un $\tau > 1$.

1. Introduction. We consider the following singularly perturbed semilinear elliptic problem

$$\epsilon^2 \Delta u - u + f(u) = 0, \quad u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $\epsilon > 0$ is a small constant, $\Delta := \sum_{j=1}^N \frac{\partial^2}{\partial x_j \partial x_j}$ denotes the Laplace operator in R^N, ν stands for the unit outer normal to $\partial\Omega$, $f(u) = u^p$ and p satisfies $1 < p < (\frac{N+2}{N-2})_+ (= \frac{N+2}{N-2}$ when $N \geq 3; = +\infty$ when $N = 1, 2)$.

Equation (1.1) arises in many branches of applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of Gierer-Meinhardt model in biological pattern formation ([7], [18]) or of parabolic equations in chemotaxis, population dynamics and phase transitions. Associated with (1.1) is the energy functional J_ϵ defined by

$$J_\epsilon[u] := \int_\Omega \left(\frac{\epsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - F(u) \right) dx \qquad \text{for } u \in H^1(\Omega), \text{ where } F(u) = \int_0^u f(s)ds.$$

In the pioneering papers [14], [15] and [16], Lin, Ni and Takagi established the existence of least-energy solutions and showed that for ϵ sufficiently small the least-energy solution has only one local maximum point P_ϵ with $P_\epsilon \in \partial\Omega$. Moreover, $H(P_\epsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\epsilon \to 0$, where $H(P)$ is the mean curvature of $\partial\Omega$ at P. Since then, many works have been devoted to finding solutions with multiple spikes for the Neumann problem as well as the Dirichlet problem. See [1], [2], [3], [4], [5], [6], [8], [9], [10], [11], [12], [13], [15], [16], [17], [19], [20], [21], and the review article [18] and the references therein.

A common tool for proving the existence of spike solutions is by energy expansion: In [15] and [16], Ni and Takagi proved, among others, that for a single boundary spike solution u_ϵ the following asymptotic expansion for $J_\epsilon[u_\epsilon]$ holds true:

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + o(\epsilon)\right], \tag{1.2}$$

where $c_1 > 0$ is a generic constant, P_ϵ is the unique local maximum point of u_ϵ, $H(P_\epsilon)$ is the mean curvature function at $P_\epsilon \in \partial\Omega$, w is the unique solution of the following ground-state problem

$$\Delta w - w + f(w) = 0, w > 0 \ \text{ in } R^N, \quad w(0) = \max_{y\in R^N} w(y), \lim_{|y|\to+\infty} w(y) = 0, \tag{1.3}$$

and $I[w]$ is the ground-state energy $I[w] = \frac{1}{2}\int_{R^N} \left(|\nabla w|^2 + \frac{1}{2} w^2 - F(w)\right) dy$. Based on (1.2), Ni and Takagi [16] concluded that the least energy solution must concentrate at a maximum point of the mean curvature function. However, if $H(P)$ has more than one maximum point on $\partial\Omega$, the asymptotic expansion (1.2) has to be refined to prove such a statement and the next order term in (1.2) becomes important. This is exactly the purpose of this paper.

We now state our main theorem. First, we introduce boundary deformations. Let $P \in \partial\Omega$. After rotation and translation of the coordinate system we may assume that the inward normal to $\partial\Omega$ at P points in the direction of the positive x_N-axis, that $P = 0$, and that there exists a constant $\delta > 0$ and a smooth function ρ such that $\Omega \cap B_\delta(P) = \{(x', x_N) | x_N > \rho(x')\}$. Moreover, we may assume that

$$\rho(x') = \frac{1}{2}\sum_{i=1}^{N-1} k_i x_i^2 + O(|x'|^3), \quad x' = (x_1, ..., x_{N-1}),$$

where $k_i, i = 1, ..., N-1$ are the principal curvatures at P. (Note that $H(P) = \frac{1}{N-1}\sum_{i=1}^{N-1} k_i$ is the mean curvature.) For $N \geq 3$, we also need to define $R(P) = \sum_{i\neq j} k_i k_j$, which is called Ricci scalar curvature at P. When $N = 2$, we let $R(P) = 0$.

Now we can state the main result of this paper.

Theorem 1. *Let u_ϵ be a single boundary spike solution of (1.1) with a unique local maximum point $P_\epsilon \in \partial\Omega$. Then, for ϵ sufficiently small, we have*

$$J_\epsilon = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 [c_2 (H(P_\epsilon))^2 + c_3 R(P_\epsilon)] + o(\epsilon^2)\right], \tag{1.4}$$

where $c_1 = \frac{N-1}{N+1}\int_{R_+^N} (w')^2 y_N dy > 0$, and c_2, c_3 are generic constants. Moreover, we have $c_3 > 0$. Here $R_+^N = \{(y', y_N) | y_N > 0\}$.

As a corollary, we give a refinement of the results of [15] and [16].

Corollary 2 *Let u_ϵ be a least energy solution of (1.1). Then, for ϵ sufficiently small, we have*

$$H(P_\epsilon) \to \max_{P\in\partial\Omega} H(P), \ R(P_\epsilon) \to \min_{Q\in\partial\Omega, H(Q)=\max_{P\in\partial\Omega} H(P)} R(Q). \tag{1.5}$$

Remarks: 1. (1.5) shows that the least energy solution will concentrate at a global maximum mean curvature point with smallest scalar curvature. For example, for $N = 3$, and suppose that the mean curvature function $H(P)$ has two global maximum points P_1 and P_2. Let the principal curvatures at P_i be given by $k_{i,j}, i = 1, 2, j = 1, 2$. Then $R(P_i) = k_{i,1}k_{i,2}, i = 1, 2$. The spike will approach the point with smaller R. However, if $N = 2$, (1.5) yields no new results. In that case, we have to expand $J_\epsilon[u_\epsilon]$ up to the order $O(\epsilon^3)$ to obtain more information on the spike locations.

2. Theorem 1 holds true if we replace $-u + u^p$ with more general nonlinearities. See [22].

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2. Two Important Lemmas. In this section we present two main lemmas needed to prove Theorem 1. We begin with the following on good approximate functions.

Lemma 3. *For each $P \in \partial\Omega$, there exists a smooth function $\tilde{w}_{\epsilon,P}$ such that*

$$\epsilon^2 \Delta \tilde{w}_{\epsilon,P} - \tilde{w}_{\epsilon,P} + f(\tilde{w}_{\epsilon,P}) = O(\epsilon^{1+\sigma}), \tag{2.1}$$

$$J_\epsilon[\tilde{w}_{\epsilon,P}] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P) + \epsilon^2 [c_2 (H(P))^2 + c_3 R(P)] + o(\epsilon^2) \right], \tag{2.2}$$

where $\sigma = \min(1, p-1)$ and c_1, c_2, c_3 are generic constants. In particular,

$$c_3 = \frac{1}{16} \int_{R^N_+} [|\nabla \Psi_0|^2 + |\Psi_0|^2 - f^{'}(w)\Psi_0^2]\, dy > 0,$$

where Ψ_0 satisfies $\Delta \Psi_0 - \Psi_0 + f^{'}(w)\Psi_0 = 0$ in R^N_+, $\frac{\partial \Psi_0}{\partial y_N} = \frac{w^{'}}{|y|}(y_1^2 - y_2^2)$ on ∂R^N_+.

The proof of Lemma 3 is technical and we refer to Section 2 and Section 3 of [22].

Our next lemma is about the expansion of u_ϵ which is a single boundary spike solution of (1.1). Let P_ϵ be its local maximum point. The key observation is that by using $\tilde{w}_{\epsilon,P_\epsilon}$ as our approximating function, we just need to expand u_ϵ up to $O(\epsilon^\tau)$ for some $\tau > 1$. In fact, we do not even need to know the exact asymptotic expansion in $O(\epsilon^\tau)$. We now choose $\tau = 1 + \frac{\sigma}{2}$. Thus we get

Lemma 4: *For ϵ sufficiently small, we have $u_\epsilon = \tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau \phi_\epsilon$, where ϕ_ϵ satisfies*

$$\|\phi_\epsilon\|_{L^\infty(\bar{\Omega})} \le C, \tag{2.3}$$

$$\epsilon^{-N} \int_\Omega (\epsilon^2 |\nabla \phi_\epsilon|^2 + |\phi_\epsilon|^2)\, dx \le C. \tag{2.4}$$

Proof: We sketch the main ideas of the proof. For details, see Section 5 of [22]. Substituting $u_\epsilon = \tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau \phi_\epsilon$ into (1.1), we see from (2.1) that ϕ_ϵ satisfies

$$\epsilon^2 \Delta \phi_\epsilon - \phi_\epsilon + f^{'}(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon = O(\epsilon^{\sigma/2}) + N_\epsilon[\phi_\epsilon] \text{ in } \Omega, \quad \frac{\partial \phi_\epsilon}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{2.5}$$

where $N_\epsilon[\phi_\epsilon] = -\epsilon^{-\tau}[f(\tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau \phi_\epsilon) - f(\tilde{w}_{\epsilon,P_\epsilon}) - \epsilon^\tau f^{'}(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon] = o(1)|\phi_\epsilon|$, by the mean value theorem.

Now we can prove (2.3). Suppose not, then there exists a sequence $\epsilon_k \to 0$ such that $M_{\epsilon_k} := \|\phi_{\epsilon_k}\|_{L^\infty(\bar{\Omega})} \to +\infty$. For simplicity of notation, we still denote ϵ_k by ϵ. Let $M_\epsilon = |\phi_\epsilon(x_\epsilon)|$, where $x_\epsilon \in \bar{\Omega}$. Without loss of generality, we may assume that x_ϵ is a maximum point of ϕ_ϵ. We proceed by proving two claims.

Claim 1: $\frac{|x_\epsilon - P_\epsilon|}{\epsilon} \le C$. Suppose not, that is $\frac{|x_\epsilon - P_\epsilon|}{\epsilon} \to +\infty$. Then $-1 + f'(\tilde{w}_{\epsilon,P_\epsilon}(x_\epsilon)) \le -\frac{1}{4}$ for ϵ small. Since $\frac{\partial \phi_\epsilon}{\partial \nu} = 0$, by the Hopf boundary Lemma, $x_\epsilon \notin \partial\Omega$. So $x_\epsilon \in \Omega$, which implies $\Delta\phi_\epsilon(x_\epsilon) \le 0$. From (2.5) we then deduce that

$$(1 - f'(\tilde{w}_{\epsilon,P_\epsilon}(x_\epsilon)))M_\epsilon + o(1)M_\epsilon + O(\epsilon^{\tau-1}) \le 0$$

and hence M_ϵ is bounded, a contradiction. Let $\hat{\phi}_\epsilon(y) = \frac{\phi_\epsilon(x)}{M_\epsilon}$, where $\epsilon y = x - P$.

Claim 2: $\hat{\phi}_\epsilon(y) \to 0$ in $C^1_{\mathrm{loc}}(R^N_+)$, as $\epsilon \to 0$. In fact, from the equation for $\hat{\phi}_\epsilon$, we see that as $\epsilon \to 0$, $\hat{\phi}_\epsilon \to \hat{\phi}_0$, where $\Delta\hat{\phi}_0 - \hat{\phi}_0 + f'(w)\hat{\phi}_0 = 0,\ |\hat{\phi}_0| \le 1,\ \text{in } R^N_+,\ \ \frac{\partial\hat{\phi}_0}{\partial y_N} = 0 \ \text{ on } \partial R^N_+$. By the nondegeneracy of w, there exist $N-1$ constants $a_1, ..., a_{N-1}$ such that $\hat{\phi}_0 = \sum_{j=1}^{N-1} a_j \frac{\partial w}{\partial y_j}$. On the other hand, we know that $\nabla_{x_k} u_\epsilon(P_\epsilon) = 0, k = 1, ..., N-1$ and hence

$$0 = \nabla_{x_k}(\tilde{w}_{\epsilon,P_\epsilon}(P_\epsilon) + \epsilon^\tau \phi_\epsilon(P_\epsilon)) = O(\epsilon) + \epsilon^{\tau-1} M_\epsilon \nabla_{y_k}\hat{\phi}_\epsilon(0).$$

Thus we have $\nabla_{y_k}\hat{\phi}_\epsilon(0) \to 0$ which shows that $\nabla_{y_k}\hat{\phi}_0(0) = 0$. This implies $\nabla_{y_k}\left(\sum_{j=1}^{N-1} a_j \frac{\partial w}{\partial y_j}\right)_{y=0} = 0$, $k = 1, ..., N-1$. Thus $a_1 = ... = a_{N-1} = 0$. This proves Claim 2.

Equation (2.3) now follows from Claim 1 and Claim 2: Let $y_\epsilon = \frac{x_\epsilon - P_\epsilon}{\epsilon}$. Then by Claim 1, $|y_\epsilon| \le C$. So we may assume that $y_\epsilon \to y_0$ as $\epsilon \to 0$. Since $\hat{\phi}_\epsilon(y_\epsilon) = 1$, we have $\hat{\phi}_0(y_0) = 1$, which contradicts Claim 2.

Multiplying (2.5) by ϕ_ϵ, integrating over Ω and using (2.3), we obtain (2.4).

□

3. Proofs of Theorem 1 and Corollary 2.

We prove Theorem 1 by using Lemma 3 and Lemma 4.

Proof of Theorem 1: Since both $\tilde{w}_{\epsilon,P_\epsilon}$ and ϕ_ϵ satisfy the Neumann boundary condition, we get

$$\begin{aligned} J_\epsilon[u_\epsilon] &= J_\epsilon[\tilde{w}_{\epsilon,P}] + \epsilon^\tau \int_\Omega (\epsilon^2 \nabla\tilde{w}_{\epsilon,P}\nabla\phi_\epsilon + \tilde{w}_{\epsilon,P}\phi_\epsilon - f(\tilde{w}_{\epsilon,P})\phi_\epsilon)\,dx \\ &+ \frac{\epsilon^{2\tau}}{2}\left(\int_\Omega (\epsilon^2|\nabla\phi_\epsilon|^2 + |\phi_\epsilon|^2)\,dx - \int_\Omega f'(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon^2\,dx\right) \\ &- \int_\Omega [F(\tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau\phi_\epsilon) - F(\tilde{w}_{\epsilon,P_\epsilon}) - \epsilon^\tau f(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon - \frac{\epsilon^{2\tau}}{2} f'(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon^2]\,dx. \end{aligned}$$

By Lemma 4, the last two terms are $o(\epsilon^{N+2})$. Now integrating by parts and using (2.1) we obtain

$$\epsilon^\tau \int_\Omega (\epsilon^2\nabla\tilde{w}_{\epsilon,P}\nabla\phi_\epsilon + \tilde{w}_{\epsilon,P}\phi_\epsilon - f(\tilde{w}_{\epsilon,P})\phi_\epsilon)\,dx = \epsilon^\tau \int_\Omega S_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}]\phi_\epsilon\,dx = O(\epsilon^{N+1+\tau+\sigma}).$$

Hence $J_\epsilon[u_\epsilon] = J_\epsilon[\tilde{w}_{\epsilon,P}] + o(\epsilon^{N+2})$ which, by Lemma 3, finishes the proof of Theorem 1. □

Next, we prove Corollary 2.

Proof of Corollary 2: Let u_ϵ be a least energy solution of (1.1). By Theorem 1, we have

$$c_\epsilon := J_\epsilon[u_\epsilon] = \epsilon^N\left[\frac{1}{2}I[w] - c_1\epsilon H(P_\epsilon) + \epsilon^2(c_2(H(P_\epsilon))^2 + c_3 R(P_\epsilon)) + o(\epsilon^2)\right]. \tag{3.1}$$

On the other hand, by using $\tilde{w}_{\epsilon,Q}$ as test function, we see that

$$c_\epsilon \le \epsilon^N\left[\frac{1}{2}I[w] - c_1\epsilon H(Q) + \epsilon^2(c_2(H(Q))^2 + c_3 R(Q)) + o(\epsilon^2)\right], \tag{3.2}$$

where we take Q such that $H(Q) = \max_{P\in\partial\Omega} H(P)$. Comparing (3.1) with (3.2), we arrive at

$$c_1 H(Q) - \epsilon(c_2(H(Q))^2 + c_3 R(Q)) + o(\epsilon) \le c_1 H(P_\epsilon) - \epsilon(c_2(H(P_\epsilon))^2 + c_3 R(P_\epsilon)) + o(\epsilon).$$

Since $c_1 > 0, c_3 > 0$, we obtain (1.5). □

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Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: wei@math.cuhk.edu.hk

Mathematisches Institut A, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany

E-mail address: winter@mathematik.uni-stuttgart.de