ASYMMETRIC SPOTTY PATTERNS FOR THE GRAY-SCOTT MODEL IN R^2

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Abstract. In this paper, we rigorously prove the existence and stability of asymmetric spotty patterns for the Gray-Scott model in a bounded two dimensional domain. We show that given any two positive integers k_1, k_2, there are asymmetric solutions with k_1 large spots (type **A**) and k_2 small spots (type **B**). We also give conditions for their location and calculate their heights. Most of these asymmetric solutions are shown to be unstable. However, in a narrow range of parameters, asymmetric solutions may be stable.

1. Introduction: The Gray-Scott Model

In this paper, we continue our study ([39]) of the Gray-Scott model in a bounded two-dimensional domain and rigorously prove existence and stability of **asymmetric spotty patterns**. These are the first results about asymmetric solutions for the Gray-Scott model.

Let us first recall the classical, irreversible Gray-Scott model [10], [11] which describes the kinetics of a simple autocatalytic reaction in an unstirred flow reactor. There is a substance V whose concentration is kept fixed outside the reactor and which is supplied through the walls into the reactor with rate F. The product P of the reaction is removed from the reactor with the same rate. Inside the reactor V undergoes a reaction involving an intermediate substance U. Furthermore, V reacts at the rate k to change into P. Both reactions are irreversible, so P is an inert product. These reactions are summarized as follows:

$$U + 2V \to 3V$$

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$$V \to P$$

The equations of chemical kinetics which describe the spatiotemporal changes of the concentrations of U and V in the reactor are given in in dimensionless units by

$$\begin{cases} V_t = D_V \Delta V - (F+k)V + UV^2 & \text{in } \Omega, \\ U_t = D_U \Delta U + F(1-U) - UV^2 & \text{in } \Omega, \\ \frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The unknowns $U = U(x,t)$ and $V = V(x,t)$ represent the concentrations of the two biochemicals at a point $x \in \Omega \subset R^2$ and at a time $t > 0$, respectively; $\Delta := \sum_{j=1}^{2} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in R^2; Ω is a bounded and smooth domain in R^2; $\nu(x)$ is the outer normal at $x \in \partial\Omega$; D_U, D_V are the diffusion coefficients of U and V, respectively.

Now we rewrite the system (1.1) in standard form. Dividing the first equation in (1.1) by $F+k$ and dividing the second equation in (1.1) by F we obtain

$$\begin{cases} \frac{1}{F+k} V_t = \frac{D_V}{F+k} \Delta V - V + \frac{1}{F+k} UV^2 & \text{in } \Omega, \\ \frac{1}{F} U_t = \frac{D_U}{F} \Delta U + 1 - U - \frac{1}{F} UV^2 & \text{in } \Omega, \\ \frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Setting $V = \sqrt{F}\hat{V}$ gives

$$\begin{cases} \frac{1}{F+k} \hat{V}_t = \frac{D_V}{F+k} \Delta \hat{V} - \hat{V} + \frac{\sqrt{F}}{F+k} U\hat{V}^2 & \text{in } \Omega, \\ \frac{1}{F} U_t = \frac{D_U}{F} \Delta U + 1 - U - U\hat{V}^2 & \text{in } \Omega, \\ \frac{\partial U}{\partial \nu} = \frac{\partial \hat{V}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Rescaling time $t = \frac{\hat{t}}{F+k}$ and introducing the variables $A = \frac{\sqrt{F}}{F+k}$, $\tau = \frac{F+k}{F} > 1$ we can rewrite

$$\begin{cases} \hat{V}_{\hat{t}} = \frac{D_V}{F+k} \Delta_{\hat{x}} \hat{V} - \hat{V} + AU\hat{V}^2 & \text{in } \Omega, \\ \tau U_{\hat{t}} = \frac{D_U}{F} \Delta_{\hat{x}} U + 1 - U - U\hat{V}^2 & \text{in } \Omega, \\ \frac{\partial U}{\partial \nu} = \frac{\partial \hat{V}}{\partial \nu} = 0 & \text{on } \partial\hat{\Omega}. \end{cases} \tag{1.4}$$

Letting $\epsilon^2 = \frac{D_V}{(F+k)}, D = \frac{D_U}{F}$ and dropping the hats we get

$$\begin{cases} v_t = \epsilon^2 \Delta v - v + Auv^2 & \text{in } \Omega, \\ \tau u_t = D\Delta u + 1 - u - uv^2 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

2. The Main Results: Spotty patterns for the Gray-Scott Model

In [39], we proved the existence and stability of **symmetric** $K-$spotty solutions in a two-dimensional domain. More precisely, we considered the stationary Gray-Scott model

$$\begin{cases} \epsilon^2 \Delta v - v + Auv^2 = 0 \quad \text{in } \Omega, \\ D\Delta u + 1 - u - uv^2 = 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.6}$$

for $\epsilon << 1$ and $D = D(\epsilon)$, where ϵ and D do not depend on $x \in \Omega$ and $\Omega \subset R^2$ is a bounded and smooth domain.

A $K-$spotty solution (v_ϵ, u_ϵ) of (2.6) is assumed to take the following form:

$$v_\epsilon \sim \sum_{j=1}^{K} \frac{1}{A\xi_{\epsilon,j}} w(\frac{x - P_j^\epsilon}{\epsilon}), \quad u_\epsilon(P_j^\epsilon) \sim \xi_{\epsilon,j}, \tag{2.7}$$

where $P_j^\epsilon, j = 1, ..., K$ are the locations of the K spots, $\xi_{\epsilon,j}$ is the height of the spot placed at P_j^ϵ, and w is the unique solution of the problem

$$\begin{cases} \Delta w - w + w^2 = 0, \quad w > 0 \quad \text{in } R^2, \\ w(0) = \max_{y \in R^2} w(y), \quad w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{2.8}$$

(For existence and uniqueness of the solutions of (2.8) we refer to [9] and [17].)

Now we introduce the two most important parameters

$$\eta_\epsilon = \frac{|\Omega|}{2\pi D} \log \frac{\sqrt{|\Omega|}}{\epsilon}, \quad L_\epsilon = \frac{\epsilon^2 \int_{R^2} w^2}{A^2 |\Omega|}. \tag{2.9}$$

Note that η_ϵ and L_ϵ are invariant under scaling of the domain.

In [39], we assumed that the $K-$spotty solution is asymptotically symmetric, i.e., as $\epsilon \to 0$, the heights of different spots are asymptotically equal,

$$\lim_{\epsilon \to 0} \frac{\xi_{\epsilon,j}}{\xi_{\epsilon,1}} = 1, \quad j = 2, ..., K \tag{2.10}$$

and showed the existence of symmetric $K-$spotty solutions which concentrate at nondegenerate critical points of a functional which is related to Green's function, provided that the following condition is satisfied

$$\lim_{\epsilon \to 0} 4(\eta_\epsilon + K)L_\epsilon < 1. \tag{2.11}$$

Concerning stability we studied the "large" eigenvalues of order $O(1)$ and the "small" eigenvalues of order $o(1)$ separately. We showed that the large eigenvalues are related to a nonlocal eigenvalue problem and the small eigenvalues are related to the second derivatives of the functional mentioned above. Suppose these small eigenvalues have negative real parts (compare Remark 2.1 below). Then for symmetric $K-$spotty solutions the following result holds true: if

$$\lim_{\epsilon\to 0}\frac{(2\eta_\epsilon+K)^2}{\eta_\epsilon}L_\epsilon<1 \tag{2.12}$$

then K-spotty solutions are stable for τ large or small. On the other hand, if

$$\lim_{\epsilon\to 0}\frac{(2\eta_\epsilon+K)^2}{\eta_\epsilon}L_\epsilon>1 \tag{2.13}$$

then $K-$spotty solutions are unstable for all $\tau>0$.

Naturally, the following questions arise:

Question: Do **asymmetric** K-spotty solutions exist? If yes, when are they stable? Can we characterize all asymmetric solutions?

In this paper we answer these questions. We first show that the heights $(\xi_{\epsilon,1},...,\xi_{\epsilon,K})$ must satisfy certain nonlinear algebraic equations which can be solved explicitly (Section 5). As a result, we show that asymmetric patterns can exist only if

$$\lim_{\epsilon\to 0}\eta_\epsilon=\eta_0\in(0,+\infty). \tag{2.14}$$

In other words, $D\sim C\log\frac{1}{\epsilon}$.

Furthermore, the heights for the asymmetric solutions are generated by exactly two kinds of spots – called type **A** and type **B**, respectively. Type **A** and type **B** spots have different heights. Fix any two integers $k_1\geq 1, k_2\geq 1$ such that $k_1+k_2=K\geq 2$. We show that if

$$\lim_{\epsilon\to 0}L_\epsilon:=L_0\leq\frac{\eta_0}{4(\eta_0+k_1)(\eta_0+k_2)}, \tag{2.15}$$

then there are asymmetric $K-$spotty solutions with k_1 type **A** spots and k_2 type **B** spots.

Let $K \geq 2$ be a positive integer and let k_1, $k_2 \geq 1$ be two integers such that

$$k_1 + k_2 = K. \tag{2.16}$$

To introduce the **heights of the** $K-$**spots**, we need to define four numbers. Set

$$\rho_+ = \frac{\eta_0 + \sqrt{\eta_0^2 - 4(k_1+\eta_0)(k_2+\eta_0)\eta_0 L}}{2(k_2+\eta_0)},$$

$$\rho_- = \frac{\eta_0 - \sqrt{\eta_0^2 - 4(k_1+\eta_0)(k_2+\eta_0)\eta_0 L_0}}{2(k_2+\eta_0)}; \tag{2.17}$$

$$\eta_+ = \frac{\eta_0 - \sqrt{\eta_0^2 - 4(k_1+\eta_0)(k_2+\eta_0)\eta_0 L_0}}{2(k_1+\eta_0)},$$

$$\eta_- = \frac{\eta_0 + \sqrt{\eta_0^2 - 4(k_1+\eta_0)(k_2+\eta_0)\eta_0 L_0}}{2(k_1+\eta_0)}. \tag{2.18}$$

Note that

$$\rho_+\eta_+ = \eta_0 L_0, \quad \rho_-\eta_- = \eta_0 L_0. \tag{2.19}$$

From now on, let either $(\rho, \eta) = (\rho_+, \eta_+)$ or $(\rho, \eta) = (\rho_-, \eta_-)$ and we drop the indices "+" or "−" if there is no confusion.

Let the heights of the $K-$spots $(\xi_1, \dots, \xi_K) \in R_+^K$ be such that

$$\xi_j = \rho \text{ or } \xi_j = \eta, \text{ and the number of } \rho\text{'s in } (\xi_1, \dots, \xi_K) \text{ is } k_1. \tag{2.20}$$

Then there are k_2 η's in $(\xi_1, \dots, \xi_K)$.

Concerning the **locations of the** $K-$**spots**, let $\mathbf{P} \in \Omega^K$, where $\mathbf{P}$ is arranged such that

$$\mathbf{P} = (P_1, P_2, \dots, P_K), \text{ with } P_i = (P_{i,1}, P_{i,2}) \quad \text{for } i = 1, 2, \dots, K.$$

For the rest of the paper, we assume that $\mathbf{P} \in \bar{\Lambda}$, where for $\delta > 0$ we define

$$\Lambda = \{(P_1, P_2, \dots, P_K) \in \Omega^K : |P_i - P_j| > 4\delta \text{ for } i \neq j$$
$$\text{and } d(P_i, \partial\Omega) > 4\delta \text{ for } i = 1, 2 \dots, K\}. \tag{2.21}$$

Let $G_0(x,\xi)$ be the Green's function

$$\begin{cases} \Delta G_0(x,\xi) - \frac{1}{|\Omega|} + \delta(x-\xi) = 0 & x,\xi \in \Omega, \\ \int_\Omega G_0(x,\xi)\,dx = 0, & \\ \frac{\partial G_0(x,\xi)}{\partial \nu_x} = 0 & x \in \partial\Omega, \xi \in \Omega \end{cases} \tag{2.22}$$

and let

$$H_0(x,\xi) = \frac{1}{2\pi} \log \frac{1}{|x-\xi|} - G_0(x,\xi)$$

be the regular part of $G_0(x,\xi)$.

For $\mathbf{P} \in \overline{\Lambda}$, we define

$$F_0(\mathbf{P}) = \sum_{k=1}^K H_0(P_k,P_k)\frac{1}{\xi_k^2} - \sum_{i,j=1,\dots,K, i\neq j} G_0(P_i,P_j)\frac{1}{\xi_i\xi_j} \tag{2.23}$$

and

$$M_0(\mathbf{P}) = \nabla^2_{\mathbf{P}} F_0(\mathbf{P}) \tag{2.24}$$

Here $M(\mathbf{P})$ is a $(2K)\times(2K)$ matrix with components $\frac{\partial^2 F(\mathbf{P})}{\partial P_{i,j}\partial P_{k,l}}, i,k = 1,...,K, j,l = 1,2$, (recall that $P_{i,j}$ is the j-th component of P_i).

Note that $F_0(\mathbf{P}) \in C^\infty(\overline{\Lambda})$.

To summarize, **throughout the paper, we assume that**

$$\epsilon << 1, \quad \tau \geq 0, \tag{2.25}$$

$$\eta_0 \in (0,+\infty), \quad L_0 \leq \frac{\eta_0}{4(\eta_0+k_1)(\eta_0+k_2)}, \tag{2.26}$$

and that the following technical condition holds

$$(T1) \qquad L_0 \neq \frac{\eta_0}{(2\eta_0+K)^2}. \tag{2.27}$$

Furthermore, let $C > 0$ be a generic constant which is independent of ϵ and D and may change from line to line and δ is a very small but fixed constant. We always assume that $\mathbf{P}, \mathbf{P}^0 \in \overline{\Lambda}$, where $\overline{\Lambda}$ was defined in (2.21) and that $|\mathbf{P}-\mathbf{P}^0| < 4\delta$. To simplify our notation, we use $e.s.t.$ to denote exponentially small terms in the corresponding norms, more precisely, $e.s.t. = O(e^{-\delta/\epsilon})$. The notation $A(\epsilon) \sim B(\epsilon)$ means that $\lim_{\epsilon\to 0} \frac{A(\epsilon)}{B(\epsilon)} = 1 > 0$, for some positive number c_0.

We shall frequently consult and use the results of the paper [37].

Our first result is on the existence of asymmetric K-spotty patterns.

Theorem 2.1. *(Existence of* **asymmetric** *solutions). Assume that* $\epsilon << 1$ *and that*

$$\lim_{\epsilon\to 0}\frac{|\Omega|}{2\pi D}\log\frac{\sqrt{|\Omega|}}{\epsilon}=\eta_0\in(0,\infty),$$

$$\lim_{\epsilon\to 0}L_\epsilon=\frac{\epsilon^2\int_{R^2}w^2(y)\,dy}{A^2|\Omega|}=L_0,$$

$$L_0\le\frac{\eta_0}{4(\eta_0+k_1)(\eta_0+k_2)},$$

$$(T1)\qquad\qquad L_0\neq\frac{\eta_0}{(2\eta_0+K)^2}.$$

Let $(\xi_1,...,\xi_K)$ *be given by (2.20) and* $\mathbf{P}_0=(P_1^0,P_2^0,\dots,P_K^0)\in\overline{\Lambda}$ *be a nondegenerate critical point of* $F_0(\mathbf{P})$ *(defined by (2.23)). Then problem (2.6) has a stationary solution* (v_ϵ,u_ϵ) *with the following properties:*

(1) $v_\epsilon(x)=\sum_{j=1}^K\frac{1}{A\xi_{\epsilon,j}}(w(\frac{x-P_j^\epsilon}{\epsilon})+O(\frac{1}{\log\frac{1}{\epsilon}}))$ *uniformly for* $x\in\bar{\Omega}$*, where* $\xi_{\epsilon,j}\to\xi_j$ *and* ξ_j *is defined by (2.20).*

(2) $u_\epsilon(P_j^\epsilon)=\xi_{\epsilon,j}(1+O(\frac{1}{\log\frac{1}{\epsilon}}))$*.*

(3) $P_j^\epsilon\to P_j^0$ *as* $\epsilon\to 0$ *for* $j=1,...,K$*.*

Several remarks are in the order.

Remark 2.1. The condition on the locations of $\mathbf{P}_0$ is not so severe. For any bounded smooth domain Ω, the functional $F_0(\mathbf{P})$ always admits a global maximum at some $\mathbf{P}_0\in\overline{\Lambda}$. In fact, this can be seen very easily: if $|P_i-P_j|\to 0$ or $d(P_i,\partial\Omega)\to 0$, then $F_0(\mathbf{P})\to-\infty$. (Note that as $d(P_i,\partial\Omega)\to 0$, $H_0(P_i,P_i)\to-\infty$.) This point $\mathbf{P}_0$ is a critical point of $F_0(\mathbf{P})$. If $\mathbf{P}_0$ is also a nondegenerate critical point of $F_0(\mathbf{P})$, then the matrix $M_0(\mathbf{P}_0)$ has only negative eigenvalues. (It is an interesting question to numerically compute the critical points of $F_0(\mathbf{P})$. Some interesting results on $F_0(\mathbf{P})$ are contained in a recent work [16].)

Remark 2.2. Note that if

$$L_0<\frac{\eta_0}{(2\eta_0+K)^2},$$

then (2.26) holds for any $k_1+k_2=K$. Thus there will be 2^{K-1} choices of $(\xi_1,...,\xi_K)$. So if the matrix $M_0(\mathbf{P}_0)$ is nondegenerate, we will have 2^{K-1}

asymmetric solutions. If $L_0 = \frac{\eta_0}{4(\eta_0+k_1)(\eta_0+k_2)}$, then there will be 2^{K-2} asymmetric solutions.

Next we study the stability and instability of the asymmetric K-spotty solutions constructed in Theorem 2.1.

Linearizing (1.5) around the equilibrium states (v_ϵ, u_ϵ)

$$\begin{cases} v = v_\epsilon + \phi_\epsilon e^{\lambda_\epsilon t}, \\ u = u_\epsilon + \psi_\epsilon e^{\lambda_\epsilon t}, \end{cases}$$

and substituting the result into (1.5) we deduce the following eigenvalue problem

$$\mathcal{L}_\epsilon \begin{pmatrix} \phi_\epsilon \\ \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} \epsilon^2 \Delta\phi_\epsilon - \phi_\epsilon + 2Au_\epsilon\phi_\epsilon + Av_\epsilon^2\psi_\epsilon, \\ \frac{1}{\tau}(D\Delta\psi_\epsilon - \psi_\epsilon - 2u_\epsilon\phi_\epsilon - v_\epsilon^2\psi_\epsilon) \end{pmatrix} = \lambda_\epsilon \begin{pmatrix} \phi_\epsilon \\ \psi_\epsilon \end{pmatrix}, \tag{2.28}$$

We say that (v_ϵ, u_ϵ) is **linearly stable** if the spectrum $\sigma(\mathcal{L}_\epsilon)$ of $\mathcal{L}_\epsilon$ lies in the left half plane $\{\lambda \in \mathcal{C} : \text{ Re } (\lambda) < 0\}$. (v_ϵ, u_ϵ) is called **linearly unstable** if there exists an eigenvalue λ_ϵ of $\mathcal{L}_\epsilon$ with Re $(\lambda_\epsilon) > 0$. (From now on, we use the notations linearly stable and linearly unstable as defined above.)

Theorem 2.2. *(Stability of* **asymmetric** *solutions). Let the assumptions of Theorem 2.1 be satisfied. Let $\mathbf{P}_0$ be a nondegenerate critical point of $F_0(\mathbf{P})$ and let (v_ϵ, u_ϵ) be the asymmetric $K-$spotty solutions constructed in Theorem 2.1 for ϵ sufficiently small, whose spots are located near $\mathbf{P}_0 \in \overline{\Lambda}$.*

*(a) (***Stability***)*

Assume that

$$\frac{\eta_0}{(2\eta_0 + K)^2} < L_0 \le \frac{\eta_0}{4(\eta_0 + k_1)(\eta_0 + k_2)} \tag{2.29}$$

and

$$k_1 > k_2, \quad (\rho, \eta) = (\rho_+, \eta_+)$$

(compare (2.20).

Suppose that $M_0(\mathbf{P}_0)$ has only negative eigenvalues. Then for τ small enough, (v_ϵ, u_ϵ) is linearly stable.

*(b) (***Instability***)*

Assume that either

$$L_0 < \frac{\eta_0}{(2\eta_0 + K)^2}$$

or

$$\tau \textit{ is large enough}$$

or

$$k_1 > k_2,\ (\rho, \eta) = (\rho_-, \eta_-).$$

Then (v_ϵ, u_ϵ) *is linearly unstable.*

Remark 2.3. By the Remark 2.1, if the global maximum point $\mathbf{P}_0$ of $F_0(\mathbf{P})$ is nondegenerate, then the matrix $M_0(\mathbf{P}_0)$ has only negative eigenvalues.

We believe that for other types of critical points of $F_0(\mathbf{P})$, such as saddle points, the solution constructed in Theorem 2.1 should be linearly unstable. We are not able to prove this at the moment, since the operator $\mathcal{L}_\epsilon$ is **not self-adjoint**.

The proof of our main results will be organized as follows:

In Section 4, we study the properties of w as well as some nonlocal eigenvalue problems (NLEPs). This section provides the key steps in the derivation of the critical thresholds for stability.

In Section 5, we formally compute the algebraic equations for the heights of the spots and then we solve them up to $o(1)$.

Sections 4 and 5 both provide some preliminary analysis which uses only the leading-order asymptotics for the steady state. Therefore this is done first.

From Section 6 to Section 8, we rigorously prove the existence result, Theorem 2.1, by the Liapunov-Schmidt reduction procedure: Section 6 contains the construction of good approximate functions, in Section 7 we perform the reduction process (the proofs of Proposition 7.1 and Proposition 7.2 have been moved to Appendix A) , and finally, in Section 8, we solve the reduced problem by Brouwer's fixed point theorem.

Section 9 provides the crucial part of the stability analysis which deals with large eigenvalues.

The analysis of the small eigenvalues including rigorous error estimates is similar to [37]. Therefore this is done in Appendix B. We will see that the asymptotic behavior of small eigenvalues can be characterized in terms of the matrix $M_0(\mathbf{P}_0)$.

We conclude the paper with a short section (Section 10) in which we summarize our results.

3. Discussion: Patterns for Turing systems

Let us compare our results with previous work on pattern formation for Turing systems, first for the Gray-Scott model, at the end of the section also for other Turing systems.

One of the most interesting phenomena related to the Gray-Scott model is the so-called "self-replicating" pattern which has been observed and explained in a number of studies. First, in 1993, Pearson [23] presented some numerical simulations on the Gray-Scott model in a square of size 2.5 in R^2 with periodic boundary conditions. By choosing $D_U = 2 \times 10^{-5}, D_V = 10^{-5}$ and varying the parameters F and k, several interesting patterns were discovered. It was shown that spots may replicate in a self-sustaining fashion and develop into a variety of time-dependent and time-independent asymptotic states. Lin, McCormick, Pearson and Swinney [18] reported their chemical experiments in a ferro-cyanide-iodate-sulfite reaction which showed strong qualitative agreement with the self-replication regimes in simulations of [23]. Moreover, those same experiments led to the discovery of other new patterns, such as annular patterns emerging from circular spots. See [19] for more details on the set-up.

In 1-D, numerical simulations were done by Reynolds, Pearson and Ponce-Dawson [25], [26], independently by Petrov, Scott and Showalter [24] and again self-replication phenomena were observed. However, in 1-D, self-replication patterns were observed when $D_U = 1, D_V = \delta^2 = 0.01$. Some formal asymptotics and dynamics in 1-D are contained in [25] and [24]. Recent numerical simulations of [6] in 1-D and [22], [20] in 2-D show that the single spot may be stable in some very narrow parameter regimes.

The first rigorous result in constructing single spot (or pulse or spike) solutions is due to Doelman, Kaper and Zegeling in 1997 [6]. Using the

Mel'nikov method, they constructed single and multiple pulse solutions for (1.1) in the case $N = 1, D_U = 1, D_V = \delta^2 << 1$. In their paper [6], it is assumed that $F \sim C\delta^2, F + k \sim C\delta^{2\alpha/3}$, where $\alpha \in [0, \frac{3}{2})$. In this case, they showed that $U = O(\delta^\alpha)$, $V = O(\delta^{-\frac{\alpha}{3}})$. Later the stability of single and multiple pulse solutions in 1-D are shown in [3], [4]. Hale, Peletier and Troy studied the case $D_U = D_V$ in 1-D and the existence of single and multiple pulse solutions are established in [13], [14]. Nishiura and Ueyama proposed a skeleton structure of self-replicating dynamics in [22]. Some related results on the existence and stability of solutions to the Gray-Scott model in 1-D can be found in [7], [8] and [25].

Muratov and Osipov have given some formal asymptotic analysis on the construction and stability of spotty solutions in R^2 and R^3 citemo. In [32], the system (1.1) in R^2 is studied for the shadow system case, namely, $D_U >> 1, D_V << 1$ and $F = O(1), F + k = O(1)$. Note that the shadow system can be reduced to a single equation. In [34], (1.1) is studied for in R^2 and rigorous results on existence and stability of single spotty ground states are established.

We now compare our results on K-spotty patterns for the Gray-Scott system with results on similar patterns for other Turing systems. Similar results have been obtained for the Gierer-Meinhardt system

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^2}{H} & \text{in } \Omega, \\ \tau H_t = D \Delta H - H + A^2 & \text{in } \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.30}$$

We now describe these results in some detail. When $\Omega = (-1, 1) \subset R^1$, I. Takagi [27] first showed the existence of symmetric $K-$spike solutions with spikes distributed at equal distance. The stability of such symmetric $K-$peaked solutions was completely characterized for τ small in [15] by using matched asymptotic analysis. Later, the authors gave a rigorous proof by using the Liapunov-Schmidt reduction method [40]. The case of finite τ has been studied recently in [30]. When $\Omega = R^1$, Doelman, Gardner and Kaper [2] studied the stability of single and multiple pulses for any $\tau > 0$. For asymmetric patterns, M. Ward and the first author in [29] showed that for $D < D_K = \frac{1}{K^2(\log[\sqrt{3}])^2}$, problem (3.30) has asymmetric $K-$spike solutions which are again generated by two types of spikes with different heights which can be arranged in any given order. Also the stability of such asymmetric $K-$spike solutions is studied in [29]. Numerical computations show that in 1-D all the asymmetric spikes are unstable with respect to the small eigenvalues. By using a different approach (geometric singular perturbation method), Doelman, Kaper and van der Ploeg [5] also established the existence of asymmetric patterns for D sufficiently small (i.e., the domain is sufficiently large). Also some other interesting asymmetric patterns such as multiple clusters of spikes are discovered in [5].

When $\Omega \subset R^2$, symmetric and asymmetric spotty solutions for (3.30) are studied by the authors in [37], [38]. It is shown that symmetric $K-$spots exist in a wide range of $D >> 1$ and these solutions are stable if and only if

$$D < D_K = \frac{|\Omega|}{2\pi K} \log \frac{\sqrt{|\Omega|}}{\epsilon}.$$

In R^2, we can completely characterize the heights of the spots of asymmetric patterns. For the Gierer-Meinhardt system we have obtained a similar phenomenon as for the Gray-Scott model: Asymmetric patterns are generated by exactly two different heights. (The reason behind this is unclear.) Furthermore, asymmetric patterns can be stable, even though the stability region given in Theorem 2.2 is rather narrow. Finally, it is found that in R^2 the stability of asymmetric patterns (in leading order) does not depend on the locations.

To the authors' knowledge, there are no results on the existence of asymmetric patterns for the Gray-Scott model in R^1. We believe that asymmetric patterns in R^1 do exist.

Finally, we remark that the Gray-Scott model and Gierer-Meinhardt system both belong to the so-called Turing systems, [28], [21]. However, they have different behavior: the Gierer-Meinhardt system is an activator-inhibitor system while the Gray-Scott model is an autocatalytic (feed-back) system, [21]. We have shown that both systems admit symmetric and asymmetric patterns. More importantly, in both systems, asymmetric patterns are generated by exactly **two** patterns. An interesting open question is: Are all asymmetric patterns in Turing systems generated by exactly two patterns? If not, what are suitable (necessary and/or sufficient) conditions for this behavior.

4. Preliminaries I: Some Properties of w and the Study of NLEPs

Let w be the unique solution of (2.8). In this section, we study some properties of w as well as some NLEPs. This section provide the key results which are necessary for the proofs of Theorem 2.1 and Theorem 2.2.

Let

$$L_0\phi = \Delta\phi - \phi + 2w\phi, \quad \phi \in H^2(R^2). \tag{4.1}$$

We first recall the following well-known result:

Lemma 4.1. *(Lemma 2.1 of* [37]*.) The eigenvalue problem*

$$L_0\phi = \nu\phi, \quad \phi \in H^2(R^2), \tag{4.2}$$

admits the following set of eigenvalues

$$\nu_1 > 0, \quad \nu_2 = \nu_3 = 0, \quad \nu_4 < 0, \dots . \tag{4.3}$$

The eigenfunction Φ_0 corresponding to ν_1 can be made positive and radially symmetric; the space of eigenfunctions corresponding to the eigenvalue 0 *is*

$$K_0 := span \left\{ \frac{\partial w}{\partial y_j}, j = 1, 2 \right\}. \tag{4.4}$$

Next, we consider the following nonlocal eigenvalue problems (NLEPs)

$$L\phi := \Delta\phi - \phi + 2w\phi - f(\tau\lambda_0)\frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = \lambda_0\phi, \phi \in H^2(R^2), \tag{4.5}$$

where w is the unique solution of (2.8), $f(\tau\lambda_0)$ is a continuous function in $\mathcal{C}$ and $f(t) \in R$ for $t \in R$.

We first have

Lemma 4.2. *If $f(0) < 1$ and $0 < c \le f(\alpha)$ for $\alpha > 0$, then there exists a positive eigenvalue of (4.5) for any $\tau > 0$.*

Proof: The proof is similar to Lemma 2.3 of [37]. For the reader's convenience, we include a proof here.

First, we may assume that ϕ is a radially symmetric function, namely, $\phi \in H_r^2(R^2) = \{u \in H^2(R^2) | u = u(|y|)\}$. Let $L_0 = \Delta - 1 + 2w$. Then L_0 is invertible in $H_r^2(R^2)$. Let us denote the inverse as L_0^{-1}. On the other hand, by Lemma 4.1, L_0 has a unique positive eigenvalue, ν_1. Moreover the corresponding eigenfunction is of constant sign. So we may assume that $f(0) \neq 0, \lambda_0 \neq \nu_1$.

Then $\lambda_0 > 0$ is an eigenvalue of (4.5) if and only if it satisfies the following algebraic equation:

$$\int_{R^2} w^2 = f(\tau\lambda_0) \int_{R^2} [((L_0 - \lambda_0)^{-1} w^2) w]. \tag{4.6}$$

Equation (4.6) can be simplified further to the following

$$\rho(\lambda_0) := (1 - f(\tau\lambda_0)) \int_{R^2} w^2 - \lambda_0 f(\tau\lambda_0) \int_{R^2} [((L_0 - \lambda_0)^{-1} w) w] = 0. \tag{4.7}$$

Note that $\rho(0) = (1 - f(0)) \int_{R^2} w^2 > 0$. As $\lambda_0 \to \nu_1$, $0 < \lambda_0 < \nu_1$, we have $\int_{R^2} ((L_0 - \lambda_0)^{-1} w) w \to +\infty$ and hence $\rho_0(\lambda_0) \to -\infty$. By continuity, there exists an $\lambda_0 \in (0, \nu_1)$ such that $\rho(\lambda_0) = 0$. Such a positive λ_0 will be an eigenvalue of L.

□

Similarly, we have

Lemma 4.3. *If $\lim_{\tau \to +\infty} f(\tau\lambda) = f_{+\infty} < 1$ and $0 < c \le f(\alpha)$ for $\alpha > 0$, then there exists a positive eigenvalue of (4.5) for $\tau > 0$ large.*

Proof: Using the same notation as in the proof of Lemma 4.2, we fix a $\lambda_1 \in (0, \nu_1)$ so that $\lambda_0 \int_{R^2} [((L_0 - \lambda_0)^{-1} w) w] < (1 - f_{+\infty}) \int_{R^2} w^2$. For τ large, it is easy to see that $\rho(\lambda_1) > 0$. Now the rest follows from the proof of Lemma 4.2.

□

Next we consider the case when $f(0) > 1$. To this end, we need the following lemma:

Lemma 4.4. *Consider the eigenvalue problem*

$$\Delta\phi - \phi + 2w\phi - \gamma \frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = \lambda_0 \phi, \phi \in H^2(R^2), \tag{4.8}$$

where w is the unique solution of (2.8) and γ is real.

(1) If $\gamma > 1$, then there exists a positive constant c_0 such that $Re(\lambda_0) \leq -c_0$ for any nonzero eigenvalue λ_0 of (4.8).

(2) If $\gamma < 1$, then there exists a positive eigenvalue λ_0 of (4.8).

(3) If $\gamma \neq 1$ and $\lambda_0 = 0$, then $\phi \in$ span $\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$.

(4) If $\gamma = 1$ and $\lambda_0 = 0$, then $\phi \in$ span $\{w, \frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$.

Proof: (1), (3) and (4) have been proved in Theorem 5.1 of [33]. (2) follows from Lemma 4.2. □

Lemma 4.5. *Suppose that $f(0) > 1$ and $|f(z)| \leq C$ for all z with $Re(z) \geq 0$. Then for τ small, there exists a positive constant c_0 such that $Re(\lambda_0) \leq -c_0$ for any nonzero eigenvalue λ_0 of (4.5).*

Proof: This follows from a standard perturbation result; for the reader's convenience we explain the details.

We apply the following inequality (Lemma 5.1 in [33]): for any (real-valued) $\phi \in H_r^2(R^2)$, we have

$$\int_{R^2} (|\nabla\phi|^2 + \phi^2 - 2w\phi^2) + 2\frac{\int_{R^2} w\phi \int_{R^2} w^2\phi}{\int_{R^2} w^2} - \frac{\int_{R^2} w^3}{(\int_{R^2} w^2)^2} (\int_{R^2} w\phi)^2 \geq 0, \tag{4.9}$$

where equality holds if and only if ϕ is a multiple of w.

Now let $\phi = \phi_R + \sqrt{-1}\phi_I$ satisfy (4.5). Then we have

$$L_0\phi - f(\tau\lambda_0) \frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = \lambda_0 \phi. \tag{4.10}$$

Multiplying (4.10) by $\bar{\phi}$ – the conjugate function of ϕ – and integrating over R^2, we obtain that

$$\int_{R^2} (|\nabla\phi|^2 + |\phi|^2 - 2w|\phi|^2) = -\lambda_0 \int_{R^2} |\phi|^2 - f(\tau\lambda_0) \frac{\int_{R^2} w\phi}{\int_{R^2} w^2} \int_{R^2} w^2 \bar{\phi}. \tag{4.11}$$

Multiplying (4.10) by w and integrating over R^2, we obtain that

$$\int_{R^2} w^2\phi = (\lambda_0 + f(\tau\lambda_0)\frac{\int_{R^2} w^3}{\int_{R^2} w^2})\int_{R^2} w\phi. \tag{4.12}$$

Hence

$$\int_{R^2} w^2\bar{\phi} = (\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{R^2} w^3}{\int_{R^2} w^2})\int_{R^2} w\bar{\phi}. \tag{4.13}$$

Substituting (4.13) into (4.11), we have that

$$\int_{R^2}(|\nabla\phi|^2 + |\phi|^2 - 2w|\phi|^2)$$

$$= -\lambda_0\int_{R^2}|\phi|^2 - f(\tau\lambda_0)(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{R^2} w^3}{\int_{R^2} w^2})\frac{|\int_{R^2} w\phi|^2}{\int_{R^2} w^2}. \tag{4.14}$$

We just need to consider the real part of (4.14). Now applying the inequality (4.9) and using (4.13) we arrive at

$$-\lambda_R \geq \mathrm{Re}(f(\tau\lambda_0)(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{R^2} w^3}{\int_{R^2} w^2})) - 2\mathrm{Re}(\bar{\lambda}_0 + f(\tau\bar{\lambda}_0)\frac{\int_{R^2} w^3}{\int_{R^2} w^2}) + \frac{\int_{R^2} w^3}{\int_{R^2} w^2}$$

where λ_R is the real part of λ_0.

Assuming that $\lambda_R \geq 0$, then we have

$$\frac{\int_{R^2} w^3}{\int_{R^2} w^2}|f(\tau\lambda_0) - 1|^2 + \mathrm{Re}(\bar{\lambda}_0(f(\tau\lambda_0) - 1)) \leq 0. \tag{4.15}$$

On the other hand, since $|f(\tau\lambda_0)| \leq C$ for some constant $C > 0$, from (4.14) see that $|\lambda_0| \leq C$ (independent of τ). Since $f(\tau\lambda_0) \to f(0)$ as $\tau \to 0$, we see that, for τ small, (4.15) can not hold, which implies that $\lambda_R \leq c < 0$. □

5. Preliminaries II: Calculating the heights of the spots

In this section we calculate the heights of the spots as needed in the sections below. It is found that the heights depend on the number of spots but not on their locations. This leading order asymptotic analysis is valid for $\epsilon \to 0$. A rigorous derivation of the heights $\xi_{\epsilon,j}$ will be given in Lemma 6.1 below.

Let

$$\beta = \frac{1}{\sqrt{D}}. \tag{5.1}$$

By assumption (2.26), $\beta \sim C\frac{1}{\sqrt{\log\frac{1}{\epsilon}}}$.

Let $G_\beta(x,\xi)$ be the Green's function

$$\begin{cases} \Delta G_\beta(x,\xi) - \beta^2 G_\beta(x,\xi) + \delta(x-\xi) = 0 & x,\xi \in \Omega, \\ \dfrac{\partial G_\beta(x,\xi)}{\partial \nu_x} = 0 & x \in \partial\Omega, \xi \in \Omega. \end{cases} \tag{5.2}$$

The relation between G_0 and G_β is given by the following lemma, whose proof is simple and is given in Section 3 of [37].

Lemma 5.1. *For $\beta << 1$, we have*

$$G_\beta(x,\xi) = \frac{\beta^{-2}}{|\Omega|} + G_0(x,\xi) + O(\beta^2) \tag{5.3}$$

in the operator norm of $L^2(\Omega) \to H^2(\Omega)$. (Note that the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$ is compact.)

We define cut-off functions as follows: Let $r_0 = \frac{\delta}{4} > 0$ and χ be a smooth cut-off function which is equal to 1 in $B_1(0)$ and equal to 0 in $R^2 \setminus \overline{B_2(0)}$.

Let us assume the following ansatz for (v_ϵ, u_ϵ):

$$\begin{cases} v_\epsilon \sim \sum_{j=1}^K \frac{1}{A\xi_{\epsilon,j}} w(\frac{x-P_j^\epsilon}{\epsilon})\chi_{\epsilon,j}(x), \\ u_\epsilon(P_j^\epsilon) \sim \xi_{\epsilon,j}, \end{cases} \tag{5.4}$$

where w is the unique solution of (2.8), $(P_1^\epsilon, ..., P_K^\epsilon) \in \overline{\Lambda}$, $\xi_{\epsilon,j}$ is the height of the spot at P_j^ϵ, and

$$\chi_{\epsilon,j}(x) = \chi\left(\frac{x-P_j^\epsilon}{r_0}\right), \qquad x \in \Omega, \quad j = 1, \ldots, K, \tag{5.5}$$

From the equation for u_ϵ in (2.6),

$$\Delta(1-u_\epsilon) - \beta^2(1-u_\epsilon) + \beta^2 u_\epsilon v_\epsilon^2 = 0,$$

we get by (5.3)

$$1 - u_\epsilon(P_i^\epsilon) = 1 - \xi_{\epsilon,i} = \int_\Omega G_\beta(P_i^\epsilon,\xi)\beta^2 u_\epsilon(\xi) v_\epsilon^2(\xi)\, d\xi$$

$$= \int_\Omega (\frac{\beta^{-2}}{|\Omega|} + G_0(P_i^\epsilon,\xi) + O(\beta^2))\beta^2 \left(\sum_{j=1}^K \frac{1}{A^2\xi_{\epsilon,j}^2} w^2(\frac{\xi - P_j^\epsilon}{\epsilon})\xi_{\epsilon,j} + e.s.t.\right) u_\epsilon\, d\xi$$

$$= \int_\Omega (\frac{1}{|\Omega|} + \beta^2 G_0(P_i^\epsilon, \xi) + O(\beta^4)) \left(\sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} w^2(\frac{\xi - P_j^\epsilon}{\epsilon}) \right) \, d\xi.$$

Thus

$$1 - \xi_{\epsilon,i} = \sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} \frac{\epsilon^2}{|\Omega|} \int_{R^2} w^2(y)\, dy + \frac{1}{A^2 \xi_{\epsilon,i}} \beta^2 \int_\Omega G_0(P_i^\epsilon, \xi) w^2(\frac{\xi - P_i^\epsilon}{\epsilon})\, d\xi$$

$$+\beta^2 \sum_{j \neq i} G_0(P_i^\epsilon, P_j^\epsilon) \frac{1}{A^2 \xi_{\epsilon,j}} \epsilon^2 \int_{R^2} w^2(y)\, dy + \sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} O(\beta^4 \epsilon^2). \tag{5.6}$$

Using the expansion for G_0 in (5.6) gives

$$1 - \xi_{\epsilon,i} = \sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} \frac{\epsilon^2}{|\Omega|} \int_{R^2} w^2(y)\, dy$$

$$+\frac{1}{A^2 \xi_{\epsilon,i}} \beta^2 \int_\Omega \left(\frac{1}{2\pi} \log \frac{1}{|P_i^\epsilon - \xi|} - H_0(P_i^\epsilon, \xi) \right) w^2(\frac{\xi - P_i^\epsilon}{\epsilon})\, d\xi$$

$$+ \sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} O(\beta^2 \epsilon^2)$$

$$= \sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} \frac{\epsilon^2}{|\Omega|} \int_{R^2} w^2(y)\, dy$$

$$+ \frac{1}{A^2 \xi_{\epsilon,i}} \frac{\beta^2}{2\pi} \epsilon^2 \log \frac{1}{\epsilon} \int_{R^2} w^2(y)\, dy + \sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} O(\beta^2 \epsilon^2). \tag{5.7}$$

Note that $H_0 \in C^2(\Omega \times \Omega)$.

Recall the definition of η_ϵ and L_ϵ in (2.9). Then from (5.7) we get the basic equation for the heights

$$1 - \xi_{\epsilon,i} - \frac{\eta_\epsilon L_\epsilon}{\xi_{\epsilon,i}} = \sum_{j=1}^K \frac{L_\epsilon}{\xi_{\epsilon,j}} + O(\sum_{j=1}^K \frac{\beta^2 L_\epsilon}{\xi_{\epsilon,j}}), \quad i = 1, ..., K. \tag{5.8}$$

Assuming asymptotically that

$$\lim_{\epsilon \to 0} \xi_{\epsilon,j} = \xi_j, \quad j = 1, ..., K, \tag{5.9}$$

we obtain the following system of algebraic equations

$$1 - \xi_i - \frac{\eta_0 L_0}{\xi_i} = \sum_{j=1}^K \frac{L_0}{\xi_j}, \quad i = 1, ..., K. \tag{5.10}$$

Since we are studying asymmetric patterns, there must be at least one $i \geq 2$ such that $\xi_i \neq \xi_1$. Without loss of generality, we may assume that $\xi_2 \neq \xi_1$. We now claim that for $i \geq 2$ we have $\xi_i \in \{\xi_1, \xi_2\}$. To this end, let

$$\rho(t) = 1 - t - \frac{\eta_0 L}{t}. \tag{5.11}$$

Then we have

$$\rho(\xi_i) = \sum_{j=1}^{K} \frac{L}{\xi_j}. \tag{5.12}$$

Hence

$$\rho(\xi_i) = \rho(\xi_j) \quad \text{for} \quad i \neq j. \tag{5.13}$$

That is

$$(\xi_i - \xi_j)\left(1 - \frac{\eta_0 L_0}{\xi_i \xi_j}\right) = 0. \tag{5.14}$$

Hence for $i \neq j$ we have

$$\xi_i - \xi_j = 0 \quad \text{ or } \ \xi_i \xi_j = \eta_0 L_0. \tag{5.15}$$

Since $\xi_1 \neq \xi_2$, we have

$$\xi_1 \xi_2 = \eta_0 L_0. \tag{5.16}$$

Let us calculate ξ_j, $j = 3, \ldots, K$. If $\xi_j \neq \xi_1$, then $\xi_j \xi_2 = \eta_0 L_0$, which implies that $\xi_j = \xi_2$. Thus for $j \geq 3$, we have either $\xi_j = \xi_1$, or $\xi_j = \xi_2$.

Let k_1 be the number of ξ_1's in $\{\xi_1, \ldots, \xi_K\}$ and k_2 the number of ξ_2's in $\{\xi_1, \ldots, \xi_K\}$. Then this implies (2.16) with $k_1 \geq 1$, $k_2 \geq 1$.

Now from (5.11) and (5.12) we have

$$1 - \xi_1 = \frac{(k_1 + \eta_0) L_0}{\xi_1} + \frac{k_2 L_0}{\xi_2}, \tag{5.17}$$

and (5.16) implies

$$\xi_2 = \frac{\eta_0 L_0}{\xi_1}. \tag{5.18}$$

Substituting (5.18) into (5.17), we obtain

$$1 - \xi_1 = \frac{(k_1 + \eta_0) L_0}{\xi_1} + \frac{k_2}{\eta_0} \xi_1$$

and therefore

$$(k_2+\eta_0)\xi_1^2-\eta_0\xi_1+(k_1+\eta_0)\eta_0 L_0=0. \tag{5.19}$$

(5.19) has a solution if and only if

$$\eta_0\geq 4(k_1+\eta_0)(k_2+\eta_0)L_0 \tag{5.20}$$

which is ensured by (2.26). It is easy to see that the solutions to (5.19) are given by $(\rho_\pm,\eta_\pm)$ (defined in (2.17) and (2.18)). Let

$$\xi_1=\rho_\pm,\quad \xi_2=\eta_\pm.$$

We conclude that: if $L_0<\frac{\eta_0}{4(k_1+\eta_0)(k_2+\eta_0)}$, there exist two solutions (ξ_1,ξ_2) to (5.19). If $L_0=\frac{\eta_0}{4(k_1+\eta_0)(k_2+\eta_0)}$, there exists one solution (ξ_1,ξ_2). If $L_0>\frac{\eta_0}{4(k_1+\eta_0)(k_2+\eta_0)}$, there are no solutions.

Let us fix the height $(\xi_1,\xi_2,\ldots,\xi_K)$. We assume that there are k_1 $\rho's$ and k_2 $\eta's$ where ρ,η satisfy (2.17), (2.18) respectively.

Remark 5.1. From equations (5.10), it is easy to see that if either $\eta_0=0$ or $\eta_0=+\infty$, asymmetric patterns do not exist.

6. Existence Proof I: Approximate Solutions

Let us start to prove Theorem 2.1. The first step is to choose a good approximate solution (Section 6). The second step is to use Liapunov-Schmidt reduction process to reduce the problem into finite dimensional problem (Section 8). The last step is to solve the reduced problem (Section 8). Such a procedure has been used in the study of Gierer-Meinhardt system (both in the strong coupling case [35], [36] and in the weak coupling case [37]). We shall sketch it in the present context and leave the details to the reader.

Motivated by the results in Section 2, we rescale

$$\hat{v}(y)=Av(\epsilon y),\quad y\in\Omega_\epsilon=\{y|\epsilon y\in\Omega\}. \tag{6.1}$$

Then an equilibrium solution $(\hat{v},\,u)$ has to solve the following rescaled Gray-Scott model:

$$\begin{cases}\Delta_y\hat{v}-\hat{v}+\hat{v}^2u=0,\quad y\in\Omega_\epsilon,\\ \Delta_x u+\beta^2(1-u)-\frac{\beta^2}{A^2}\hat{v}^2u=0,\quad x\in\Omega,\end{cases} \tag{6.2}$$

where

$$\hat{v} \in H^2_N(\Omega_\epsilon) = \{u \in H^2(\Omega_\epsilon) | \frac{\partial u}{\partial \nu_\epsilon} = 0 \text{ on } \partial\Omega_\epsilon\},$$

$$\hat{v} \in H^2_N(\Omega) = \{u \in H^2(\Omega) | \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}.$$

(Here the index N represents Neumann boundary condition. ν_ϵ, ν are the corresponding boundary normal derivatives of Ω_ϵ, Ω, respectively.)

For a function $v \in H^2_N(\Omega_\epsilon)$, let $T[v]$ be the unique solution of the following problem

$$\Delta T[v] + \beta^2(1 - T[v]) - \frac{\beta^2}{A^2} v^2 T[v] = 0 \text{ in } \Omega, \ \frac{\partial T[v]}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{6.3}$$

In other words, we have

$$1 - T[v](x) = \int_\Omega G_\beta(x, \xi) \frac{\beta^2}{A^2} v(\frac{\xi}{\epsilon})^2 T[v](\xi) \, d\xi. \tag{6.4}$$

System (6.2) is equivalent to the following equation in operator form:

$$S_\epsilon(\hat{v}, u) = \begin{pmatrix} S_1(\hat{v}, u) \\ S_2(\hat{v}, u) \end{pmatrix} = 0, \tag{6.5}$$

where

$$S_1(\hat{v}, u) = \Delta_y \hat{v} - \hat{v} + \hat{v}^2 u, \quad H^2_N(\Omega_\epsilon) \to L^2(\Omega_\epsilon),$$

$$S_2(\hat{v}, u) = \Delta_x \hat{u} + \beta^2(1 - \hat{u}) - \frac{\beta^2}{A^2} \hat{v}^2 u, \quad H^2_N(\Omega) \to L^2(\Omega).$$

Let $\mathbf{P} \in \overline{\Lambda}$ and $(\xi_1, ..., \xi_K)$ be the vector which satisfies (2.20).

We now determine a good approximate function. Therefore will choose suitable $(\xi_{\epsilon,1}, ..., \xi_{\epsilon,K})$ such that $|\xi_{\epsilon,j} - \xi_j| \le \delta_0$ for δ_0 small and set

$$\hat{v}_{\epsilon,j}(y) := \frac{1}{\xi_{\epsilon,j}} w(\frac{\epsilon y - P_j}{\epsilon}) \chi(\frac{\epsilon y - P_j}{r_0}), \quad y \in \Omega_\epsilon. \tag{6.6}$$

Note that the $\xi_{\epsilon,j}$ are undetermined. Then we will choose the following approximate solutions:

$$v_{\epsilon,\mathbf{P}}(y) := \sum_{j=1}^K \hat{v}_{\epsilon,j}(y), \quad u_{\epsilon,\mathbf{P}}(x) := T[v_{\epsilon,\mathbf{P}}](x) \tag{6.7}$$

for

$$x \in \Omega, \quad y \in \Omega_\epsilon = \{y \in R^2 | \epsilon y \in \Omega\}.$$

Note that $u_{\epsilon,\mathbf{P}}$ satisfies

$$\Delta u_{\epsilon,\mathbf{P}}+\beta^2(1-u_{\epsilon,\mathbf{P}})-\frac{\beta^2}{A^2}v_{\epsilon,\mathbf{P}}^2u_{\epsilon,\mathbf{P}}$$
$$=\Delta u_{\epsilon,\mathbf{P}}+\beta^2(1-u_{\epsilon,\mathbf{P}})-\frac{\beta^2}{A^2}\sum_{j=1}^K\hat{v}_{\epsilon,j}^2u_{\epsilon,\mathbf{P}}+e.s.t.$$

Let $\hat{\xi}_{\epsilon,j}=u_{\epsilon,\mathbf{P}}(P_j)$. Then we have

$$1-\hat{\xi}_{\epsilon,i}=\frac{\beta^2}{A^2}\int_\Omega G_\beta(P_i^\epsilon,\xi)\sum_{j=1}^K\hat{v}_{\epsilon,j}^2(\frac{\xi}{\epsilon})u_{\epsilon,\mathbf{P}}d\xi+e.s.t.,\quad i=1,...,K.$$

Similar to the computations in Section 5, we obtain

$$1-\hat{\xi}_{\epsilon,i}=\sum_{j=1}^K\frac{L_\epsilon\hat{\xi}_{\epsilon,j}}{\xi_{\epsilon,j}^2}+\frac{\eta_\epsilon L_\epsilon\hat{\xi}_{\epsilon,i}}{\xi_{\epsilon,i}^2}+O(\sum_{j=1}^K\frac{\beta^2L_\epsilon\hat{\xi}_{\epsilon,j}}{\xi_{\epsilon,j}^2}),\quad i=1,...,K. \tag{6.8}$$

Now we have

Lemma 6.1. *Let $(\xi_1,...,\xi_K)$ be given in (2.20). Then for ϵ sufficiently small, there exists a unique solution $(\xi_{\epsilon,1},...,\xi_{\epsilon,K})$ such that*

$$\hat{\xi}_{\epsilon,j}=\xi_{\epsilon,j},\quad j=1,...,K, \tag{6.9}$$

and $\xi_{\epsilon,j}=\xi_j+O(\beta^2)$.

Proof: Let $\xi=(\xi_1,...,\xi_K),\xi_\epsilon=(\xi_{\epsilon,1},...,\xi_{\epsilon,K})$ and $\hat{\xi}_\epsilon=(\hat{\xi}_{\epsilon,1},...,\xi_{\epsilon,K})$. Note that $\hat{\xi}_\epsilon$ is a function of ξ_ϵ. We write (6.8) as a functional equation

$$\mathbf{G}(\epsilon,\xi_\epsilon,\hat{\xi}_\epsilon)=0,\quad \|\xi-\xi_\epsilon\|<\delta_0, \tag{6.10}$$

where

$$\mathbf{G}(\epsilon,\xi_\epsilon,\hat{\xi}_\epsilon)=r.h.s.\text{ of }(6.8)-l.h.s.\text{ of }(6.8)$$

and the norm is the vector norm. Note that $\mathbf{G}(0,\xi,\hat{\xi})|_{\xi=\hat{\xi}=(\xi_1,\cdots,\xi_K)}=0$. Now we claim that $\nabla_{\hat{\xi}}\mathbf{G}(0,\xi,\hat{\xi})|_{\xi=\hat{\xi}=(\xi_1,\cdots,\xi_K)}$ is nonsingular. Once this is proved, then the implicit function theorem gives the result.

Now it follows that

$$-\nabla_{\hat{\xi}}\mathbf{G}(0,\xi,\hat{\xi})|_{\xi=\hat{\xi}=(\xi_1,\cdots,\xi_K)}=\begin{pmatrix}1+\frac{L_0\eta_0}{\xi_1^2} & & \\ & \ddots & \\ & & 1+\frac{L_0\eta_0}{\xi_K^2}\end{pmatrix}+\begin{pmatrix}\frac{L_0}{\xi_1^2} & \cdots & \frac{L_0}{\xi_K^2}\\ \vdots & \vdots & \vdots\\ \frac{L_0}{\xi_1^2} & \cdots & \frac{L_0}{\xi_K^2}\end{pmatrix}.$$

Since $\nabla_{\hat{\xi}}\mathbf{G}(0,\xi,\hat{\xi})|_{\xi=\hat{\xi}=(\xi_1,\cdots,\xi_K)}$ is strictly negative definite it is nonsingular.

□

The following lemma shows that the functions in (6.7) are good approximations to K-spots is since they solve (6.5) reasonably well. We substitute (6.7) into (6.5) and calculate

$$S_2(v_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}}) = 0, \tag{6.11}$$

$$\begin{aligned}
S_1(v_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}}) &= \Delta_y v_{\epsilon,\mathbf{P}} - v_{\epsilon,\mathbf{P}} + v^2_{\epsilon,\mathbf{P}} u_{\epsilon,\mathbf{P}} \\
&= \sum_{j=1}^K \frac{1}{\xi_{\epsilon,j}} \Big[\Delta_y w(y - \frac{P_j}{\epsilon}) - w(y - \frac{P_j}{\epsilon})\Big] \\
&\quad + \sum_{j=1}^K \frac{1}{\xi^2_{\epsilon,j}} w^2(y - \frac{P_j}{\epsilon}) u_{\epsilon,\mathbf{P}} + e.s.t. \\
&= \sum_{j=1}^K \frac{1}{\xi^2_{\epsilon,j}} w^2(y - \frac{P_j}{\epsilon})(u_{\epsilon,\mathbf{P}} - \xi_{\epsilon,j}) + e.s.t. \\
&= \sum_{j=1}^K \frac{1}{\xi^2_{\epsilon,j}} w^2(y - \frac{P_j}{\epsilon})(\hat{\xi}_{\epsilon,j} - \xi_{\epsilon,j}) \\
&\quad + \sum_{j=1}^K \frac{1}{\xi^2_{\epsilon,j}} w^2(y - \frac{P_j}{\epsilon})(u_{\epsilon,\mathbf{P}}(x) - \hat{\xi}_{\epsilon,j}) + e.s.t. \\
&= \sum_{j=1}^K \frac{1}{\xi^2_{\epsilon,j}} w^2(y - \frac{P_j}{\epsilon})(u_{\epsilon,\mathbf{P}}(x)) - u_{\epsilon,\mathbf{P}}(P_j)) + O(\beta^2)
\end{aligned}$$

by Lemma 6.1.

On the other hand, from (2.23) and (5.3), we calculate for $i = 1, ..., K$ and $x = P_i + \epsilon z$:

$$\begin{aligned}
u_{\epsilon,\mathbf{P}}(x) - u_{\epsilon,\mathbf{P}}(P_i) &= u_{\epsilon,\mathbf{P}}(P_i + \epsilon z) - u_{\epsilon,\mathbf{P}}(P_i) \\
&= \frac{\beta^2}{A^2} \int_\Omega (G_\beta(P_i, \xi) - G_\beta(P_i + \epsilon z, \xi)) \sum_{j=1}^K \hat{v}^2_{\epsilon,j}(\frac{\xi}{\epsilon}) u_{\epsilon,\mathbf{P}}(\xi) d\xi + e.s.t. \\
&= \frac{\beta^2}{A^2} \int_\Omega (G_\beta(P_i, \xi) - G_\beta(P_i + \epsilon z, \xi)) \hat{v}^2_{\epsilon,i}(\frac{\xi}{\epsilon}) u_{\epsilon,\mathbf{P}}(\xi) d\xi \\
&\quad + \frac{\beta^2}{A^2} \int_\Omega (G_\beta(P_i, \xi) - G_\beta(P_i + \epsilon z, \xi)) \sum_{j \neq i} \hat{v}^2_{\epsilon,j}(\frac{\xi}{\epsilon}) u_{\epsilon,\mathbf{P}}(\xi) d\xi + e.s.t. \\
&= |\Omega| \beta^2 L_\epsilon \xi_{\epsilon,i} (\epsilon \frac{1}{2} \nabla_{P_i} F_0(\mathbf{P}) \cdot z + O(\epsilon |z| \beta^2)) \\
&\quad + \frac{|\Omega| \beta^2 L_\epsilon}{\xi_{\epsilon,i} \int_{R^2} w^2} \int_{R^2} \log \frac{|z - \zeta|}{|\zeta|} w^2(\zeta)\, d\zeta (1 + O(\beta^2)),
\end{aligned} \tag{6.12}$$

where the last line is radially symmetric in z. (Recall the definition of F_0 in (2.23).)

Therefore we have the following key estimate

Lemma 6.2. *For $x = P_i + \epsilon z$, $|\epsilon z| < \delta$, we have*

$$S_1(v_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}}) = S_{1,1} + S_{1,2}, \tag{6.13}$$

where

$$S_{1,1}(z) = |\Omega|\beta^2 L_\epsilon \frac{1}{\xi_{\epsilon,i}} w^2(z)(\epsilon \nabla_{P_i} F_0(\mathbf{P}) \cdot z + O(\epsilon|z|\beta^2)) \tag{6.14}$$

and

$$S_{1,2}(z) = \frac{|\Omega|\beta^2 L_\epsilon}{\xi_{\epsilon,i}^3 \int_{R^2} w^2} w^2(z) \int_{R^2} \log \frac{|z-\zeta|}{|\zeta|} w^2(\zeta)\, d\zeta (1 + O(\beta^2)), \tag{6.15}$$

where $S_{1,2}(z)$ is radially symmetric in z. Furthermore, $S_1(v_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}}) = e.s.t.$ for $|x - P_j| \geq \delta$, $j = 1, 2, ..., K$.

7. Existence proof II: Reduction to finite dimensions

In this section, we use the Liapunov-Schmidt method to reduce the problem of finding an equilibrium state to a finite-dimensional problem.

We first study the linearized operator defined by

$$\tilde{L}_{\epsilon,\mathbf{P}} := S'_\epsilon \begin{pmatrix} v_{\epsilon,\mathbf{P}} \\ u_{\epsilon,\mathbf{P}} \end{pmatrix},$$

$$\tilde{L}_{\epsilon,\mathbf{P}} : H^2_N(\Omega_\epsilon) \times H^2_N(\Omega) \to L^2(\Omega_\epsilon) \times L^2(\Omega),$$

where $\epsilon > 0$ is small and $\mathbf{P} \in \bar{\Lambda}$.

Similar to [37], we define the approximate kernel and cokernel as follows:

$$K_{\epsilon,\mathbf{P}} := \text{span}\,\{\frac{\partial v_{\epsilon,\mathbf{P}}}{\partial P_{j,l}} | j = 1, \ldots, K,\, l = 1, 2\} \subset H^2_N(\Omega_\epsilon)$$

and

$$C_{\epsilon,\mathbf{P}} := \text{span}\,\{\frac{\partial v_{\epsilon,\mathbf{P}}}{\partial P_{j,l}} | j = 1, \ldots, K,\, l = 1, 2\} \subset L^2(\Omega_\epsilon),$$

$$\mathcal{K}_{\epsilon,\mathbf{P}} := K_{\epsilon,\mathbf{P}} \oplus \{0\} \subset H^2_N(\Omega_\epsilon) \times H^2_N(\Omega),$$

$$\mathcal{C}_{\epsilon,\mathbf{P}} := C_{\epsilon,\mathbf{P}} \oplus \{0\} \subset L^2(\Omega_\epsilon) \times L^2(\Omega).$$

We then define

$$\mathcal{K}^{\perp}_{\epsilon,\mathbf{P}} := K^{\perp}_{\epsilon,\mathbf{P}} \oplus H^2_N(\Omega) \subset H^2_N(\Omega_\epsilon) \times H^2_N(\Omega),$$

$$\mathcal{C}^{\perp}_{\epsilon,\mathbf{P}} := C^{\perp}_{\epsilon,\mathbf{P}} \oplus L^2(\Omega) \subset L^2(\Omega_\epsilon) \times L^2(\Omega),$$

where $C_{\epsilon,\mathbf{P}}$ and $K_{\epsilon,\mathbf{P}}$ denote the orthogonal complement with the scalar product of $L^2(\Omega_\epsilon)$ in $H^2_N(\Omega_\epsilon)$ and $L^2(\Omega_\epsilon)$, respectively.

Let $\pi_{\epsilon,\mathbf{P}}$ denote the projection in $L^2(\Omega_\epsilon) \times L^2(\Omega)$ onto $\mathcal{C}^{\perp}_{\epsilon,\mathbf{P}}$. (Here the second component of the projection is the identity map.) We are going to show that the equation

$$\pi_{\epsilon,\mathbf{P}} \circ S_\epsilon \begin{pmatrix} v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ u_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} = 0$$

has the unique solution $\Sigma_{\epsilon,\mathbf{P}} = \begin{pmatrix} \Phi_{\epsilon,\mathbf{P}}(y) \\ \Psi_{\epsilon,\mathbf{P}}(x) \end{pmatrix} \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}}$ if ϵ is small enough. That is equivalent to the following equation

$$S_1(v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, T[v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}]) \in C_{\epsilon,\mathbf{P}}, \quad \Phi_{\epsilon,\mathbf{P}} \in K^{\perp}_{\epsilon,\mathbf{P}}. \tag{7.1}$$

The following two propositions show the invertibility of the corresponding linearized operator.

Proposition 7.1. *Suppose that (2.27) holds. Let* $\mathcal{L}_{\epsilon,\mathbf{P}} = \pi_{\epsilon,\mathbf{P}} \circ \tilde{L}_{\epsilon,\mathbf{P}}$. *There exist positive constants* $\bar{\epsilon}, C$ *such that for all* $\epsilon \in (0, \bar{\epsilon})$,

$$\|\mathcal{L}_{\epsilon,\mathbf{P}}\Sigma\|_{L^2(\Omega_\epsilon)\times L^2(\Omega)} \geq C\|\Sigma\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)} \tag{7.2}$$

for arbitrary $\mathbf{P} \in \bar{\Lambda}$, $\Sigma \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}}$.

Proposition 7.2. *Suppose (2.27) holds. There exist positive constant* $\bar{\bar{\epsilon}}$ *such that for all* $\epsilon \in (0, \bar{\bar{\epsilon}})$ *the map*

$$\mathcal{L}_{\epsilon,\mathbf{P}} = \pi_{\epsilon,\mathbf{P}} \circ \tilde{L}_{\epsilon,\mathbf{P}} : \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}} \to \mathcal{C}^{\perp}_{\epsilon,\mathbf{P}}$$

is surjective for arbitrary $\mathbf{P} \in \bar{\Lambda}$.

Similarly by using Contraction Mapping Principle, we get

Lemma 7.3. *There exist $\bar{\epsilon} > 0, C > 0$ such that for every pair $(\epsilon, \mathbf{P})$ with $0 < \epsilon < \bar{\epsilon}$, $\mathbf{P} \in \bar{\Lambda}$ there exists a unique $(\Phi_{\epsilon,\mathbf{P}}, \Psi_{\epsilon,\mathbf{P}}) \in \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp}$ satisfying $S_\epsilon(\begin{pmatrix} v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ u_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix}) \in \mathcal{C}_{\epsilon,\mathbf{P}}$ and*

$$\|(\Phi_{\epsilon,\mathbf{P}}, \Psi_{\epsilon,\mathbf{P}})\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)} \le C \frac{1}{\log \frac{1}{\epsilon}}. \tag{7.3}$$

More refined estimates for $\Phi_{\epsilon,\mathbf{P}}$ are needed. Recall that in Lemma (6.2) we have found a decomposition of S_1 into two parts, $S_{1,1}, S_{1,2}$, where $S_{1,1}$ is an odd function in y and $S_{1,2}$ is a radially symmetric function in y for $|\epsilon y| < \delta$. Similarly, we can decompose $\Phi_{\epsilon,\mathbf{P}}$:

Lemma 7.4. *Let $\Phi_{\epsilon,\mathbf{P}}$ be defined as in Lemma 7.3. Then for $x = P_i + \epsilon z$, $|\epsilon z| < \delta$, we have*

$$\Phi_{\epsilon,\mathbf{P}} = \Phi^1_{\epsilon,\mathbf{P}} + \Phi^2_{\epsilon,\mathbf{P}}, \tag{7.4}$$

where $\Phi^2_{\epsilon,\mathbf{P}}$ is a radially symmetric function in z and

$$\|\Phi^1_{\epsilon,\mathbf{P}}\|_{H^2(\Omega_\epsilon)} = O(\epsilon\beta^2). \tag{7.5}$$

Proof: Let $S[v] := S_1(v, T[v])$. Then we first solve

$$S[v_{\epsilon,\mathbf{P}} + \Phi^2_{\epsilon,\mathbf{P}}] - S[v_{\epsilon,\mathbf{P}}] + \sum_{j=1}^{K} S_{1,2}(y - \frac{P_j}{\epsilon}) \in C_{\epsilon,\mathbf{P}}, \Phi^2_{\epsilon,\mathbf{P}} \in K^{\perp}_{\epsilon,\mathbf{P}}. \tag{7.6}$$

Then we solve

$$S[v_{\epsilon,\mathbf{P}} + \Phi^2_{\epsilon,\mathbf{P}} + \Phi^1_{\epsilon,\mathbf{P}}] - S[v_{\epsilon,\mathbf{P}} + \Phi^2_{\epsilon,\mathbf{P}}] + \sum_{j=1}^{K} S_{1,1}(y - \frac{P_j}{\epsilon}) \in C_{\epsilon,\mathbf{P}} \tag{7.7}$$

for $\Phi^1_{\epsilon,\mathbf{P}} \in K^{\perp}_{\epsilon,\mathbf{P}}$. Using the same proof as in Lemma 7.3, both equations (7.7) and (7.6) have unique solutions for $\epsilon << 1$. By uniqueness, $\Phi_{\epsilon,\mathbf{P}} = \Phi^1_{\epsilon,\mathbf{P}} + \Phi^2_{\epsilon,\mathbf{P}}$. Since $S_{11} = S^0_{11} + S^{\perp}_{11}$, where $\|S^0_{11}\|_{H^2(\Omega_\epsilon)} = O(\epsilon\beta^2)$ and $S^{\perp}_{11} \in C^{\perp}_{\epsilon,\mathbf{P}}$, it follows that $\Phi^1_{\epsilon,\mathbf{P}}$ and $\Phi^2_{\epsilon,\mathbf{P}}$ have the required properties. □

8. Existence proof III: The reduced problem

In this section, we solve the reduced problem and prove our main theorem on the existence of asymmetric solutions, Theorem 2.1.

Let $\mathbf{P}^0$ be a nondegenerate critical point of $F_0(\mathbf{P})$.

By Lemma 7.3, for each $\mathbf{P} \in B_\delta(\mathbf{P}^0)$, there exists a unique solution $(\Phi_{\epsilon,\mathbf{P}}, \psi_{\epsilon,\mathbf{P}}) \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$ such that

$$S_\epsilon \begin{pmatrix} v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ u_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} = \begin{pmatrix} e_{\epsilon,\mathbf{P}} \\ 0 \end{pmatrix} \in \mathcal{C}_{\epsilon,\mathbf{P}}.$$

Our idea is to find $\mathbf{P} = \mathbf{P}^\epsilon \in B_\delta(\mathbf{P}^0)$ such that

$$S_\epsilon \begin{pmatrix} v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ u_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} \perp \mathcal{C}_{\epsilon,\mathbf{P}}.$$

Let

$$W_{\epsilon,j,i}(\mathbf{P}) := \frac{2\xi^2_{\epsilon,j}}{L_\epsilon |\Omega| \beta^2 \epsilon^2} \int_{\Omega_\epsilon} (S_1(v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}}) \frac{\partial v_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}),$$

$$W_\epsilon(\mathbf{P}) := (W_{\epsilon,1,1}(\mathbf{P}), ..., W_{\epsilon,K,2}(\mathbf{P})),$$

where $\xi_{\epsilon,j}$ is given by Lemma 6.1. Recall that $P_{j,i}$ denotes the i-th component of the j-th point. Then $W_\epsilon(\mathbf{P})$ is a map which is continuous in $\mathbf{P}$ and our problem is reduced to finding a zero of the vector field $W_\epsilon(\mathbf{P})$.

To simplify our computation, we let $\tilde{u}_{\epsilon,\mathbf{P}} = u_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} = T[v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}]$ and

$$\Omega_{\epsilon,P_j} = \{z | \epsilon z + P_j \in \Omega\}. \tag{8.1}$$

We calculate

$$\int_{\Omega_\epsilon} S_1(v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, \tilde{u}_{\epsilon,\mathbf{P}}) \frac{\partial v_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}$$

$$= \int_{\Omega_\epsilon} S_1(v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, \tilde{u}_{\epsilon,\mathbf{P}}(P_j)) \frac{\partial v_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}$$

$$+ \int_{\Omega_\epsilon} (v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2 (\tilde{u}_{\epsilon,\mathbf{P}}(x) - \tilde{u}_{\epsilon,\mathbf{P}}(P_j)) \frac{\partial v_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}$$

$$= I_1 + I_2,$$

where I_1 and I_2 are defined by the last equality.

For I_1, we have

$$I_1=\epsilon\int_{\Omega_{\epsilon,P_j}}[\Delta(v_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})-(v_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})+(v_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})^2(\tilde{u}_{\epsilon,\mathbf{P}}(P_j)](-\frac{1}{\xi_{\epsilon,j}}\frac{\partial w}{\partial z_i})dz$$

$$=\epsilon\int_{\Omega_{\epsilon,P_j}}[(v_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})^2(\tilde{u}_{\epsilon,\mathbf{P}}(P_j))-2w(v_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,P})](-\frac{1}{\xi_{\epsilon,j}}\frac{\partial w}{\partial z_i})\,dz+e.s.t.$$

$$=\epsilon\int_{\Omega_{\epsilon,P_j}}(\Phi^2_{\epsilon,\mathbf{P}})^2(\tilde{u}_{\epsilon,\mathbf{P}}(P_j))(-\frac{1}{\xi_{\epsilon,j}}\frac{\partial w}{\partial z_i})\,dz+e.s.t.$$

$$=O(\epsilon^2\beta^2)$$

by Lemma 7.4.

For I_2, we have similar to the computation in (6.12):

$$\tilde{u}_{\epsilon,\mathbf{P}}(P_j+\epsilon z)-\tilde{u}_{\epsilon,\mathbf{P}}(P_j)=|\Omega|\beta^2L_\epsilon\xi_{\epsilon,j}(\epsilon\frac{1}{2}\nabla_{P_j}F_0(\mathbf{P})\cdot z+O(\epsilon|z|\beta^2))$$

$$+\frac{|\Omega|\beta^2L_\epsilon}{\xi_{\epsilon,j}\int_{R^2}w^2}\int_{R^2}\log\frac{|z-\zeta|}{|\zeta|}w^2(\zeta)\,d\zeta(1+O(\beta^2)),$$

where the last line is a function, which is rotationally symmetric in z. Hence

$$I_2=|\Omega|\beta^2L_\epsilon\epsilon^2\int_{\Omega_{\epsilon,P_j}}(\frac{1}{\xi_{\epsilon,j}}w+\Phi_{\epsilon,\mathbf{P}})^2(\nabla_{P_j}F_0(\mathbf{P})\cdot z+O(\epsilon|z|\beta^2))(-\frac{\partial w}{\partial z_i}+O(\beta^2))$$

$$=-\frac{|\Omega|\beta^2L_\epsilon\epsilon^2}{2\xi^2_{\epsilon,j}}[\int_{R^2}w^2\frac{\partial w}{\partial z_i}z_i\nabla_{P_{j,i}}F_0(\mathbf{P})+O(\beta^2)]. \tag{8.2}$$

Combining I_1 and I_2, we obtain

$$W_\epsilon(\mathbf{P})=c_0\nabla_{\mathbf{P}}F_0(\mathbf{P})+o(1),$$

where

$$c_0=-\int_{R^2}w^2\frac{\partial w}{\partial z_i}z_i=\frac{1}{3}\int_{R^2}w^3$$

and $o(1)$ is a continuous function of $\mathbf{P}$ which goes to 0 as $\epsilon\to 0$.

Since we assume that $\mathbf{P}_0$ is a nondegenerate critical point of $F_0(\mathbf{P})$, we have $\nabla_{\mathbf{P}}F_0(\mathbf{P}_0)=0,\det(\nabla_{\mathbf{P}}\nabla_{\mathbf{P}}(F_0(\mathbf{P}_0))\neq 0$. Thus, since W_ϵ is continuous in $\mathbf{P}$, and for ϵ, β small enough maps balls $B_\delta(\mathbf{P}_0)$ into (possibly larger) balls, the standard Brouwer fixed point theorem implies that for $\epsilon<<1$ there exists $\mathbf{P}^\epsilon\in\mathbf{B}_\delta(\mathbf{P_0})$ such that $W_\epsilon(\mathbf{P}^\epsilon)=0$ and $\mathbf{P}^\epsilon\to\mathbf{P}_0$.

Thus we have proved the following proposition.

Proposition 8.1. *For ϵ sufficiently small, there exist points $\mathbf{P}^\epsilon$ with $\mathbf{P}^\epsilon\to\mathbf{P}_0$ such that $W_\epsilon(\mathbf{P}^\epsilon)=0$.*

Finally, we prove Theorem 2.1.

Proof of Theorem 2.1: By Proposition 8.1, there exists $\mathbf{P}^\epsilon \to \mathbf{P}_0$ such that $W_\epsilon(\mathbf{P}^\epsilon) = 0$. In other words, $S_1(v_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}, u_{\epsilon,\mathbf{P}^\epsilon} + \Psi_{\epsilon,\mathbf{P}^\epsilon}) = 0$. Let $v_\epsilon = \frac{1}{A}(v_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}), u_\epsilon = u_{\epsilon,\mathbf{P}^\epsilon} + \Psi_{\epsilon,\mathbf{P}^\epsilon}$. It is easy to see that $u_\epsilon = \xi_{\epsilon,j}(1 + O(\beta^2))$ and hence $v_\epsilon \geq 0$. By the Maximum Principle, $v_\epsilon > 0$. Therefore (v_ϵ, u_ϵ) satisfies Theorem 2.1.

□

9. Stability Proof: Large Eigenvalues

In this section, we study the eigenvalue problem (2.28) for the solutions which we have rigorously constructed in Sections 6–8. Let $v_\epsilon = v_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}$, $\hat{v}_\epsilon = Av_\epsilon, u_\epsilon = T[\hat{v}_\epsilon]$. (2.28) is equivalent to the following eigenvalue problem

$$\begin{cases} \Delta_y \phi_\epsilon - \phi_\epsilon + 2\hat{v}_\epsilon u_\epsilon \phi_\epsilon + \hat{v}_\epsilon^2 \psi_\epsilon = \lambda_\epsilon \phi_\epsilon, \quad y \in \Omega_\epsilon, \\ \frac{1}{\beta^2}\Delta_x \psi_\epsilon - \psi_\epsilon - \frac{2}{A^2}\hat{v}_\epsilon u_\epsilon \phi_\epsilon - \frac{1}{A^2}\hat{v}_\epsilon^2 \psi_\epsilon = \tau \lambda_\epsilon \psi_\epsilon, \quad x \in \Omega, \end{cases} \tag{9.1}$$

where $\lambda_\epsilon \in \mathcal{C}$ and

$$\phi_\epsilon \in H_N^2(\Omega_\epsilon), \quad \psi_\epsilon \in H_N^2(\Omega).$$

We study **two cases separately**: $\lambda_\epsilon \to \lambda_0 \neq 0$ **(large eigenvalues)** and $\lambda_\epsilon \to 0$ **(small eigenvalues)**. In this section, we study the large eigenvalue case. The small eigenvalues will be considered in Appendix B.

Note that since the operator is not self-adjoint, λ_0 may be complex. We will see that in leading order the large eigenvalues are independent of the locations $P_j^\epsilon, j = 1, ..., K$.

We assume that $|\lambda_\epsilon| \geq c > 0$ for ϵ small. If $\text{Re}(\lambda_\epsilon) \leq -c$, we are done. (Then λ_ϵ is a stable large eigenvalue.) Therefore we may assume that $\text{Re}(\lambda_\epsilon) \geq -c$. Let $\lambda_\epsilon \to \lambda_0 \neq 0$ as $\epsilon \to 0$.

The second equation in (9.1) is equivalent to

$$\Delta_x \psi_\epsilon - \beta^2(1 + \tau\lambda_\epsilon)\psi_\epsilon - \frac{2\beta^2}{A^2} u_\epsilon \hat{v}_\epsilon \phi_\epsilon - \frac{\beta^2}{A^2}\hat{v}_\epsilon^2 \psi_\epsilon = 0. \tag{9.2}$$

We introduce the following

$$\beta_{\lambda_\epsilon} = \beta\sqrt{1 + \tau\lambda_\epsilon} \tag{9.3}$$

where in $\sqrt{1+\tau\lambda_\epsilon}$ we take the principal part. (This means that the real part of $\sqrt{1+\tau\lambda_\epsilon}$ is positive, which is possible because $\text{Re}(1+\tau\lambda_\epsilon) \geq \frac{1}{2}$.)

Let us assume that

$$\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)} = 1.$$

We cut off ϕ_ϵ as follows: Introduce

$$\phi_{\epsilon,j}(y - \frac{P_j^\epsilon}{\epsilon}) = \phi_\epsilon(y)\chi_{\epsilon,j}(x),$$

where $\chi_{\epsilon,j}(x)$ was introduced in (5.5).

From (9.1) using the fact that $\text{Re}(\lambda_\epsilon) \geq -c$ and the exponential decay of w it follows that

$$\phi_\epsilon = \sum_{j=1}^K \phi_{\epsilon,j} + e.s.t. \quad \text{ in } H^2(\Omega_\epsilon).$$

Then by a standard procedure we extend $\phi_{\epsilon,j}$ to a function defined on R^2 such that

$$\|\phi_{\epsilon,j}\|_{H^2(R^2)} \leq C\|\phi_{\epsilon,j}\|_{H^2(\Omega_\epsilon)}, \qquad j = 1, \dots, K.$$

Then $\|\phi_{\epsilon,j}\|_{H^2(\Omega_\epsilon)} \leq C$. By taking a subsequence of ϵ, we may also assume that $\phi_{\epsilon,j} \to \phi_j$ as $\epsilon \to 0$ in $H^1(R^2)$ for $j = 1, \dots, K$.

We have by (9.2)

$$\psi_\epsilon(x) = -\frac{\beta^2}{A^2}\int_\Omega G_{\beta_{\lambda_\epsilon}}(x,\xi)(u_\epsilon(\xi)2\hat{v}_\epsilon(\xi)\phi_\epsilon(\frac{\xi}{\epsilon}) + \psi_\epsilon(\xi)\hat{v}_\epsilon^2(\xi))\,d\xi. \tag{9.4}$$

At $x = P_i^\epsilon$, $i = 1, \dots, K$, we calculate

$$\psi_\epsilon(P_i^\epsilon) = -\frac{\beta^2}{A^2}\int_\Omega \left(\frac{(\beta_{\lambda_\epsilon})^{-2}}{|\Omega|} + G_0(P_i^\epsilon,\xi) + O(\beta^2)\right)$$

$$\times\left(\sum_{j=1}^K 2w(\frac{\xi - P_j^\epsilon}{\epsilon})\phi_{\epsilon,j}(\frac{\xi-P_j^\epsilon}{\epsilon}) + \psi_\epsilon(P_j^\epsilon)\frac{1}{\xi_{\epsilon,j}^2}w^2(\frac{\xi-P_j^\epsilon}{\epsilon})\right)d\xi + o\left(\frac{\epsilon^2}{A^2}\right)$$

$$= \frac{1}{1+\tau\lambda_\epsilon}\left(-\frac{2\epsilon^2}{A^2|\Omega|}\sum_{j=1}^K\int_{R^2} w\phi_j - \frac{\epsilon^2\int_{R^2}w^2}{A^2|\Omega|}\sum_{j=1}^K \psi_\epsilon(P_j^\epsilon)\frac{1}{\xi_j^2}\right)$$

$$+\frac{\beta^2\epsilon^2\log\frac{1}{\epsilon}}{2\pi}\left(-\frac{2\epsilon^2}{A^2}\int_{R^2} w\phi_i - \frac{\epsilon^2\int w^2}{A^2}\psi_\epsilon(P_i^\epsilon)\frac{1}{\xi_i^2}\right) + o\left(\frac{\epsilon^2}{A^2}\right)$$

$$= \frac{L_\epsilon}{1+\tau\lambda_\epsilon}\left(-2\frac{\sum_{j=1}^K\int_{R^2}w\phi_j}{\int_{R^2}w^2} - \sum_{j=1}^K\psi_\epsilon(P_j^\epsilon)\frac{1}{\xi_j^2}\right)$$

$$+L_\epsilon \eta_\epsilon \left(-2 \frac{\int_{R^2} w\phi_i}{\int_{R^2} w^2} - \psi_\epsilon(P_i^\epsilon) \frac{1}{\xi_i^2} \right) + o(L_\epsilon).$$

Let

$$\psi_\epsilon(P_j^\epsilon) \frac{1}{\xi_j^2} = \hat{\psi}_{\epsilon,j}, \quad \hat{\Psi}_\epsilon = (\hat{\psi}_{\epsilon,1}, ..., \hat{\psi}_{\epsilon,K}). \tag{9.5}$$

Then we have

$$\xi_i^2 \hat{\psi}_{\epsilon,i} = \frac{L_\epsilon}{(1+\tau\lambda_0)} \left(-2 \frac{\sum_{j=1}^K \int_{R^2} w\phi_{\epsilon,j}}{\int_{R^2} w^2} - \sum_{j=1}^K \hat{\psi}_{\epsilon,j} \right)$$

$$+L_\epsilon \eta_\epsilon \left(-2 \frac{\int_{R^2} w\phi_i}{\int_{R^2} w^2} - \hat{\psi}_{\epsilon,i} \right) + o(L_\epsilon).$$

Writing this system in matrix form, we obtain

$$\left[\mathcal{F} + \frac{L_0}{1+\tau\lambda_0} \mathcal{E} \right] \lim_{\epsilon \to 0} \hat{\Psi}_\epsilon = -2L_0 \left(\eta_0 \mathcal{I} + \frac{1}{1+\tau\lambda_0} \mathcal{E} \right) \frac{\int_{R^2} w\Phi}{\int_{R^2} w^2},$$

where

$$\mathcal{F} = \begin{pmatrix} \xi_1^2 + L_0\eta_0 & & \\ & \ddots & \\ & & \xi_K^2 + L_0\eta_0 \end{pmatrix}, \tag{9.6}$$

$$\mathcal{E} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \tag{9.7}$$

and $\mathcal{I}$ is the identity matrix.

Thus, in the limit $\epsilon \to 0$, we obtain the following nonlocal eigenvalue problem (NLEP):

$$\Delta\Phi - \Phi + 2w\Phi - 2\mathcal{B} \frac{\int_{R^2} w\Phi}{\int_{R^2} w^2} w^2 = \lambda_0 \Phi, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_K \end{pmatrix} \in (H^2(R^2))^K, \tag{9.8}$$

where

$$\mathcal{B} = L_0 \left(\mathcal{F} + \frac{L_0}{1+\tau\lambda_0} \mathcal{E} \right)^{-1} \left(\eta_0 \mathcal{I} + \frac{1}{1+\tau\lambda_0} \mathcal{E} \right). \tag{9.9}$$

More precisely, we have the following statement:

Theorem 9.1. *Assume that* (v_ϵ, u_ϵ) *is a solution constructed in Theorem 2.1.*

Let λ_ϵ *be an eigenvalue of (9.1) such that* $Re(\lambda_\epsilon) > -a_0$ *for some* $a_0 > 0$.

(1) Suppose that (for suitable sequences $\epsilon_n \to 0$*) we have* $\lambda_{\epsilon_n} \to \lambda_0 \neq 0$. *Then* λ_0 *is an eigenvalue of the problem (NLEP) given in (9.8).*

(2) Let $\lambda_0 \neq 0, Re(\lambda_0) > 0$ *be an eigenvalue of the (NLEP) problem given in (9.8). Then for* ϵ *sufficiently small, there is an eigenvalue* λ_ϵ *of (9.1) with* $\lambda_\epsilon \to \lambda_0$ *as* $\epsilon \to 0$.

Proof:

(1) of Theorem 9.1 follows the asymptotic analysis at the beginning of this section.

The proof of (2) is similar to that of Case 1 of Section 6 in [37]. We omit the details here.

□

Therefore, the study of large eigenvalues can be reduced to the study of the system of nonlocal eigenvalue problems (9.8). We can further reduce the problem by computing the eigenvalues of $\mathcal{B}$.

Let $\mathbf{q} = (q_1, \dots, q_K)^T$ be an eigenvector of $\mathcal{B}$ with eigenvalue μ. Then we have

$$\mathcal{B}\mathbf{q} = \mu\mathbf{q}, \tag{9.10}$$

which is equivalent to

$$L_0\left(\eta_0\mathcal{I} + \frac{1}{1+\tau\lambda_0}\mathcal{E}\right)\mathbf{q} = \mu\left(\mathcal{F} + \frac{1}{1+\tau\lambda_0}L_0\mathcal{E}\right)\mathbf{q},$$

$$(\eta_0 L_0\mathcal{I} - \mu\mathcal{F})\mathbf{q} = \frac{\mu-1}{1+\tau\lambda_0}L_0\mathcal{E}\mathbf{q}.$$

Writing down the above equation in components, we obtain

$$(\eta_0 L_0 - \mu(\xi_i^2 + L_0\eta_0))q_i = \frac{\mu-1}{1+\tau\lambda_0}L_0\sum_{j=1}^K q_j, \quad i = 1, ..., K.$$

Hence characteristic equation is

$$\frac{(\mu-1)k_1L_0}{\mu\rho^2 + (\mu-1)\eta_0L_0} + \frac{(\mu-1)k_2L_0}{\mu\eta^2 + (\mu-1)\eta_0L_0} + 1 + \tau\lambda_0 = 0. \tag{9.11}$$

Problem (9.11) is a quadratic equation and there are two roots $\mu_i = \mu_i(\tau\lambda_0)$, $i = 1, 2$. (The expression of μ_i is complicated.)

Therefore, by the diagonalization, problem (9.8) is reduced to two NLEPs:

$$\Delta\phi - \phi + 2w\phi - 2\mu_i(\tau\lambda_0)\frac{\int_{R^2} w\phi}{\int_{R^2} w^2}w^2 = \lambda_0\phi,\ i = 1, 2,\ \phi \in H^2(R^2), \tag{9.12}$$

which have been studied in Section 4.

Using the results of Section 4, we are now ready to finish the study of (9.12).

Completion of the study of (9.12):

We first consider the case when τ is large. If $\tau = \infty$, then the eigenvalues of $\mathcal{B}$ are

$$\mu_1 = \frac{\eta_0 L_0}{\rho^2 + \eta_0 L_0} \quad \text{and} \quad \mu_2 = \frac{\eta_0 L_0}{\eta^2 + \eta_0 L_0}.$$

Since $\rho\eta = \eta_0 L_0$ and if we assume $\rho < \eta$ then we have $\eta^2 > \eta_0 L_0$ and therefore $2\mu_2 < 1$. Therefore we have instability for τ large by Lemma 4.3.

When $\tau = 0$, by simple computations, (9.11) is equivalent to

$$\mu^2 - \left(1 + \frac{KL_0}{\rho + \eta}\right)\mu + \frac{L_0(\eta_0 + K)}{\rho + \eta} = 0. \tag{9.13}$$

It is easy to see that $2\mu_1 > 1$, $2\mu_2 > 1$ if and only if

$$\rho + \eta < (4\eta_0 + 2K)L_0. \tag{9.14}$$

Note that after some straightforward computations in (9.14) ρ, η can be eliminated and we get

$$k_1 > k_2, (\rho, \eta) = (\rho_+, \eta_+), \tag{9.15}$$

and

$$(k_1 - k_2)^2(\eta_0 - L_0(2\eta_0 + K)^2) > (2\eta_0 + K)^2(\eta_0 - L_0(2\eta_0 + K)^2). \tag{9.16}$$

If $L_0 < \frac{\eta_0}{(2\eta_0 + K)^2}$, then we must have

$$(k_1 - k_2)^2 > (2\eta_0 + K)^2,$$

which is clearly impossible. Therefore, we must have

$$L_0 > \frac{\eta_0}{(2\eta_0 + K)^2},$$

and then the condition

$$(k_1 - k_2)^2 < (2\eta_0 + K)^2$$

implies the validity of (9.16).

By Lemma 4.2, we conclude that for $\eta_0 > L_0(2\eta_0 + K)^2$ all asymmetric patterns are unstable for all τ. If $\eta_0 < L_0(2\eta_0 + K)^2$ and $\eta_0 > 4L_0(\eta_0 + k_1)(\eta_0 + k_2)$ and if we choose $k_1 > k_2, (\rho, \eta) = (\rho_+, \eta_+)$, then the asymmetric pattern is stable for τ small enough by Lemma 4.5.

We note that to establish stability one also has to study the small eigenvalues. Since this analysis is mainly parallel to [37], we have moved it to Appendix B.

Combining the results for the large eigenvalues in this section with the result for the small eigenvalues in Appendix B, we have completed the proof of our main stability theorem, Theorem 2.2.

□

10. Concluding Section: Summary of our results

Combining the results for the symmetric ([39]) and asymmetric K-spotty solutions, we summarize them as follows:

For the existence of symmetric K-spotty patterns, we need

$$L_0 \le \frac{1}{4(\eta_0 + K)}. \tag{10.17}$$

For the stability of symmetric K-spotty patterns, we need

$$L_0 < \frac{\eta_0}{(2\eta_0 + K)^2}, \ \tau \text{ small or large.} \tag{10.18}$$

For the existence of asymmetric K-spotty patterns, we need

$$L_0 \le \frac{\eta_0}{4(\eta_0 + k_1)(\eta_0 + k_2)} \tag{10.19}$$

For the stability of asymmetric K-spotty patterns, we need

$$\frac{\eta_0}{(2\eta_0 + K)^2} < L_0 \le \frac{\eta_0}{4(\eta_0 + k_1)(\eta_0 + k_2)}, \tau \text{ small.} \tag{10.20}$$

We observe a remarkable phenomenon: If symmetric $K-$spots are stable then asymmetric ones are unstable and vice versa. Note also that because of

$$\eta_0(\eta_0+K) < (\eta_0+k_1)(\eta_0+k_2)$$

whenever $k_1>1$ or $k_2>1$ (and $k_1+k_2=K$), the domain of existence for symmetric patterns is strictly larger than for asymmetric patterns. On the other hand, for τ large all asymmetric solutions are unstable. We believe that the asymmetric patterns which we obtained in this paper play an important role in the study of "self-replicating" phenomena in R^2 as they may provide the connecting orbits between symmetric K-spotty solutions. In fact, we conjecture that an asymmetric (k_1,k_2)-spotty solution may provide the right link between the symmetric (k_1+k_2)-spotty solution and symmetric k_1- or k_2-spotty solutions.

11. Appendix A: Proofs of Propositions 7.1 and 7.2

In this appendix, we prove the two propositions 7.1 and 7.2. Since the proofs are quite similar to that of Appendix A of [37] we shall be sketchy.

To obtain the asymptotic form of $\tilde{L}_{\epsilon,\mathbf{P}}$, suppose

$$\tilde{L}_{\epsilon,\mathbf{P}}\begin{pmatrix}\phi_\epsilon\\ \psi_\epsilon\end{pmatrix}=0.$$

Similar to Section 9, we cut off ϕ_ϵ as follows: Define

$$\phi_{\epsilon,j}(y-\frac{P_i}{\epsilon}) := \phi_\epsilon(y)\chi_{\epsilon,j}(x),$$

where $\chi_{\epsilon,j}(x)$ was introduced in (5.5) and $y\in\Omega_\epsilon$. By taking a subsequence of ϵ, we may also assume that $\phi_{\epsilon,j}\to\phi_j$ as $\epsilon\to 0$ in $H^1(R^2)$ for $j=1,\dots,K$.

Similar to the estimate leading to (9.8), the asymptotic limit of $\tilde{L}_{\epsilon,\mathbf{P}}$ is the following system of linear operators

$$\mathcal{L}\Phi := \Delta\Phi-\Phi+2w\Phi-2\mathcal{B}_0\frac{\int_{R^2}w\Phi}{\int_{R^2}w^2}w^2,\ \Phi=\begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_K\end{pmatrix}\in (H^2(R))^K, \tag{11.1}$$

where

$$\mathcal{B}_0 = L_0(\mathcal{F} + L_0\mathcal{E})^{-1}(\eta_0\mathcal{I} + \mathcal{E}) \tag{11.2}$$

and where $\mathcal{F}$ and $\mathcal{E}$ are defined in Section 9. The eigenvalues $\mu_i, i = 1, 2$ of $\mathcal{B}_0$ satisfy equation (9.13) with $\lambda_0 = 0$. Hence

$$2(\mu_1 + \mu_2) = 1 + \frac{KL_0}{\rho + \eta}$$

We see that $2\mu_i = 1$ if and only if $L_0 = \frac{\eta_0}{(2\eta_0+K)^2}$. In other words, if $L_0 \neq \frac{\eta_0}{(2\eta_0+K)^2}$, then $2\mu_i \neq 1, i = 1, 2$.

Now we have the following key lemma which reduces the infinite dimensional problem to a finite dimensional one.

Lemma 11.1. *Assume that assumption (2.27) holds. Then*

$$Ker(\mathcal{L}) = Ker(\mathcal{L}^*) = X_0 \oplus X_0 \oplus \cdots \oplus X_0, \tag{11.3}$$

where

$$X_0 = span\left\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\right\}$$

and $\mathcal{L}^*$ is the conjugate operator of $\mathcal{L}$ under the $(L^2(R^2))^K$ inner product.

As a consequence, the operator

$$\mathcal{L} : (H^2(R^2))^K \to (L^2(R^2))^K$$

is an invertible operator if it is restricted as follows

$$\mathcal{L} : (X_0 \oplus \cdots \oplus X_0)^\perp \cap (H^2(R^2))^K \to (X_0 \oplus \cdots \oplus X_0)^\perp \cap (L^2(R^2))^K.$$

Moreover, $\mathcal{L}^{-1}$ is bounded.

Proof: By (2.27) and the argument above, we see that $2\mu_i \neq 1$. If $\mathcal{L}\Phi = 0$, then by diagonalization, this can be reduced to (9.12) with $\lambda_0 = 0$. By Lemma 4.4(3), $\Phi \in X_0 \oplus X_0 \oplus \cdots \oplus X_0$.

Next, let $\Psi \in Ker(\mathcal{L}^*)$. Then we have

$$\Delta\Psi - \Psi + 2w\Psi - 2\mathcal{B}_0^t \frac{\int_{R^2} w^2\Psi}{\int_{R^2} w^2} w = 0, \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_K \end{pmatrix} \in (H^2(R))^K, \tag{11.4}$$

Multiplying the above equation by w (componentwise), we obtain

$$(\mathcal{I} - 2\mathcal{B}_0^t) \int_{R^2} w^2 \Psi = 0 \tag{11.5}$$

Since the matrix $\mathcal{B}_0^t$ has the same eigenvalues as $\mathcal{B}_0$ we know that $\mathcal{I} - 2\mathcal{B}_0^t$ is nonsingular. This implies that $\int_{R^2} w^2 \Psi = 0$. Thus all the nonlocal terms vanish and $\Psi \in X_0 \oplus X_0 \oplus \cdots \oplus X_0$. The rest follows from the Fredholm Alternatives Theorem. □

The rest of the proof is as in Appendix A of [37]. We omit the details.

12. Appendix B: Study of the Small Eigenvalues

It remains to study the small ($o(1)$) eigenvalues. Namely, we assume that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. We shall prove that the small eigenvalues are related to the matrix $M_0(\mathbf{P}^0)$ given in (2.24).

The analysis is the similar to that in [37]. To save space, we shall only give a sketch.

Let us define

$$\tilde{v}_{\epsilon,j}(y - \frac{P_j^\epsilon}{\epsilon}) = \chi_{\epsilon,j}(x)\hat{v}_\epsilon(y), \quad j = 1, ..., K, \quad y \in \Omega_\epsilon,$$

where $\chi_{\epsilon,j}$ was defined in (5.5).

Then it is easy to see that

$$\hat{v}_\epsilon(y) = \sum_{j=1}^K \tilde{v}_{\epsilon,j}(y - \frac{P_j^\epsilon}{\epsilon}) + e.s.t. \quad \text{in } H^2(\Omega_\epsilon).$$

We decompose ϕ_ϵ as follows:

$$\phi_\epsilon = \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \frac{\partial \tilde{v}_{\epsilon,j}}{\partial y_k} + \phi_\epsilon^\perp \tag{12.1}$$

with real numbers $a_{j,k}^\epsilon$, where

$$\phi_\epsilon^\perp \perp \tilde{K}_\epsilon = \text{span}\left\{ \frac{\partial \tilde{v}_{\epsilon,j}}{\partial y_k} | j = 1, \ldots, K, \, k = 1, 2 \right\} \subset H_N^2(\Omega_\epsilon)$$

Accordingly, we put

$$\psi_\epsilon(x) = \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \psi_{\epsilon,j,k} + \psi_\epsilon^\perp,$$

where $\psi_{\epsilon,j,k}$ is the unique solution of the problem

$$\begin{cases} \frac{1}{\beta^2}\Delta\psi_{\epsilon,j,k} - \psi_{\epsilon,j,k} - \frac{2}{A^2}\hat{v}_\epsilon u_\epsilon \frac{\partial \tilde{v}_{\epsilon,j}}{\partial y_k} - \frac{1}{A^2}\hat{v}_\epsilon^2 \psi_{\epsilon,j,k} = 0 \qquad \text{in } \Omega, \\ \frac{\partial \psi_{\epsilon,j,k}}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

and $\psi_\epsilon^\perp$ satisfies

$$\begin{cases} \frac{1}{\beta^2}\Delta\psi_\epsilon^\perp - \psi_\epsilon^\perp - \frac{2}{A^2}\hat{v}_\epsilon u_\epsilon \phi_\epsilon^\perp - \frac{1}{A^2}\hat{v}_\epsilon^2 \psi_\epsilon^\perp = 0 \qquad \text{in } \Omega, \\ \frac{\partial \psi_\epsilon^\perp}{\partial \nu} = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Suppose that $\|\phi_{\epsilon,j}\|_{H^2(\Omega_\epsilon)} = 1$. Then $|a_{j,k}^\epsilon| \le C$.

Our main idea consists of two steps: First, we show that the error $\phi_\epsilon^\perp$ is small in a suitable norm and thus can be neglected. Second, we derive algebraic equations for $a_{j,k}^\epsilon$ which are related to the matrix $M_0(\mathbf{P}_0)$.

Substituting the decompositions of ϕ_ϵ and ψ_ϵ into (9.1) we have

$$\begin{aligned} &\sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon (\tilde{v}_{\epsilon,j})^2 \left[\psi_{\epsilon,j,k} - \epsilon \frac{\partial u_\epsilon}{\partial x_k}\right] \\ &+\Delta_y \phi_\epsilon^\perp - \phi_\epsilon^\perp + 2\hat{v}_\epsilon u_\epsilon \phi_\epsilon^\perp + (\hat{v}_\epsilon)^2 \psi_\epsilon^\perp - \lambda_\epsilon \phi_\epsilon^\perp \\ &= \lambda_\epsilon \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \frac{\partial \tilde{v}_{\epsilon,j}}{\partial y_k} \quad \text{in } \Omega_\epsilon. \end{aligned} \tag{12.2}$$

Set

$$I_1 = \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon (\tilde{v}_{\epsilon,j})^2 \left[\psi_{\epsilon,j,k} - \epsilon \frac{\partial u_\epsilon}{\partial x_k}\right]$$

and

$$I_2 = \Delta_y \phi_\epsilon^\perp - \phi_\epsilon^\perp + 2\hat{v}_\epsilon u_\epsilon \phi_\epsilon^\perp + (\hat{v}_\epsilon)^2 \psi_\epsilon^\perp - \lambda_\epsilon \phi_\epsilon^\perp.$$

We divide our proof into two steps.

Step 1: Estimates for $\phi_\epsilon^\perp$.

Since $\phi_\epsilon^\perp \perp \tilde{K}_\epsilon$, then similar to the proof of Proposition 7.2 it follows that

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} \le C\|I_1\|_{L^2(\Omega_\epsilon)}. \tag{12.3}$$

Let us now compute I_1. The key is to estimate $\psi_{\epsilon,l,k} - \epsilon \frac{\partial u_\epsilon}{\partial x_k}$ near $x \in B_{r_0}(P_l^\epsilon)$.

From the equation for $\psi_{\epsilon,j,k}$, we obtain that

$$\psi_{\epsilon,j,k}(x) = -\frac{\beta^2}{A^2}\int_\Omega G_\beta(P_l^\epsilon, \xi)[2\hat{v}_\epsilon u_\epsilon \frac{\partial \tilde{v}_{\epsilon,j}}{\partial y_k} + \hat{v}_\epsilon^2 \psi_{\epsilon,j,k}]. \tag{12.4}$$

Similar to Section 6, we have

$$\psi_{\epsilon,j,k}(P_l^\epsilon) = O(\beta^2 L_\epsilon \epsilon) - \sum_{s=1}^K \frac{L_\epsilon}{\xi_{\epsilon,s}^2}\psi_{\epsilon,j,k}(P_s^\epsilon) - \frac{\eta_\epsilon L_\epsilon}{\xi_{\epsilon,l}^2}\psi_{\epsilon,j,k,}(P_l^\epsilon), \quad l = 1, ..., K$$

which implies that

$$\psi_{\epsilon,j,k}(P_l^\epsilon) = O(\beta^2 L_\epsilon \epsilon), \quad l = 1, ..., K. \tag{12.5}$$

For $x = P_l^\epsilon + \epsilon z \in B_{r_0}(P_l^\epsilon)$ we calculate

$$\psi_{\epsilon,j,k}(P_l^\epsilon + \epsilon z) - \psi_{\epsilon,j,k}(P_l^\epsilon)$$
$$= \frac{\beta^2}{A^2}\int_\Omega (G_\beta(P_l^\epsilon, \xi) - G_\beta(P_l^\epsilon + \epsilon z, \xi))[\hat{v}_\epsilon u_\epsilon \frac{\partial \tilde{v}_{\epsilon,j}}{\partial z_k} + \hat{v}_\epsilon^2 \psi_{\epsilon,j,k}]d\xi$$
$$= \frac{\beta^2}{A^2}\int_\Omega (G_\beta(P_l^\epsilon, \xi) - G_\beta(P_l^\epsilon + \epsilon z, \xi))[\tilde{v}_{\epsilon,j} u_\epsilon \frac{\partial \tilde{v}_{\epsilon,j}}{\partial z_k}] + O(\beta^2 L_\epsilon \epsilon |z| \sum_{l=1}^K |\psi_{\epsilon,j,k}(P_l^\epsilon)|)$$
$$= \frac{\beta^2}{A^2}\int_\Omega (G_\beta(P_l^\epsilon, \xi) - G_\beta(P_l^\epsilon + \epsilon z, \xi))[\tilde{v}_{\epsilon,j} u_\epsilon \frac{\partial \tilde{v}_{\epsilon,j}}{\partial z_k}] + O(\beta^4 L_\epsilon^2 \epsilon^2 |z|).$$

If $l \neq j$, then we have

$$\psi_{\epsilon,j,k}(P_l^\epsilon + \epsilon z) - \psi_{\epsilon,j,k}(P_l^\epsilon)$$
$$= -\frac{\beta^2}{A^2}\nabla_{P_l^\epsilon}\nabla_{P_j^\epsilon} G_\beta(P_l^\epsilon, P_j^\epsilon)\epsilon^2 z \frac{1}{\xi_{\epsilon,j}}\int_{R^2} z w(z)\frac{\partial w}{\partial z_k} dz + O(\beta^4 L_\epsilon^2 \epsilon^2 |z|).$$
$$= \frac{\beta^2 |\Omega| L_\epsilon}{2\xi_{\epsilon,j}}\epsilon^2 \nabla_{P_l^\epsilon}\nabla_{P_j^\epsilon} G_0(P_l^\epsilon, P_j^\epsilon) + O(\beta^4 L_\epsilon^2 \epsilon^2 |z|) \tag{12.6}$$

For $l = j$, similar calculations show that

$$\psi_{\epsilon,j,k}(P_j^\epsilon + \epsilon z) - \psi_{\epsilon,j,k}(P_l^\epsilon) = -\frac{\beta^2 |\Omega| L_\epsilon}{2\xi_{\epsilon,j}}\epsilon^2 \nabla_{P_j^\epsilon}\nabla_{P_j^\epsilon} H_0(P_j^\epsilon, P_j^\epsilon) + O(\beta^4 L_\epsilon^2 \epsilon^2 |z|)$$
$$+ \frac{\beta^2}{A^2 \xi_{\epsilon,j}}\epsilon^2 \int_{R^2} \log\frac{|z-\zeta|}{|\zeta|} w(\zeta)\frac{\partial w}{\partial \zeta_k}(\zeta)\, d\zeta. \tag{12.7}$$

Next, we compute $\epsilon \frac{\partial u_\epsilon}{\partial x_k}(x)$ for $x = P_l^\epsilon + \epsilon z \in B_{r_0}(P_l^\epsilon)$:

$$\epsilon \frac{\partial u_\epsilon}{\partial x_k}(x) = -\frac{\beta^2}{A^2}\int_\Omega \frac{\partial}{\partial x_k} G_\beta(x, \xi)(\epsilon \hat{v}_\epsilon^2 u_\epsilon)\, d\xi.$$

So

$$\epsilon(\frac{\partial u_\epsilon}{\partial x_k}(x) - \frac{\partial u_\epsilon}{\partial x_k}(P_l^\epsilon)) = -\frac{\beta^2}{A^2}\int_\Omega [\frac{\partial}{\partial x_k} G_\beta(x, \xi) - \frac{\partial}{\partial x_k} G_\beta(x, \xi)|_{x = P_l^\epsilon}](\epsilon \hat{v}_\epsilon^2 u_\epsilon)\, d\xi$$
$$+ \frac{\beta^2}{A^2 \xi_{\epsilon,j}}\epsilon^2 \int_{R^2} \log\frac{|z-\zeta|}{|\zeta|} w \frac{\partial w}{\partial \zeta_k}\, d\zeta + o(\beta^2 L_\epsilon^2 \epsilon^2 |z|) \tag{12.8}$$

since

$$\nabla_{P_j^\epsilon} F_0(\mathbf{P}^\epsilon) = o(1).$$

Combining (12.7) and (12.8), we obtain that

$$[\psi_{\epsilon,j,k}(P_l^\epsilon + \epsilon z) - \epsilon \frac{\partial u_\epsilon}{\partial x_k}(P_l^\epsilon + \epsilon z)] - [\psi_{\epsilon,j,k}(P_l^\epsilon) - \epsilon \frac{\partial u_\epsilon}{\partial x_k}(P_l^\epsilon)]$$

$$= -\frac{\beta^2 |\Omega| L_\epsilon \xi_{\epsilon,l}}{2} \epsilon^2 \nabla_{P_j^\epsilon} \nabla_{P_l^\epsilon} F_0(\mathbf{P}^\epsilon) z_k + o(\beta^2 L_\epsilon^2 \epsilon^2 |z|). \tag{12.9}$$

Hence we have

$$\|I_1\|_{L^2(\Omega_\epsilon)} = o(\beta^2 \epsilon^2 \sum_{j=1}^K \sum_{k=1}^2 |a_{j,k}^\epsilon|)$$

and

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} = o(\beta^2 \epsilon^2 \sum_{j=1}^K \sum_{k=1}^2 |a_{j,k}^\epsilon|). \tag{12.10}$$

It is easy to show that

$$\int_{\Omega_{\epsilon,P_j^\epsilon}} (I_2 \frac{\partial \tilde{v}_{\epsilon,l}}{\partial z_m}) d\xi = \int_\Omega \tilde{v}_{\epsilon,l}^2 (\epsilon \frac{\partial u_\epsilon}{\partial x_m} \phi_\epsilon^\perp - \frac{\partial \tilde{v}_{\epsilon,l}}{\partial x_m} \psi_\epsilon^\perp)) d\xi$$

$$= o(\beta^2 \epsilon^2 \sum_{j=1}^K \sum_{k=1}^2 |a_{j,k}^\epsilon|)$$

since

$$\frac{\partial u_\epsilon}{\partial x_m} = O(\beta^2) \quad \text{in } \Omega.$$

Step 2: Algebraic equations for $a_{j,k}^\epsilon$.

Multiplying both sides of (12.2) by $-\frac{\partial \tilde{v}_{\epsilon,l}}{\partial z_m}$ and integrating over $\Omega_{\epsilon,P_l^\epsilon}$, we obtain

$$r.h.s. = \lambda_\epsilon \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \int_{\Omega_{\epsilon,P_l^\epsilon}} \frac{\partial \tilde{v}_{\epsilon,j}}{\partial z_k} \frac{\partial \tilde{v}_{\epsilon,l}}{\partial z_m}$$

$$= \frac{1}{\xi_{\epsilon,l}^2} \lambda_\epsilon \sum_{j,k} a_{j,k}^\epsilon \delta_{jl} \delta_{km} \int_{R^2} \left(\frac{\partial w}{\partial z_1} \right)^2 dz (1 + O(\log \frac{1}{\epsilon}))$$

$$= \frac{1}{\xi_{\epsilon,l}^2} \lambda_\epsilon a_{l,m}^\epsilon \int_{R^2} \left(\frac{\partial w}{\partial z_1} \right)^2 (1 + O(\log \frac{1}{\epsilon}))$$

and

$$l.h.s. = \epsilon^2 \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \int_{\Omega_{\epsilon,P_l^\epsilon}} (\tilde{v}_{\epsilon,j})^2 \left[\psi_{\epsilon,j,k} - \epsilon \frac{\partial u_\epsilon}{\partial x_k} \right] \frac{\partial \tilde{v}_{\epsilon,l}}{\partial z_m}$$

$$+\int_{\Omega_{\epsilon,P_l^\epsilon}}(I_2\frac{\partial\tilde{v}_{\epsilon,l}}{\partial z_m})d\xi$$

$$=\epsilon^2\sum_{j=1}^K\sum_{k=1}^2 a_{j,k}^\epsilon\int_{\Omega_{\epsilon,P_l^\epsilon}}(\tilde{v}_{\epsilon,j})^2\left[\psi_{\epsilon,j,k}-\epsilon\frac{\partial u_\epsilon}{\partial x_k}\right]\frac{\partial\tilde{v}_{\epsilon,l}}{\partial z_m}$$

$$+o(\beta^2\epsilon^2\sum_{j=1}^K\sum_{k=1}^2|a_{j,k}^\epsilon|).$$

Using (12.9), we obtain

$$l.h.s.=\frac{\epsilon^2|\Omega|\beta^2L_\epsilon}{2\xi_{\epsilon,l}}\sum_{j=1}^K\sum_{k=1}^2 a_{j,k}^\epsilon$$

$$\times\int_{\Omega_{\epsilon,P_l^\epsilon}}w^2(-\frac{1}{\xi_{\epsilon,j}}\frac{\partial}{\partial P_{l,m}^\epsilon}\frac{\partial}{\partial P_{j,k}^\epsilon}F_0(\mathbf{P}^\epsilon)\epsilon z_m)\frac{\partial w}{\partial z_m}$$

$$+o(\epsilon^2\beta^2\sum_{j=1}^K\sum_{k=1}^2|a_{j,k}^\epsilon|)$$

$$=\frac{\epsilon^2|\Omega|\beta^2L_\epsilon}{2\xi_{\epsilon,l}}\int_{R^2}w^2\frac{\partial w}{\partial z_m}z_m\sum_{k=1}^2 a_{l,k}^\epsilon\left(-\frac{1}{\xi_{\epsilon,j}}\frac{\partial}{\partial P_{l,m}^\epsilon}\frac{\partial}{\partial P_{j,k}^\epsilon}F_0(\mathbf{P}^\epsilon)\right)$$

$$+o(\epsilon^2\beta^2\sum_{j=1}^K\sum_{k=1}^2|a_{j,k}^\epsilon|).$$

Note that

$$\int_{R^2}w^2\frac{\partial w}{\partial z_m}z_m=\int_{R^2}w^2w^{'}\frac{z_m^2}{|z|}$$

$$=\frac{1}{2}\int_{R^2}w^2w^{'}|z|<0.$$

Thus we have

$$l.h.s.=\frac{\epsilon^2\Omega|\beta^2L_\epsilon}{4\xi_{\epsilon,l}}(-\int_{R^2}w^2w^{'}|z|)\sum_{j=1}^K\sum_{k=1}^2 a_{j,k}^\epsilon\left(\frac{1}{\xi_{\epsilon,j}}\frac{\partial}{\partial P_{l,m}^\epsilon}\frac{\partial}{\partial P_{j,k}^\epsilon}F(\mathbf{P}^\epsilon)\right)$$

$$+o(\epsilon^2\beta^2\sum_{j=1}^K\sum_{k=1}^2|a_{j,k}^\epsilon|).$$

Combining the $l.h.s.$ and $r.h.s$, we have

$$\frac{\epsilon^2|\Omega|\beta^2L_\epsilon}{4}\xi_{\epsilon,l}(-\int_{R^2}w^2w^{'}|z|)\sum_{j=1}^K\sum_{k=1}^2 a_{j,k}^\epsilon\left(\frac{1}{\xi_{\epsilon,j}}\frac{\partial}{\partial P_{l,m}^\epsilon}\frac{\partial}{\partial P_{j,k}^\epsilon}F_0(\mathbf{P}^\epsilon)\right)$$

$$+o(\epsilon^2\beta^2\sum_{j=1}^K\sum_{k=1}^2|a_{j,k}^\epsilon|)$$

$$= \lambda_\epsilon a_{l,m}^\epsilon \int_{R^2} \left(\frac{\partial w}{\partial z_1}\right)^2. \tag{12.11}$$

We have shown that the small eigenvalues with $\lambda_\epsilon \to 0$ satisfy $\lambda_\epsilon \sim C\epsilon^2\beta^2$ with some $C \neq 0$. Furthermore, (asymptotically) they are eigenvalues of the matrix $\mathcal{X} M_0(\mathbf{P}_0)\mathcal{X}^{-1}$, where

$$\mathcal{X} = \begin{pmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_K \end{pmatrix}$$

and the coefficients $a_{j,k}^\epsilon$ are the corresponding eigenvectors.

If the matrix $M_0(\mathbf{P}_0)$ is strictly negative definite, as $\mathcal{X}$ is strictly positive definite, it follows that $\mathrm{Re}(\lambda_\epsilon) \le c < 0$, where c is independent of ϵ. Therefore the small eigenvalues λ_ϵ are stable if ϵ is small enough.

□

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