CRITICAL THRESHOLD AND STABILITY OF CLUSTER SOLUTIONS FOR LARGE REACTION-DIFFUS ION SYSTEMS IN R^1

JUNCHENG WEI * AND MATTHIAS WINTER †

Abstract. We study a large reaction-diffusion system which arises in the modeling of catalytic networks and describes the emerging of cluster states. We construct single cluster solutions on the real line and then establish their stability or instability in terms of the number N of components and the connection matrix. We provide a rigorous analysis around the single cluster solutions, which is new for systems of this kind. Our results show that for $N \leq 4$ the hypercycle system is linearly stable while for $N \geq 5$ the hypercycle system is linearly unstable.

Key words. Pattern Formation, Stability, Cluster Solutions, Reaction-Diffusion System, Catalytic Network, Hypercycle

AMS subject classifications. Primary 35B35, 92C40; Secondary 35B40

1. Introduction: The Model. In this paper, we continue our study [61] on the cluster solutions for large reaction-diffusion systems. A typical example is the hypercyclical reaction-diffusion system which arises as a spatial model concerning the origin of life similar to the one introduced by Eigen and Schuster [18] - [20], [21]. For more background on the concept of the hypercycle see also [35], [36]. It arises in the modeling of catalytic networks in the case that a number of RNA-like polymers ("components") catalyse the replication of each other in a cyclic way. Examples in nature include the Krebs cycle for biosynthesis in the living cell and the Bethe-Weizsäcker cycle for high rate energy production in massive stars. Eigen and Schuster argue that the hypercycle satisfies important criteria of natural selection: 1. Selective stability of each component due to favorable competition with error copies, 2. Cooperative behavior of the components integrated into the hypercycle, and 3. Favorable competition of the hypercycle unit with other less efficient systems.

We show rigorously that this may lead to compartmentation (i.e., the build-up of spatially small and essentially closed subsystems) due to spontaneous formation of clusters (also called "spots" or "spikes").

We first study a general system of $N+1$ equations, where N may be any fixed positive integer representing the number of components. For this general system we first prove the existence of solutions with clusters which for the different components have the same location and different heights.

Then we study the stability question for some particularly important examples. At this point we should like to emphasize that we provide a rigorous analysis around cluster solutions, not around *constant states*. We also establish a threshold size for the system such that smaller systems are stable and larger one are unstable. This type of result is new for the kind of $(N+1)$-systems under investigation.

*Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong (wei@math.cuhk.edu.hk).

†Mathematisches Institut, Universität Stuttgart, D-70511 Stuttgart, Germany (winter@mathematik.uni-stuttgart.de)

We now proceed to write down the reaction-diffusion system explicitly and define the biological terms in a mathematically rigorous way. As suggested in [8], [9] we study the following:

$$(1.1)\begin{cases} \frac{\partial X_i}{\partial t} = D_X \Delta X_i - g_X X_i + M \sum_{j=1}^N k_{ij} X_i X_j, \quad i = 1, 2, \ldots, N, \quad x \in R, \\ \frac{\partial M}{\partial t} = D_M \Delta M + k_M - g_M M - LM \sum_{i,j=1}^N k_{ij} X_i X_j, \quad x \in R, \end{cases}$$

where N is the number of different polymer species, X_i denotes the concentration of the polymers, and M is the concentration of activated monomers. The replication of each polymer X_i is catalysed by each X_j at a non-negative rate constant k_{ij}. Linear (non-catalytic) growth terms are neglected. The activated monomers are produced at constant rate, k_M; g_X and g_M are decay rate constants. L is the number of monomers in each polymer, and D_X and D_M are constant diffusion coefficients.

A typical example of the matrix k_{ij} is a hypercyclical $N \times N$ matrix, namely

$$(1.2) \qquad (k_{ij}^{hyper}) = \begin{pmatrix} 0 & 0 & 0 & ... & k_0 \\ k_0 & 0 & 0 & ... & 0 \\ 0 & k_0 & 0 & ... & 0 \\ ... & ... & ... & ... & 0 \\ 0 & 0 & ... & k_0 & 0 \end{pmatrix}_{N \times N}, \quad k_0 > 0.$$

The system (1.1) with the matrix (k_{ij}^{hyper}) is called "elementary hypercycle" by Eigen and Schuster [21] as the polymers interact in pairs only. There are more complex hypercycles if the polymers interact in triples, quadruples, etc. However, more complex hypercycles are likely to be of less importance for an efficient start of evolution than elementary hypercycles since they are more difficult to form in the first place.

While Eigen and Schuster [21] use an assumption of constant organization, meaning that the total sum of all polymer concentrations is kept constant, in system (1.1) another mechanism for bounding the polymer concentrations is present: Since each polymer consists of L monomers the polymer concentrations are bounded by the limited supply of activated monomers. This is a nonlocal coupling in contrast to the local coupling in the model of Eigen and Schuster.

We pose the problem in one-dimensional space which on the one hand allows a rigorous analysis and on the other hand is relevant if the early biochemical reactions take place in very thin lines like for example on the edges of rocks.

A cluster may loosely be defined as a region of high concentrations X_i of the polymers and low concentration M of the monomer, as monomers are consumed by the replication of polymers. A rigorous definition of cluster is given by the solution in the existence theorem (Theorem 2.1).

In this paper, we study the existence and stability of a single-cluster solution in R^1. Let us first reduce the system (1.1) to standard form. Dividing by g_X and g_M, respectively, gives

$$\frac{1}{g_X} \partial_t X_i = \frac{D_X}{g_X} \Delta X_i - X_i + \frac{M}{g_X} \sum_{j=1}^N k_{ij} X_i X_j,$$

$$\frac{1}{g_M}\partial_t M = \frac{D_M}{g_M}\Delta M + \frac{k_M}{g_M} - M - \frac{LM}{g_M}\sum_{ji,=1}^{N} k_{ij}X_iX_j.$$

Rescaling $M = (k_M/g_M)\hat{M}$, $X_i = \sqrt{g_M/L}\hat{X}_i$, we get

$$\frac{1}{g_X}\partial_t \hat{X}_i = \frac{D_X}{g_X}\Delta\hat{X}_i - \hat{X}_i + \frac{1}{g_X}\frac{k_M}{g_M}\hat{M}\sqrt{\frac{g_M}{L}}\sum_{j=1}^{N} k_{ij}\hat{X}_i\hat{X}_j,$$

$$\frac{1}{g_M}\partial_t\hat{M} = \frac{D_M}{g_M}\Delta\hat{M} + 1 - \hat{M} - \hat{M}\sum_{i,j=1}^{N} k_{ij}\hat{X}_i\hat{X}_j.$$

Rescaling space variables x and time variable t:

$$x = \sqrt{\frac{D_M}{g_M}}\hat{x}, \quad t = \frac{1}{g_X}\hat{t},$$

renaming constants:

$$A = \frac{k_M}{g_X g_M}\sqrt{\frac{g_M}{L}}, \quad \epsilon^2 = \frac{D_X}{D_M}\frac{g_M}{g_X}, \quad \tau = \frac{g_X}{g_M}$$

and dropping the hats, we finally arrive at the following standard form

$$\left\{\begin{array}{l} \partial_t X_i = \epsilon^2\Delta X_i - X_i + AM\sum_{i=1}^N k_{ij}X_iX_j, \\ \tau\partial_t M = \Delta M + 1 - M - M\sum_{i,j=1}^N k_{ij}X_iX_j. \end{array}\right. \tag{1.3}$$

We shall study (1.3) on the real line R for $\epsilon > 0$ small. Different choices of A and τ might distinguish between stability and instability. Therefore we will treat them as parameters. We look for solutions of (1.3) which are even:

$$X_i = X_i(|x|) \in H^1(R), \quad i = 1, \ldots, N,$$

$$1 - M = 1 - M(|x|) \in H^1(R).$$

The stationary equation corresponding to (1.3) becomes

$$\left\{\begin{array}{c} \epsilon^2\Delta X_i - X_i + AM\sum_{j=1}^N k_{ij}X_iX_j = 0, \quad i = 1, ..., N, \\ \Delta M + 1 - M - M\sum_{i,j=1}^N k_{ij}X_iX_j = 0, \\ X_i(|x|) > 0, 0 < M(|x|) < 1, x \in R. \end{array}\right. \tag{1.4}$$

¿From now on, we shall concentrate on (1.3) and (1.4).

2. Main Results: Existence and Stability. We now state our main results of this paper. We first construct cluster solutions to (1.4). To this end, we need to introduce some assumptions and notations.

Let w be the unique solution of the following problem

$$\left\{\begin{array}{l} \Delta w - w + w^2 = 0, w > 0 \ \text{ in } R, \\ w(0) = \max_{y\in R} w(y), w(y) \to 0 \text{ as } |y| \to +\infty. \end{array}\right. \tag{2.1}$$

Since (2.1) is an ODE, we can write w explicitly

$$w(y) = \frac{3}{2\cosh^2 \frac{y}{2}}. \tag{2.2}$$

We now state the existence result. In fact, this is quite easy. We search for solutions of the following type

$$X_i = \xi_i X_0, \xi_i > 0, \quad i = 1, \ldots, N, \tag{2.3}$$

where ξ_i are positive constants which satisfy

$$\sum_{j=1}^{N} k_{ij}\xi_j = 1, \quad i = 1, \ldots, N. \tag{2.4}$$

Our first assumption is that

(H1) there exists a unique solution $(\xi_1, ..., \xi_N)$ of (2.4).

Suppose (H1) holds true. Substituting (2.3) into (1.4), we see that (X_0, M) must satisfy

$$\begin{cases} \epsilon^2 \Delta X_0 - X_0 + AMX_0^2 = 0, \text{ in } R, \\ \Delta M + 1 - M - M(\sum_{i=1}^{N} \xi_i) X_0^2 = 0, \text{ in } R. \end{cases} \tag{2.5}$$

In the case $N = 1$ problem (2.5) becomes the standard Gray-Scott model [23], [24], [58]. The existence of single-pulse solutions for the Gray-Scott model in one dimension has been studied in [14] and in two dimension in [58].

Following the same proof as in Theorem 2.1 of [58], we define

$$L = L(A, \epsilon) := \frac{1}{2A^2 \sum_{i=1}^{N} \xi_i} \epsilon \int_R (w(y))^2 dy. \tag{2.6}$$

If $0 < L < \frac{1}{4}$, then the following equation has two solutions:

$$\eta(1 - \eta) = L. \tag{2.7}$$

We denote the smaller one by η^s, where $0 < \eta^s < \frac{1}{2}$ and the larger one by η^l, where $1 > \eta^l > \frac{1}{2}$.

Now we have

THEOREM 2.1.

Suppose that (H1) holds.

Assume that

$$\epsilon << 1 \tag{2.8}$$

and

$$\epsilon << L < \frac{1}{4} - \delta_0, \tag{2.9}$$

more precisely, for $L = L(A,\epsilon)$ there are positive numbers δ_0, δ_1 and ϵ_0 such that for all ϵ and A with $0 < \epsilon < \epsilon_0$ we have $L < \frac{1}{4} - \delta_0$ and $\epsilon/L(A,\epsilon) < \delta_1$.

Then problem (1.4) admits two "single-cluster" solutions $(X^s_\epsilon, M^s_\epsilon) = (X^s_{\epsilon,1}, \ldots, X^s_{\epsilon,N}, M^s_\epsilon)$ and $(X^l_\epsilon, M^l_\epsilon) = (X^l_{\epsilon,1} \ldots, X^l_{\epsilon,N}, M^l_\epsilon)$ with the following properties:

(1) all components are even functions.

(2) $X^s_{\epsilon,i} = \frac{\xi_i}{AM^s_\epsilon(0)}(1+o(1))w(\frac{|x|}{\epsilon})$, $i = 1,\ldots,N$, $X^l_{\epsilon,i} = \frac{\xi_i}{AM^l_\epsilon(0)}(1+o(1))w(\frac{|x|}{\epsilon})$, $i = 1,\ldots,N$, where w is the unique solution of (2.1).

(3) $M^s_\epsilon(x) \to 1$ $M^l_\epsilon(x) \to 1$ for all $x \neq 0$ and $M^s_\epsilon(0)$, $M^l_\epsilon(0)$ satisfy

$$
\begin{gathered}
M^s_\epsilon(0) \sim \eta^s, \quad M^l_\epsilon(0) \sim \eta^l, \\
0 < M^s_\epsilon(0) < M^l_\epsilon(0) < 1.
\end{gathered}
\tag{2.10}
$$

(4) There exist $a > 0, b > 0$ such that

$$
\begin{gathered}
0 < 1 - M^s_\epsilon(x) \le Ce^{-a|x|}, \quad 0 < 1 - M^l_\epsilon(x) \le Ce^{-a|x|}, \\
0 < X^s_{\epsilon,i}(x) \le C(AM^s_\epsilon(0))^{-1}e^{-b\frac{|x|}{\epsilon}}, \quad 0 < X^l_{\epsilon,i}(x) \le C(AM^l_\epsilon(0))^{-1}e^{-b\frac{|x|}{\epsilon}}
\end{gathered}
\tag{2.11}
$$

Finally, if ϵ is small enough and $L > \frac{1}{4} + \delta_0$ (in the same sense as in (2.9)) then there are no single-cluster solutions.

The proof of Theorem 2.1 is exactly the same as in the proof of Theorem 2.1 of [58] or Theorem 1.1 of [61]. We omit the details here.

The main goal of this paper is to study the stability and instability of the cluster solution constructed in Theorem 2.1. To this end, we first linearize the equations (1.3) around $(X^s_\epsilon, M^s_\epsilon)$ or $(X^l_\epsilon, M^l_\epsilon)$, respectively. From now on we omit the superscripts s or l where this is possible without confusing the reader. The linearized operator is as follows:

$$
\mathcal{L}_\epsilon \begin{pmatrix} \phi_{\epsilon,i} \\ \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} \epsilon^2\Delta\phi_{\epsilon,i} - \phi_{\epsilon,i} + AM_\epsilon \sum_{j=1}^N k_{ij}(\phi_{\epsilon,j}X_{\epsilon,i} + X_{\epsilon,j}\phi_{\epsilon,i}) \\ +A\psi_\epsilon \sum_{j=1}^N k_{ij}X_{\epsilon,i}X_{\epsilon,j} \\ \Delta\psi_\epsilon - \psi_\epsilon - \psi_\epsilon \sum_{i,j=1}^N k_{ij}X_{\epsilon,i}X_{\epsilon,j} \\ -M_\epsilon \sum_{i,j=1}^N k_{ij}(\phi_{\epsilon,j}X_{\epsilon,i} + \phi_{\epsilon,i}X_{\epsilon,j}) \end{pmatrix},
\tag{2.12}
$$

where $i = 1,\ldots,N$. The eigenvalue problem becomes

$$
\mathcal{L}_\epsilon \begin{pmatrix} \phi_{\epsilon,i} \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} \lambda_\epsilon\phi_{\epsilon,i} \\ \tau\lambda_\epsilon\psi_\epsilon \end{pmatrix}, \ i = 1,\ldots,N.
\tag{2.13}
$$

We assume that the domain of $\mathcal{L}_\epsilon$ is $(H^2(R))^N$ and $\lambda_\epsilon \in \mathcal{C}$ – the set of complex numbers.

Certainly 0 is an eigenvalue of $\mathcal{L}_\epsilon$. We say that a cluster solution is **linearly stable** if the spectrum $\sigma(\mathcal{L}_\epsilon)$ of $\mathcal{L}_\epsilon$ (except for 0) lies in a left half plane $\{\lambda \in \mathcal{C} : \text{Re}\,(\lambda) < -c_0\}$ where $c_0 > 0$, and that 0 is a simple eigenvalue. A cluster solution is called **linearly unstable** if there exists an eigenvalue λ_ϵ of $\mathcal{L}_\epsilon$ with Re $(\lambda_\epsilon) > 0$. (From now on, we use the notations linearly stable and linearly unstable as defined above.)

Before we state our results on the stability, we introduce two more assumptions on the connection matrix (k_{ij}).

The second assumption is the following:

$$\text{(H2)} \qquad \textstyle\sum_{i=1}^N k_{ij}\xi_i = 1,\ j = 1, ..., N,$$

where ξ_j is given (2.4).

Note that Assumption (H2) imposes a certain symmetry on the connection matrix (k_{ij}).

The last assumption concerns the following eigenvalue problem:

$$\text{(EVP)} \qquad \begin{cases} \Delta\phi - \phi + \mu w\phi = 0, \\ \phi \in H^1(R). \end{cases}$$

By Lemma 4.1 of [51], (EVP) admits the following set of eigenvalues

$$\mu_1 = 1,\ \mu_2 = 2,\ 2 < \mu_3 \le \mu_4 \le \dots. \tag{2.14}$$

(In fact, we have the following explicit values of μ_n (see Appendix A):

$$\mu_n = \frac{(1+n)(2+n)}{6}, \qquad n = 1, 2, 3, \dots, .) \tag{2.15}$$

Put

$$\mathcal{B} = (b_{ij}),\ \text{where} \qquad b_{ij} = (\xi_i k_{ij}). \tag{2.16}$$

Observe that by (2.4) and (H1) the matrix $\mathcal{B}$ has an eigenvalue 1 and the associated eigenvector is $\xi = (\xi_1, \xi_2, \dots, \xi_N)^\tau$, i.e. we have $\mathcal{B}\xi = \xi$.

We take the Jordan decomposition of $\mathcal{B}$

$$\mathcal{B} = \mathcal{P}\mathcal{D}\mathcal{P}^{-1}, \tag{2.17}$$

where $\mathcal{P}$ is an invertible matrix and $\mathcal{D}$ is the Jordan form. Namely, we have

$$b_{ij} = \sum_{k,l=1}^N p_{ik} d_{kl} p_{lj}^{-1},$$

where d_{kl} has Jordan form (i.e., it is composed of Jordan blocks

$$\begin{pmatrix} \sigma_k & 1 & 0 & \cdots & 0 \\ 0 & \sigma_k & 1 & \cdots & 0 \\ 0 & 0 & \sigma_k & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 1 \\ 0 & 0 & 0 & \cdots & \sigma_k \end{pmatrix}$$

with eigenvalues $\sigma_k \in \mathcal{C}$) and $\sum_{k=1}^N p_{ik}p_{kj} = \delta_{ij}$.

We now assume that

$$\text{(H3)} \quad \begin{cases} [1 + \text{spec}(\mathcal{B})] \cap \text{spec(EVP)} = \{2\}, \\ 1 \text{ is a simple eigenvalue of } \mathcal{B}. \end{cases}$$

Assumption (H3) means the following: Let us denote the eigenvalues of $\mathcal{B}$ by:

$$\sigma_1 = 1, \sigma_2, ..., \sigma_N, \tag{2.18}$$

where σ_j may be complex. Then assumption (H3) is equivalent to

$$\sigma_j \neq \frac{(1+n)(2+n)}{6} - 1 \quad \text{for } j \geq 2, n = 1, 2, \dots . \tag{2.19}$$

Since $\xi = (\xi_1, ..., \xi_N)^\tau$ is an eigenvector of $\mathcal{B}$ with eigenvalue 1, by assumption (H3), we may assume that

$$\mathcal{P} = (\mathbf{p}_1, ..., \mathbf{p}_N), \mathbf{p}_1 = \frac{1}{\|\xi\|}\xi, \|\xi\| = \sqrt{\sum_{i=1}^{N} \xi_i^2}. \tag{2.20}$$

The following is our main result on stability.

THEOREM 2.2. *Suppose that the matrix (k_{ij}) satisfies (H1), (H2) and (H3). Assume that*

$$\epsilon << 1, \quad \epsilon << L < \frac{1}{4} - \delta_0, \tag{2.21}$$

in the same sense as in (2.9).

Let $(X_\epsilon^s, M_\epsilon^s)$ and $(X_\epsilon^l, X_\epsilon^l)$ be the solutions constructed in Theorem 2.1.

Let $\sigma = \sigma_R + i\sigma_I$ be an eigenvalue of $\mathcal{B}$ and let

$$f(\sigma) := (12\sigma_R + 5)^2(3\sigma_R^2 + 2\sigma_R) - 3\sigma_I^2. \tag{2.22}$$

Then we have the following:

(1) (Stability) Suppose that $0 \leq \tau < \tau_0$, where $\tau_0 > 0$ may be chosen independent of ϵ. Assume that for all eigenvalues σ of $\mathcal{B}$ with $\sigma \neq 1$ and $\sigma_R > 0$, we have $f(\sigma) < 0$. Then $(X_\epsilon^s, M_\epsilon^s)$ is linearly stable.

(2) (Instability) Assume that there exists an eigenvalue σ of $\mathcal{B}$ with $\sigma \neq 1$ and $\sigma_R > 0$ such that $f(\sigma) > 0$. Then $(X_\epsilon^s, M_\epsilon^s)$ is linearly unstable for all $\tau > 0$.

(3) (Instability) $(X_\epsilon^l, M_\epsilon^l)$ is linearly unstable for all $\tau > 0$.

Theorem 2.2 applies to many matrices. In Section 4, we shall apply Theorem 2.2 to some specific examples which include the N-hypercycle case, $(k_{ij}) = (k_{ij}^{hyper})$, where (k_{ij}^{hyper}) is given by (1.2). In this case, we have

THEOREM 2.3. *Consider the hypercycle case, i.e. let (k_{ij}) be given in (1.2). Assume that (2.21) holds. Let $(X_\epsilon^s, M_\epsilon^s)$ and $(X_\epsilon^l, X_\epsilon^l)$ be the solutions constructed in Theorem 2.1.*

Then we have the following:

(1) (Stability) Assume that $N \leq 4$ and $0 < \tau < \tau_0$ for some small $\tau_0 > 0$ which is independent of ϵ. Then $(X_\epsilon^s, M_\epsilon^s)$ is linearly stable.

(2) (Instability) Assume that $N > 4$. Then $(X_\epsilon^s, M_\epsilon^s)$ is linearly unstable for all $\tau > 0$

(3) (Instability) $(X_\epsilon^l, M_\epsilon^l)$ is linearly unstable for all $\tau > 0$.

The proof of Theorem 2.3 is based on Theorem 2.2 and will be given in Section 4.

Some remarks on the stability results – Theorem 2.2 and Theorem 2.3 – are in order.

Remarks:

1). For existence (Theorem 2.1), only assumption (H1) is needed. For the stability results (Theorem 2.2), we need all three assumptions (H1) – (H3). Conditions (H2) and (H3) are needed in the reduction process (Section 6, Lemma 6.4) and in the study of vectorial nonlocal eigenvalue problem (Section 7). These conditions enable us to **decouple** the system. It is an interesting open problem to study the case when assumptions (H2) and (H3) are dropped.

Note also that it is allowed that $\xi_i \neq \xi_j$ for $i \neq j$. So we may have clusters with **different heights**.

2). In (1) of Theorem 2.2, we have assumed that τ is small. In the case that τ is large, we can show that the stability of $(X_\epsilon^s, M_\epsilon^s)$ can be reduced to the study of an algebraic equation (Section 5). More precisely, one can use hypergeometric functions and generalized hypergeometric functions to reduce the stability of the nonlocal eigenvalue problem (NLEP) given in (5.2) to the algebraic equation which is given in Lemma 5.4 and derived in Appendix B.

3). The threshold of stability at $N = 4$ for the hypercycle system (Theorem 2.3) has far-reaching consequences for biological applications. It implies that the underlying biological system can only be stable if it does not have too many constituents. This shows that pre-biotic evolution might fail if the system becomes too large.

This is qualitatively the same result as has been established by the authors in the two-dimensional system. However, in two dimensions we were not able to establish the exact threshold [61].

Knowing the exact threshold size for stability is also important to verify the validity of our model by experiments: Now the question can be studied if the thresholds given by theory and the one determined by experiments are the same. Furthermore, the agreement between theoretical values and numerically calculated ones for related models play an important role in finding which model to choose preferably. (We refer to the works quoted at the end of the introduction for related numerical investigations, in particular in [7], where among others multi-cluster states in one space dimension have been computed numerically).

Our critical threshold is in correspondence with the result of Eigen and Schuster [21] that the **constant** non-trivial steady state for the hypercycle is stable if and only if $N \leq 4$.

To see quickly how the magic number 4 comes into play, we have to study an eigenvalue problem with complex coefficients:

$$\Delta\phi - \phi + (1 + e^{\sqrt{-1}\theta})w\phi = \lambda\phi, \phi \in H^2(R), \tag{2.23}$$

where $\theta = \frac{2\pi}{N}$. By using hypergeometric functions, we show (Section 5) that problem (2.23) is stable if and only if $\theta > \theta^h \sim \arccos(0.0455)$. Substituting the expression for $\theta = \frac{2\pi}{N}$, we see that $N \leq 4$.

Let us conclude this section by mentioning some related results.

In [8] the parameter dependence of stability of clusters and spirals against parasites (i.e., rival polymers which receive catalytic support from the hypercycle but do not contribute to the catalysis of any other polymer) is studied numerically. Mathematically speaking, occurrence of a parasite means that there exists $i_0 \in \{1, 2, \ldots, N\}$ such that $k_{i_0,j} > 0$ for some $j \neq i_0$ but $k_{j,i_0} = 0$ for all j. A parasite may or may not destroy the hypercycle depending on the rate constants. In [9] clusters (for $N = 5$) are established numerically for the elementary N-hypercycle system in two space dimensions.

It is known numerically ([8], [9]) that parasites may destroy stable cluster states. Our results complement the picture by the rigorously proved fact that even pure cluster states may turn unstable if they become two large. This implies that the hypercycle although it has some very preferable properties (see the beginning of the introduction) on the other hand it has an inherent instability behavior which may act as an obstruction to the evolution of large biological systems.

In [7] for a closely related reaction-diffusion model in one and two space dimensions the dependence of various properties of cluster states on diffusivities is shown numerically including the cluster size, their shape, and the distance between different clusters.

The effect of faulty replication on the hypercycle has been studied by an analysis of the geometry of bifurcations around steady states and numerical computations in the framework of an ODE reaction model [1].

For a cellular automata model it was shown numerically that a spiral wave structure may be stable against parasites [5]. The chaotic dynamics for this type of model has been investigated numerically in [34], [46].

There are a number of recent results on the Gray-Scott model, which we would like to recall here. In [14], by using Mel'nikov method, Doelman, Kaper and Zegeling constructed single and multiple pulse solutions for (1.1) in the one-dimensional case with $D_M = 1, D_X = \delta^2 << 1$, where $X_i = X$. In their paper [14], it is assumed that $k_M = g_M = \delta^2, g_X = \delta^{2\alpha/3}$, $k_{11} = 1$, $L = 1$, where $\alpha \in [0, \frac{3}{2})$. In this case, they showed that $M = O(\delta^\alpha), X = O(\delta^{-\frac{\alpha}{3}})$. Later the stability of single and multiple pulse solutions in 1-D are obtained in [12], [13]. (The techniques are extended to other reaction-diffusion equations in [15].) We note that in their scaling, $\tau = \delta^{\frac{2\alpha}{3}-2}$. Their scaling is chosen in order to obtain $X = O(1), M = O(1)$. Since they choose two scaling parameters accordingly they can achieve their goal. In our standard formulation of the system (1.3) we have only the scaling parameter A so that we cannot obtain $X = O(1), M = O(1)$. On the other hand, the homoclinic solution in their scaling corresponds exactly to our cluster solution in (1.3) which is given in Theorem 1.1. For the stability results it is important to notice that the results of the system for the general N case are much more complicated than for $N = 1$. The main reason is that the behaviour for the N-system cannot be reduced to the case $N = 1$ in contract to the existence issue and therefore a new analysis is needed.

Some related results on the existence and stability of solutions to the Gray-Scott model in 1-D can be found in [16], [29], [30], [42], [43], and [47].

In R^2 and R^3, Muratov and Osipov [37] have given some formal asymptotic analysis on the construction and stability of spiky solution. In [57], the system (1.1) for $N = 1$ is studied on the real axis in the shadow system case, namely, $D_M >> 1, D_X << 1$ and $k_M = g_M = O(1), g_X = O(1)$, $k_{11} = 1$, $L = 1$. The shadow system can be reduced to a single equation. For spike solutions for single equations as well as other systems please see [3], [4], [11], [22], [25], [27], [28], [32], [33], [38], [39], [40], [41], [44], [44], [50], [51], [52], [53], [54], [55], [56], [59], [60], and the references therein.

In the two-dimensional case rigorous existence and stability results on the Gray-Scott system have been established in [58]. The existence of one-spike solutions is proved. Their stability is established and rests upon the derivation and analysis of a related nonlocal eigenvalue problem (NLEP).

3. Outline of the proof of Theorem 2.2. We outline the proof of Theorem 2.2, which is our main theorem. It is divided into four steps. We need to analyze the eigenvalue problem (2.12). We consider two cases: small eigenvalues ($\lambda_\epsilon = o(1)$) and large eigenvalues ($|\lambda_\epsilon| \geq C > 0$ for some positive constant $C > 0$).

Step 1. (Small Eigenvalue Case) We show that in the small eigenvalue case, λ_ϵ must be zero and the corresponding eigenfunction must be translations of (X_ϵ, M_ϵ). This is done in Theorem 6.1 (1).

Step 2. (Large Eigenvalue Case). We show that in the large eigenvalue case, problem (2.12) can be reduced to a vectorial nonlocal eigenvalue problems (NLEP). This is done in Theorem 6.1 (2) and (3).

Step 3. (Study of Vectorial NLEP). We show under the assumptions (H2) and (H3), the study of the vectorial NLEP can be decoupled to the study of two eigenvalue problems – one is scalar eigenvalue problem but with complex coefficients and the other one is a scalar NLEP. This is done in Section 7.

Step 4. (Study of Two Eigenvalue Problems) We study the two reduced eigenvalue problems in Section 5. This analysis provides the key estimates in this paper.

The structure of the paper is as follows:

In Section 4, we consider the applications of Theorem 2.2. In particular, we consider several interesting matrices (k_{ij}) including the hypercycle matrix and symmetric matrices.

In Section 5, we study some scalar local and nonlocal eigenvalue problems associated with w.

In Section 6, we separate the eigenvalue problem into two cases: small eigenvalues and large eigenvalues. The case of large eigenvalues is then linked to a vectorial NLEP given in (6.9).

In Section 7, we reduce the vectorial NLEP given in (6.9) to a local eigenvalue problem with complex coefficients given in (5.1) and a scalar NLEP given in (5.2).

Throughout this paper, the letter C will always denote various generic constants which are independent of ϵ, for ϵ sufficiently small. The notation $A \sim B$ means that $\lim_{\epsilon\to 0} \frac{A}{B} = 1$ and $A = O(B)$ is defined as $|A| \leq C|B|$ for some $C > 0$.

4. Effect of the Connection Matrix (k_{ij}). In this section, we apply the stability results of Theorem 2.2 to some specific examples. We would like to point out

that there are many matrices which satisfy assumptions (H1) – (H3) in Theorem 2.2.
Example 1. (**Proof of Theorem 2.3**:)

For the hypercyclical network we have

$$\xi_1 = ... = \xi_N = \frac{1}{k_0},$$

$$b_{ij}^{hyper} = \delta_{i,j+1} \quad \text{modulo } N,$$

The eigenvalues are $\sigma = e^{2\pi j\sqrt{-1}/N}$, $\quad j = 1, \ldots, N$ and they are all simple. In this case, it is easy to see that (H1) – (H3) are satisfied. By Theorem 2.2, we just need to find the zeroes of the following function

$$f(\sigma) := (12\sigma_R + 5)^2(3\sigma_R^2 + 2\sigma_R) - 3\sigma_I^2, \quad \sigma_R^2 + \sigma_I^2 = 1, \quad 0 < \sigma_R < 1. \tag{4.1}$$

It is easy to check that the solution to (4.1) is

$$\sigma_R^0 = 0.0455...$$

Note that $\cos(\frac{2\pi}{5}) > \sigma_R^0$.

By Theorem 2.2 (1), we obtain the stability of the small cluster solution for $N = 1, 2, 3, 4$. By Theorem 2.2 (2), we obtain the instability of small solutions for $N \geq 5$.

We conclude that the critical threshold size for the hypercycle system is 4. When the system size exceeds 4, then a parasite appears: there is an eigenvector $c = (c_1, ..., c_N)^\tau$ of (k_{ij}) such that $\sum_{j=1}^N c_j X_j$ vanishes quickly.
Example 2. We consider the case when the connection matrix (k_{ij}) is symmetric, i.e.

$$k_{ij} = k_{ji}.$$

In this case, it is easy to see that the matrix $\mathcal{B} = (k_{ij}\xi_i)$ has only real eigenvalues. Let the eigenvalues of $\mathcal{B}$ be

$$\sigma_1 = 1, \sigma_2, ..., \sigma_N.$$

The first eigenvalue $\sigma_1 = 1$ is guaranteed by (2.4).

Assumption (H2) is satisfied if we further assume that $\xi_1 = ... = \xi_N$.

Assumption (H3) says that

$$\sigma_j \neq \frac{(1+k)(2+k)}{6} - 1, j = 2, ..., N, k = 1, 2, ... \tag{4.2}$$

Since $f(\sigma) > 0$ if $\sigma = \sigma_R > 0$, Theorem 2.2 shows that $(X_\epsilon^s, M_\epsilon^s)$ is linearly stable if

$$\sigma_j < 0, j = 2, ..., N. \tag{4.3}$$

On the other hand, if there exists $\sigma_j > 0$ for some $j \geq 2$, we have instability. (Assumption (H3) implies that $\sigma_j \neq 0$.)

Example 3. For the (cyclical) bidiagonal matrix

$$(k_{ij}) = k_0 \begin{pmatrix} 1-\alpha & \alpha & 0 & ... & 0 \\ 0 & 1-\alpha & \alpha & ... & 0 \\ 0 & 0 & 1-\alpha & ... & 0 \\ ... & ... & ... & ... & \alpha \\ \alpha & 0 & ... & 0 & 1-\alpha \end{pmatrix}_{N\times N}, \quad k_0 > 0$$

with $0 \le \alpha < 1$ it is easy to see that conditions (H1) – (H3) are satisfied. In this case, $\xi_1 = ... = \xi_N = \frac{1}{k_0}$. The eigenvalues are computed as $\sigma = 1 - \alpha(1 - e^{2\pi j\sqrt{-1}/N})$, $j = 1, \ldots, N$ and are all simple.

We substitute σ into the polynomial and compute the critical threshold $N_{critical}$. It turns out that $N_{critical}$ depends on the value of α: $N_{critical}$ will increase of the order α as α increases. The following is a table of $N_{critical}$ for small α:

α	$N_{critical}$
0.5	3
1	4
1.5	5
2	6

¿From all the previous examples, we see as a general trend that if the system is not too much dominated by diagonal terms we have stability. Otherwise, a parasite emerges. This means that cooperative behavior and not self-enhancement is needed to stabilize the cluster.

We point to the last example where the stability is especially strong if the parameter α gets large. In the case $\alpha > 1$ (which means that the diagonal becomes negative and the off-diagonal elements are positive and bigger than the diagonal), this describes self-inhibition coupled with cooperative enhancement and leads to particularly good stability.

5. Two eigenvalue problems. In this section, we study two eigenvalue problems. The first is a local eigenvalue problem with complex coefficients

$$\begin{cases} \Delta\phi - \phi + w\phi + \sigma w\phi = \lambda\phi, \\ \sigma = \sigma_R + i\sigma_I = |\sigma|e^{i\theta}, |\sigma| > 0, \theta \in (-\pi, \pi], \quad \phi \in H^1(R), \end{cases} \tag{5.1}$$

where w is defined by (2.1).

The second is a scalar nonlocal eigenvalue problem (NLEP):

$$\Delta\phi - \phi + 2w\phi - \frac{2(1-\eta)}{\eta\sqrt{1+\tau\lambda} + 1 - \eta} \frac{\int_R w\phi}{\int_R w^2} w^2 = \lambda\phi, \phi \in H^2(R), \tag{5.2}$$

where

$$0 < \eta < 1, \tau \ge 0, \lambda \in \mathcal{C}, \lambda = \lambda_R + i\lambda_I, \lambda_R \ge 0$$

and we take the principal branch for $\sqrt{1+\tau\lambda}$.

The analysis presented in this section provides the key estimates for this paper.

To study (5.1) and (5.2), we first collect some important properties associated with the function w.

LEMMA 5.1. *(1) The linear operator*

$$\begin{cases} L_0\phi := \Delta\phi - \phi + 2w\phi, \\ \phi \in H^1(R) \end{cases}$$

has the kernel

$$Ker\ (L_0) = span\ \left\{w^{'}(y)\right\}.$$

(2) The eigenvalue problem (EVP)

$$(EVP) \qquad \begin{cases} \Delta\phi - \phi + \mu w\phi = 0, \\ \phi \in H^1(R) \end{cases}$$

admits the following set of eigenvalues

$$\mu_1 = 1,\ v_1 = span\ \{w\},$$

$$\mu_2 = 2,\ v_2 = Ker\ (L_0),$$

$$\mu_n = \frac{(1+n)(2+n)}{6} > 2,\ \ for\ n \geq 3.$$

(3) If $\mu_R > 0$, then the following eigenvalue problem

$$\begin{cases} \Delta\phi - \phi + w\phi + \mu_R w\phi = \lambda\phi, \\ \mu_R > 0,\ \phi \in H^1(R) \end{cases}$$

admits a positive (principal) eigenvalue λ_1 such that

$$-\lambda_1 = \inf_{\phi\in H^1(R)\setminus\{0\}} \frac{\int_R (\phi')^2 + \phi^2 - (1+\mu_R)w\phi^2}{\int_R \phi^2} < 0.$$

Moreover when $\mu_R = 1$, there is only one positive eigenvalue (which is the principal one).

(4) Let ϕ (complex-valued) satisfy the following eigenvalue problem

$$\begin{cases} \Delta\phi - \phi + w\phi + \sigma w\phi = \lambda\phi \\ Re\ (\sigma) \leq 0, \quad \phi \in H^1(R), \quad \lambda \neq 0. \end{cases}$$

Then

$$Re\ (\lambda) \leq -c_0 < 0.$$

Proof: The proof will be given in Appendix A. The proof of (2) follows the lines of Lemma 5.2. Some of the results have been proved in previous work. For the convenience of the reader we recall the proofs of (3) and (4). □

We are ready to study the first eigenvalue problem (5.1). By symmetry, we may assume that $\theta \in [0, \frac{\pi}{2}]$. We consider θ as a parameter. By Lemma 5.1 (3) and a perturbation argument, for $|\theta|$ near 0, there is an unstable eigenvalue λ for problem (5.1), i.e. $\lambda = \lambda_R + i\lambda_I$ where $\lambda_R > 0$. On the other hand, by Lemma 5.1 (4), for $|\theta| \geq \frac{\pi}{2}$, problem (5.1) has only stable eigenvalues, i.e. $\lambda = \lambda_R + i\lambda_I$ where $\lambda_R < 0$. Now if we vary θ, then there must be a point $\theta^h \in (0, \frac{\pi}{2})$ such that for $\theta = \theta^h$, problem (5.1) has a Hopf bifurcation, i.e. there is an eigenvalue $\lambda = \sqrt{-1}\lambda_I$. Let us now compute θ^h. It turns out that unlike in the 2-D case [61], we can now obtain the exact value for θ^h in 1-D. That is

LEMMA 5.2.

Let ϕ (complex-valued) satisfy the eigenvalue problem (5.1) with $\sigma = \sigma_R + \sqrt{-1}\sigma_I, \sigma_R > 0$. Then

(1) If $f(\sigma) < 0$, then problem (5.1) is stable.

(2) If $f(\sigma) > 0$, then problem (5.1) is unstable.

(3) If $f(\sigma) = 0$, then there exists an eigenvalue λ with $\lambda = \sqrt{-1}\lambda_I$.

Here $f(\sigma) := (12\sigma_R + 5)^2(3\sigma_R^2 + 2\sigma_R) - 3\sigma_I^2$.

Proof: We are looking for a Hopf bifurcation for problem (5.1). Therefore we have to solve

$$\Delta\phi - \phi + (1+\sigma)w\phi = \lambda\phi \tag{5.3}$$

with

$$\lambda = \sqrt{-1}\lambda_I$$

(i.e. the real part λ_R of λ vanishes) and

$$\sigma = \sigma_R + i\sigma_I.$$

As in [12], let

$$\gamma = \sqrt{1+\lambda}, \quad \mu = 1+\sigma, \quad \phi = w^\gamma F.$$

Then F satisfies

$$F^{''} + 2\gamma\frac{w^{'}}{w}F^{'} + (\mu - (\gamma + \frac{2}{3}\gamma(\gamma-1)))w^{p-1}F = 0. \tag{5.4}$$

Next we introduce the following new variable

$$z = \frac{1}{2}(1 - \frac{w^{'}}{w}). \tag{5.5}$$

Then

$$\frac{w^{'}}{w} = 1 - 2z, \quad w = 6z(1-z), \quad \frac{dz}{dx} = z(1-z).$$

This yields the following equation for F as a function of z:

$$z(1-z)F^{''} + (c - (a+b+1)z)F^{'} - abF = 0, \tag{5.6}$$

where

$$a+b+1=2+4\gamma, \quad ab=2(2\gamma(\gamma-1)-3(\mu-\gamma)), \quad c=1+2\gamma. \tag{5.7}$$

The solutions to (5.6) are standard hypergeometric functions. See [49] for more details. Now there are two solutions to (5.6):

$$F(a,b;c;z), \quad z^{1-c}F(a-c+1,b-c+1;2-c;z).$$

By our construction F is regular at $z=0$. At $z=1$, $F(a,b;c;z)$ has a singularity

$$\lim_{z\to 1}(1-z)^{-(c-a-b)}F(a,b;c;z)=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)},$$

where $c-a-b=-2\gamma<0$. Note that since $\gamma=\sqrt{1+\sqrt{-1}\lambda_I}$, the real part of γ is positive. So a solution that is regular at both $z=0$ and $z=1$ can exist only if $\Gamma(x)$ has a pole at a or b, respectively. In other words, $a,b=0,-1,-2,....$

¿From (5.7), we compute that

$$a=2\gamma-\alpha \text{ or } b=2\gamma-\alpha,$$

where α satisfies

$$\alpha^2+\alpha-6\mu=0. \tag{5.8}$$

By symmetry we may assume that $a=2\gamma-\alpha=-l, l\geq 0$ and $\alpha=\alpha_R+\sqrt{-1}\alpha_I$. So we have to solve the system

$$\begin{cases} \alpha_R^2+\alpha_R-\alpha_I^2-6(1+\sigma_R)=0, \\ 2\gamma=\alpha-l, \quad l=0,1,2,\dots. \end{cases} \tag{5.9}$$

Since we take the principal branch for $\gamma=\sqrt{1+\sqrt{-1}\lambda_I}$, it follows that

$$\alpha_R>l.$$

Moreover we have

$$4=(\alpha_R-l)^2-\alpha_I^2$$

which implies that

$$\alpha_R\geq l+2. \tag{5.10}$$

On the other hand, we have

$$4=(\alpha_R-l)^2-\alpha_I^2=\alpha_R^2-\alpha_I^2-2l\alpha_R+l^2$$

$$=-(2l+1)\alpha_R+l^2+6(1+\sigma_R).$$

So we obtain

$$\alpha_R=\frac{1}{2l+1}(l^2+2+6\sigma_R).$$

By (5.10), we have

$$\frac{1}{2l+1}(l^2+2+6\sigma_R) \geq l+2.$$

which is impossible unless $l=0$ or $l=1$. For $l=1$ we just recover the case $\lambda=0$ with the eigenfunction w' given by Lemma 5.1 (1). This clearly does not correspond to Hopf bifurcation.

In conclusion, for Hopf bifurction we must have $a=0$ or $b=0$. In this case, we have

$$\alpha_R = 2+6\sigma_R, \alpha_I = \frac{6}{(2\alpha_R+1)}\sigma_I. \tag{5.11}$$

Substituting this relation into (5.9) we obtain that (σ_R, σ_I) must be a zero of the polynomial f defined by (2.22).

In summary, Hopf bifurcation can occur only at the point (σ_R^h, σ_I^h) such that $f(\sigma)=0$.

Note that the set $\{f(\sigma)=0\}$ defines a monotone curve within the set $\{(\sigma_R,\sigma_I)|\sigma_R > 0, \sigma_I > 0\}$. Since $f(0,\sigma_I)<0$ for $\sigma_I>0$ and $f(\sigma_R,0)>0$ for $\sigma_R>0$, we see that if $f(\sigma)<0$, then problem (5.1) is stable and if $f(\sigma)>0$, then problem (5.1) is unstable. □

We next study the scalar NLEP (5.2). We first state the following lemma.

LEMMA 5.3. *Consider the nonlocal eigenvalue problem (5.2).*

(1) Suppose that $0 \leq \tau < \tau_0$ *where* τ_0 *is sufficiently small and* $0<\eta<\frac{1}{2}$. *Let* $\lambda_0 \neq 0$ *be an eigenvalue of (5.2). Then we have* $Re(\lambda_0) \leq -c_1$ *for some* $c_1>0$.

(2) Suppose that $\tau>0$ *and* $\frac{1}{2}<\eta<1$, *then problem (5.2) admits a real eigenvalue* λ_0 *with* $\lambda_0 \geq c_2 > 0$ *for some* $c_2>0$.

Proof:

(1). When $\tau=0$, we have

$$\frac{2(1-\eta)}{\eta\sqrt{1+\tau\lambda}+1-\eta} = 2(1-\eta) > 1$$

if $0<\eta<\frac{1}{2}$. By Theorem 2.1 of [57], we have that $\lambda_R < -c_1 < 0$.

To show that the same thing is true when τ is small, we have to show that if $\lambda_R \geq 0$ and $0<\tau<1$, then $|\lambda| \leq C$, where C is a generic constant (independent of τ). In fact, multiplying (5.2) by $\bar{\phi}$ – the conjugate of ϕ – and integrating by parts, we obtain that

$$\int_R (|\nabla\phi|^2+|\phi|^2-2w|\phi|^2) = -\lambda\int_R |\phi|^2 - f(\tau\lambda)\frac{\int_R w\phi}{\int_R w^2}\int_R w^2\bar{\phi}, \tag{5.12}$$

where $f(\tau\lambda)=\frac{2(1-\eta)}{\eta\sqrt{1+\tau\lambda}+1-\eta}$. From the imaginary part of (5.12), we obtain that

$$|\lambda_I| \leq C_1|f(\tau\lambda)|$$

where $\lambda=\lambda_R+\sqrt{-1}\lambda_I$ and C_1 is a positive constant (independent of τ). Note that the real part of $\sqrt{1+\tau\lambda}$ is positive. Hence $|f(\tau\lambda)| \leq 2$ and so $|\lambda_I| \leq 2C_1$. Taking the real part of (5.12), we obtain that $\lambda_R \le C_2$, where C_2 is a positive constant (independent of $\tau > 0$). Therefore, we have $|\lambda|$ is uniformly bounded and hence a perturbation argument gives the desired conclusion.

(2). Assume that $\frac{1}{2} < \eta < 1$. We now show that (5.2) admits a positive eigenvalue for all $\tau > 0$.

By Lemma 5.1 (3), L_0 has only one positive eigenvalue $\lambda_1 > 0$. Consider the following function

$$h(\alpha) = \int_R ((L_0 - \alpha)^{-1} w) w, \quad 0 < \alpha < \lambda_1. \tag{5.13}$$

It is easy to see that

$$h^{'}(\alpha) = \int_R ((L_0 - \alpha)^{-2} w) w = \int_R [(L_0 - \alpha)^{-1} w]^2 > 0,$$

and

$$\lim_{\alpha \to \lambda_1} h(\alpha) = +\infty.$$

Next we consider the following function

$$\rho(\lambda) = \frac{\eta\sqrt{1+\tau\lambda} + 1 - \eta}{2(1-\eta)} - 1 - (\int_R w^2)^{-1} \lambda h(\lambda). \tag{5.14}$$

Note that

$$\rho(0) = \frac{1}{2(1-\eta)} - 1 > 0$$

since $\frac{1}{2} < \eta < 1$. On the other hand,

$$\lim_{\lambda \to \lambda_1 -} \rho(\lambda) = -\infty.$$

Hence there must exist an $\lambda_0 \in (0, \lambda_1)$ such that $\rho(\lambda_0) = 0$. It is easy to see that this $\lambda_0 > 0$ is an eigenvalue of (5.2), which proves (2) of Lemma 5.3.

□

In the general case when τ is large and $0 < \eta < \frac{1}{2}$, there are no analytic results for problem (5.2) available. Fortunately, we can use hypergeometric functions and generalized hypergeometric functions to reduce problem (5.2) to a computable one. Such an idea has already been used in [12]. However, our transformation is different and the eigenvalue problem becomes computable more easily. We recall that by Lemma 5.3 (1) for $\tau = 0$ all eigenvalues are stable. So if we vary τ, either we obtain stability or Hopf bifurcation. All we need is to compute when Hopf bifurcation happens.

Let us first introduce the so-called generalized Gauss function. Let $a_1, a_2, ..., a_A$ and $b_1, b_2, ..., b_B$ be two sequences of numbers. Consider the following series

$$1 + \frac{a_1 a_2 ... a_A}{b_1 b_2 ... b_B} \frac{z}{1!} + \frac{(a_1+1)(a_2+1)...(a_A+1)}{(b_1+1)(b_2+1)...(b_B+1)} \frac{z^2}{2!} + ... \tag{5.15}$$

$$\equiv {}_AF_B \left\{ \begin{matrix} a_1, & a_2, & ..., & a_A & ; \\ & & & & z \\ b_1, & b_2, & ..., & b_B & ; \end{matrix} \right\}$$

${}_AF_B$ is called generalized Gauss function or generalized hypergeometric function. For more details on such functions, we refer to [49].

Now we have the following lemma, whose proof is technical and thus is delayed to Appendix B.

LEMMA 5.4. *Let $\lambda = \sqrt{-1}\lambda_I$ be an eigenvalue of problem (5.2). Then λ is a solution of the following algebraic equation*

$$\frac{5}{6}(3-\lambda)\frac{\eta\sqrt{1+\tau\lambda}+1-\eta}{2(1-\eta)}$$

$$= {}_4F_3 \left\{ \begin{matrix} 1, & \frac{1}{2}, & 4, & 3 & ; \\ & & & & 1 \\ & 3-\sqrt{1+\lambda}, & 3+\sqrt{1+\lambda}, & \frac{7}{2} & ; \end{matrix} \right\}. \tag{5.16}$$

By Lemma 5.4, problem (5.2) can be solved by using Mathematica. We will not produce any numerical results here. The readers are referred to [12] for some numerical results the case $N = 1$.

6. Derivation of the vectorial NLEP and Reduction Process. In this section we study the eigenvalue problem (2.12) and show that it can be reduced to a vectorial nonlocal eigenvalue problem (NLEP).

Let (X_ϵ, M_ϵ) be one of the two solutions constructed in Section 2. We now study the eigenvalue problem associated with (X_ϵ, M_ϵ). We assume that

$$\epsilon << L < \frac{1}{4} - \delta_0,$$

(in the same sense as in (2.9), where $\delta_0 > 0$ is a small but fixed constant and that $0 \le \tau < \tau_0$, where τ_0 is given by Lemma 5.3 and is independent of ϵ.

We need to analyze the following eigenvalue problem (letting $x = \epsilon y$)

$$\left\{ \begin{array}{l} \Delta_y \phi_{\epsilon,i} - \phi_{\epsilon,i} + AM_\epsilon \sum_{j=1}^N k_{ij}(X_{\epsilon,j}\phi_{\epsilon,i} + \phi_{\epsilon,j}X_{\epsilon,i}) \\ +A\psi_\epsilon \sum_{j=1}^N k_{ij}X_{\epsilon,i}X_{\epsilon,j} = \lambda_\epsilon \phi_{\epsilon,i},\ y \in R, i = 1, ..., N, \\ \Delta_x \psi_\epsilon - \psi_\epsilon - \psi_\epsilon \sum_{i,j=1}^N k_{ij}X_{\epsilon,i}X_{\epsilon,j} \\ -M_\epsilon \sum_{i,j=1}^N k_{ij}(X_{\epsilon,j}\phi_{\epsilon,i} + X_{\epsilon,i}\phi_{\epsilon,j}) = \tau\lambda_\epsilon\psi_\epsilon,\ x \in R, \\ \lambda_\epsilon \in \mathcal{C}. \end{array} \right. \tag{6.1}$$

We assume that $(\phi_{\epsilon,1}, ..., \phi_{\epsilon,N}, \psi_\epsilon) \in (H^2(R))^N \oplus H^2(R)$. Here we equip $(H^2(R))^N \oplus H^2(R)$ with the following norm

$$\|(X,u)\|^2_{(H^2(R))^N \oplus H^2(R)} = \|X(y)\|^2_{(H^2(R))^N} + \|u(x)\|^2_{H^2(R)}.$$

Since $X_{\epsilon,i} = \xi_i X_{0,\epsilon}$, problem (6.1) becomes

$$\left\{ \begin{array}{l} \Delta_y \phi_{\epsilon,i} - \phi_{\epsilon,i} + AM_\epsilon X_{0,\epsilon} \sum_{j=1}^N k_{ij}(\xi_i\phi_{\epsilon,j} + \xi_j\phi_{\epsilon,i}) \\ +A\psi_\epsilon \xi_i X_{0,\epsilon}^2 = \lambda_\epsilon \phi_{\epsilon,i}, \\ \Delta\psi_\epsilon - \psi_\epsilon - \psi_\epsilon(\sum_{i=1}^N \xi_i)X_{0,\epsilon}^2 \\ -M_\epsilon \sum_{i,j=1}^N k_{ij}(\xi_i\phi_{\epsilon,j} + \xi_j\phi_{\epsilon,i})X_{0,\epsilon} = \tau\lambda_\epsilon\psi_\epsilon. \end{array} \right. \tag{6.2}$$

Let us first formally derive the limiting eigenvalue problems.

Since $(X_{0,\epsilon}, M_\epsilon)$ satisfies (2.5), we have

$$X_{0,\epsilon}(y) \sim (AM_\epsilon(0))^{-1}(1+o(1))w(y), \tag{6.3}$$

and

$$M_\epsilon(0)(1-M_\epsilon(0)) \sim L := \frac{1}{2A^2(\sum_{i=1}^N \xi_i)}\epsilon \int_R w(y)^2 dy. \tag{6.4}$$

By the assumptions (H1) and (H2), $\sum_{j=1}^N k_{ij}\xi_j = \sum_{i=1}^N k_{ij}\xi_i = 1$, the eigenvalue problem is changed into

$$\begin{cases} \Delta_y \phi_{\epsilon,i} - \phi_{\epsilon,i} + w\phi_{\epsilon,i} + w\sum_{j=1}^N b_{ij}\phi_{\epsilon,j} \\ +\frac{1}{AM_\epsilon(0)^2}\xi_i\psi_\epsilon w^2 = \lambda_\epsilon\phi_{\epsilon,i}, \quad i=1,...,N, \\ \Delta_x\psi_\epsilon - \psi_\epsilon - \frac{1}{A^2M_\epsilon(0)^2(\sum_{i=1}^N \xi_i)}\psi_\epsilon w^2 \\ -\frac{M_\epsilon}{AM_\epsilon(0)} 2\sum_{j=1}^N \phi_{\epsilon,j} w = \tau\lambda_\epsilon\psi_\epsilon. \end{cases} \tag{6.5}$$

Let $\beta_\epsilon = \sqrt{1+\tau\lambda_\epsilon}$. Here we take principal branch of $1+\tau\lambda_\epsilon$. Since we are interested only in the unstable eigenvalues of λ_ϵ (otherwise it is stable), we may assume that $\mathrm{Re}(\lambda_\epsilon) \geq -a_0$ for some small number $a_0 > 0$ so that $1+\tau a_0 > \frac{1}{2}$. Following the same proof as for (1) of Lemma 5.3 (that is, multiplying the equations for $\phi_{\epsilon,i}$ by $\bar{\phi}_{\epsilon,i}$, integrating by parts and summing up), we see that

$$|\lambda_\epsilon| \leq C, \quad \text{if } \mathrm{Re}(\lambda_\epsilon) \geq -a_0, \tag{6.6}$$

where $C > 0$ is a positive constant (independent of $\epsilon > 0$).

¿From the second equation in (6.5), we calculate using the fact that the Green's function of

$$\Delta G(x,\xi) - \beta^2 G(x,\xi) + \delta(\xi) = 0 \quad \text{in } R$$

is

$$G(x,\xi) = \frac{1}{2\beta}e^{-\beta|x-\xi|},$$

the relation

$$\psi_\epsilon(0) = \frac{1}{2\beta_\epsilon}\int_R e^{-\beta_\epsilon|x|}$$

$$\times\left(-\frac{1}{A^2M_\epsilon(0)^2(\sum_{i=1}^N \xi_i)}\psi_\epsilon w^2 - \frac{M_\epsilon}{AM_\epsilon(0)} 2\sum_{j=1}^N \phi_{\epsilon,j} w\right) dx$$

$$= \frac{1}{2\beta_\epsilon}\epsilon\left[-\frac{\psi_\epsilon(0)}{A^2M_\epsilon(0)^2(\sum_{i=1}^N \xi_i)}\int_R w^2(y)\,dy\right.$$

$$-\frac{1}{A}\sum_{j=1}^{N}\int_R (\sum_{j=1}^{N}\phi_j) w\,dy + o(\epsilon)\Big], \tag{6.7}$$

where

$$\phi_{\epsilon,i}(x) = \phi_i(\frac{x}{\epsilon}), \quad x = \epsilon y, \quad i = 1, \ldots, N. \tag{6.8}$$

By (6.4) and (6.7), we have

$$\frac{\psi_\epsilon(0)}{AM_\epsilon^2(0)}$$

$$= -\frac{\epsilon}{\beta_\epsilon A^2 M_\epsilon(0)^2}\int_R (\sum_{j=1}^{N}\phi_j) w \left[1 + \frac{\epsilon}{2\beta_\epsilon(\sum_{i=1}^N \xi_i)A^2 M_\epsilon(0)^2}\int_R w^2(y)\,dy\right]^{-1}$$

$$= -\int_R (\sum_{j=1}^{N}\phi_j) w \left[\frac{\beta_\epsilon A^2 M_\epsilon(0)^2}{\epsilon} + \frac{1}{2(\sum_{i=1}^N \xi_i)}\int_R w^2(y)\,dy\right]^{-1}$$

$$= -\int_R (\sum_{j=1}^{N}\phi_j) w \left[\frac{\beta_\epsilon M_\epsilon(0)(\sum_{i=1}^N \xi_i)}{2(1-M_\epsilon(0))}\int_R w^2(y)\,dy + \frac{(\sum_{i=1}^N \xi_i)}{2}\int_R w^2(y)\,dy\right]^{-1}$$

$$= -\frac{2(1-M_\epsilon(0))}{1-M_\epsilon(0)+\beta_\epsilon M_\epsilon(0)}\frac{\int_R(\sum_{j=1}^N \phi_j) w}{(\sum_{i=1}^N \xi_i)\int_R w^2(y)\,dy}.$$

Substituting this relation into the first equation in (6.5) and taking the limit $\epsilon \to 0$, we obtain the following nonlocal eigenvalue problem (NLEP):

$$\Delta\phi_i - \phi_i + \phi_i w + \sum_{j=1}^{N} b_{ij}\phi_j \tag{6.9}$$

$$-\xi_i \frac{2(1-\eta)}{\eta\beta_0 + 1 - \eta}\frac{\sum_{j=1}^N \int_R \phi_j w}{(\sum_{i=1}^N \xi_i)\int_R w^2} w^2 = \lambda_0\phi_i, \phi_i \in H^2(R), i = 1, ..., N,$$

where $\eta = \lim_{\epsilon\to 0} M_\epsilon(0), \lambda_0 = \lim_{\epsilon\to 0}\lambda_\epsilon, \beta_0 = \lim_{\epsilon\to 0}\beta_\epsilon = \sqrt{1+\tau\lambda_0}$. (Here, we have assumed that all the limits exist. Otherwise, we take a subsequence $\epsilon_n \to 0$.)

Though the derivations above are formal, we can rigorously prove the following separation of eigenvalues.

THEOREM 6.1.

Suppose that the assumptions (H1) – (H3) are satisfied.

Let λ_ϵ be an eigenvalue of (6.2) such that $Re(\lambda_\epsilon) > -a_0$.

(1) Suppose that (for suitable sequences $\epsilon_n \to 0$) we have $\lambda_{\epsilon_n} \to 0$ as $n \to \infty$. Then for n sufficiently large, it follows that $\lambda_{\epsilon_n} = 0$ and

$$(\phi_{\epsilon_n,1}, ..., \phi_{\epsilon_n,N}, \psi_{\epsilon_n}) \in span\ \{(X'_{\epsilon_n}, M'_{\epsilon_n})\}.$$

(2) Suppose that (for suitable sequences $\epsilon_n \to 0$) we have $\lambda_{\epsilon_n} \to \lambda_0 \neq 0$. Then λ_0 is an eigenvalue of the problem (NLEP) given in (6.9).

(3) Let $\lambda_0 \neq 0$ be an eigenvalue of the (NLEP) problem given in (6.9). Then for ϵ sufficiently small, there is an eigenvalue λ_ϵ of (6.2) with $\lambda_\epsilon \to \lambda_0$ as $\epsilon \to 0$.

¿From Theorem 6.1 (1) and (3), we see that problem (6.2) is reduced to the study of the vectorial NLEP (6.9).

In the rest of this section, we prove Theorem 6.1.

Proof of Theorem 6.1:

For (1), the proof is very delicate. We can proceed as in the proof of Theorem 2.2 (3) in Section 6 of [58], where existence and stability of single cluster state for the Gray-Scott system in 2-D are studied. We first prove the analogies of Lemma 3.1 and Lemma 3.2 of [58] in 1-D. We begin with

LEMMA 6.2. *Let $g(y)$ be a function in $L^2(R^1)$ such that*

$$|g(y)| \leq Ce^{-c|y|}$$

where c is a positive constant. Then we have

$$\left| \int_R (|y-z| - |z|)|g(z)|dz \right| \leq C|y|, \tag{6.10}$$

where C depends on $\int_R |z||g(z)|dz$.

Proof: This follows from standard potential analysis. □

Next, we study the asymptotic behavior of ψ_ϵ. We have

LEMMA 6.3. *Let $(\phi_{\epsilon,1}, \dots, \phi_{\epsilon,N}, \psi_\epsilon)$ satisfy (6.5). Then we have*

$$\frac{1}{AM_\epsilon(0)^2}\psi_\epsilon(0) = -\frac{1-M_\epsilon(0)}{\beta_\epsilon M_\epsilon(0) + 1 - M_\epsilon(0)} \frac{2\sum_{j=1}^N \int_R \sum_{j=1}^N \phi_{\epsilon,j} w}{(\sum_{i=1}^N \xi_i)\int_R w^2} + o(1), \tag{6.11}$$

and

$$\frac{1}{AM_\epsilon(0)^2(\sum_{i=1}^N \xi_i)}(\psi_\epsilon(x) - \psi_\epsilon(0))$$

$$= O\left(\frac{2}{\epsilon(1-\sqrt{1-4L})}(1 + \sum_{i=1}^N \|\phi_{\epsilon,i}\|_{L^2_y})(1 + \frac{|x|}{\epsilon}) \right), \tag{6.12}$$

where $x = \epsilon y$ and

$$\|\phi\|^2_{L^2_y} = \int_R \phi^2(y)dy.$$

Proof: Relation (6.11) follows from representation formula. To prove (6.12), we note that by the representation formula we calculate

$$\psi_\epsilon(x) - \psi_\epsilon(0) = \frac{1}{2\beta}\int_R (e^{-\beta_\epsilon|z-x|} - e^{-\beta_\epsilon|z|})$$

$$\times \left(-\frac{\psi_\epsilon X^2_{0,\epsilon}}{(\sum_{i=1}^N \xi_i)} - 2M_\epsilon (\sum_{j=1}^N \phi_{\epsilon,j})X_{0,\epsilon} \right) dz.$$

Let $x = \epsilon y, z = \epsilon \tilde{z}$. It is easy to see that

$$e^{-\beta_\epsilon |z-x|} - e^{-\beta_\epsilon |z|} = e^{-\beta_\epsilon \epsilon |y-\tilde{z}|} - e^{-\beta_\epsilon \epsilon |\tilde{z}|}$$

$$= -\beta_\epsilon \epsilon (|y-\tilde{z}| - |\tilde{z}|) + O(\beta_\epsilon^2 \epsilon^2 (|y|^2 + |\tilde{z}|^2)).$$

(6.12) now follows from Lemma 6.2.

□

Finally we need the analogue of Lemma 4.2 of [58].

Let us denote the linear operator on the left hand side of (6.9) as $\mathcal{L}$, where $\mathcal{L} : (H^2(R))^N \to (L^2(R))^N$. Then we have

LEMMA 6.4.

Assume that assumptions (H1) – (H3) hold true.

(1). Let ϕ be an eigenfunction of (6.9) with $\lambda_0 = 0$. Then we have

$$\phi \in \mathcal{K}_0 := span\ \{w^{'}(y)\vec{e}_0\}, \tag{6.13}$$

where $\vec{e}_0 = (1, \ldots, 1)^\tau$. (This implies that Ker $(\mathcal{L}) = \mathcal{K}_0$.)

(2). The operator $\mathcal{L}$ is an invertible operator if restricted as follows

$$\mathcal{L} : \mathcal{K}_0^{\perp,1} \to \mathcal{K}_0^{\perp,2},$$

where

$$\mathcal{K}_0^{\perp,1} = \{u \in (H^2(R))^N | \int_R u w^{'}(y)\vec{e_0} = 0\},$$

$$\mathcal{K}_0^{\perp,2} = \{u \in (L^2(R))^N | \int_R u w^{'}(y)\vec{e_0} = 0\}.$$

The proof of Lemma 6.4 is technical and is delayed to Appendix C.

Now Theorem 6.1 (1) follows from Lemma 6.4, by the same proof as for Theorem 2.2 (3) of [58].

(2) of Theorem 6.1 follows the asymptotic analysis done at the beginning of this section.

To prove (3) of Theorem 6.1, we use the same argument as given in Section 2 of [10], where the following eigenvalue problem was studied:

$$\begin{cases} \epsilon^2 \Delta h - h + p u_\epsilon^{p-1} h - \frac{qr}{s+1+\tau\lambda_\epsilon} \frac{\int_\Omega u_\epsilon^{r-1} h}{\int_\Omega u_\epsilon^r} u_\epsilon^p = \lambda_\epsilon h \text{ in } \Omega, \\ h = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.14}$$

where u_ϵ is a solution of the single equation

$$\begin{cases} \epsilon^2 \Delta u_\epsilon - u_\epsilon + u_\epsilon^p = 0 \text{ in } \Omega, \\ u_\epsilon > 0 \quad \text{in } \Omega,\ u_\epsilon = 0 \text{ on } \partial\Omega. \end{cases}$$

Here $1 < p < \frac{n+2}{n-2}$ if $n \geq 3$ and $1 < p < +\infty$ if $n = 1, 2$, $\frac{qr}{(s+1)(p-1)} > 1$ and $\Omega \subset R^n$ is a smooth bounded domain. If u_ϵ is a single interior peak solution, then it can be shown that ([56]) the limiting eigenvalue problem is a NLEP

$$\Delta\phi - \phi + p w^{p-1}\phi - \frac{qr}{s+1+\tau\lambda_0} \frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r} w^p = \lambda_0 \phi \tag{6.15}$$

where w is the corresponding ground state solution in R^n:

$$\Delta w - w + w^p = 0, w > 0 \text{ in } R^n, w = w(|y|) \in H^1(R^n).$$

Dancer in [10] showed that if $\lambda_0 \neq 0$, $\mathrm{Re}(\lambda_0) > 0$ is an unstable eigenvalue of (6.15), then there exists an eigenvalue λ_ϵ of (6.14) such that $\lambda_\epsilon \to \lambda_0$.

We now follow his idea. Let $\lambda_0 \neq 0$ be an eigenvalue of problem (6.9) with $\mathrm{Re}(\lambda_0) > 0$. We first note that from the equation for ψ_ϵ, we can express ψ_ϵ in terms of $(\phi_{\epsilon,1}, ..., \phi_{\epsilon,N})$. Now we write the first equation for $(\phi_{\epsilon,1}, \phi_{\epsilon,2}, ..., \phi_{\epsilon,N})$ as follows:

$$\phi_{\epsilon,i} = -R_\epsilon(\lambda_\epsilon)\left[AM_\epsilon \sum_{j=1}^N k_{ij}(X_{\epsilon,j}\phi_{\epsilon,i} + \phi_{\epsilon,j}X_{\epsilon,i}) + A\psi_\epsilon \sum_{j=1}^N k_{ij}X_{\epsilon,i}X_{\epsilon,j}\right], i = 1, ..., N \tag{6.16}$$

where $R_\epsilon(\lambda)$ is the inverse of $-\Delta + (1 + \lambda_\epsilon)$ in $H^2(R)$ (which exists if $\mathrm{Re}(\lambda_\epsilon) > -1$ or $\mathrm{Im}(\lambda_\epsilon) \neq 0$). The important thing is that $R_\epsilon(\lambda_\epsilon)$ is a compact operator if ϵ is sufficiently small. The rest of the argument follows exactly that in [10]. For the sake of limited space, we omit the details here.

□

7. Analysis of the vectorial NLEP and The proof of Theorem 2.2. In this section we analyze the vectorial nonlocal eigenvalue problem (NLEP) which we have obtained in (6.9):

$$\Delta\phi_i - \phi_i + \phi_i w + \sum_{j=1}^N b_{ij}\phi_j \tag{7.1}$$

$$-\frac{\xi_i}{\sum_{i=1}^N \xi_i}\frac{2(1-\eta)}{1-\eta+\eta\beta_0}\frac{\sum_{j=1}^N \int_R \phi_j w}{\int_R w^2} w^2 = \lambda_0\phi_i, i = 1, ..., N, \phi_i \in H^2(R).$$

We will decouple it to a local eigenvalue problem with complex coefficients given in (5.1) and a scalar nonlocal eigenvalue problem given in (5.2). Here assumptions (H2) and (H3) play a very important role. By Lemma 5.2, Lemma 5.3 and Theorem 6.1, we finish the proof of Theorem 2.2.

Proof of (1) of Theorem 2.2:

Consider the case for $(X_\epsilon^s, M_\epsilon^s)$ and τ small. In this case, $0 \leq \eta = \lim_{\epsilon\to 0} M_\epsilon(0) < \frac{1}{2}$. By Theorem 6.1 (1), if $\lambda_\epsilon = o(1)$, then $\lambda_\epsilon = 0$ and 0 is a simple eigenvalue (the eigenspace is one-dimensional). We only need to consider large eigenvalues. Let us assume that for a subsequence $\epsilon_n \to 0$ we have $\lambda_{\epsilon_n} \to \lambda_0$ where $\mathrm{Re}(\lambda_0) \geq 0$ and $\lambda_0 \neq 0$. We shall derive a contradiction.

By Theorem 6.1 (2), λ_0 is an eigenvalue of (7.1). We first take care of the nonlocal terms in (7.1). Adding the equations for $i = 1, \ldots, N$ (using the assumption (H2)), we get

$$\Delta(\sum_{i=1}^N \phi_i) - (\sum_{i=1}^N \phi_i) + 2w(\sum_{i=1}^N \phi_i)$$

$$-\frac{2(1-\eta)}{\beta_0\eta+1-\eta}\frac{\int_R(\sum_{i=1}^N\phi_i)w}{\int_R w^2}w^2=\lambda_0\sum_{i=1}^N\phi_i.$$

¿From Lemma 5.3 (1) we know that for $0<\eta=\lim_{\epsilon\to 0}M_\epsilon(0)<\frac{1}{2}$ and τ small we have

$$\sum_{i=1}^N\phi_i=0 \qquad \text{if Re }(\lambda_0)\geq 0,\lambda_0\neq 0. \tag{7.2}$$

Therefore the nonlocal terms in (7.1) all vanish and (7.1) reduces to the following vectorial local eigenvalue problem:

$$\Delta\phi_i-\phi_i+w\phi_i+w\sum_{j=1}^N b_{ij}\phi_j=\lambda_0\phi_i,\phi_i\in H^1(R),i=1,...,N. \tag{7.3}$$

To finish the proof we have to transform this to Jordan form, we decompose

$$b_{ij}=\sum_{k,l=1}^N p_{ik}d_{kl}p_{lj}^{-1},$$

as in (2.17) of Section 2, where d_{kl} has Jordan form.

Set

$$\Phi_i=\sum_{j=1}^N p_{ij}^{-1}\phi_j. \tag{7.4}$$

Then (7.3) can be expressed in terms of Φ as follows:

$$\Delta\Phi_i-\Phi_i+w\Phi_i+\sum_{j=1}^N d_{ij}\Phi_j w=\lambda_0\Phi_i,i=1,...,N.. \tag{7.5}$$

We have to study the eigenvalue problems for each Jordan block separately.

Let σ be an eigenvalue of $\mathcal{B}$. By assumption (H3), $\sigma=1$ is a simple eigenvalue of $\mathcal{B}$. Assume also that for those $\sigma\neq 1$ with $\sigma_R>0$, it holds that $f(\sigma)<0$.

For those eigenvalues $\sigma_k\neq 1,k>1$, then the corresponding i-th component Φ_i of the eigenfunction satisfies

$$\Delta\Phi_i-\Phi_i+(1+\sigma_k)w\Phi_i=\lambda_0\Phi_i \tag{7.6}$$

with Re $(\lambda_0)\geq 0$.

By Lemma 5.2 (1), $\Phi_i=0$, by our assumption on σ_k. Substituting this into the $(i-1)$-th equation we get (for the eigenfunction Φ_{i-1})

$$\Delta\Phi_{i-1}-\Phi_{i-1}+(1+\sigma_k)w\Phi_{i-1}=\lambda_0\Phi_{i-1} \tag{7.7}$$

and by Lemma 5.2 (1) again, we conclude that $\Phi_{i-1}=0$. Continuing in this way we see that those components of Φ corresponding to the Jordan block of σ_k all vanish.

Since $\sigma_1 = 1$ is a simple eigenvalue, we are left with the only possibility that $\Phi_1 \neq 0$. On the other hand, we have that

$$\sum_{j=1}^{N} \phi_j = \sum_{j=1}^{N} c_j \Phi_j, \tag{7.8}$$

where $c_j =< \vec{e}_0, \mathbf{p}_j >$, where $\vec{e}_0 = (1, ..., 1)^\tau$ and $\mathbf{p}_j$ is the $j-$th column of $\mathcal{P}$. Note that $c_1 = \frac{\sum_{i=1}^N \xi_i}{\|\xi\|} > 0$.

Since $\sum_{j=1}^N \phi_j = 0$ and $\Phi_j = 0, j = 2, ..., N$, we conclude from (7.8) that $\sum_{j=1}^N \phi_j = c_1 \Phi_1 = 0$ and hence $\Phi_1 = ... = \Phi_N = 0$. A contradiction.

Therefore $\mathrm{Re}(\lambda_0) \geq 0$ is not possible. Thus we have Re $(\lambda_0) \leq -c_0 < 0$.

This proves (1) of Theorem 2.2.

□

Proofs of (2) and (3) of Theorem 2.2:

As before, we decompose $\mathcal{B} = \mathcal{P}\mathcal{D}\mathcal{P}^{-1}$ and let $\phi = \mathcal{P}\Phi$. The problem (7.1) is equivalent to the following:

$$\Delta\Phi - \Phi + w\Phi + w\mathcal{D}\Phi - \mathcal{P}^{-1}\xi \frac{2(1-\eta)}{(\beta_0\eta + 1 - \eta)(\sum_{i=1}^N \xi_i)} \frac{\sum_{i=1}^N \int_R w\phi_i}{\int_R w^2} w^2 = \lambda_0 \Phi. \tag{7.9}$$

Note that

$$\mathcal{P}^{-1}\xi = \|\xi\| \vec{e}_1 \tag{7.10}$$

since ξ is the first eigenvector of $\mathcal{B}$, where $\vec{e}_1 = (1, 0, ..., 0)^\tau$.

Therefore (7.9) is decoupled into

$$\Delta\Phi_1 - \Phi_1 + 2w\Phi_1 - \frac{2\|\xi\|(1-\eta)}{(\beta_0\eta + 1 - \eta)(\sum_{i=1}^N \xi_i)} \frac{\sum_{i=1}^N \int_R w\phi_i}{\int_R w^2} w^2 = \lambda_0 \Phi_1, \tag{7.11}$$

and

$$\Delta\Phi_i - \Phi_i + w\Phi_i + \sum_{j=1}^{N} d_{ij} \Phi_j w = \lambda_0 \Phi_i, i = 2, .., K. \tag{7.12}$$

By (7.8), we have that

$$\int_R \sum_{i=1}^{N} w\phi_i = \sum_{j=1}^{N} c_j \int_R w\Phi_j. \tag{7.13}$$

We first prove (3) of Theorem 2.2. We consider $(X_\epsilon^l, M_\epsilon^l)$. In this case, $2(1-\eta) < 1$. By Lemma 5.3 (2), for any $\tau > 0$, there exists an eigenvalue $\lambda_0 > 0$ and an eigenfunction Φ_0 such that

$$\Delta\Phi_0 - \Phi_0 + 2w\Phi_0 - \frac{2(1-\eta)}{(\beta_0\eta + 1 - \eta)} \frac{\int_R w\Phi_0}{\int_R w^2} w^2 = \lambda_0 \Phi_0, \lambda_0 > 0.$$

Now we choose

$$\Phi_1 = \Phi_0, \Phi_j = 0, j = 2, ..., K. \tag{7.14}$$

then $\mathbf{\Phi} = (\Phi_1, ..., \Phi_N)$ is a solution of (7.9) with $\lambda_0 > 0$. The corresponding $\phi = \mathcal{P}\Phi$ is a solution of (7.1) with $\lambda_0 > 0$. By Theorem 6.1 (3), we have the instability of $(X_\epsilon^l, M_\epsilon^l)$ for any $\tau > 0$.

This proves (3) of Theorem 2.2.

Finally, we prove (2) of Theorem 2.2. Consider $(X_\epsilon^s, M_\epsilon^s)$. Assume that there exists $\sigma_k \neq 1$ with $\mathrm{Re}(\sigma_k) > 0$ such that $f(\sigma_k) > 0$. By Lemma 5.2 (2), there exists an eigenvalue λ_0 with $\mathrm{Re}(\lambda_0) > 0$ and an eigenfunction Φ_0 such that

$$\Delta\Phi_0 - \Phi_0 + (1 + \sigma_k)w\Phi_0 = \lambda_0\Phi_0. \tag{7.15}$$

If σ_k is positive, we may choose λ_0 to be the principal eigenvalue given by Lemma 5.1 (3).

We choose $\Phi_k = \Phi_0$ and $\Phi_j = 0$ for $j \neq k$, $j \neq 1$. To choose Φ_1, we see that we have to solve equation (7.11) which becomes

$$\Delta\Phi_1 - \Phi_1 + 2w\Phi_1 - \frac{2(1-\eta)}{(\beta_0\eta + 1 - \eta)} \frac{\int_R w\Phi_1}{\int_R w^2} w^2 - \lambda_0\Phi_1 \tag{7.16}$$

$$= c_k \frac{2\|\xi\|(1-\eta)}{(\beta_0\eta + 1 - \eta)(\sum_{i=1}^N \xi_i)} \frac{\int_R w\Phi_0}{\int_R w^2} w^2.$$

To see that (7.16) is solvable, we note that (7.16) is equivalent to

$$\Delta\tilde{\Phi}_1 - \tilde{\Phi}_1 + 2w\tilde{\Phi}_1 - \lambda_0\tilde{\Phi}_1 = -\Lambda\lambda_0 w, \tag{7.17}$$

where

$$\tilde{\Phi}_1 = \Phi_1 - \Lambda w,$$

$$\Lambda = \frac{2(1-\eta)}{(\beta_0\eta + 1 - \eta)} \frac{\int_R w\Phi_1}{\int_R w^2} + c_k \frac{2\|\xi\|(1-\eta)}{(\beta_0\eta + 1 - \eta)(\sum_{i=1}^N \xi_i)} \frac{\int_R w\Phi_0}{\int_R w^2}.$$

If σ_k is not real, then $\mathrm{Im}(\lambda_0) \neq 0$ and so $L_0 - \lambda_0$ is invertible, where $L_0 = \Delta - 1 + 2w$. If σ_k is positive then $\sigma_k \neq 1$ and $L_0 - \lambda_0$ is invertible. Thus (7.17) is solvable and hence (7.16) is solvable. Going backwards, we see that there exists a solution to (7.1) with $\mathbf{\Phi} = (\Phi_1, 0, .., 0, \Phi_0, 0, ..., 0)$ and $\mathrm{Re}(\lambda_0) > 0$. Hence $(X_\epsilon^s, M_\epsilon^s)$ is unstable.

(2) of Theorem 2.2 is thus proved.

□

Acknowledgments: This research is supported by an Earmarked Research Grant from RGC of Hong Kong. MW thanks the Department of Mathematics at The Chinese University of Hong Kong for their kind hospitality. We thank two anonymous referees for their helpful comments which lead us to improve this paper.

8. Appendix A: Proof of Lemma 5.1 . For (1), please see Lemma 4.1 of [51].

For (2), the fact that $\mu_1 = 1, \mu_2 = 2$ has already been proved in Lemma 4.1 of [51]. The exact value of μ_n can be computed along the same line as in the proof of Lemma 5.2. In fact, in this case, $\lambda = 0, \gamma = 1$ and hence the eigenvalues are given by

$$a = 2\gamma - \alpha = -(n-1),\ n = 1, 2, 3...$$

where $\alpha^2 + \alpha - 6\mu = 0$. Thus $\mu_n = \frac{\alpha^2+\alpha}{6}, \alpha = n+1$.

(3) follows by the variational characterization of the eigenvalues:

$$-\lambda_1 = \inf_{\phi \in H^1(R), \phi \not\equiv 0} \frac{\int_R (\phi')^2 + \phi^2 - (1+\mu_R) w\phi^2}{\int_R \phi^2} < 0$$

since by the last inequality for $\phi = w$

$$-\lambda_1 \leq -\mu_R \frac{\int_R w^3}{\int_R w^2} < 0.$$

This is the same analysis as in [61].

When $\mu_R = 1$, there exists only one positive eigenvalue (which is the principal one). See Lemma 1.2 of [56].

To prove (4) note that

$$\sigma = \sigma_R + i\sigma_I,\ \phi = \phi_R + i\phi_I,\ \lambda = \lambda_R + i\lambda_I$$

and write the eigenvalue problem for real and imaginary parts separately:

$$\Delta\phi_R - \phi_R + (1+\sigma_R)w\phi_R - \sigma_I w\phi_I = \lambda_R\phi_R - \lambda_I\phi_I, \tag{8.1}$$

$$\Delta\phi_R - \phi_I + (1+\sigma_R)w\phi_I + \sigma_I w\phi_R = \lambda_R\phi_I + \lambda_I\phi_R. \tag{8.2}$$

Multiplying (8.1) by ϕ_R, (8.2) by ϕ_I, integrating over R, and adding up, we get

$$\int_R [-({\phi_R}')^2 - \phi_R^2 + (1+\sigma_R)w\phi_R^2] + \int_R [-|{\phi_I}'|^2 - \phi_I^2 + (1+\sigma_R)w\phi_I^2]$$

$$= \lambda_R \int_R \phi_R^2 + \phi_I^2.$$

Since in the last equation l.h.s.≤ 0 we also get r.h.s.≤ 0. Therefore $\lambda_R \leq 0$. Now assume that $\lambda_R = 0$. Then by (2) we get $\phi_R = c_1 w$, $\phi_I = c_2 w$ (with $c_1, c_2 \in R$) and $\sigma_R = 0$. But this implies $\lambda_I = 0$, $\sigma_I = 0$ and we get $\lambda = 0$, contrary to what we assumed. Therefore λ_R can not be zero and we conclude Re $\lambda \leq -c_0 < 0$.

9. Appendix B : Proof of Lemma 5.4 . In this appendix, we show how problem (5.2) can be reduced to (5.16).

Let ${}_AF_B$ be defined by (5.15). An important property of ${}_AF_B$ is the following integral property, whose proof can be found in [49]:

$${}_{A+1}F_{B+1} \left\{ \begin{matrix} a_1, & a_2, & ..., & a_A, & c, & ; & \\ & & & & & & z \\ b_1, & b_2, & ..., & b_B, & d & ; & \end{matrix} \right\} \tag{9.1}$$

$$= \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^1 t^{c-1}(1-t)^{d-c-1} {}_AF_B \left\{ \begin{matrix} a_1, & a_2, & ..., & a_A & ; & \\ & & & & & tz \\ b_1, & b_2, & ..., & b_B & ; & \end{matrix} \right\} dt.$$

Let

$$f(\lambda) = \frac{2(1-\eta)}{\eta\sqrt{1+\tau\lambda}+1-\eta}$$

and w be the unique solution of (2.1). Integrating (2.1) it follows that

$$w^{'} = -\sqrt{w^2 - \frac{2}{3}w^3}.$$

Let us first solve the following problem

$$\Delta\phi_0 - \phi_0 + 2w\phi_0 = w^2 + \lambda\phi_0, \quad \phi_0 \in H^2(R). \tag{9.2}$$

Since w is an even function, we may assume that ϕ_0 is also an even function. Let us denote the variable by t. Note that ϕ_0 is unique.

Set

$$\gamma = \sqrt{1+\lambda},$$

where we take the principal branch of $\sqrt{1+\lambda}$.

Then it is easy to see that problem (5.2) becomes

$$\frac{1}{f(\lambda)} = \frac{\int_R w\phi_0}{\int_R w^2} = \frac{\int_0^{+\infty} w\phi_0 dt}{\int_0^{+\infty} w^2 dt}. \tag{9.3}$$

Let us first set

$$\phi_0 = w^\gamma G.$$

Then by some simple computations, G satisfies

$$\frac{d^2G}{dt^2} + 2\gamma\frac{w^{'}}{w}\frac{dG}{dt} + (2 - \frac{\gamma}{3}(1+2\gamma))wG = w^{1-\gamma}. \tag{9.4}$$

Next we perform the following change of variables

$$z = \frac{2}{3}w. \tag{9.5}$$

Note that $w(0) = \frac{3}{2}$ and so z is a homeomorphism from $[0,+\infty]$ to $[0,1]$.

(We remark that here we take a different transformation as in [12]. Our transformation can be considered as a quadratic transformation for hypergeometric functions.)

By some lengthy computations, we obtain the following equation for $G(z)$:

$$z(1-z)G^{''} + (c-(a+b+1)z)G^{'} - abG = (\frac{3}{2})^{2-\gamma}z^{1-\gamma} \tag{9.6}$$

where

$$a = 2+\gamma, \quad b = \gamma - \frac{3}{2}, \quad c = 1+2\gamma. \tag{9.7}$$

To solve (9.6), we take a power series

$$G(z) = z^s \sum_{k=0}^{+\infty} c_k z^k.$$

and substituting it into (9.6), we obtain that

$$\sum_{k=0}^{+\infty} c_k z^{s+k-1}(s+k)(s+k-1+c) - \sum_{k=1}^{+\infty} c_k z^{s+k}(s+k+a)(s+k+b) = (\frac{3}{2})^{2-\gamma} z^{1-\gamma}.$$

So

$$s-1 = 1-\gamma, \quad c_0 s(s-1+c) = (\frac{3}{2})^{2-\gamma},$$

$$c_k(s+k)(s+k-1+c) = c_{k-1}(s+k-1+a)(s+k-1+b).$$

By regrouping the coefficients, we have that

$$G(z) = (\frac{3}{2})^{2-\gamma}(4-\gamma^2)^{-1} z^{2-\gamma} {}_3F_2 \left\{ \begin{matrix} 1, & \frac{1}{2}, & 4 & ; & \\ & & & & z \\ & 3-\gamma, & 3+\gamma & ; & \end{matrix} \right\}. \tag{9.8}$$

Now we can compute

$$\int_0^{+\infty} w\phi_0 dt = \frac{3}{2}\int_0^1 w^{1+\gamma} G(z) \frac{dz}{-w'}$$

$$= (\frac{3}{2})^{1+\gamma} \int_0^1 z^{\gamma}(1-z)^{-\frac{1}{2}} G(z) dz$$

$$= (\frac{3}{2})^3 (4-\gamma^2)^{-1} \int_0^1 z^2 (1-z)^{-\frac{1}{2}} {}_3F_2 \left\{ \begin{matrix} 1, & \frac{1}{2}, & 4 & ; & \\ & & & & z \\ & 3-\gamma, & 3+\gamma & ; & \end{matrix} \right\} dz.$$

By (9.1), we obtain that

$$\int_0^{+\infty} w\phi_0 dt$$

$$= (\frac{3}{2})^3 (4-\gamma^2)^{-1} \frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})} {}_4F_3 \left\{ \begin{matrix} 1, & \frac{1}{2}, & 4, & 3 & ; & \\ & & & & & 1 \\ & 3-\gamma, & 3+\gamma, & \frac{7}{2} & ; & \end{matrix} \right\}. \tag{9.9}$$

On the other hand it is easy to compute that

$$\int_0^{+\infty} w^2 dt = (\frac{3}{2})^2 \int_0^1 z^2 (1-z)^{-\frac{1}{2}} dz = (\frac{3}{2})^2 \frac{\Gamma(2)\Gamma(\frac{1}{2})}{\Gamma(2+\frac{1}{2})}. \tag{9.10}$$

By (9.9), (9.10) and (9.3), we obtain (5.16).

10. Appendix C: proof of Lemma 6.4. We prove Lemma 6.4 in this appendix. We assume that the assumptions (H1)-(H3) are satisfied.

Proof of Lemma 6.4 (1): Recall that $L_0 = \Delta - 1 + 2w$. It is easy to check that $w^{'}\vec{e}_0 \in$ Ker $(\mathcal{L})$. All we need to show is that the dimension of Ker $(\mathcal{L})$ is at most 1. To this end, let $\phi \in$ Ker $(\mathcal{L})$. We first show that the nonlocal term vanishes. In fact, summing up all the equations and using the assumptions (H1) and (H2), we obtain

$$\Delta(\sum_{j=1}^N \phi_j) - (\sum_{j=1}^N \phi_j) + 2w(\sum_{j=1}^N \phi_j) - 2(1-\eta)\frac{\int_R w(\sum_{j=1}^N \phi_j)}{\int_R w^2} w^2 = 0,$$

since $\beta_0 = \sqrt{1+\tau\lambda_0} = 1$.

That is

$$\Delta(\sum_{j=1}^N \phi_j - cw) - (\sum_{j=1}^N \phi_j - cw) + 2w(\sum_{j=1}^N \phi_j - cw) = 0, \tag{10.1}$$

where

$$c = 2(1-\eta)\frac{\int_R w(\sum_{j=1}^N \phi_j)}{\int_R w^2}. \tag{10.2}$$

By Lemma 5.1 (1)

$$\sum_{j=1}^N \phi_j - cw \in \text{Ker } (L_0) = \text{span}\{w'\}.$$

So we have

$$\int_R w(\sum_{j=1}^N \phi_j - cw) = 0.$$

Substituting this relation into (10.2) we get

$$\int_R w \sum_{j=1}^N \phi_j = 0$$

since

$$2(1-\eta) \neq 1.$$

Thus in $\mathcal{L}$ the nonlocal term vanishes and we obtain the following system of equations

$$\Delta\phi_i - \phi_i + w\phi_i + \sum_{j=1}^N b_{ij} w\phi_j = 0, \quad i = 1, ..., N.$$

We decompose

$$\mathbf{B} = \mathcal{P}\mathcal{D}\mathcal{P}^{-1}$$

as in (2.17) of Section 2.

Set

$$\Phi_i = \sum_{j=1}^{N} p_{ij}^{-1} \phi_j.$$

Then the operator L can be expressed in terms of Φ as follows:

$$\Delta\Phi_i - \Phi_i + w\Phi_i + \sum_{j=1}^{N} d_{ij}\Phi_j w = 0.$$

If $1+\sigma \notin$ spec (EVP) (recall that (EVP) was defined in Lemma 5.1 (2)), then by the last line of the Jordan block corresponding to σ we get $\Phi_i = 0$ using Lemma 5.1. Using this in the previous line we get $\Phi_{i-1} = 0$, etc. This implies all components of Φ in the Jordan block corresponding to σ vanish.

If $1+\sigma \in$ spec (EVP) then by hypothesis (H3) we have $\sigma = 1$. By assumption (H3), the eigenvalue $\sigma = 1$ is simple. Since $\Phi_j = 0, j = 2, ..., K$, we are left with Φ_1 only.

Now by Lemma 5.1 (1) we get $\Phi_1 \in$ Ker $(L_0) =$ span$\{w'\}$.

In conclusion, we have proved that except for $i = 1$, where $\Phi_1 = cw'$, $c \in R$, for all other $i = 2, ..., N$, it holds that $\Phi_i = 0$. This implies that the dimension of Ker$\mathcal{L}$ is at most 1.

This finishes the proof of Lemma 6.4 (1).

□

Proof of Lemma 6.4 (2): To show that $\mathcal{L}$ is invertible from $\mathcal{K}_0^{\perp,1} \to \mathcal{K}_0^{\perp,2}$, we just need to show that the conjugate operator of $\mathcal{L}$ – denoted by $\mathcal{L}^*$ – has the kernel $\mathcal{K}_0$. In fact, let $\phi \in ker(\mathcal{L}^*)$. Then we have

$$\Delta\phi_i - \phi_i + \phi_i w + \sum_{j=1}^{N} b_{ji}\phi_j w$$

$$-2(1-\eta)\frac{\int_R w^2 \sum_{i=1}^N \xi_i\phi_i}{\int_R w^2(\sum_{i=1}^N \xi_i)} w = 0, \quad i = 1, ..., N.$$

Multiplying the $i-$th equation by ξ_i and summing up all the equations, by (H1) we have the following:

$$(10.3)\Delta(\sum_{i=1}^{N} \xi_i\phi_i) - (\sum_{i=1}^{N} \xi_i\phi_i) + 2w(\sum_{i=1}^{N} \xi_i\phi_i) - 2(1-\eta)\frac{\int_R w^2(\sum_{i=1}^N \xi_i\phi_i)}{\int_R w^2} w = 0.$$

Multiplying (10.3) by w and then integrating over R, we obtain

$$(1-2(1-\eta))\int_R w^2 \sum_{i=1}^{N} \xi_i\phi_i = 0$$

Since $2(1-\eta) \neq 1$, it is easy to deduce that

$$\int_R w^2 \sum_{i=1}^N \xi_i \phi_i = 0.$$

This means that the nonlocal term vanishes. The rest of the proof of Lemma 6.4 (2) is similar to Lemma 6.4 (1) since spec $(\mathcal{B})$ = spec $(\mathcal{B}^\tau)$ and may be omitted. □

REFERENCES

[1] M. A. Andrade, J. C. Nuño, F. Morán, F. Montero, and G. J. Mpitsos, Complex dynamics of a catalytic network having faulty replication into error-species, *Physica D* 63 (1993), 21-40.

[2] R. A. Barrio, C. Varea, J. L. Aragòn, P. K. Maini, A two-dimensional numerical study of spatial pattern formation in interacting Turing systems, *Bull. Math. Biol.* 61 (1999), 483-505.

[3] P. Bates and G. Fusco, Equilibria with many nuclei for the Cahn-Hilliard equation, *J. Diff. Eqns.*, 160 (2000), 283-356.

[4] P. Bates, E.N. Dancer and J. Shi, Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability, *Adv. Diff. Eqns.* 4 (1999), 1-69.

[5] M.C. Boerlijst and P. Hogeweg, Spiral wave structure in pre-biotic evolution: hypercycles stable against parasites, *Physica D* 48 (1991), 17-28.

[6] K.-S. Cheng and W.-M. Ni, On the structure of the conformal Gaussian curvature equation on R, *Duke Math. J.* 62 (1991), 721-737.

[7] P. Chacón and J.C. Nuño, Spatial dynamics of a model for prebiotic evolution, *Physica D* 81 (1995), 398-410.

[8] M. B. Cronhjort and C. Blomberg, Hypercycles versus parasites in a two-dimensional partial differential equations model, *Journal Theoret. Biol.* 169 (1994), 31-49.

[9] M. B. Cronhjort and C. Blomberg, Cluster compartmentalization may provide resistance to parasites for catalytic networks, *Physica D* 101 (1997), 289-298.

[10] E.N. Dancer, On stability and Hopf bifurcations for chemotaxis systems, to appear in *Methods Appl. Anal.* (2001).

[11] M. del Pino, P. Felmer, and J. Wei, On the role of mean curvature in some singularly perturbed Neumann problems, *SIAM J. Math. Anal.* 31 (1999), 63-79.

[12] A. Doelman, A. Gardner, and T.J. Kaper, Stability analysis of singular patterns in the 1-D Gray-Scott model: A matched asymptotic approach, *Physica D.* 122 (1998), 1-36.

[13] A. Doelman, A. Gardner, and T.J. Kaper, A stability index analysis of 1-D patterns of the Gray-Scott model, *Methods Appl. Anal.* 7 (2000).

[14] A. Doelman, T. Kaper, and P. A. Zegeling, Pattern formation in the one-dimensional Gray-Scott model, *Nonlinearity* 10 (1997), 523-563.

[15] A. Doelman, R. Gardner, and T. J. Kaper, Large stable pulse solutions in reaction-diffusion equations, *Indiana Univ. Math. J.* 49, no. 4 (2000).

[16] S.-I. Ei, Y. Nishiura and B. Sandstede, in preparation.

[17] S.-I. Ei and Y. Nishiura, private communications.

[18] M. Eigen and P. Schuster, The hypercycle. A principle of natural self organisation. Part A. Emergence of the hypercycle. *Naturwissenschaften* 64 (1977), 541-565.

[19] M. Eigen and P. Schuster, The hypercycle. A principle of natural self organisation. Part B. The abstract hypercycle. *Naturwissenschaften* 65 (1978), 7-41.

[20] M. Eigen and P. Schuster, The hypercycle. A principle of natural self organisation. Part C. The realistic hypercycle. *Naturwissenschaften* 65 (1978), 341-369.

[21] M. Eigen and P. Schuster, The hypercycle: A principle of natural selforganisation (Springer, Berlin, 1979).

[22] C. Gui and N. Ghoussoub, Multi-peak solutions for a semilinear Neumann problem involving the critical Sobolev exponent, *Math. Zeit.* 229 (1998), 443-474.

[23] P. Gray and S.K. Scott, Autocatalytic reactions in the isothermal, continuous stirred tank reactor: isolas and other forms of multistability, *Chem. Eng. Sci.* 38 (1983), 29-43.

[24] P. Gray and S.K. Scott, Autocatalytic reactions in the isothermal, continuous stirred tank reactor: oscillations and instabilities in the system $A + 2B \to 3B, B \to C$, *Chem. Eng. Sci.* 39 (1984), 1087-1097.

[25] C. Gui, Multi-peak solutions for a semilinear Neumann problem, *Duke Math. J.* 84 (1996), 739-769.

[26] B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in R^N, *Adv. Math. Suppl Stud.* 7A (1981), 369-402.

[27] C. Gui and J. Wei, Multiple interior peak solutions for some singularly perturbed Neumann problems, *J. Differential Equations* 158 (1999), 1-27.

[28] C. Gui, J. Wei and M. Winter, Multiple boundary peak solutions for some singularly perturbed Neumann problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 17 (2000), 249-289.

[29] J.K. Hale, L.A. Peletier and W.C. Troy, Exact homoclinic and heteroclinic solutions of the Gray-Scott model for autocatalysis, *SIAM J. Appl. Math.* 61 (2000), 102-130.

[30] J.K. Hale, L.A. Peletier and W.C. Troy, Stability and instability of Gray-Scott model: the case of equal diffusivities, *Appl. Math. Lett.* 12 (1999), 59-65.

[31] M.K. Kwong and L. Zhang, Uniqueness of positive solutions of $\Delta u + f(u) = 0$ in an annulus, *Differential Integral Equations* 4 (1991), 583-599.

[32] M. Kowalczyk, Multiple spike layers in the shadow Gierer-Meinhardt system: existence of equilibria and approximate invariant manifold, *Duke Math. J.* 98 (1999), 59-111.

[33] Y.-Y. Li, On a singularly perturbed equation with Neumann boundary condition, *Comm. Partial Differential Equations* 23 (1998), 487-545.

[34] K. Lindgren and M.G. Nordahl, Evolutionary dynamics of spatial games, *Physica D* 75 (1994), 292-309.

[35] R. M. May, Hypercycles spring to life. *Nature* 353 (1991), 607-608.

[36] J. Maynard Smith Hypercycles and the origin of life. *Nature* 280 (1979), 445-446.

[37] C.B. Muratov, V.V. Osipov, Spike autosolitons in Gray-Scott model, Los Alamose-print, patt-sol/9804001, submitted.

[38] W.-M. Ni and I. Takagi, On the shape of least energy solution to a semilinear Neumann problem, *Comm. Pure Appl. Math.* 41 (1991), 819-851.

[39] W.-M. Ni and I. Takagi, Locating the peaks of least energy solutions to a semilinear Neumann problem, *Duke Math. J.* 70 (1993), 247-281.

[40] W.-M. Ni and I. Takagi, Point-condensation generated by a reaction-diffusion system in axially symmetric domains, *Japan J. Industrial Appl. Math.* 12 (1995), 327-365.

[41] W.-M. Ni, I. Takagi and E. Yanagida, *Tohoku Math. J.*, to appear.

[42] Y. Nishiura and D. Ueyama, A skeleton structure of self-replicating dynamics, *Physcia D.* 130 (1999), 73-104.

[43] Y. Nishiura and D. Ueyama, A skeleton structure of self-replicating dynamics, *Physcia D.* 150 (2001), 137-162.

[44] W.-M. Ni and J. Wei, On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems, *Comm. Pure Appl. Math.* 48 (1995), 731-768.

[45] W.-M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states, *Notices Amer. Math. Soc.* 45 (1998), 9-18.

[46] M. A. Nowak and R. M. May, Evolutionary games and spatial chaos, *Nature* 359 (1993), 826-829.

[47] J. Reynolds, J. Pearson and S. Ponce-Dawson, Dynamics of self-replicating spots in reaction-diffusion systems, *Phy. Rev. E* 56 (1) (1997), 185-198.

[48] J. Reynolds, J. Pearson and S. Ponce-Dawson, Dynamics of self-replicating patterns in reaction diffusion systems, *Phy. Rev. Lett.* 72 (1994), 2797-2800.

[49] L.J. Slater, Generalized hypergeometric functions, Cambridge University Press, 1966.

[50] M. J. Ward, An asymptotic analysis of localized solutions for some reaction-diffusion models in multi-dimensional domains, *Studies in Appl. Math.* 97 (1996), 103-126.

[51] J. Wei, On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problem, *J. Differential Equations* 129 (1996), 315-333.

[52] J. Wei, On the boundary spike layer solutions of singularly perturbed semilinear Neumann problem, *J. Differential Equations* 134 (1997), 104-133.

[53] J. Wei, On the interior spike layer solutions for some singular perturbation problems, *Proc. Royal Soc. Edinburgh, Section A (Mathematics)* 128 A (1998), 849-874.

[54] J. Wei, On the interior spike layer solutions of singularly perturbed semilinear Neumann problem, *Tohoku Math. J.* 50 (1998), 159-178.

[55] J. Wei, Uniqueness and critical spectrum estimates of boundary spike solutions, *Proc. Royal Soc. Edinburgh, Section A (Mathematics)*, to appear.

[56] J. Wei, On single interior spike solutions of Gierer-Meinhardt system: uniqueness, spectrum estimates and stability analysis, *Euro. J. Appl. Math.* 10 (1999), 353-378.

[57] J. Wei, Existence, stability and metastability of point condensation patterns generated by Gray-Scott system, *Nonlinearity* 12 (1999), 593-616.

[58] J. Wei, On two dimensional Gray-Scott model: existence of single pulse solutions and their stability, *Physica D*, 148 (2001), 20-48.

[59] J. Wei and M. Winter, Stationary solutions for the Cahn-Hilliard equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 15 (1998), 459-492.

[60] J. Wei and M. Winter, Multiple boundary spike solutions for a wide class of singular perturbation problems, *J. London Math. Soc.* 59 (1999), 585-606.

[61] J. Wei and M. Winter, On a two dimensional reaction-diffusion system with hypercyclical structure, *Nonlinearity* 13 (2000), 2005-2032.