HIGHER-ORDER ENERGY EXPANSIONS AND SPIKE LOCATIONS

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ABSTRACT. We consider the following singularly perturbed semilinear elliptic problem:

$$(I) \qquad \begin{cases} \epsilon^2 \Delta u - u + f(u) = 0 \ \text{ in } \Omega, \\ u > 0 \ \text{ in } \ \Omega \ \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $\epsilon > 0$ is a small constant and f is some superlinear but subcritical nonlinearity. Associated with (I) is the energy functional J_ϵ defined by

$$J_\epsilon[u] := \int_\Omega \left(\frac{\epsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - F(u) \right) dx \qquad \text{for } u \in H^1(\Omega),$$

where $F(u) = \int_0^u f(s) ds$. Ni and Takagi ([24], [25]) proved that for a single boundary spike solution u_ϵ, the following asymptotic expansion holds:

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + o(\epsilon) \right],$$

where $c_1 > 0$ is a generic constant, P_ϵ is the unique local maximum point of u_ϵ and $H(P_\epsilon)$ is the boundary mean curvature function at $P_\epsilon \in \partial\Omega$. In this paper, we obtain a higher-order expansion of $J_\epsilon[u_\epsilon]$:

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 [c_2 (H(P_\epsilon))^2 + c_3 R(P_\epsilon)] + o(\epsilon^2) \right]$$

where c_2, c_3 are generic constants and $R(P_\epsilon)$ is the Ricci scalar curvature at P_ϵ. In particular $c_3 > 0$. Some applications of this expansion are given.

1. INTRODUCTION

We consider the following singularly perturbed semilinear elliptic problem:

$$\begin{cases} \epsilon^2 \Delta u - bu + f(u) = 0 \ \text{ in } \Omega, \\ u > 0 \ \text{ in } \ \Omega \ \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $\epsilon > 0$ is a small constant, $\Delta := \sum_{j=1}^N \frac{\partial^2}{\partial x_j \partial x_j}$ denotes the Laplace operator in R^N, ν stands for the unit outer normal to $\partial\Omega$ and $\partial/\partial\nu$ for the normal derivative, $b > 0$ is a positive constant and $f(t)$ is a function in $C^{1+\sigma}(R) \cap C^2_{\text{loc}}(0, +\infty)$ such that $f(0) = f'(0) = 0$. Typical examples of the function $-bu + f(u)$ are

$$-bu + f(u) = -u + u_+^p \text{ with } u_+ = \max(0, u), \quad b = 1, \tag{1.2}$$

$$-bu + f(u) = u(u-a)(1-u) \text{ with } 0 < a < \frac{1}{2}, \quad b = a, \tag{1.3}$$

where

$$1 < p < \left(\frac{N+2}{N-2}\right)_+ \left(= \frac{N+2}{N-2} \text{ when } N \geq 3; = +\infty \text{ when } N = 1, 2\right). \tag{1.4}$$

Equation (1.1) with (1.2) or (1.3) arises in many branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in biological pattern formation ([13], [29], [35]) or of parabolic equations in chemotaxis, population dynamics and phase transitions ([2], [3],[23], [27]).

Without loss of generality, we may assume that $b = 1$.

Associated with (1.1) is the energy functional J_ϵ defined by

$$J_\epsilon[u] := \int_\Omega \left(\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{1}{2}u^2 - F(u)\right) dx \qquad \text{for } u \in H^1(\Omega), \tag{1.5}$$

where $F(u) = \int_0^u f(s)ds$.

It is known that any solution u of (1.1) is a critical point of J_ϵ and vice versa. In this paper, we restrict ourselves to families of solutions $\{u_\epsilon\}_{0<\epsilon<\epsilon_0}$ of (1.1) with **finite** energy, i.e.

$$\epsilon^{-N} J_\epsilon[u_\epsilon] < +\infty \qquad \text{for } 0 < \epsilon < \epsilon_0. \tag{1.6}$$

It can be proved that for ϵ sufficiently small, any family of solutions of (1.1) satisfying (1.6) can have at most a finite number of local maximum points (see [24]). Let the local maximum points be $\{P_1^\epsilon, ..., P_K^\epsilon\} \subset \bar{\Omega}$. If $P_j^\epsilon \in \partial\Omega, j = 1, ..., K$, we call u_ϵ a $K-$boundary spike solution. If $K = 1$, we call u_ϵ a single boundary spike solution.

In the pioneering papers [23], [24] and [25], Lin, Ni and Takagi established the existence of least-energy solutions and showed that for ϵ sufficiently small the least-energy solution is a single boundary spike solution and has only one local maximum point P_ϵ with $P_\epsilon \in \partial\Omega$. Moreover, $H(P_\epsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\epsilon \to 0$, where $H(P)$ is the mean curvature of $\partial\Omega$ at P.

Since then many works have been devoted to finding solutions with multiple spikes for the Neumann problem as well as the Dirichlet problem. See [1], [2], [3], [4], [6], [7], [8], [9], [10], [11], [12], [15], [16], [17], [18], [19], [21], [22], [24], [25], [26], [27], [28], [31], [32], [36], [37], and the references therein. Recent surveys can be found in [29], [35].

A common tool for proving the existence of spike solutions is the energy expansion: In [24] and [25], Ni and Takagi proved, among others, that for a single boundary spike solution u_ϵ, the following asymptotic expansion for $J_\epsilon[u_\epsilon]$ holds:

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + o(\epsilon) \right], \tag{1.7}$$

where $c_1 > 0$ is a generic constant, P_ϵ is the unique local maximum point of u_ϵ, $H(P_\epsilon)$ is the mean curvature function at $P_\epsilon \in \partial\Omega$, w is the unique solution of the following ground-state problem:

$$\begin{cases} \Delta w - w + f(w) = 0, \quad w > 0 \ \text{ in } R^N, \\ w(0) = \max_{y\in R^N} w(y), \quad \lim_{|y|\to+\infty} w(y) = 0 \end{cases} \tag{1.8}$$

and $I[w]$ is the ground-state energy

$$I[w] = \frac{1}{2}\int_{R^N} |\nabla w|^2\,dy + \frac{1}{2}\int_{R^N} w^2\,dy - \int_{R^N} F(w)\,dy. \tag{1.9}$$

(Note that Ni and Takagi ([24], [25]) proved (1.7) for least-energy solutions. But it is easy to see that it also holds for any single boundary spike solution.)

Based on (1.7), Ni and Takagi [25] showed that the least energy solution must concentrate at a maximum point of the mean curvature function.

If $H(P)$ has more than one maximum points on $\partial\Omega$, the asymptotic expansion (1.7) is no longer sufficient to derive the spike location and the next order term in (1.7) becomes important. This is exactly the purpose of this paper.

Before stating our main result, we introduce some notation.

First we give some conditions on the function $f(t)$:

(f1) $f \in C^{1+\sigma}(R) \cap C^2_{\rm loc}(0,+\infty)$ with $0<\sigma\le 1$, $f(0)=0$, $f^{'}(0)=0$ and $f(t)\equiv 0$ for $t\le 0$.

(f2) The problem (1.8) in the whole space has a unique solution w, which is nondegenerate, i.e.

$$\text{Kernel } (\Delta - 1 + f^{'}(w)) = \text{span}\left\{\frac{\partial w}{\partial y_1},...,\frac{\partial w}{\partial y_N}\right\}. \tag{1.10}$$

By the well-known result of Gidas, Ni, and Nirenberg [14], w is radially symmetric: $w(y)=w(|y|)$ and strictly decreasing: $w^{'}(r)<0$ for $r>0, r=|y|$. Moreover, we have the following asymptotic behavior of w:

$$w(r) = A_N r^{-\frac{N-1}{2}} e^{-r}\left(1+O\left(\frac{1}{r}\right)\right),$$

$$w^{'}(r) = -A_N r^{-\frac{N-1}{2}} e^{-r}\left(1+O\left(\frac{1}{r}\right)\right) \tag{1.11}$$

as $r\to\infty$, where $A_N>0$ is a generic constant.

The uniqueness of w is proved in [20] for the case $f(u)=u^p$. For a general nonlinearity, see [5]. For $f(u)$ defined by (1.3), the uniqueness of the entire solution was proved by Peletier and Serrin [30].

In what follows we always assume that $f(t)$ satisfies (f1) and (f2).

Next, we introduce boundary deformations.

Let $P\in\partial\Omega$. We can define a diffeomorphism straightening the boundary in a neighborhood of P. After rotation and translation of the coordinate system we may assume that the inward normal to $\partial\Omega$ at P points in the direction of the positive x_N-axis and that $P=0$. Denote $x'=(x_1,\ldots,x_{N-1})$, $B'(\delta)=\{x^{'}\in R^{N-1}\ :\ |x'|<\delta\}$, and $\Omega_1=\Omega\cap B(P,\delta)$, where $B(P,\delta)=\{x\in R^N\ :\ \ |x-P|<\delta\}$.

Then, since $\partial\Omega$ is smooth, we can find a constant $\delta>0$ such that $\partial\Omega\cap B(P,\delta)$ can be represented by the graph of a smooth function

$$\rho_P : B'(\delta)\to R, \text{ where } \rho_P(0)=0, \nabla\rho_P(0)=0, \text{ and}$$

$$\Omega\cap B(P,\delta)=\{(x',x_N)\in B(P,\delta)\ :\ x_N-P_N>\rho(x'-P')\}.$$

Moreover, we may assume that

$$\rho_P(x'-P') = \frac{1}{2}\sum_{i=1}^{N-1} k_i(x_i-P_i)^2$$

$$+\frac{1}{6}\sum_{i,j,k=1}^{N-1}\rho_{ijk}(0)(x_i-P_i)(x_j-P_j)(x_k-P_k)+O(|x^{'}-P^{'}|^4),$$

where

$$\rho_{ijk}(0)=\frac{\partial^3\rho_P(0)}{\partial x_i\partial x_j\partial x_k}, \qquad i,j,k=1,\dots,N-1.$$

From now on we omit the P of ρ_P and write ρ instead if this can be done without causing confusion.

Here $k_i, i=1,...,N-1$, are the principal curvatures at P. Furthermore, the average of the principal curvatures of $\partial\Omega$ at P is the mean curvature $H(P)=\frac{1}{N-1}\sum_{i=1}^{N-1}k_i$.

For $N\geq 3$, we also need to define

$$R(P)=\sum_{i\neq j}k_ik_j, \tag{1.12}$$

which is called Ricci scalar curvature at P (up to a constant). When $N=2$, we let $R(P)=0$.

Throughout the paper, we use the following notation:

$$y=(y^{'},y_N), \quad y^{'}=(y_1,...,y_{N-1}), \quad R^N_+=\{y\in R^N\,:\,y_N>0\}. \tag{1.13}$$

Now we can state the main result of this paper.

Theorem 1.1. *Let u_ϵ be a single boundary spike solution of (1.1) with local maximum point $P_\epsilon\in\partial\Omega$. Then, for ϵ sufficiently small, we have*

$$J_\epsilon[u_\epsilon]=\epsilon^N\left[\frac{1}{2}I[w]-c_1\epsilon H(P_\epsilon)+\epsilon^2[c_2(H(P_\epsilon))^2+c_3R(P_\epsilon)]+o(\epsilon^2)\right], \tag{1.14}$$

where

$$c_1=\frac{N-1}{N+1}\int_{R^N_+}(w^{'}(|y|))^2y_Ndy>0 \tag{1.15}$$

and c_2,c_3 are generic constants to be defined later (see (3.26) of Section 3). Moreover, we have $c_3>0$.

For multiple boundary spike solutions, we have a similar asymptotic expansion:

Theorem 1.2. *Let u_ϵ be a K-boundary spike solution of (1.1) with local maximum point $P_1^\epsilon, ..., P_K^\epsilon \in \partial\Omega$. Let $P_j^\epsilon \to P_j^0 \in \partial\Omega$. Suppose that $P_i^0 \neq P_j^0$ for $i \neq j$. Then, for ϵ sufficiently small, we have*

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{K}{2} I[w] - c_1 \epsilon \sum_{j=1}^{K} H(P_j^\epsilon) + \epsilon^2 \sum_{j=1}^{K} [c_2 (H(P_j^\epsilon))^2 + c_3 R(P_j^\epsilon)] + o(\epsilon^2) \right], \tag{1.16}$$

From Theorem 1.1, we can give a refinement of the results of [24] and [25]. To this end, we assume that f satisfies (f1) and

(f3) For $t \geq 0$, f admits the following decomposition in $C^{1+\sigma}(R)$:

$$f(t) = f_1(t) - f_2(t)$$

where $(i) f_1(t) \geq 0$ and $f_2(t) \geq 0$ with $f_1(0) = f_1'(0) = 0$, whence it follows that $f_2(0) = f_2'(0) = 0$ by (f1); and (ii) there is a $q \geq 1$ such that $f_1(t)/t^q$ is nondecreasing in $t > 0$, whereas $f_2(t)/t^q$ is nonincreasing in $t > 0$, and in case $q = 1$ we require further that the above monotonicity condition for $f_1(t)/t$ is strict,

(f4) $f(t) = O(t^p)$ as $t \to +\infty$ where p satisfies (1.4),

(f5) There exists a constant $\theta \in (0, \frac{1}{2})$ such that $F(t) \leq \theta t f(t)$ for $t \geq 0$.

By taking a function $e(x) \equiv k$ for some constant k in Ω, and choosing k large enough, we have $J_\epsilon[e] < 0$, for all $\epsilon \in (0, 1)$. Then for each $\epsilon \in (0, 1)$, we can define the so-called mountain-pass value

$$c_\epsilon = \inf_{h \in \Gamma} \max_{0 \leq t \leq 1} J_\epsilon[h(t)] \tag{1.17}$$

where $\Gamma = \{h : [0, 1] \to H^1(\Omega) | h(t) \text{ is continuous },\ h(0) = 0, h(1) = e\}$.

In [24] and [25], it is proved that there exists a mountain-pass solution u_ϵ which is also a least-energy solution. Moreover, as $\epsilon \to 0$, u_ϵ develops a spike layer behavior near a maximum point of the mean curvature function. Now we have

Corollary 1.3. *Suppose that $f(u)$ satisfies (f1), (f3), (f4) and (f5). Let u_ϵ be a least energy solution of (1.1) (constructed in* [24]*) and let P_ϵ be the* *unique local maximum point of u_ϵ. Then, for ϵ sufficiently small, we have*

$$H(P_\epsilon) \to \max_{P \in \partial\Omega} H(P), \quad R(P_\epsilon) \to \min_{Q \in \partial\Omega, H(Q)=\max_{P\in\partial\Omega} H(P)} R(Q). \tag{1.18}$$

Remark: 1. If $N = 2$, (1.18) yields no new result. In that case, we have to expand $J_\epsilon[u_\epsilon]$ up to the order $O(\epsilon^3)$ to obtain more information on the spike locations.

2. The asymptotic expansion (1.14) shows that the Ricci scalar curvature can play an important role in the case of constant mean curvature boundary.

The proof of Theorem 1.1 is divided into three steps:

Step 1: We choose a good approximate function, concentrating at a boundary point P and called $\tilde{w}_{\epsilon,P}$, such that

$$\epsilon^2 \Delta \tilde{w}_{\epsilon,P} - \tilde{w}_{\epsilon,P} + f(\tilde{w}_{\epsilon,P}) = O(\epsilon^{1+\sigma}), \tag{1.19}$$

where σ is the Holder exponent of f' (see assumption (f1)).

This is done in Section 2.

Step 2: Our key observation is that in order to obtain the term of order ϵ^2 in the asymptotic expansion of $J_\epsilon[u_\epsilon]$, we do not need to expand u_ϵ up to the order $O(\epsilon^2)$. In fact, it is enough to have

$$u_\epsilon = \tilde{w}_{\epsilon,P_\epsilon} + O(\epsilon^\tau) \tag{1.20}$$

for some $\tau > 1$. We choose $\tau = 1 + \frac{\sigma}{2}$. We do not even need to know the term of order ϵ^τ in the asymptotic expansion of u_ϵ. From (1.20) we derive that

$$J_\epsilon[u_\epsilon] = J_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}] + o(\epsilon^{N+2}). \tag{1.21}$$

This is proved in Section 5.

Step 3: It then remains to compute the energy of $\tilde{w}_{\epsilon,P}$. A higher-order energy expansion is derived Section 3 and in Section 4 it is shown that $c_1 > 0$ and $c_3 > 0$.

Finally, the proofs of Theorem 1.1, Theorem 1.2, and Corollary 1.3 are contained in Section 6.

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2. A suitable approximate function $\tilde{w}_{\epsilon,P}$

In this section, we introduce a suitable approximate function $\tilde{w}_{\epsilon,P}$.

Let Ω be a smooth domain in R^N and w be the unique solution of (1.8). For $P \in \partial\Omega$, we define $w_{\epsilon,P}(x)$ to be the unique solution of the following problem:

$$\begin{cases} \epsilon^2 \Delta w_{\epsilon,P} - w_{\epsilon,P} + f(w(\frac{x-P}{\epsilon})) = 0 \text{ in } \Omega, \\ \frac{\partial w_{\epsilon,P}}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

The function $w_{\epsilon,P}$ was first introduced and studied in [36]. It can be considered as a projection of $w(\frac{x-P}{\epsilon}) \in H^1(\Omega)$ into

$$H^1_\nu(\Omega) = \left\{ u \in H^1(\Omega) \,:\, \frac{\partial u}{\partial \nu} = 0 \,\text{at}\, \partial\Omega \right\}.$$

Set

$$w_{\epsilon,P} = w\left(\frac{x-P}{\epsilon}\right) - h_{\epsilon,P}(x).$$

Then $h_{\epsilon,P}$ satisfies

$$\begin{cases} \epsilon^2 \Delta h_{\epsilon,P} - h_{\epsilon,P} = 0 \text{ in } \Omega, \\ \frac{\partial h_{\epsilon,P}}{\partial \nu} = \frac{\partial w(\frac{x-P}{\epsilon})}{\partial \nu} \text{ on } \partial\Omega. \end{cases} \tag{2.2}$$

We deform the boundary near P as in Section 1. For $x \in \Omega_1 = \Omega \cap B(P,\delta)$, set now

$$\epsilon y' = x' - P', \quad \epsilon y_N = x_N - P_N - \rho(x' - P'). \tag{2.3}$$

This transformation is denoted as $y = T_\epsilon(x)$. Note that the Jacobian of T_ϵ equals ϵ^{-N}. Its inverse is called $x = T_\epsilon^{-1}(y)$. One computes that

$$x' = P' + \epsilon y', \quad x_N = P_N + \epsilon y_N + \rho(\epsilon y'). \tag{2.4}$$

In our coordinate system, for $x \in \omega_1 := \partial\Omega \cap B(P, \delta)$, we have

$$\nu(x) = \frac{1}{\sqrt{1 + |\nabla_{x'}\rho|^2}}(\nabla_{x'}\rho, -1),$$

$$\frac{\partial}{\partial\nu} = \frac{1}{\sqrt{1 + |\nabla_{x'}\rho|^2}} \left\{ \sum_{j=1}^{N-1} \rho_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_N} \right\} \Bigg|_{x_N - P_N = \rho(x' - P')},$$

and the Laplace operator becomes

$$\epsilon^2 \Delta_x = \Delta_y + |\nabla_{x'}\rho|^2 \frac{\partial^2}{\partial y_N^2} - 2\sum_{i=1}^{N-1} \rho_i \frac{\partial^2}{\partial y_i \partial y_N} - \epsilon \Delta_{x'}\rho \frac{\partial}{\partial y_N}. \tag{2.5}$$

We need to analyze the behavior of $h_{\epsilon,P}$ up to the order $O(\epsilon^3)$. To this end, we recall the following three functions introduced in [36].

Let v_1 be the unique solution of

$$\begin{cases} \Delta v_1 - v_1 = 0 & \text{in } R^N_+, \\ \frac{\partial v_1}{\partial y_N} = -\frac{w'(|y|)}{2|y|} \sum_{i=1}^{N-1} k_i y_i^2 & \text{on } \partial R^N_+, \end{cases} \tag{2.6}$$

v_2 be the unique solution of

$$\begin{cases} \Delta v_2 - v_2 - 2\sum_{i=1}^{N-1} k_i y_i \frac{\partial^2 v_1}{\partial y_i \partial y_N} - (\sum_{i=1}^{N-1} k_i) \frac{\partial v_1}{\partial y_N} = 0 & \text{in } R^N_+, \\ \frac{\partial v_2}{\partial y_N} = \sum_{i=1}^{N-1} k_i y_i \frac{\partial v_1}{\partial y_i} & \text{on } \partial R^N_+, \end{cases} \tag{2.7}$$

and v_3 be the unique solution of

$$\begin{cases} \Delta v_3 - v_3 = 0 & \text{in } R^N_+, \\ \frac{\partial v_3}{\partial y_N} = -\frac{w'}{3|y|} \sum_{i,j,k=1}^{N-1} \rho_{ijk} y_i y_j y_k & \text{on } \partial R^N_+. \end{cases} \tag{2.8}$$

Note that v_1, v_2 are even functions in $y' = (y_1, ..., y_{N-1})$ and v_3 is an odd function in $y' = (y_1, ..., y_{N-1})$ (i.e. $v_1(y', y_N) = v_1(-y', y_N), v_3(y', y_N) = -v_3(-y', y_N)$). Moreover, it is easy to see that $|v_1|, |v_2|, |v_3| \le Ce^{-a|y|}$ for some $a > 0$.

Let $\chi(x)$ be a smooth cut-off function such that $\chi(x) = 1$ for $x \in B(0, \frac{\delta}{2})$ and $\chi(x) = 0$ for $x \notin B(0, \delta)$.

Set

$$h_{\epsilon,P}(x) = \epsilon v_1(T_\epsilon(x))\chi(x - P) + \epsilon^2[v_2(T_\epsilon(x))\chi(x - P)$$

$$+v_3(T_\epsilon(x))\chi(x-P)] + \epsilon^3\Psi_{\epsilon,P}(x), \tag{2.9}$$

where $y = T_\epsilon(x)$ is given in (2.3).

Then we have the following asymptotic expansion, whose proof can be found in Proposition 2.1 of [36].

Proposition 2.1. *For ϵ sufficiently small,*

$$w_{\epsilon,P}(x) = w\left(\frac{x-P}{\epsilon}\right) - \epsilon v_1(T_\epsilon(x))\chi(x-P)$$

$$-\epsilon^2(v_2(T_\epsilon(x)) + v_3(T_\epsilon(x)))\chi(x-P) + \epsilon^3\Psi_{\epsilon,P}(x), \tag{2.10}$$

where $\Psi_{\epsilon,P}$ satisfies

$$\epsilon^{-N}\int_\Omega \left(\epsilon^2|\nabla\Psi_{\epsilon,P}|^2 + |\Psi_{\epsilon,P}|^2\right) dx \le C, \tag{2.11}$$

$$|\Psi_{\epsilon,P}(T_\epsilon^{-1}(y))| \le Ce^{-a|y|} \tag{2.12}$$

for some constant $a > 0$.

Next we study the properties of the following linear operator:

$$L_0 := \Delta - 1 + f'(w): \quad H^2(R^N) \to L^2(R^N). \tag{2.13}$$

By assumption (f2),

$$\text{Kernel } (L_0) = \text{span}\left\{\frac{\partial w}{\partial y_j} \,:\, j = 1, ..., N\right\}.$$

If we restrict L_0 to

$$H^2_\nu(R^N_+) = H^2(R^N_+) \cap \left\{\frac{\partial u}{\partial y_N} = 0 \text{ on } \partial R^N_+\right\}$$

then we have

$$\text{Kernel } (L_0) \cap H^2_\nu(R^N_+) = \text{span}\left\{\frac{\partial w}{\partial y_j} \,:\, j = 1, ..., N-1\right\}. \tag{2.14}$$

By (2.14), there is a unique solution to

$$\begin{cases} \Delta\Phi_0 - \Phi_0 + f'(w)\Phi_0 - f'(w)v_1 = 0 & \text{in } R^N_+, \\ \frac{\partial \Phi_0}{\partial y_N} = 0 & \text{on } \partial R^N_+, \\ \Phi_0 \text{ is even in } y'. \end{cases} \tag{2.15}$$

We call this solution Φ_0. We modify Φ_0 to a new function $\Phi_{\epsilon,P}$ which satisfies the Neumann boundary condition. To this end, let $\phi_{\epsilon,P}$ be the solution of

$$\begin{cases} \epsilon^2\Delta\phi_{\epsilon,P} - \phi_{\epsilon,P} = 0 \quad \text{in } \Omega, \\ \frac{\partial\phi_{\epsilon,P}}{\partial\nu} = \frac{\partial(\Phi_0(T_\epsilon(x))\chi(x-P))}{\partial\nu} \quad \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

Put

$$\Phi_{\epsilon,P}(x) = \Phi_0(T_\epsilon(x))\chi(x-P) - \phi_{\epsilon,P}(x). \tag{2.17}$$

It is easy to see that $\Phi_{\epsilon,P}$ satisfies the Neumann boundary condition and $\Phi_{\epsilon,P}(T_\epsilon^{-1}(y)) = \Phi_0(y) + O(\epsilon e^{-a|y|})$. Furthermore, $|\Phi_{\epsilon,P}(T_\epsilon^{-1}(y))| \le Ce^{-a|y|}$ for some $a > 0$.

Finally, we introduce the following approximate function:

$$\tilde{w}_{\epsilon,P}(x) = w_{\epsilon,P}(x) + \epsilon\Phi_{\epsilon,P}(x), \quad x \in \Omega. \tag{2.18}$$

Note that $\tilde{w}_{\epsilon,P}(x)$ satisfies the Neumann boundary condition.

Our next lemma says that $\tilde{w}_{\epsilon,P}$ satisfies the equation (1.1) up to the order $O(\epsilon^{1+\sigma})$.

Lemma 2.2. *Let*

$$S_\epsilon[\tilde{w}_{\epsilon,P}] := \epsilon^2\Delta\tilde{w}_{\epsilon,P} - \tilde{w}_{\epsilon,P} + f(\tilde{w}_{\epsilon,P}). \tag{2.19}$$

Then, for ϵ sufficiently small, we have

$$|S_\epsilon[\tilde{w}_{\epsilon,P}]| \le C\epsilon^{1+\sigma}e^{-a|y|}. \tag{2.20}$$

Proof: We expand $S_\epsilon[\tilde{w}_{\epsilon,P}]$:

$$\begin{aligned} S_\epsilon[\tilde{w}_{\epsilon,P}] &= S_\epsilon[w_{\epsilon,P}] + \epsilon[\epsilon^2\Delta\Phi_{\epsilon,P} - \Phi_{\epsilon,P} + f^{'}(w_{\epsilon,P})\Phi_{\epsilon,P}] \\ &+[f(w_{\epsilon,P} + \epsilon\Phi_{\epsilon,P}) - f(w_{\epsilon,P}) - \epsilon f^{'}(w_{\epsilon,P})\Phi_{\epsilon,P}] = S_1 + S_2 + S_3, \end{aligned} \tag{2.21}$$

where S_1, S_2 and S_3 are defined by the last equality.

By (2.1), Proposition 2.1 and (2.15),

$$\begin{aligned} S_1 + S_2 &= f(w_{\epsilon,P}) - f\left(w\left(\frac{x-P}{\epsilon}\right)\right) + \epsilon[\epsilon^2\Delta\Phi_{\epsilon,P} - \Phi_{\epsilon,P} + f^{'}(w_{\epsilon,P})\Phi_{\epsilon,P}] \\ &= \left[f(w_{\epsilon,P}) - f\left(w\left(\frac{x-P}{\epsilon}\right)\right) + \epsilon v_1\chi f^{'}\left(w\left(\frac{x-P}{\epsilon}\right)\right)\right] \\ &+\epsilon\left[\epsilon^2\Delta\Phi_{\epsilon,P} - \Phi_{\epsilon,P} + f^{'}(w_{\epsilon,P})\Phi_{\epsilon,P} - f^{'}\left(w\left(\frac{x-P}{\epsilon}\right)\right)v_1\chi\right] \end{aligned}$$

$$= O(\epsilon^2 e^{-a|y|}).$$

On the other hand, it follows by the mean-value theorem that

$$|f(a+b) - f(a) - f'(a)b| \le C|a|^\sigma |b|^{1+\sigma} \tag{2.22}$$

for any a, b such that $|b| \le 2|a| \le C$. Thus

$$S_3 = O(\epsilon^{1+\sigma}|w_{\epsilon,P}|^\sigma |\Phi_{\epsilon,P}|^{1+\sigma}) = O(\epsilon^{1+\sigma} e^{-a|y|}). \tag{2.23}$$

This proves the lemma.

□

3. The computation of $J_\epsilon[\tilde{w}_{\epsilon,P}]$

In this section, we compute the energy of the approximate function $\tilde{w}_{\epsilon,P}$. In the next section, we will show that $\tilde{w}_{\epsilon,P}$ contributes the energy expansion up to the order $o(\epsilon^2)$.

We begin with

$$\begin{aligned} J_\epsilon[\tilde{w}_{\epsilon,P}] &= J_\epsilon[w_{\epsilon,P} + \epsilon\Phi_{\epsilon,P}] \\ &= J_\epsilon[w_{\epsilon,P}] + \epsilon \int_\Omega (\epsilon^2 \nabla w_{\epsilon,P} \nabla \Phi_{\epsilon,P} + w_{\epsilon,P}\Phi_{\epsilon,P} - f(w_{\epsilon,P})\Phi_{\epsilon,P})\, dx \\ &+\epsilon^2 \left(\frac{\epsilon^2}{2} \int_\Omega |\nabla \Phi_{\epsilon,P}|^2\, dx + \frac{1}{2}\int_\Omega |\Phi_{\epsilon,P}|^2\, dx - \frac{1}{2}\int_\Omega f'(w_{\epsilon,P})\Phi_{\epsilon,P}^2\, dx \right) \\ &- \int_\Omega \left[F(w_{\epsilon,P} + \epsilon\Phi_{\epsilon,P}) - F(w_{\epsilon,P}) - \epsilon f(w_{\epsilon,P})\Phi_{\epsilon,P} - \frac{\epsilon^2}{2} f'(w_{\epsilon,P})|\Phi_{\epsilon,P}|^2 \right] dx. \end{aligned} \tag{3.1}$$

The last term in (3.1) can be estimated using (2.22):

$$\begin{aligned} &\int_\Omega \left| F(w_{\epsilon,P} + \epsilon\Phi_{\epsilon,P}) - F(w_{\epsilon,P}) - \epsilon f(w_{\epsilon,P})\Phi_{\epsilon,P} - \frac{\epsilon^2}{2} f'(w_{\epsilon,P})|\Phi_{\epsilon,P}|^2 \right| dx \\ &\le C\epsilon^{2+\sigma} \int_\Omega w_{\epsilon,P}^\sigma |\Phi_{\epsilon,P}|^{2+\sigma}\, dx \le C\epsilon^{N+2+\sigma}. \end{aligned} \tag{3.2}$$

Using (2.1) and (3.2), we see that

$$\begin{aligned} J_\epsilon[\tilde{w}_{\epsilon,P}] &= J_\epsilon[w_{\epsilon,P}] + \epsilon \int_\Omega \left(f\left(w\left(\frac{x-P}{\epsilon} \right) \right) - f(w_{\epsilon,P}) \right) \Phi_{\epsilon,P}\, dx \\ &+ \frac{\epsilon^2}{2}\left[\epsilon^2 \int_\Omega |\nabla\Phi_{\epsilon,P}|^2\, dx + \int_\Omega |\Phi_{\epsilon,P}|^2\, dx - \int_\Omega f'(w_{\epsilon,P})\Phi_{\epsilon,P}^2\, dx \right] + o(\epsilon^{N+2}) \\ &= I_1 + I_2 + I_3 + o(\epsilon^{N+2}), \end{aligned} \tag{3.3}$$

where I_1, I_2 and I_3 are defined by the last equality.

We compute I_3 first. In fact, it is easy to see that

$$\epsilon^{-N-2} I_3 \to \frac{1}{2}\int_{R_+^N} \left(|\nabla \Phi_0|^2 + |\Phi_0|^2 - f^{'}(w)\Phi_0^2\right) dy \tag{3.4}$$

$$= -\frac{1}{2}\int_{R_+^N} f^{'}(w) v_1 \Phi_0 \, dy.$$

The last equality follows from equation (2.15).

Next, for I_2 we get:

$$\epsilon^{-N-2} I_2 \to \int_{R_+^N} f^{'}(w) v_1 \Phi_0 \, dy. \tag{3.5}$$

Combining (3.4) and (3.5), we deduce that

$$I_2 + I_3 = \frac{\epsilon^{N+2}}{2}\int_{R_+^N} f^{'}(w) v_1 \Phi_0 \, dy + o(\epsilon^{N+2}). \tag{3.6}$$

Now it remains to compute I_1. Using equation (2.1) and Proposition 2.1, we deduce that

$$I_1 = \frac{\epsilon^2}{2}\int_\Omega |\nabla w_{\epsilon,P}|^2 \, dx + \frac{1}{2}\int_\Omega w_{\epsilon,P}^2 \, dx - \int_\Omega F(w_{\epsilon,P}) \, dx$$

$$= \frac{1}{2}\int_\Omega f(w) w_{\epsilon,P} \, dx - \int_\Omega F(w_{\epsilon,P}) \, dx$$

$$= \frac{1}{2}\int_\Omega f(w)(w - \epsilon v_1 \chi - \epsilon^2 (v_2 + v_3)\chi) \, dx$$

$$- \int_\Omega F(w - \epsilon v_1 \chi - \epsilon^2 (v_2 + v_3)\chi) \, dx + o(\epsilon^{N+2})$$

$$= \int_\Omega \left[\frac{1}{2} w f(w) - F(w)\right] dx + \frac{\epsilon}{2}\int_\Omega f(w) v_1 \, dx$$

$$+ \frac{\epsilon^2}{2}\int_\Omega (f(w) v_2 - f^{'}(w) v_1^2) \, dx + o(\epsilon^{N+2}). \tag{3.7}$$

Here we have used the fact that v_3 is odd in $y^{'}$ and hence $\int_{R_+^N} f(w) v_3 \, dy = 0$.

Let

$$I_{1,1} = \int_\Omega \left[\frac{1}{2} w f(w) - F(w)\right] dx, \quad I_{1,2} = \int_\Omega f(w) v_1 \, dx.$$

Now we compute these two terms up to $o(\epsilon^2)$. To this end, let us calculate $\frac{|x-P|}{\epsilon}$ under the transformation (2.3):

$$\frac{|x - P|}{\epsilon} = \frac{1}{\epsilon}\sqrt{\epsilon^2 |y^{'}|^2 + (\epsilon y_N + \rho(\epsilon y^{'}))^2}$$

$$= \sqrt{|y|^2 + \epsilon \sum_{i=1}^{N-1} k_i y_i^2 y_N + \frac{\epsilon^2}{3} \sum_{i,j,k=1}^{N-1} \rho_{ijk} y_i y_j y_k y_N + \frac{\epsilon^2}{4} \left(\sum_{i=1}^{N-1} k_i y_i^2 \right)^2} + O(\epsilon^3 |y|^5). \tag{3.8}$$

We state the following useful lemma.

Lemma 3.1. *Suppose that $A(|y|)$ is a radially symmetric function such that*

$$|A'(|y|)| + |A''(|y|)| + |A'''(|y|)| \le Ce^{-a|y|}$$

for some $a > 0$. Then, for ϵ sufficiently small, we have

$$A\left(\frac{|x-P|}{\epsilon}\right) = A(|y|) + \epsilon \frac{A'(|y|)}{2|y|} \sum_{i=1}^{N-1} k_i y_i^2 y_N$$
$$+\epsilon^2 \left[\frac{A'(|y|)}{2|y|} \left(\frac{1}{3} \sum_{i,j,k=1}^{N-1} \rho_{ijk} y_i y_j y_k y_N + \frac{1}{4} \left(\sum_{i=1}^{N-1} k_i y_i^2 \right)^2 \right) \right]$$
$$+ \epsilon^2 \left[\frac{A''(|y|)}{8|y|^2} - \frac{A'(|y|)}{8|y|^3} \right] \left(\sum_{i=1}^{N-1} k_i y_i^2 \right)^2 y_N^2 + O(\epsilon^3 e^{-a|y|/2}) \tag{3.9}$$

and

$$\int_\Omega A\left(\frac{|x-P|}{\epsilon}\right) dx = \epsilon^N \int_{R_+^N} A(|y|) dy - \frac{1}{2} \epsilon^{N+1} H(P) \int_{\partial R_+^N} A(|y|)|y| \, dy' + o(\epsilon^{N+2}). \tag{3.10}$$

Proof: Equation (3.9) follows by using Taylor expansion.

By (3.9), we have

$$\int_\Omega A\left(\frac{|x-P|}{\epsilon}\right) dx = \epsilon^N \int_{R_+^N} A(|y|) dy + \epsilon^{N+1} \int_{R_+^N} \frac{A'(|y|)}{2|y|} \left(\sum_{i=1}^{N-1} k_i y_i^2 y_N \right) dy$$

$$+ \epsilon^{N+2} \int_{R_+^N} \left[\frac{A'(|y|)}{8|y|} \left(\sum_{i=1}^{N-1} k_i y_i^2 \right)^2 + \frac{(A'(|y|)/|y|)'}{8|y|} \left(\sum_{i=1}^{N-1} k_i y_i^2 y_N \right)^2 \right] dy + o(\epsilon^{N+2}). \tag{3.11}$$

The last term can be estimated as follows:

$$\int_{R_+^N} \left[\frac{A'(|y|)}{8|y|} \left(\sum_{i=1}^{N-1} k_i y_i^2 \right)^2 + \frac{(A'(|y|)/|y|)'}{8|y|} \left(\sum_{i=1}^{N-1} k_i y_i^2 y_N \right)^2 \right] dy$$
$$= \frac{1}{8} \int_{R_+^N} \frac{A'(|y|)}{|y|} \left(\sum_{i=1}^{N-1} k_i y_i^2 \right)^2 dy + \frac{1}{8} \int_{R_+^N} y_N \frac{\partial (A'(|y|)/|y|)}{\partial y_N} \left(\sum_{i=1}^{N-1} k_i y_i^2 \right)^2 dy \Bigg]$$
$$= \frac{1}{8} \int_{R_+^N} \frac{\partial}{\partial y_N} \left(\frac{A'(|y|)}{|y|} y_N \left(\sum_{i=1}^{N-1} k_i y_i^2 \right)^2 \right) dy = 0. \tag{3.12}$$

Substituting (3.12) into (3.11), we obtain the lemma. □

From Lemma 3.1, it follows that

$$
\begin{aligned}
I_{1,1} &= \epsilon^N \int_{R_+^N} \left[\frac{1}{2} w f(w) - F(w)\right] dy \\
&+ \epsilon^{N+1} \sum_{i=1}^{N-1} k_i \int_{R_+^N} \left[\frac{1}{4} w f^{'}(w) - \frac{1}{4} f(w)\right] \frac{w^{'}}{|y|} y_N y_i^2 dy + o(\epsilon^{N+2}) \\
&= \epsilon^N \frac{1}{2} I[w] - \epsilon^{N+1} \frac{H(P)}{4} \int_{\partial R_+^N} [w f(w) - 2F(w)] |y|^2 dy^{'} + o(\epsilon^{N+2}).
\end{aligned}
\tag{3.13}
$$

Using Lemma 3.1 and (2.6), we see that

$$
\begin{aligned}
I_{1,2} &= \epsilon^N \int_{R_+^N} f(w) v_1 dy + \epsilon^{N+1} \int_{R_+^N} \frac{f^{'}(w) w^{'}}{2|y|} \left(\sum_{i=1}^{N-1} k_i y_i^2 y_N\right) v_1(y) dy + O(\epsilon^{N+2}) \\
&= \epsilon^N \frac{H(P)}{2} \int_{\partial R_+^N} w w^{'} |y| dy^{'} + \epsilon^{N+1} \int_{R_+^N} \frac{f^{'}(w) w^{'}}{2|y|} \left(\sum_{i=1}^{N-1} k_i y_i^2 y_N\right) v_1(y) dy + O(\epsilon^{N+2}).
\end{aligned}
\tag{3.14}
$$

Combining the estimates for $I_{1,1}$, $I_{1,2}$, I_2, I_3, we arrive at

$$
J_\epsilon[\tilde{w}_{\epsilon,P}] = \frac{\epsilon^N}{2} I(w) - c_1 \epsilon^{N+1} H(P) + \epsilon^{N+2} A_0 + o(\epsilon^{N+2}),
\tag{3.15}
$$

where

$$
c_1 = \frac{1}{4} \int_{\partial R_+^N} \left[w f(w) - 2F(w) - 2\frac{w w^{'}}{|y|}\right] |y|^2 dy^{'}
\tag{3.16}
$$

and

$$
\begin{aligned}
A_0 &= \frac{1}{2} \int_{R_+^N} f^{'}(w) v_1 (\Phi_0 - v_1)\, dy + \frac{1}{2} \int_{R_+^N} f(w) v_2\, dy \\
&+ \frac{1}{4} \sum_{i=1}^{N-1} k_i \int_{R_+^N} \frac{f^{'}(w) w^{'}}{|y|} y_i^2 y_N v_1(y) dy.
\end{aligned}
\tag{3.17}
$$

Now we are going to simplify A_0. Let $\Phi_i, i = 1, ..., N-1$, be the unique solution of the following problem:

$$
\begin{cases}
\Delta \Phi_i - \Phi_i + f^{'}(w) \Phi_i = 0 & \text{in } R_+^N, \\
\frac{\partial \Phi_i}{\partial y_N} = \frac{w^{'}(|y|)}{|y|} y_i^2 & \text{on } \partial R_+^N, \\
\Phi_i \text{ is even in } & y^{'}.
\end{cases}
\tag{3.18}
$$

Note that $\Phi_i, i = 2, ..., N-1$, can be obtained from Φ_1 by rotation. This fact will be used frequently.

We claim that

Lemma 3.2.
$$A_0 = \frac{1}{8}\left(\sum_{i=1}^{N-1} k_i\right)^2 \int_{\partial R^N_+} \Phi_1 \frac{\partial \Phi_1}{\partial y_N}\, dy' + \frac{1}{8}\sum_{i\neq j} k_i k_j \int_{\partial R^N_+} \Phi_1 \frac{\partial(\Phi_2 - \Phi_1)}{\partial y_N} dy'. \tag{3.19}$$

Proof: First, using the equations (2.6) and (2.15), we obtain
$$\frac{1}{2}\int_{R^N_+} f'(w) v_1 (\Phi_0 - v_1)\, dy = -\frac{1}{2}\int_{R^N_+} v_1(\Delta \Phi_0 - \Phi_0)\, dy$$
$$= -\frac{1}{2}\int_{R^N_+} (\Delta v_1 - v_1)\Phi_0\, dy - \frac{1}{2}\int_{\partial R^N_+} \Phi_0 \frac{\partial v_1}{\partial y_N}\, dy' = \frac{1}{4}\sum_{i=1}^{N-1} k_i \int_{\partial R^N_+} \frac{w'}{|y|}\Phi_0 y_i^2\, dy'. \tag{3.20}$$

Next, using (2.7),
$$\frac{1}{2}\int_{R^N_+} f(w) v_2\, dy = -\frac{1}{2}\int_{R^N_+} (\Delta w - w) v_2\, dy$$
$$= -\frac{1}{2}\int_{R^N_+} (\Delta v_2 - v_2) w\, dy + \frac{1}{2}\int_{\partial R^N_+} \left(v_2 \frac{\partial w}{\partial y_N} - w\frac{\partial v_2}{\partial y_N}\right) dy'$$
$$= -\frac{1}{2}\int_{R^N_+} \left(2\sum_{i=1}^{N-1} k_i y_i \frac{\partial^2 v_1}{\partial y_i y_N} + \sum_{i=1}^{N-1} k_i \frac{\partial v_1}{\partial y_N}\right) w\, dy - \frac{1}{2}\int_{\partial R^N_+} w \sum_{i=1}^{N-1} k_i y_i \frac{\partial v_1}{\partial y_i} dy'$$
$$= \sum_{i=1}^{N-1} k_i \int_{R^N_+} \frac{\partial (y_i w)}{\partial y_i}\frac{\partial v_1}{\partial y_N}\, dy - \frac{1}{2}\sum_{i=1}^{N-1} k_i \int_{R^N_+} w \frac{\partial v_1}{\partial y_N}\, dy + \frac{1}{2}\sum_{i=1}^{N-1} k_i \int_{\partial R^N_+} v_1 \frac{\partial (w y_i)}{\partial y_i}\, dy'$$
$$= \sum_{i=1}^{N-1} k_i \int_{R^N_+} y_i \frac{\partial w}{\partial y_i}\frac{\partial v_1}{\partial y_N}\, dy + \frac{1}{2}\sum_{i=1}^{N-1} k_i \int_{R^N_+} w \frac{\partial v_1}{\partial y_N}\, dy + \frac{1}{2}\sum_{i=1}^{N-1} k_i \int_{\partial R^N_+} v_1 \frac{\partial (w y_i)}{\partial y_i}\, dy'$$
$$= \sum_{i=1}^{N-1} k_i \int_{R^N_+} y_i \frac{\partial w}{\partial y_i}\frac{\partial v_1}{\partial y_N}\, dy - \frac{1}{2}\sum_{i=1}^{N-1} k_i \int_{R^N_+} v_1 \frac{\partial w}{\partial y_N}\, dy + \frac{1}{2}\sum_{i=1}^{N-1} k_i \int_{\partial R^N_+} v_1 y_i \frac{\partial w}{\partial y_i}\, dy'$$
$$= -\sum_{i=1}^{N-1} k_i \int_{R^N_+} y_i \frac{\partial^2 w}{\partial y_i \partial y_N} v_1\, dy - \frac{1}{2}\sum_{i=1}^{N-1} k_i \int_{R^N_+} v_1 \frac{\partial w}{\partial y_N}\, dy - \frac{1}{2}\sum_{i=1}^{N-1} k_i \int_{\partial R^N_+} v_1 \frac{w'}{|y|} y_i^2\, dy'. \tag{3.21}$$

Finally,
$$\frac{1}{4}\sum_{i=1}^{N-1} k_i \int_{R^N_+} f'(w) \frac{w'}{|y|} y_i^2 y_N v_1\, dy = \frac{1}{4}\sum_{i=1}^{N-1} k_i \int_{R^N_+} y_i^2 \frac{\partial f(w)}{\partial y_N} v_1\, dy$$
$$= -\frac{1}{4}\sum_{i=1}^{N-1} k_i \int_{R^N_+} y_i^2 \left(\Delta \frac{\partial w}{\partial y_N} - \frac{\partial w}{\partial y_N}\right) v_1\, dy$$

$$= \frac{1}{4}\sum_{i=1}^{N-1} k_i \int_{\partial R^N_+} v_1 y_i^2 \frac{w'}{|y|}\, dy' - \frac{1}{4}\sum_{i=1}^{N-1} k_i \int_{R^N_+} \left[4y_i \frac{\partial v_1}{\partial y_i} + 2v_1\right] \frac{\partial w}{\partial y_N}\, dy$$

$$= \frac{1}{4}\sum_{i=1}^{N-1} k_i \int_{\partial R^N_+} v_1 \frac{w'}{|y|} y_i^2\, dy' + \sum_{i=1}^{N-1} k_i \int_{R^N_+} y_i \frac{\partial^2 w}{\partial y_i \partial y_N} v_1\, dy$$

$$+ \frac{\sum_{i=1}^{N-1} k_i}{2} \int_{R^N_+} v_1 \frac{\partial w}{\partial y_N}\, dy. \tag{3.22}$$

Combining (3.20), (3.21) and (3.22), we have

$$A_0 = \frac{1}{4}\sum_{i=1}^{N-1} k_i \int_{\partial R^N_+} (\Phi_0 - v_1)\frac{w'}{|y|} y_i^2\, dy'$$

$$= \frac{1}{8}\sum_{i,j=1}^{N-1} k_i k_j \int_{\partial R^N_+} \Phi_j \frac{\partial \Phi_i}{\partial y_N} dy'. \tag{3.23}$$

By symmetry, we have

$$\int_{\partial R^N_+} \Phi_i \frac{\partial \Phi_i}{\partial y_N}\, dy' = \int_{\partial R^N_+} \Phi_1 \frac{\partial \Phi_1}{\partial y_N}\, dy', \quad i = 1, ..., N-1,$$

$$\int_{\partial R^N_+} \Phi_k \frac{\partial \Phi_l}{\partial y_N}\, dy' = \int_{\partial R^N_+} \Phi_1 \frac{\partial \Phi_2}{\partial y_N}\, dy', \quad k, l = 1, ..., N-1, k \neq l. \tag{3.24}$$

Hence

$$A_0 = \frac{1}{8}\left(\sum_{i=1}^{N-1} k_i\right)^2 \int_{\partial R^N_+} \Phi_1 \frac{\partial \Phi_1}{\partial y_N}\, dy' + \frac{1}{8}\sum_{i\neq j} k_i k_j \int_{\partial R^N_+} \Phi_1 \frac{\partial(\Phi_2 - \Phi_1)}{\partial y_N}\, dy'$$

$$= c_2 (H(P))^2 + c_3 R(P), \tag{3.25}$$

where

$$c_2 = \frac{(N-1)^2}{8} \int_{\partial R^N_+} \Phi_1 \frac{\partial \Phi_1}{\partial y_N}\, dy', \quad c_3 = \frac{1}{8} \int_{\partial R^N_+} \Phi_1 \frac{\partial(\Phi_2 - \Phi_1)}{\partial y_N}\, dy'. \tag{3.26}$$

□

In summary, we have derived the following proposition.

Proposition 3.3. *Let $P \in \partial\Omega$ and $\tilde{w}_{\epsilon,P}$ be defined at (2.18). Then, for ϵ sufficiently small, we have*

$$J_\epsilon[\tilde{w}_{\epsilon,P}] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P) + \epsilon^2 [c_2 (H(P))^2 + c_3 R(P)] + o(\epsilon^2) \right], \tag{3.27}$$

where c_1, c_2, c_3 are the generic constants defined by (3.16) and (3.26), respectively.

4. The signs of c_1 and c_3

In this section, we study the constants c_1 and c_3. Even though we can not compute them explicitly, we can determine their signs.

We begin with c_1. Since w is radially symmetric, integration by parts gives

$$c_1 = \frac{1}{4} \int_{\partial R^N_+} \left[(w^{'})^2 + w^2 - 2F(w) \right] |y|^2 \, dy^{'}.$$

By Lemma 3.3 of [24],

$$c_1 = \frac{N-1}{4} \int_{R^N_+} \left[(w^{'})^2 + w^2 - 2F(w) \right] y_N \, dy^{'}$$

$$= \frac{N-1}{N+1} \int_{R^N_+} (w^{'}(|y|))^2 y_N \, dy > 0. \tag{4.1}$$

The sign of c_3 is more difficult to determine. To this end, we begin with the following lemma.

Lemma 4.1. *Consider the following eigenvalue problem:*

$$\begin{cases} \Delta\phi - \phi + f^{'}(w)\phi = \lambda\phi, & \phi \in H^2(R^N_+), \\ \frac{\partial\phi}{\partial y_N} = 0 \text{ on } \partial R^N_+. \end{cases} \tag{4.2}$$

Then we can arrange the eigenvalues in such a way that

$$\lambda_1 > 0 = \lambda_2 > \lambda_3 > ...,$$

where the eigenspace to λ_1 is spanned by a radially symmetric eigenfunction Ψ_1 which can be made positive. The eigenspace to $\lambda_2 = 0$ is $(N-1)$-dimensional and is spanned by $\frac{\partial w}{\partial y_j}$, $j = 1, ..., N-1$.

Proof: The fact that the eigenspace to $\lambda_2 = 0$ is spanned by $\frac{\partial w}{\partial y_j}$, $j = 1, ..., N-1$ follows from assumption (f2). The first eigenvalue λ_1 is called principal eigenvalue and it is a standard result that the corresponding eigenspace is spanned by a radially symmetric eigenfunction which can be made positive.

The fact that $\lambda_1 > \lambda_2 = 0$ follows from Proposition 1.3 of [2]. □

We define the following quadratic form:

$$Q[\phi] := \frac{\int_{R^N_+} (|\nabla\phi|^2 + \phi^2 - f'(w)\phi^2)\, dy}{\int_{R^N_+} \phi^2\, dy} \text{ for } \phi \in H^1(R^N_+), \phi \not\equiv 0. \tag{4.3}$$

Lemma 4.1 implies the following inequality.

Lemma 4.2. *We have*

$$-\lambda_3 = \inf_{\int_{R^N_+} \phi\Psi_1\, dy = \int_{R^N_+} \phi \frac{\partial w}{\partial y_j}\, dy = 0,\, j=1,...,N-1} Q[\phi] > 0. \tag{4.4}$$

Now we claim

Lemma 4.3. *We have $c_3 > 0$.*

Proof: Since Φ_i is even in y', we see that

$$\int_{R^N_+} (\Phi_1 - \Phi_2) \frac{\partial w}{\partial y_j}\, dy = 0, \quad j = 1, ..., N-1, \tag{4.5}$$

Since Ψ_1 is radially symmetric, we also get

$$\int_{R^N_+} (\Phi_1 - \Phi_2)\Psi_1\, dy = 0. \tag{4.6}$$

Now we compute

$$\int_{R^N_+} [|\nabla(\Phi_1 - \Phi_2)|^2 + |\Phi_1 - \Phi_2|^2 - f'(w)(\Phi_1 - \Phi_2)^2]\, dy$$

$$= -\int_{R^N_+} (\Phi_1 - \Phi_2) \frac{\partial(\Phi_1 - \Phi_2)}{\partial y_N}\, dy$$

$$= \int_{R^N_+} \Phi_1 \frac{\partial(\Phi_2 - \Phi_1)}{\partial y_N}\, dy + \int_{R^N_+} \Phi_2 \frac{\partial(\Phi_1 - \Phi_2)}{\partial y_N}\, dy.$$

By symmetry of Φ_1 and Φ_2, we see that

$$\int_{R^N_+} \Phi_1 \frac{\partial(\Phi_2 - \Phi_1)}{\partial y_N}\, dy + \int_{R^N_+} \Phi_2 \frac{\partial(\Phi_1 - \Phi_2)}{\partial y_N}\, dy$$

$$= 2\int_{R^N_+} \Phi_1 \frac{\partial(\Phi_2-\Phi_1)}{\partial y_N}\,dy = 16c_3. \tag{4.7}$$

By (4.5), (4.6) and Lemma 4.2, we have

$$16c_3 = (\int_{R^N_+} |\Phi_1-\Phi_2|^2\,dy\,)\ \ Q[\Phi_1-\Phi_2] > 0. \tag{4.8}$$

□

5. The asymptotic behavior of u_ϵ and $J_\epsilon[u_\epsilon]$

Let u_ϵ be a single boundary spike solution of (1.1) and P_ϵ be its local maximum point. In this section, we compute the energy of u_ϵ. The key observation is that by using $\tilde{w}_{\epsilon,P_\epsilon}$ as our approximating function, we just need to expand u_ϵ up to $O(\epsilon^\tau)$ for some $\tau > 1$. Now we choose $\tau = 1 + \frac{\sigma}{2}$.

We first prove the following theorem.

Theorem 5.1. *For ϵ sufficiently small, we have*

$$u_\epsilon = \tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau \phi_\epsilon, \tag{5.1}$$

where ϕ_ϵ satisfies

$$\|\phi_\epsilon\|_{L^\infty(\bar{\Omega})} + \epsilon^{-N}\int_\Omega (\epsilon^2|\nabla\phi_\epsilon|^2 + |\phi_\epsilon|^2) \le C. \tag{5.2}$$

Let us first assume that Theorem 5.1 holds. We then have

Lemma 5.2. *For ϵ sufficiently small, we have*

$$J_\epsilon[u_\epsilon] = J_\epsilon[\tilde{w}_{\epsilon,P}] + o(\epsilon^{N+2}). \tag{5.3}$$

Proof of Lemma 5.2: Note that both $\tilde{w}_{\epsilon,P_\epsilon}$ and ϕ_ϵ satisfy the Neumann boundary condition. So we have

$$J_\epsilon[u_\epsilon] = J_\epsilon[\tilde{w}_{\epsilon,P}]$$
$$+\epsilon^\tau \int_\Omega (\epsilon^2 \nabla\tilde{w}_{\epsilon,P}\nabla\phi_\epsilon + \tilde{w}_{\epsilon,P}\phi_\epsilon - f(\tilde{w}_{\epsilon,P})\phi_\epsilon)\,dx$$
$$+\frac{\epsilon^{2\tau}}{2}\left(\int_\Omega (\epsilon^2|\nabla\phi_\epsilon|^2 + |\phi_\epsilon|^2)\,dx - \int_\Omega f'(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon^2\,dx\right)$$
$$-\int_\Omega \left[F(\tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau\phi_\epsilon) - F(\tilde{w}_{\epsilon,P_\epsilon}) - \epsilon^\tau f(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon - \frac{\epsilon^{2\tau}}{2} f'(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon^2\right] dx.$$

By Theorem 5.1, the last two terms are $O(\epsilon^{N+2\tau})$. Now, integrating by parts, we obtain that

$$\epsilon^\tau \int_\Omega (\epsilon^2 \nabla \tilde{w}_{\epsilon,P} \nabla \phi_\epsilon + \tilde{w}_{\epsilon,P}\phi_\epsilon - f(\tilde{w}_{\epsilon,P})\phi_\epsilon)\, dx$$
$$= \epsilon^\tau \int_\Omega S_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}]\phi_\epsilon \, dx = O(\epsilon^{N+\tau+1+\sigma}).$$

This finishes the proof of Lemma 5.2. □

We are now ready to prove Theorem 5.1. The key step is the following lemma.

Lemma 5.3. *For ϵ sufficiently small, we have*

$$\|\phi_\epsilon\|_{L^\infty(\bar{\Omega})} \le C. \tag{5.4}$$

Proof: Recall

$$S_\epsilon[u] = \epsilon^2 \Delta u - u + f(u), \quad S_\epsilon^{'}[u](\phi) = \epsilon^2 \Delta \phi - \phi + f^{'}(u)\phi.$$

Then, substituting $u_\epsilon = \tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau \phi_\epsilon$ into equation (1.1), we see that ϕ_ϵ satisfies

$$\begin{cases} \epsilon^2 \Delta \phi_\epsilon - \phi_\epsilon + f^{'}(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon = -\epsilon^{-\tau} S_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}] + N_\epsilon[\phi_\epsilon] \text{ in } \Omega, \\ \frac{\partial \phi_\epsilon}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.5}$$

where

$$N_\epsilon[\phi_\epsilon] = -\epsilon^{-\tau}[f(\tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau \phi_\epsilon) - f(\tilde{w}_{\epsilon,P_\epsilon}) - \epsilon^\tau f^{'}(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon]. \tag{5.6}$$

From Lemma 2.2, we have

$$\epsilon^{-\tau} S_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}] = O(\epsilon^{\sigma/2}). \tag{5.7}$$

By the mean value theorem, we get

$$|N_\epsilon[\phi_\epsilon]| = o(1)|\phi_\epsilon|. \tag{5.8}$$

Now we can prove Lemma 5.2. Suppose not, that is there exists a sequence $\epsilon_k \to 0$ such that $\|\phi_{\epsilon_k}\|_{L^\infty(\bar{\Omega})} \to +\infty$. For simplicity of notation, we still denote ϵ_k as ϵ. Set

$$M_\epsilon = \|\phi_\epsilon\|_{L^\infty(\bar{\Omega})} \to +\infty.$$

Let $M_\epsilon = |\phi_\epsilon(x_\epsilon)|$, where $x_\epsilon \in \bar{\Omega}$. Without loss of generality, we may assume that x_ϵ is a maximum point of ϕ_ϵ.

We proceed in two claims.

Claim 1: $\frac{|x_\epsilon - P_\epsilon|}{\epsilon} \le C$.

In fact, suppose not. That is $\frac{|x_\epsilon - P_\epsilon|}{\epsilon} \to +\infty$. Then $-1 + f'(\tilde{w}_{\epsilon,P_\epsilon}(x_\epsilon)) \le -\frac{1}{4}$ for ϵ small. Since $\frac{\partial \phi_\epsilon}{\partial \nu} = 0$, by the Hopf boundary Lemma, $x_\epsilon \notin \partial\Omega$. So $x_\epsilon \in \Omega$, which implies that

$$\Delta \phi_\epsilon(x_\epsilon) \le 0.$$

From (5.5) we deduce that

$$(1 - f'(\tilde{w}_{\epsilon,P_\epsilon}(x_\epsilon)))M_\epsilon + o(1)M_\epsilon + O(\epsilon^{\tau-1}) \le 0$$

and hence M_ϵ is bounded. A contradiction.

This proves Claim 1.

Let

$$\hat{\phi}_\epsilon(y) = \frac{\phi_\epsilon(x)}{M_\epsilon}\chi(x - P_\epsilon), \quad y = T_\epsilon(x), \tag{5.9}$$

where $y = T_\epsilon(x)$ is given in (2.3) (replacing P by P_ϵ).

Claim 2: $\hat{\phi}_\epsilon(y) \to 0$ in $C^1_{\text{loc}}(R^N_+)$ as $\epsilon \to 0$.

In fact, from the equation for $\hat{\phi}_\epsilon$, we see that as $\epsilon \to 0$, $\hat{\phi}_\epsilon \to \hat{\phi}_0$ which satisfies

$$\Delta \hat{\phi}_0 - \hat{\phi}_0 + f'(w)\hat{\phi}_0 = 0, \ |\hat{\phi}_0| \le 1 \text{ in } R^N_+,$$

$$\frac{\partial \hat{\phi}_0}{\partial y_N} = 0 \ \text{ on } \partial R^N_+.$$

By the nondegeneracy of w (see (2.14)), there exist $N-1$ constants $a_1, ..., a_{N-1}$ such that

$$\hat{\phi}_0 = \sum_{j=1}^{N-1} a_j \frac{\partial w}{\partial y_j}. \tag{5.10}$$

On the other hand, we know that $\nabla_{x_k} u_\epsilon(P_\epsilon) = 0$, $k = 1, ..., N-1$ and hence

$$0 = \nabla_{x_k}(\tilde{w}_{\epsilon,P_\epsilon}(P_\epsilon) + \epsilon^\tau \phi_\epsilon(P_\epsilon))$$

$$= O(\epsilon^2) + \nabla_{x_k}\left(w\left(\frac{x - P_\epsilon}{\epsilon}\right) - \epsilon v_1 \chi - \epsilon^2(v_2 + v_3)\chi\right) + \epsilon^{\tau-1} M_\epsilon \nabla_{y_k} \hat{\phi}_\epsilon(0)$$

$$= O(\epsilon) + \epsilon^{\tau-1} M_\epsilon \nabla_{y_k} \hat{\phi}_\epsilon(0).$$

(Note that $\nabla_{y_k} v_1(0) = \nabla_{y_k} v_2(0) = 0$.) Thus we have $\nabla_{y_k}\hat{\phi}_\epsilon(0) \to 0$ which shows that $\nabla_{y_k}\hat{\phi}_0(0) = 0$, $k = 1, ..., N-1$. This implies that

$$\nabla_{y_k}\left(\sum_{j=1}^{N-1} a_j \frac{\partial w}{\partial y_j}\right)\Bigg|_{y=0} = 0, \quad k = 1, ..., N-1.$$

Thus $a_1 = ... = a_{N-1} = 0$.

This proves Claim 2.

Lemma 5.3 now follows from Claim 1 and Claim 2: Let $y_\epsilon = \frac{x_\epsilon - P_\epsilon}{\epsilon}$, then by Claim 1, $|y_\epsilon| \le C$. So we may assume that $y_\epsilon \to y_0$ as $\epsilon \to 0$. Since $\hat{\phi}_\epsilon(y_\epsilon) = 1$, we have $\hat{\phi}_0(y_0) = 1$ which contradicts Claim 2.

□

Theorem 5.1 now follows from Lemma 5.3: In fact, multiplying (5.5) by ϕ_ϵ and integrating over Ω, we obtain

$$\begin{aligned}
&\epsilon^2 \int_\Omega |\nabla\phi_\epsilon|^2\,dx + \int_\Omega |\phi_\epsilon|^2\,dx \\
&= \int_\Omega f'(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon\,dx - \int_\Omega N_\epsilon[\phi_\epsilon]\phi_\epsilon\,dx + \epsilon^{-\tau}\int_\Omega S_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}]\phi_\epsilon\,dx \\
&\le C\epsilon^N + o(1)\int_\Omega |\phi_\epsilon|^2\,dx.
\end{aligned}$$

This finishes the proof of Theorem 5.1.

□

6. The proofs of Theorem 1.1, Theorem 1.2, and Corollary 1.3

Theorem 1.1 follows from Lemma 5.2 and Proposition 3.2.

To prove Theorem 1.2, we follow the proof of Theorem 1.1: first we note that

$$S_\epsilon[\sum_{j=1}^{K} \tilde{w}_{\epsilon,P_j^\epsilon}] = \sum_{j=1}^{K} S_\epsilon[\tilde{w}_{\epsilon,P_j^\epsilon}] + O(e^{-\delta/\epsilon}) \tag{6.1}$$

for some $\delta > 0$, since $\min_{i\neq j}|P_i^\epsilon - P_j^\epsilon| \ge \delta$. Then we decompose

$$u_\epsilon = \sum_{j=1}^{K} \tilde{w}_{\epsilon,P_j^\epsilon} + \epsilon^\tau \phi_\epsilon$$

and show that $\|\phi_\epsilon\|_{L^\infty(\bar{\Omega})} \le C$. The rest of the proof is exactly the same.

Finally, we prove Corollary 1.3.

Proof of Corollary 1.3: Let u_ϵ be a least energy solution of (1.1). By Theorem 1.1, we have

$$\begin{aligned} c_\epsilon &:= J_\epsilon[u_\epsilon] \\ &= \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 (c_2 (H(P_\epsilon))^2 + c_3 R(P_\epsilon)) + o(\epsilon^2) \right]. \end{aligned} \tag{6.2}$$

On the other hand, let

$$\beta(t) = J_\epsilon[t \tilde{w}_{\epsilon,P}], \quad t > 0, \tag{6.3}$$

where $\tilde{w}_{\epsilon,P}$ is given by (2.18).

By Lemma 3.1 of [24],

$$c_\epsilon \le \max_{t>0} \beta(t). \tag{6.4}$$

By assumption (f3) (see (3.16) of [24]), there exists a unique $t = t_{\epsilon,P}$ such that

$$\beta'(t_{\epsilon,P}) = 0, \quad \beta(t_{\epsilon,P}) = \max_{t>0} \beta(t).$$

Note that

$$\begin{aligned} \beta'(1) &= \int_\Omega [\epsilon^2 \nabla \tilde{w}_{\epsilon,P} \nabla \tilde{w}_{\epsilon,P} + \tilde{w}_{\epsilon,P}^2 - f(\tilde{w}_{\epsilon,P}) \tilde{w}_{\epsilon,P}]\, dx \\ &= \int_\Omega S_\epsilon[\tilde{w}_{\epsilon,P}] \tilde{w}_{\epsilon,P}\, dx = O(\epsilon^{N+1+\sigma}). \end{aligned}$$

Similar to (3.16) of [24], one can show that

$$t_{\epsilon,P} = 1 + O(\epsilon^{1+\sigma}). \tag{6.5}$$

Then

$$\begin{aligned} \beta(t_{\epsilon,P}) &= \beta(1) + \beta'(1)(t_{\epsilon,P} - 1) + O(\epsilon^N |t_{\epsilon,P} - 1|^2) \\ &= \beta(1) + o(\epsilon^{N+2}) \end{aligned}$$

which implies that

$$\begin{aligned} c_\epsilon &\le \max_{t>0} \beta(t) = J_\epsilon[t_{\epsilon,P} \tilde{w}_{\epsilon,P}] = J_\epsilon[\tilde{w}_{\epsilon,P}] + o(\epsilon^{N+2}) \\ &\le \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P) + \epsilon^2 (c_2 (H(P))^2 + c_3 R(P)) + o(\epsilon^2) \right] \end{aligned} \tag{6.6}$$

for any $P \in \partial\Omega$.

Now we take $P = Q_0$ such that

$$H(Q_0) = \max_{P \in \partial\Omega} H(P), R(Q_0) = \min_{Q \in \partial\Omega, H(Q) = \max_{P \in \partial\Omega} H(P)} R(Q).$$

Comparing (6.6) with (6.2), we arrive at

$$c_1 H(Q_0) - \epsilon[c_2(H(Q_0))^2 + c_3 R(Q_0)] + o(\epsilon)$$

$$\leq c_1 H(P_\epsilon) - \epsilon[c_2(H(P_\epsilon))^2 + c_3 R(P_\epsilon)] + o(\epsilon).$$

Since $c_1 > 0, c_3 > 0$, (the sign of c_2 is not important), we conclude that

$$H(P_\epsilon) \to \max_{P \in \partial\Omega} H(P), \quad R(P_\epsilon) \to \min_{Q \in \partial\Omega, H(Q) = \max_{P \in \partial\Omega} H(P)} R(Q)$$

as $\epsilon \to 0$.

□

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