EXISTENCE AND STABILITY ANALYSIS OF ASYMMETRIC PATTERNS FOR THE GIERER-MEINHARDT SYSTEM

JUNCHENG WEI AND MATTHIAS WINTER

ABSTRACT. In this paper, we rigorously prove the existence and stability of K-peaked asymmetric patterns for the Gierer-Meinhardt system in a two dimensional domain which are far from spatial homogeneity. We show that given any positive integers k_1, $k_2 \geq 1$ with $k_1 + k_2 = K$, there are asymmetric patterns with k_1 large peaks and k_2 small peaks. Most of these asymmetric patterns are shown to be unstable. However, in a narrow range of parameters, asymmetric patterns may be stable (in contrast to the one-dimensional case).
RÉSUMÉ. Nous prouvons l'existence et la stabilité de les structures asymétriques pour le systéme de Gierer-Meinhardt dans un domaine ouvert deux-dimensionnel qui sont distantes de la homogénéité spatiale. Pour $k_2 \geq 1$, $k_1 \geq 1$ il y a des structures avec k_1 grands et k_2 petits pics. La plupart des solutions asymétriques sont instables. Pour un région petit des paramètres les solutions asymétriques pouvons ětre stables (en contraste d'une dimension).

1. INTRODUCTION

Turing in his pioneering work in 1952 [30] proposed that a patterned distribution of two biochemical substances, called the morphogens, could trigger the emergence of a cell structure. He also gives the following explanation for the formation of the morphogenetic pattern: He assumes that one of the morphogens, the activator, diffuses slowly and the other, the inhibitor, diffuses much faster. In the mathematical framework of a coupled system of reaction-diffusion equations with very different diffusion coefficients he shows by linear stability analysis that the homogeneous state can be unstable. In particular, a small perturbation of spatially homogeneous initial data can evolve to a stable spatially complex pattern of the morphogens.

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Since the work of Turing, a lot of models have been proposed and analyzed to explore this phenomenon, which is now called Turing instability, and its implications for the understanding of various patterns more fully. One of the most famous of these models is the Gierer-Meinhardt system [11], [22]. In two dimensions, after rescaling and considering a special case, it is as follows:

$$(GM) \qquad \begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^2}{H}, \quad A > 0 \quad \text{in } \Omega, \\ \tau H_t = D\Delta H - H + A^2, \quad H > 0 \quad \text{in } \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The unknowns $A = A(x,t)$ and $H = H(x,t)$ represent the concentrations of two morphogens called activator and inhibitor, respectively, at a point $x \in \Omega \subset R^2$ and at a time $t > 0$, respectively; $\Delta := \sum_{j=1}^2 \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in R^2; Ω is a bounded and smooth domain in R^2; $\nu(x)$ is the outer normal at $x \in \partial\Omega$. Throughout this paper, we assume that

$$\epsilon << 1, \ \epsilon \text{ does not depend on } \ x, \ t,$$

$$\tau \geq 0 \ \text{ does not depend on } \ x, \ t, \ \text{ or } \ \epsilon,$$

$$D > 0 \ \text{ does not depend on } \ x, \ t, \ \text{ but it depends on } \ \epsilon.$$

In this paper, we further define

$$\beta^2 = \frac{1}{D}, \quad \eta_\epsilon = \frac{\beta^2 |\Omega|}{2\pi} \log \frac{\sqrt{|\Omega|}}{\epsilon}, \tag{1.1}$$

where $|\Omega|$ denotes the area of Ω, and assume that

$$\lim_{\epsilon \to 0} \eta_\epsilon = \eta_0 \in (0, +\infty). \tag{1.2}$$

Note that (1.2) implies that

$$D \to \infty \quad \text{ and } \quad \beta \to 0 \quad \text{ as } \epsilon \to 0.$$

More precisely, we have

$$D \sim \frac{|\Omega| \log \frac{\sqrt{|\Omega|}}{\epsilon}}{2\pi\eta_0} \tag{1.3}$$

and

$$\beta^2 \sim \frac{2\pi\eta_0}{|\Omega| \log \frac{\sqrt{|\Omega|}}{\epsilon}}. \tag{1.4}$$

Numerical studies by Meinhardt [22] and more recently by Holloway [14] and McInerney [20] have revealed that when ϵ is small and D is finite, (GM) seems to have stable stationary states with the property that the activator concentration is localized in K peaks which are located near certain K points in $\overline{\Omega}$. Moreover, as $\epsilon \to 0$ the pattern exhibits a *"point condensation phenomenon"*. By this we mean that these peaks become narrower and narrower and eventually shrink to the set of points itself. In fact, their spatial extension is of the order $O(\epsilon)$. Furthermore, the maximum value of the inhibitor concentration diverges to $+\infty$. Numerically, it has been observed that these patterns are stable.

In [42], we considered the existence and stability of *symmetric* $K-$peaked solutions of the following stationary Gierer-Meinhardt system:

$$\begin{cases} \epsilon^2 \Delta A - A + \frac{A^2}{H} = 0, \quad A > 0 \quad \text{in } \Omega, \\ D\Delta H - H + A^2 = 0, \quad H > 0 \quad \text{in } \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.5}$$

in the case $D = D(\epsilon) \to \infty$ as $\epsilon \to 0$ and for ϵ small enough (which is called "weak coupling case").

A $K-$peaked solution (A_ϵ, H_ϵ) of (1.5) is assumed to take the following form:

$$A_\epsilon(x) \sim \sum_{j=1}^{N} \xi_{\epsilon,j} w(\frac{x - P_j^\epsilon}{\epsilon}), \quad H_\epsilon(P_j^\epsilon) \sim \xi_{\epsilon,j}, \tag{1.6}$$

where $\xi_{\epsilon,j}$ is the height of the peak at the location P_j^ϵ, $j = 1, ..., K$, and w is the unique solution of the problem

$$\begin{cases} \Delta w - w + w^2 = 0, \quad w > 0 \quad \text{in } R^2, \\ w(0) = \max_{y \in R^2} w(y), \quad w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{1.7}$$

For existence and uniqueness of the solutions of (1.7) we refer to [18]. We also recall that

$$w(y) \sim |y|^{-1/2} e^{-|y|} \quad \text{as } |y| \to \infty. \tag{1.8}$$

In [42], we assumed that the $K-$peaked solution is asymptotically symmetric, i.e., as $\epsilon \to 0$,

$$\lim_{\epsilon\to 0} \frac{\xi_{\epsilon,j}}{\xi_{\epsilon,1}} = 1, \quad j = 2, ..., K. \tag{1.9}$$

Under the condition (1.9), we showed the existence of symmetric $K-$peaked solutions whose peaks converge to a nondegenerate critical point of a functional involving a certain Green's function and its derivatives. For the stability, we proved that there are stability thresholds

$$D_1(\epsilon) > D_2(\epsilon) > D_3(\epsilon) > \ldots > D_N(\epsilon) > \ldots$$

such that for $D < D_K(\epsilon)$ the symmetric K-peaked solution is stable and for $D > D_K(\epsilon)$ the symmetric N-peaked solution is unstable if ϵ is small enough. Furthermore, we showed that

$$D_K(\epsilon) \sim \frac{|\Omega|}{2\pi K} \log \frac{\sqrt{|\Omega|}}{\epsilon} \quad \text{as } \epsilon \to 0.$$

Naturally, the following questions can be raised:

Question: Are there solutions which are not symmetric (i.e, (1.9) does not hold)? If yes, are they stable? Can we characterize all asymmetric solutions?

In this paper we solve these questions affirmatively. We show that the heights $(\xi_{\epsilon,1}, ..., \xi_{\epsilon,K})$ must satisfy a certain nonlinear algebraic system which can be solved explicitly (Sections 2 and 3). As a result, we show that the asymmetric patterns are generated by peaks of exactly two different heights. We then give a rigorous construction of asymmetric K-peaked stationary states by using the powerful method of Liapunov-Schmidt reduction. This enables us to reduce the infinite-dimensional problem of finding an equilibrium state of (GM) to the finite-dimensional one of locating the K points at which the peaks concentrate. We give a sufficient condition for these points in terms of a Green's function and its derivatives.

Concerning stability, one has to study the eigenvalues of the order $O(1)$, which are called "large eigenvalues", and the eigenvalues of the order $o(1)$, which are called "small eigenvalues", separately. We show that the small eigenvalues are related to a Green's function and its derivatives. Suppose that these small eigenvalues all have negative real part. We then show that stable asymmetric $K-$peaked solutions exist only in a very narrow range of D, namely for

$$\frac{1}{2\pi K}\log\frac{\sqrt{|\Omega|}}{\epsilon} < \frac{D}{|\Omega|} < \frac{1}{4\pi\sqrt{k_1k_2}}\log\frac{\sqrt{|\Omega|}}{\epsilon} \tag{1.10}$$

and ϵ small enough, where k_1 and k_2 are two integers satisfying $k_1+k_2=K, k_1\geq 1, k_2\geq 1$.

We now describe the results of the paper in detail.

Let $K\geq 2$ be a positive integer. Let k_1, $k_2\geq 1$ be two integers such that

$$k_1+k_2=K. \tag{1.11}$$

Let η_0 (defined in (1.2)) be such that

$$\eta_0>2\sqrt{k_1k_2}. \tag{1.12}$$

Set

$$\rho_+=\frac{2k_2+\eta_0+\sqrt{\eta_0^2-4k_1k_2}}{2\eta_0(\eta_0+K)},\quad \rho_-=\frac{2k_2+\eta_0-\sqrt{\eta_0^2-4k_1k_2}}{2\eta_0(\eta_0+K)}, \tag{1.13}$$

$$\eta_+=\frac{2k_1+\eta_0-\sqrt{\eta_0^2-4k_1k_2}}{2\eta_0(\eta_0+K)},\quad \eta_-=\frac{2k_1+\eta_0+\sqrt{\eta_0^2-4k_1k_2}}{2\eta_0(\eta_0+K)}. \tag{1.14}$$

Note that

$$\rho_++\eta_+=\frac{1}{\eta_0},\quad \rho_-+\eta_-=\frac{1}{\eta_0}. \tag{1.15}$$

Let $(\rho,\eta)=(\rho_+,\eta_+)$ or $(\rho,\eta)=(\rho_-,\eta_-)$. We drop "$\pm$" if there is no confusion.

Let $(\hat{\xi}_1,\dots,\hat{\xi}_K)\in R_+^K$ be such that

$$\hat{\xi}_j\in\{\rho,\eta\},\text{ and the number of }\rho\text{'s in }(\hat{\xi}_1,\dots,\hat{\xi}_K)\text{ is }k_1. \tag{1.16}$$

Then there are k_2 η's in $(\hat{\xi}_1,\dots,\hat{\xi}_K)$.

For $\delta>0$ and δ small enough we define

$$\Lambda_\delta=\{\mathbf{P}=(P_1,P_2,\dots,P_K)\in\Omega^K:|P_i-P_j|>4\delta\text{ for }i\neq j$$
$$\text{and }d(P_i,\partial\Omega)>4\delta\text{ for }i=1,2\dots,K\}, \tag{1.17}$$

where

$$P_i = (P_{i,1}, P_{i,2}) \quad \text{for } i = 1, 2, \dots, K.$$

Let $G_0(x, \xi)$ be the Green's function

$$\begin{cases} \Delta G_0(x,\xi) - \frac{1}{|\Omega|} + \delta_\xi(x) = 0 & \text{in } \Omega, \\ \int_\Omega G_0(x,\xi)\, dx = 0, \\ \dfrac{\partial G_0(x,\xi)}{\partial \nu_x} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.18}$$

and let

$$H_0(x,\xi) = \frac{1}{2\pi} \log \frac{1}{|x-\xi|} - G_0(x,\xi) \tag{1.19}$$

be the regular part of $G_0(x, \xi)$. Here $\delta_\xi(x)$ means the Dirac measure at $x = \xi$.

For $\mathbf{P} \in \overline{\Lambda_\delta}$ we define

$$F(\mathbf{P}) = \sum_{k=1}^{K} H_0(P_k, P_k)\hat{\xi}_k^4 - \sum_{i,j,=1,\dots,K, i\neq j} G_0(P_i, P_j)\hat{\xi}_i^2 \hat{\xi}_j^2 \tag{1.20}$$

and

$$M(\mathbf{P}) = \nabla^2_{\mathbf{P}} F(\mathbf{P}). \tag{1.21}$$

Note that $F(\mathbf{P}) \in C^\infty(\overline{\Lambda_\delta})$.

Then we have our first theorem, which is on the existence of asymmetric $K-$peaked solutions.

Theorem 1.1. *Let $K \geq 2$ be a positive integer. Let $k_1, k_2 \geq 1$ be two integers such that $k_1 + k_2 = K$. Let*

$$\beta^2 = \frac{1}{D}, \quad \eta_\epsilon = \frac{\beta^2 |\Omega|}{2\pi} \log \frac{\sqrt{|\Omega|}}{\epsilon},$$

where $|\Omega|$ denotes the area of Ω, Assume that (1.2) and (1.12) hold.

Assume that

$$\text{(T1)} \qquad \eta_0 \neq K$$

and let

(T2) $\mathbf{P}^0 = (P_1^0, P_2^0, \dots, P_K^0) \in \overline{\Lambda_\delta}$ *be a nondegenerate critical point of* $F(\mathbf{P})$ *(defined by (1.20)).*

Then for ϵ sufficiently small problem (1.5) has a solution (A_ϵ, H_ϵ) with the following properties:

(1) $A_\epsilon(x) = \sum_{j=1}^K \xi_{\epsilon,j}(w(\frac{x-P_j^\epsilon}{\epsilon}) + O(\frac{1}{D}))$ uniformly for $x \in \bar{\Omega}$, where w is the unique solution of (1.7) and

$$\xi_{\epsilon,j} = \xi_\epsilon \hat{\xi}_{\epsilon,j}, \quad \xi_\epsilon = \frac{|\Omega|}{\epsilon^2 \int_{R^2} w^2}. \tag{1.22}$$

Further, $(\hat{\xi}_{\epsilon,1}, ..., \hat{\xi}_{\epsilon,K}) \to (\hat{\xi}_1, ..., \hat{\xi}_K)$ which is given by (1.16).

(2) $H_\epsilon(P_j^\epsilon) = \xi_{\epsilon,j}(1 + \frac{1}{D})$ for $j = 1, ..., K$.

(3) $P_j^\epsilon \to P_j^0$ as $\epsilon \to 0$ for $j = 1, ..., K$.

Remark:

1.1). Condition (T1) of Theorem 1.1 is a technical condition which will be used in the Liapunov-Schmidt reduction process. See Lemma 7.2.

Next we study the stability or instability of the asymmetric K-peaked solutions constructed in Theorem 1.1.

Theorem 1.2. *Assume that (1.2) and (1.12) hold. Further, assume that (T1) and (T2) of Theorem 1.1 hold and let (A_ϵ, H_ϵ) be the $K-$peaked solutions constructed in Theorem 1.1 for ϵ sufficiently small, whose peaks converge to $\mathbf{P}^0 \in \overline{\Lambda_\delta}$. Further assume that*

$$(*) \qquad \mathbf{P}^0 \text{ is a nondegenerate local maximum point of } F(\mathbf{P}).$$

Then we have:

*(a) (**Stability**)*

Assume that

$$2\sqrt{k_1 k_2} < \eta_0 < K \tag{1.23}$$

and

$$k_1 > k_2, \quad (\rho, \eta) = (\rho_+, \eta_+).$$

Then, for τ small enough, (A_ϵ, H_ϵ) is linearly stable.

*(b) (**Instability**)*

Assume that either

$$\eta_0 > K$$

or

$$\tau \text{ is large enough.}$$

Then (A_ϵ, H_ϵ) *is linearly unstable.*

The condition on the locations of $\mathbf{P}^0$ is not so severe. For any bounded smooth domain Ω, the functional $F(\mathbf{P})$ always admits a global maximum at some $\mathbf{P}^0 \in \overline{\Lambda_\delta}$. In fact, this can be seen very easily: if $|P_i - P_j|$ or $d(P_i, \partial\Omega)$ goes to 0, then $F(\mathbf{P})$ goes to $-\infty$. (Note that $H(P_i, P_i) \to -\infty$ as $d(P_i, \partial\Omega) \to 0$.) This global maximum point $\mathbf{P}^0$ is a critical point of $F(\mathbf{P})$. If $\mathbf{P}^0$ is also a nondegenerate critical point of $F(\mathbf{P})$ which should be a generic condition, then the matrix $M(\mathbf{P}^0)$ has only negative eigenvalues. It is an interesting question to numerically compute the critical points of $F(\mathbf{P})$. For recent progress in this direction see [21].

Let us now compare the results about existence and stability of asymmetric patterns in R^2 to those in R^1.

In R^1, we assume that $\Omega = (-1, 1)$. In 1986, I. Takagi [29] first showed the existence of symmetric $K-$peaked solutions with spikes centered at

$$x_j = -1 + \frac{2j-1}{N}, \quad j = 1, \ldots, N,$$

for $\epsilon << 1$, $\frac{\epsilon}{\sqrt{D}} << 1$.

Such solutions are obtained from a single spike by symmetric reflection. His main idea is to use symmetry and the implicit function theorem.

Using matched asymptotic analysis, D. Iron, M. Ward, and the first author [16] studied the stability of the symmetric K-peaked solutions for τ small and showed (formally) that there exists a sequence of numbers $D_1 > D_2 > ... > D_K > ...$ such that for $\epsilon << 1$: if $D < D_K$, the symmetric K-peaked solutions are stable, while for $D > D_K$, the symmetric K-peaked solutions are unstable.

M. Ward and the first author in [32] showed (formally) that for $D < D_K$, problem (1.5) has asymmetric $K-$peaked solutions. Such asymmetric solutions are generated by two types of spikes – called type **A** and type **B**, respectively. Type **A** and type **B** spikes have different heights and for any given order

ABAABBB...ABBBA...B

there is a corresponding $K-$peaked solution. The stability of such asymmetric $K-$peaked solutions is studied also in [32], through a formal approach.

Later, in [41], by using the Liapunov-Schmidt reduction method, a rigorous study of the existence and stability of both symmetric and asymmetric solutions is given. It is proved that the stability and existence can be reduced to the study of two $K \times K$ matrices. The results of [16] and [32] are then rigorously established.

By using a different approach (geometric singular perturbation method), Doelman, Kaper and van der Ploeg [10] also established the existence of asymmetric patterns for D sufficiently small (i.e., for fixed D the domain is sufficiently large). Moreover, some other asymmetric patterns are also discovered in [10].

Though it has not been proved rigorously, it is a numerical observation (by studying the two matrices of [32], [41]) that asymmetric patterns are all unstable in R^1.

In R^2, we can completely characterize the heights and thus the possible types of asymmetric patterns: asymmetric patterns are generated by exactly **two** different heights. (The reason behind this is unclear.) Furthermore, asymmetric patterns can be **stable**, even though the stability region given in Theorem 1.2 is rather narrow. In most cases, asymmetric patterns are unstable.

In terms of the heights, the results in R^2 are more explicit than R^1. However, the characterization in R^1 is the same.

Another remark is that in R^2, by our analysis of the algebraic system of the heights, as D decreases (e.g., $D = 1$), asymmetric patterns disappear. This is in contrast to the R^1 case [10], [32].

We now comment on some other related work.

Generally speaking, system (1.5) is quite difficult to solve since it does neither have a variational structure nor a priori estimates. One way to study (1.5) is to examine the so-called *shadow system*. Namely, we let $D \to +\infty$

first. It is known (see [17], [26], [28]) that the study of the shadow system amounts to the study of the following single equation for $p = 2$:

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0, \quad u > 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.24}$$

Equation (1.24) has a variational structure and has been studied by numerous authors. It is known that equation (1.24) has both boundary spike solutions and interior spike solutions. For boundary spike solutions, see [1], [6], [12], [24], [25], [26], [38], and the references therein. For interior spike solutions, see [2], [7], [13] and the references therein. For stability of spike solutions, see [1], [3], [5], [15], [27], [34], [36]. For dynamics we refer to [4].

Finally, we remark that some of the results of Theorem 1.1 and Theorem 1.2 may be extended to the following generalized Gierer-Meinhardt system

$$\text{(Generalized GM)} \qquad \begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^p}{H^q}, \quad A > 0 \quad \text{in } \Omega, \\ \tau H_t = D\Delta H - H + \frac{A^r}{H^s}, \quad H > 0 \quad \text{in } \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where the exponents (p, q, r, s) satisfy the following conditions

$$p > 1, \quad q > 0, \quad r > 0, \quad s \geq 0, \quad \frac{qr}{(p-1)(s+1)} > 1.$$

For example, the existence result Theorem 1.1 can be proved for the generalized Gierer-Meinhardt system without any technical difficulty. For the stability result, Theorem 1.2, there should be some restrictions on the (p, q, r, s). (The results in [5], [35], and [43] concerning stability of nonlocal eigenvalue problems in the general case may be useful.) We shall leave this to further investigations.

Other work on concentrated solutions for reaction-diffusion systems includes [8], [29], [31], and the survey [23].

To simplify our notation, we use $e.s.t.$ to denote exponentially small terms in the corresponding norms, more precisely, $e.s.t. = O(e^{-\delta/\epsilon})$ as $\epsilon \to 0$, where δ is defined in (1.17). Throughout the paper $C > 0$ is a generic constant which is independent of ϵ and may change from line to line. We always assume that $\mathbf{P}$, $\mathbf{P}^0 \in \overline{\Lambda_\delta}$, where $\overline{\Lambda_\delta}$ is defined in (1.17) and that $|\mathbf{P} - \mathbf{P}^0| < 4\delta$.

The structure of the paper is as follows:

In Section 2, we derive an algebraic system for the heights of the peaks.

In Section 3, we completely solve the nonlinear algebraic system for the heights.

In Section 4, we study some nonlocal eigenvalue problems in the whole R^2, which will be used in Section 7.

In Section 5, we start the existence proof by reducing the problem to finite dimensions.

In Section 6, we complete the existence proof by solving the reduced problem.

In Section 7, we use the results of Section 4 to study the stability of large eigenvalues.

Finally, in Section 8, we study the small eigenvalues.

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2. Preliminaries I: A system for the heights of the peaks

In this section we calculate the heights of the peaks which are needed in the sections below. It is found that the heights depend on the number of peaks but not on their locations. This is a leading order asymptotic statement that is valid for $\epsilon \to 0$ and $D \to \infty$.

For $\beta > 0$ let $G_\beta(x,\xi)$ be the Green's function given by

$$\begin{cases} \Delta G_\beta - \beta^2 G_\beta + \delta_\xi = 0 & \text{in } \Omega, \\ \dfrac{\partial G_\beta}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Recall that $\beta^2 = \frac{1}{D}$ and therefore $\beta \sim \frac{1}{\sqrt{\log \frac{1}{\epsilon}}}$. Let $G_0(x,\xi)$ be the Green's function defined in (1.18).

In Section 2 of [42] we have derived a relation between G_0 and G_β as follows:

$$G_\beta(x,\xi) = \frac{\beta^{-2}}{|\Omega|} + G_0(x,\xi) + O(\beta^2) \tag{2.2}$$

in the operator norm of $L^2\Omega) \to H^2(\Omega)$. (Note that the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$ is compact.)

We define cut-off functions as follows: Let $\mathbf{P} \in \overline{\Lambda_\delta}$. Introduce

$$\chi_{\epsilon,P_j}(x) = \chi\left(\frac{x-P_j}{\delta}\right), \qquad x \in \Omega, \quad j = 1,\dots,K, \tag{2.3}$$

where χ is a smooth cut-off function which is equal to 1 in $B_1(0)$ and equal to 0 in $R^2 \setminus \overline{B_2(0)}$, where for $r > 0, x \in R^2$ we set $B_r(x) = \{y \in R^2 \,:\, |y| \le r\}$.

Let us assume the following ansatz for a multiple-spike solution (A_ϵ, H_ϵ) of (1.5):

$$\begin{cases} A_\epsilon \sim \sum_{i=1}^K \xi_{\epsilon,i} w(\frac{x-P_i^\epsilon}{\epsilon})\chi_{\epsilon,P_i}(x), \\ H_\epsilon(P_i^\epsilon) \sim \xi_{\epsilon,i}, \end{cases} \tag{2.4}$$

where w is the unique solution of (1.7), $\xi_{\epsilon,i}, i = 1,...,K$ are the heights of the peaks, to be determined later, and $\mathbf{P}^\epsilon = (P_1^\epsilon,...,P_K^\epsilon) \in \overline{\Lambda_\delta}$ are the locations of the peaks.

Then we can make the following calculations.

From the equation for H_ϵ,

$$\Delta H_\epsilon - \beta^2 H_\epsilon + \beta^2 A_\epsilon^2 = 0,$$

we get, using (2.2),

$$H_\epsilon(P_i^\epsilon) = \int_\Omega G_\beta(P_i^\epsilon,\xi)\beta^2 A_\epsilon^2(\xi)\, d\xi$$

$$= \int_\Omega \left(\frac{1}{|\Omega|} + \beta^2 G_0(P_i^\epsilon,\xi) + O(\beta^4)\right)\left(\sum_{j=1}^K \xi_{\epsilon,j}^2 w^2(\frac{\xi-P_j^\epsilon}{\epsilon})\chi_{\epsilon,P_j}(\xi)\right)\, d\xi.$$

Thus

$$\xi_{\epsilon,i} = \xi_{\epsilon,i}^2 \frac{\epsilon^2}{|\Omega|}\int_{R^2} w^2(y)\,dy + \xi_{\epsilon,i}^2\beta^2\int_\Omega G_0(P_i^\epsilon,\xi)w^2(\frac{\xi-P_i^\epsilon}{\epsilon})\chi_{\epsilon,P_i}(\xi)\,d\xi$$

$$+\sum_{j\neq i}\left(\frac{1}{|\Omega|} + \beta^2 G_0(P_i^\epsilon,P_j^\epsilon)\right)\xi_{\epsilon,j}^2\epsilon^2\int_{R^2} w^2(y)\,dy + \sum_{j=1}^K \xi_{\epsilon,j}^2(O(\beta^2\epsilon^4) + O(\beta^4\epsilon^2)). \tag{2.5}$$

Here we have used that for $j \neq i$

$$\int_\Omega G_0(P_i^\epsilon,\xi)w^2(\frac{\xi-P_j^\epsilon}{\epsilon})\chi_{\epsilon,P_j}(\xi)\,d\xi$$
$$=\epsilon^2\int_{R^2} G_0(P_i^\epsilon,\epsilon y+P_j^\epsilon)w^2(y)\,dy+e.s.t.$$
$$=\epsilon^2 G_0(P_i^\epsilon,P_j^\epsilon)\int_{R^2} w^2(y)\,dy+\epsilon^3\sum_{l=1}^K\frac{\partial G_0(P_i^\epsilon,P_j^\epsilon)}{\partial P_{j,l}^\epsilon}\int_{R^2} w^2(y)y_l\,dy+O(\epsilon^4)$$
$$=\epsilon^2 G_0(P_i^\epsilon,P_j^\epsilon)\int_{R^2} w^2(y)\,dy+O(\epsilon^4).$$

(Here we have set $y=\frac{\xi-P_j^\epsilon}{\epsilon}$ and we have used the relation

$$\int_{R^2} w^2(y)y_l\,dy=0$$

which holds since w is radially symmetric).

Using (1.19) in (2.5) gives

$$\xi_{\epsilon,i}=\xi_{\epsilon,i}^2\frac{\epsilon^2}{|\Omega|}\int_{R^2} w^2(y)\,dy$$
$$+\xi_{\epsilon,i}^2\beta^2\int_\Omega\left(\frac{1}{2\pi}\log\frac{1}{|P_i^\epsilon-\xi|}-H_0(P_i^\epsilon,\xi)\right)w^2(\frac{\xi-P_i^\epsilon}{\epsilon})\chi_{\epsilon,P_i^\epsilon}(\xi)\,d\xi$$
$$+\sum_{j\neq i}\left(\frac{1}{|\Omega|}+\beta^2 G_0(P_i^\epsilon,P_j^\epsilon)\right)\xi_{\epsilon,j}^2\epsilon^2\int_{R^2} w^2(y)\,dy+\sum_{j=1}^K\xi_{\epsilon,j}^2(O(\beta^2\epsilon^4)+O(\beta^4\epsilon^2))$$
$$=\xi_{\epsilon,i}^2\frac{\epsilon^2}{|\Omega|}\int_{R^2} w^2(y)\,dy+\xi_{\epsilon,i}^2\frac{\beta^2}{2\pi}\epsilon^2\log\frac{1}{\epsilon}\int_{R^2} w^2(y)\,dy$$
$$+\xi_{\epsilon,i}^2\beta^2\epsilon^2\left(\frac{1}{2\pi}\int_{R^2} w^2(y)\log\frac{1}{|y|}\,dy-H_0(P_i^\epsilon,P_i^\epsilon)\int_{R^2} w^2(y)\,dy\right)$$
$$+\sum_{j\neq i}(\frac{1}{|\Omega|}+\beta^2 G_0(P_i^\epsilon,P_j^\epsilon))\xi_{\epsilon,j}^2\epsilon^2\int_{R^2} w^2(y)\,dy+\sum_{j=1}^K\xi_{\epsilon,j}^2(O(\beta^2\epsilon^4)+O(\beta^4\epsilon^2)). \quad (2.6)$$

Recall that $H_0\in C^2(\bar\Omega\times\Omega)$.

Considering only the leading terms in (2.6) we get following

$$\xi_{\epsilon,i}=\sum_{j=1}^K\xi_{\epsilon,j}^2\frac{\epsilon^2}{|\Omega|}\int_{R^2} w^2(y)\,dy+\xi_{\epsilon,i}^2\frac{\beta^2}{2\pi}\epsilon^2\log\frac{1}{\epsilon}\int_{R^2} w^2(y)\,dy$$
$$+\sum_{j=1}^K\xi_{\epsilon,j}^2 O(\beta^2\epsilon^2). \quad (2.7)$$

Recall from (1.22) that

$$\xi_{\epsilon,i} = \xi_\epsilon \hat{\xi}_{\epsilon,i}, \qquad \text{where } \ \xi_\epsilon = \frac{|\Omega|}{\epsilon^2 \int_{R^2} w^2}.$$

Then from (2.7) we get

$$\xi_{\epsilon,i} = \left(\frac{1}{|\Omega|} + \frac{\eta_\epsilon}{|\Omega|} \right) \xi_{\epsilon,i}^2 \epsilon^2 \int_{R^2} w^2(y)\, dy + \sum_{j\neq i} \xi_{\epsilon,j}^2 \frac{\epsilon^2}{|\Omega|} \int_{R^2} w^2(y)\, dy + \sum_{j=1}^K \xi_{\epsilon,j}^2 O(\beta^2 \epsilon^2),$$

where η_ϵ was introduced in (1.1). Assuming that

$$\hat{\xi}_{\epsilon,i} \to \hat{\xi}_i, \quad \eta_\epsilon \to \eta_0, \tag{2.8}$$

we obtain the following system of algebraic equations

$$\hat{\xi}_i = \sum_{j=1}^K \hat{\xi}_j^2 + \hat{\xi}_i^2 \eta_0, \quad i = 1, \ldots, K. \tag{2.9}$$

We will solve (2.9) in the next section.

3. Analyzing the algebraic system (2.9)

In this section, we completely determine the solutions of $\hat{\xi}_i, i = 1, ..., K$ for the algebraic system (2.9). To this end, set

$$\rho(t) = t - \eta_0 t^2. \tag{3.1}$$

Then (2.9) is equivalent to

$$\rho(\hat{\xi}_i) = \sum_{j=1}^K \hat{\xi}_j^2, \quad i = 1, ..., K \tag{3.2}$$

which implies that

$$\rho(\hat{\xi}_i) = \rho(\hat{\xi}_j) \quad \text{for} \quad i \neq j. \tag{3.3}$$

That is

$$(\hat{\xi}_i - \hat{\xi}_j)(1 - \eta_0(\hat{\xi}_i + \hat{\xi}_j)) = 0. \tag{3.4}$$

Hence for $i \neq j$ we have

$$\hat{\xi}_i - \hat{\xi}_j = 0 \quad \text{or} \quad \hat{\xi}_i + \hat{\xi}_j = \frac{1}{\eta_0}. \tag{3.5}$$

The case of symmetric solutions ($\hat{\xi}_i = \hat{\xi}_1$, $i = 2, ..., N$) has been studied in [42]. Let us now consider asymmetric solutions, i.e., we assume that there exists an $i \in \{2, ..., N\}$ such that $\hat{\xi}_i \neq \hat{\xi}_1$. Without loss of generality, let us assume that

$$\hat{\xi}_2 \neq \hat{\xi}_1,$$

which implies that

$$\hat{\xi}_1 + \hat{\xi}_2 = \frac{1}{\eta_0}. \tag{3.6}$$

Let us calculate $\hat{\xi}_j$, $j = 3, \ldots, K$. If $\hat{\xi}_j \neq \hat{\xi}_1$, then by (3.5), $\hat{\xi}_j + \hat{\xi}_1 = \frac{1}{\eta_0}$, which implies that $\hat{\xi}_j = \hat{\xi}_2$.

Thus for $j \geq 3$, we have either $\hat{\xi}_j = \hat{\xi}_1$ or $\hat{\xi}_j = \hat{\xi}_2$.

Let k_1 be the number of $\hat{\xi}_1$'s in $\{\hat{\xi}_1, \ldots, \hat{\xi}_K\}$ and k_2 be the number of $\hat{\xi}_2$'s in $\{\hat{\xi}_1, \ldots, \hat{\xi}_K\}$. Then we have $k_1 \geq 1$, $k_2 \geq 1$, $k_1 + k_2 = K$.

This gives

$$\hat{\xi}_1 - \eta_0 \hat{\xi}_1^{\;2} = \sum_{j=1}^{K} \hat{\xi}_j^2 = k_1 \hat{\xi}_1^2 + k_2 \hat{\xi}_2^2, \tag{3.7}$$

$$\hat{\xi}_2 = \frac{1}{\eta_0} - \hat{\xi}_1. \tag{3.8}$$

Substituting (3.8) into (3.7), we obtain

$$\hat{\xi}_1 - \eta_0 \hat{\xi}_1^2 = k_1 \hat{\xi}_1^2 + k_2 \left(\frac{1}{\eta_0} - \hat{\xi}_1 \right)^2$$

and therefore

$$(k_1 + k_2 + \eta_0) \hat{\xi}_1^2 - \frac{2k_2 + \eta_0}{\eta_0} \hat{\xi}_1 + \frac{k_2}{\eta_0^2} = 0. \tag{3.9}$$

Equation (3.9) has a solution if and only if

$$(2k_2 + \eta_0)^2 \geq 4k_2 (k_1 + k_2 + \eta_0). \tag{3.10}$$

The strict inequality of (3.10) is equivalent to (1.12).

It is easy to see that if (1.12) holds, then there are two different solutions to (3.9) which are given by $(\rho_\pm, \eta_\pm)$.

Therefore we arrive at the following conclusion.

Lemma 3.1. *Let $\eta_0 > 2\sqrt{k_1 k_2}$. Then the solutions of (2.9) are given by $(\hat{\xi}_1, ..., \hat{\xi}_N) \in (\{\rho_\pm, \eta_\pm\})^K$ where the number of $\rho'_\pm s$ is k_1 and the number of $\eta'_\pm s$ is k_2.*

If $\eta_0 > 2\sqrt{k_1 k_2}$*, there exist two solutions* $(\rho_\pm, \eta_\pm)$*.*
If $\eta_0 = 2\sqrt{k_1 k_2}$*, there exists one solution* $(\rho_\pm, \rho_\pm)$*.*
If $\eta_0 < 2\sqrt{k_1 k_2}$*, there are no solutions* $(\rho_\pm, \rho_\pm)$*.*

In general, if $\eta_0 > K$, then $\eta_0^2 > 4k_1k_2$ for all k_1, k_2 such that $k_1 + k_2 = K$, $k_1 \geq 1$, $k_2 \geq 1$ since $4k_1k_2 \leq (k_1 + k_2)^2 = K^2$. Hence if $\eta_0 > K$ there exist $2 \cdot 2^{K-2} = 2^{K-1}$ solutions to (2.9).

From now on, let us assume that (1.12) holds and we fix the heights $(\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_K)$ given by Lemma 3.1.

4. Preliminaries II: The Study of a Nonlocal Eigenvalue Problem (NLEP)

In this section, we consider the following nonlocal eigenvalue problem (NLEP):

$$L\phi := \Delta\phi - \phi + 2w\phi - f(\tau\lambda_0)\frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = \lambda_0\phi, \ \phi \in H^2(R^2), \tag{4.1}$$

where f is a continuous complex function with $f(\alpha)$ real for α real and $f(\alpha) > 0$ for $\alpha > 0$. Further, $\tau \geq 0$ is fixed.

We first recall the following well-known result

Lemma 4.1. *Let*

$$L_0 = \Delta - 1 + 2w, \quad \phi \in H^2(R^2). \tag{4.2}$$

The eigenvalue problem

$$L_0\phi = \mu\phi, \quad \phi \in H^2(R^2), \tag{4.3}$$

admits the following set of eigenvalues

$$\mu_1 > 0, \quad \mu_2 = \mu_3 = 0, \quad \mu_4 < 0, \ldots. \tag{4.4}$$

The eigenfunction Φ_0 *corresponding to* μ_1 *can be made positive and radially symmetric; the space of eigenfunctions corresponding to the eigenvalue* 0 *is*

$$K_0 := span\ \left\{\frac{\partial w}{\partial y_j}, j = 1, 2\right\}. \tag{4.5}$$

Proof: This lemma follows from Theorem 2.1 of [19] and Lemma C of [25]. □

Theorem 4.2. *If $f(0) < 1$, then for all $\tau \geq 0$ there exists a positive real eigenvalue of (4.1).*

Proof: By arguments similar to [5] or [43], we may assume that ϕ is a radially symmetric function, namely, $\phi \in H_r^2(R^2) = \{u \in H^2(R^2) | u = u(|y|)\}$. Let L_0 be given by (4.2). Then, by Lemma 4.1, L_0 is invertible in $H_r^2(R^2)$. Let us denote the inverse by L_0^{-1}. By Lemma 4.1, L_0 has a unique positive real eigenvalue μ_1 with eigenfunction Φ_0. It is easy to see that $\lambda_0 \neq \mu_1$ since $\int_{R^2} w\Phi_0 > 0$.

Then λ_0 is an eigenvalue of (4.1) if and only if

$$(L_0 - \lambda_0)\phi = f(\tau\lambda_0)\frac{\int w\phi}{\int w^2}w^2.$$

By the invertibility of $L_0 - \lambda_0$, this holds if and only if

$$\phi = f(\tau\lambda_0)\frac{\int w\phi}{\int w^2}(L_0 - \lambda_0)^{-1}w^2. \tag{4.6}$$

Note that (4.6) says that ϕ *must be* a multiple of $(L_0 - \lambda_0)^{-1}w^2$. Multiplying (4.6) on both sides by w and integrating over R^2 shows that λ_0 is an eigenvalue if and only if it satisfies the following agebraic equation:

$$\int_{R^2} w^2 = f(\tau\lambda_0)\int_{R^2}[((L_0 - \lambda_0)^{-1}w^2)w]. \tag{4.7}$$

(Here we have used the fact that $\int w\phi \neq 0$. Otherwise $\phi = \Phi_0$ and $\lambda_0 = \mu_1$, a contradiction). Now, using the relation

$$(L_0 - \lambda_0)^{-1}w^2 = w + \lambda_0(L_0 - \lambda_0)^{-1}w,$$

it follows that equation (4.7) is equivalent to the following:

$$\rho(\lambda_0) := (1 - f(\tau\lambda_0))\int_{R^2} w^2 - \lambda_0 f(\tau\lambda_0)\int_{R^2}[((L_0 - \lambda_0)^{-1}w)w] = 0. \tag{4.8}$$

Note that $\rho(0) = (1 - f(0))\int_{R^2} w^2 > 0$ by assumption. Then, as $\lambda_0 \to \mu_1, \lambda_0 < \mu_1$, we have $\int_{R^2}((L_0 - \lambda_0)^{-1}w)w \to +\infty$ and hence $\rho(\lambda_0) \to -\infty$. By continuity, there exists an $\lambda_0 \in (0, \mu_1)$ such that $\rho(\lambda_0) = 0$. This positive real number λ_0 is an eigenvalue of (4.1).

□

Now we need the following lemma:

Lemma 4.3. *Consider the eigenvalue problem*

$$\Delta\phi - \phi + 2w\phi - \gamma \frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = \lambda_0 \phi, \ \phi \in H^2(R^2), \tag{4.9}$$

where w is the unique solution of (1.7) and γ is real.

(1) If $\gamma > 1$, there exists a positive constant c_0 such that $Re(\lambda_0) \leq -c_0$ for any nonzero eigenvalue λ_0 of (4.9).

(2) If $\gamma < 1$, there exists a positive eigenvalue λ_0 of (4.9).

(3) If $\gamma \neq 1$ and $\lambda_0 = 0$, then $\phi \in$ span $\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$.

(4) If $\gamma = 1$ and $\lambda_0 = 0$, then $\phi \in$ span $\{w, \frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$.

Proof: (1), (3) and (4) have been proved in Theorem 5.1 of [34]. (2) follows from Theorem 4.2. □

With the help of Lemma 4.3, we can prove the following.

Theorem 4.4. *If $\lim_{\tau\lambda\to+\infty} f(\tau\lambda) := f_{+\infty} < 1$, there exists a positive real eigenvalue of (4.1) for $\tau > 0$ large enough.*

Proof:

By Lemma 4.3 (2), problem (4.1) with $\mu = f_\infty$ has a positive real eigenvalue α_1. Now by perturbation arguments (similar to those in [5]), for τ large enough, problem (4.1) has an eigenvalue near $\alpha_1 > 0$. This implies that for τ large enough, problem (4.1) is unstable.

□

Finally, we consider the case $f(0) > 1$ for τ small.

Theorem 4.5. *Suppose that $f(0) > 1$ and $|f(z)| \leq C$ for all z with $Re(z) \geq -\delta$. Then for τ small, there exists a positive constant c_0 such that $Re(\lambda_0) \leq -c_0$ for any nonzero eigenvalue λ_0 of (4.1).*

Proof: Although this follows from a standard perturbation argument, using (1) of Lemma 4.3, we give a different proof here as it will give us an explicit upper bound on how small c_0 and τ must be to obtain stability.

We apply the following inequality (Lemma 5.1 in [34]): for any (real function) $\phi \in H_r^2(R^2)$, we have

$$\int_{R^2}(|\nabla\phi|^2+\phi^2-2w\phi^2)+2\frac{\int_{R^2} w\phi\int_{R^2} w^2\phi}{\int_{R^2} w^2}-\frac{\int_{R^2} w^3}{(\int_{R^2} w^2)^2}(\int_{R^2} w\phi)^2 \geq 0, \tag{4.10}$$

where equality holds if and only if ϕ is a multiple of w.

Now let $\phi = \phi_R + \sqrt{-1}\phi_I$ be an eigenfunction of (4.1) such that the corresponding eigenvalue λ satisfies $\mathrm{Re}(\lambda) \leq -c_0$. Then we have

$$L_0\phi - f(\tau\lambda)\frac{\int_{R^2} w\phi}{\int_{R^2} w^2}w^2 = \lambda\phi. \tag{4.11}$$

Multiplying (4.11) by $\bar{\phi}$ — the conjugate function of ϕ — and integrating over R^2, we obtain that

$$\int_{R^2}(|\nabla\phi|^2+|\phi|^2-2w|\phi|^2) = -\lambda\int_{R^2}|\phi|^2 - f(\tau\lambda)\frac{\int_{R^2} w\phi}{\int_{R^2} w^2}\int_{R^2} w^2\bar{\phi}. \tag{4.12}$$

Multiplying (4.11) by w and integrating over R^2, we obtain that

$$\int_{R^2} w^2\phi = (\lambda + f(\tau\lambda)\frac{\int_{R^2} w^3}{\int_{R^2} w^2})\int_{R^2} w\phi. \tag{4.13}$$

Hence

$$\int_{R^2} w^2\bar{\phi} = (\bar{\lambda} + f(\tau\bar{\lambda})\frac{\int_{R^2} w^3}{\int_{R^2} w^2})\int_{R^2} w\bar{\phi}. \tag{4.14}$$

Substituting (4.14) into (4.12), we have that

$$\int_{R^2}(|\nabla\phi|^2+|\phi|^2-2w|\phi|^2) = -\lambda\int_{R^2}|\phi|^2 - f(\tau\lambda)(\bar{\lambda}+f(\tau\bar{\lambda})\frac{\int_{R^2} w^3}{\int_{R^2} w^2})\frac{|\int_{R^2} w\phi|^2}{\int_{R^2} w^2}. \tag{4.15}$$

To study the real part λ_R of λ, we just need to consider the real part of (4.15). Now, applying the inequality (4.10) and using (4.14), we arrive at

$$-\lambda_R \geq \mathrm{Re}(f(\tau\lambda)(\bar{\lambda}+f(\tau\bar{\lambda})\frac{\int_{R^2} w^3}{\int_{R^2} w^2})) - 2\mathrm{Re}(\bar{\lambda}+f(\tau\bar{\lambda})\frac{\int_{R^2} w^3}{\int_{R^2} w^2}) + \frac{\int_{R^2} w^3}{\int_{R^2} w^2}.$$

Assuming that $\lambda_R \geq -c_0$, we have

$$\frac{\int_{R^2} w^3}{\int_{R^2} w^2}|f(\tau\lambda)-1|^2 + \mathrm{Re}(\bar{\lambda}(f(\tau\lambda)-1)) \leq c_0. \tag{4.16}$$

On the other hand, since $|f(\tau\lambda)| \leq C$ for some constant $C > 0$, (4.15) implies that $|\lambda| \leq C$ (independent of τ).

Therefore for τ small, (4.16) implies that

$$-2c_0(f(0)-1) \le \mathrm{Re}(\bar{\lambda}(f(\tau\lambda)-1))$$
$$-\frac{1}{2}\frac{\int_{R^2} w^3}{\int_{R^2} w^2}(f(0)-1)^2 + c_0$$

for τ small enough. This gives a contradiction if we choose $c_0 < \frac{\int_{R^2} w^3}{\int_{R^2} w^2}\frac{(f(0)-1)^2}{4f(0)-2}$ and if τ is small enough. This finishes the proof. The inequality (4.16) may also be used to get an explicit bound on τ.

□

5. Existence I: Reduction to finite dimensions

Let us begin with the proof of Theorem 1.1.

In this section, we use the Liapunov-Schmidt method to reduce the problem of finding an equilibrium state to a finite-dimensional problem. We shall follow Section 4 of [42] and give a sketch of the proof.

Motivated by the results in Section 2, we rescale

$$x = \epsilon y, \quad x \in \Omega, \quad y \in \Omega_\epsilon = \{y | \epsilon y \in \Omega\}, \tag{5.1}$$
$$\hat{A}(y) = \frac{1}{\xi_\epsilon} A(\epsilon y),$$
$$\hat{H}(x) = \frac{1}{\xi_\epsilon} H(x), \quad x \in \Omega,$$

where ξ_ϵ is given by (1.22). Then an equilibrium solution $(\hat{A},\, \hat{H})$ has to solve the following rescaled Gierer-Meinhardt system:

$$\begin{cases} \Delta_y \hat{A} - \hat{A} + \frac{\hat{A}^2}{\hat{H}} = 0, \quad y \in \Omega_\epsilon, \\ \Delta_x \hat{H} - \beta^2 \hat{H} + \beta^2 \xi_\epsilon \hat{A}^2 = 0, \quad x \in \Omega. \end{cases} \tag{5.2}$$

(This rescaling is chosen to achieve $\hat{A} = O(1)$, $\hat{H} = O(1)$ in $L^\infty(\Omega)$.)

For a function $\hat{A} \in H^1(\Omega_\epsilon)$, let $T[\hat{A}]$ be the unique solution of the following problem

$$\Delta T[\hat{A}] - \beta^2 T[\hat{A}] + \beta^2 \xi_\epsilon \hat{A}^2 = 0 \text{ in } \Omega, \ \frac{\partial T[\hat{A}]}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{5.3}$$

which is equivalent to

$$T[\hat{A}](x) = \int_\Omega G_\beta(x,\xi)\beta^2 \xi_\epsilon \hat{A}^2(\frac{\xi}{\epsilon})\, d\xi, \tag{5.4}$$

where G_β is defined in (2.1).

System (5.2) is equivalent to the following equation in operator form:

$$S_\epsilon(\hat{A},\hat{H}) = \begin{pmatrix} S_1(\hat{A},\hat{H}) \\ S_2(\hat{A},\hat{H}) \end{pmatrix} = 0, \quad H^2_N(\Omega_\epsilon)\times H^2_N(\Omega) \to L^2(\Omega_\epsilon)\times L^2(\Omega), \tag{5.5}$$

where

$$S_1(\hat{A},\hat{H}) = \Delta_y \hat{A} - \hat{A} + \frac{\hat{A}^2}{\hat{H}} : \quad H^2_N(\Omega_\epsilon)\times H^2_N(\Omega) \to L^2(\Omega_\epsilon),$$

$$S_2(\hat{A},\hat{H}) = \Delta_x \hat{H} - \beta^2\hat{H} + \beta^2\xi_\epsilon \hat{A}^2 : \quad H^2_N(\Omega_\epsilon)\times H^2_N(\Omega) \to L^2(\Omega).$$

Here the index N indicates that the functions satisfy the Neumann boundary conditions

$$\frac{\partial \hat{A}}{\partial\nu} = 0,\ y \text{ on } \partial\Omega_\epsilon, \qquad \frac{\partial \hat{H}}{\partial\nu} = 0,\ x \text{ on } \partial\Omega.$$

Let $\mathbf{P} = (P_1,...,P_K) =\in \overline{\Lambda_\delta}$ and $j = 1,...,K$. Then we define

$$w_{\epsilon,j}(y) := w(y - \frac{P_j}{\epsilon})\chi_{\epsilon,P_j}(\epsilon y), \quad y\in\Omega_\epsilon, \tag{5.6}$$

where w is the unique solution of (1.7) and χ_{ϵ,P_j} is defined in (2.3).

We choose our approximate solution $(\hat{A},\hat{H})$ as follows:

$$A_{\epsilon,\mathbf{P}}(y) := \sum_{i=1}^K \hat{\xi}_{\epsilon,i} w_{\epsilon,i}(y), \qquad H_{\epsilon,\mathbf{P}}(x) := T[A_{\epsilon,\mathbf{P}}](x), \quad x = \epsilon y\in\Omega. \tag{5.7}$$

Note that $H_{\epsilon,\mathbf{P}}$ satisfies

$$0 = \Delta_x H_{\epsilon,\mathbf{P}} - \beta^2 H_{\epsilon,\mathbf{P}} + \beta^2\xi_\epsilon A^2_{\epsilon,\mathbf{P}}$$

$$= \Delta_x H_{\epsilon,\mathbf{P}} - \beta^2 H_{\epsilon,\mathbf{P}} + \beta^2\xi_\epsilon \sum_{j=1}^K \hat{\xi}^2_{\epsilon,j} w^2_{\epsilon,j} + e.s.t.$$

Hence

$$H_{\epsilon,\mathbf{P}}(P_j) = \beta^2\xi_\epsilon \int_\Omega G_\beta(x,\xi)\sum_{j=1}^K \hat{\xi}^2_{\epsilon,j} w^2_{\epsilon,j}(\frac{\xi}{\epsilon})\,d\xi + e.s.t.$$

Similar to the computation in Section 2 (using the definition (1.22) of ξ_ϵ), we obtain

$$H_{\epsilon,\mathbf{P}}(P_j) = \hat{\xi}_{\epsilon,j}) + O\left(\frac{1}{\log\frac{1}{\epsilon}}\right), \quad j = 1,...,K. \tag{5.8}$$

We substitute (5.7) into (5.5) and calculate

$$S_2(A_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}}) = 0, \tag{5.9}$$

$$
\begin{aligned}
S_1(A_{\epsilon,\mathbf{P}},H_{\epsilon,\mathbf{P}}) &= \Delta_y A_{\epsilon,\mathbf{P}} - A_{\epsilon,\mathbf{P}} + \frac{A^2_{\epsilon,\mathbf{P}}}{H_{\epsilon,\mathbf{P}}} \\
&= \sum_{i=1}^K \left[\hat{\xi}_{\epsilon,i}\Delta_y w\left(y-\frac{P_i}{\epsilon}\right) - \hat{\xi}_{\epsilon,i} w\left(y-\frac{P_i}{\epsilon}\right)\right] \\
&\quad + \sum_{i=1}^K \hat{\xi}^2_{\epsilon,i} w^2\left(y-\frac{P_i}{\epsilon}\right) H^{-1}_{\epsilon,\mathbf{P}} + e.s.t. \\
&= \sum_{i=1}^K w^2\left(y-\frac{P_i}{\epsilon}\right)\hat{\xi}_{\epsilon,i}(\hat{\xi}_{\epsilon,i}H^{-1}_{\epsilon,\mathbf{P}}-1) + e.s.t. \\
&= \sum_{i=1}^K w^2\left(y-\frac{P_i}{\epsilon}\right)\hat{\xi}_{\epsilon,i}(\hat{\xi}_{\epsilon,i}H_{\epsilon,\mathbf{P}}(P_i)^{-1}-1) \\
&\quad + \sum_{i=1}^K w^2\left(y-\frac{P_i}{\epsilon}\right)\hat{\xi}^2_{\epsilon,i}(H_{\epsilon,\mathbf{P}}(x)^{-1} - H_{\epsilon,\mathbf{P}}(P_i)^{-1}) + e.s.t.
\end{aligned}
\tag{5.10}
$$

Now we compute for $i=1,...,K$ and $x=P_i+\epsilon z$, $|\epsilon z|<\delta$:

$$
\begin{aligned}
&H_{\epsilon,\mathbf{P}}(P_i+\epsilon z) - H_{\epsilon,\mathbf{P}}(P_i) \\
&= \beta^2 \int_\Omega [G_\beta(P_i+\epsilon z,\xi) - G_\beta(P_i,\xi)]\xi_\epsilon A^2_{\epsilon,\mathbf{P}} d\xi \\
&= \beta^2\xi_\epsilon \int_\Omega [G_0(P_i+\epsilon z,\xi) - G_0(P_i,\xi) + O(\beta^2\epsilon|z|)] A^2_{\epsilon,\mathbf{P}} d\xi \quad \text{(by (2.2))} \\
&= \beta^2\xi_\epsilon \int_\Omega [G_0(P_i+\epsilon z,\xi) - G_0(P_i,\xi) + O(\beta^2\epsilon|z|)] \sum_{j=1}^K \hat{\xi}^2_{\epsilon,j} w^2_{\epsilon,j} d\xi \quad \text{(by (5.7))} \\
&= \beta^2\xi_\epsilon \int_\Omega [G_0(P_i+\epsilon z,\xi) - G_0(P_i,\xi) + O(\beta^2\epsilon|z|)] \hat{\xi}^2_{\epsilon,i} w^2_{\epsilon,i} d\xi \\
&\quad + \beta^2\xi_\epsilon \int_\Omega [G_0(P_i+\epsilon z,\xi) - G_0(P_i,\xi) + O(\beta^2\epsilon|z|)] \sum_{j\neq i} \hat{\xi}^2_{\epsilon,j} w^2_{\epsilon,j} d\xi \\
&= \beta^2\epsilon^2\xi_\epsilon\hat{\xi}^2_{\epsilon,i} \int_{R^2} \frac{1}{2\pi}[\log|\rho| - \log|z-\rho|] w^2(\rho) d\rho \\
&\quad - \beta^2\epsilon^2\xi_\epsilon(\hat{\xi}_{\epsilon,i})^{-2} \sum_{k=1}^2 \frac{1}{2}\frac{\partial F(\mathbf{P})}{\partial P_{i,k}}\epsilon z_k \int_{R^2} w^2 + O(\beta^4\epsilon^3\xi_\epsilon|z|),
\end{aligned}
\tag{5.11}
$$

where $\epsilon\rho = \xi - P_i$, $|\epsilon\rho|<\delta$, and F is defined in (1.20). Here we have used the expansions

$$
\begin{aligned}
&G_0(P_i+\epsilon z,\xi) - G_0(P_i,\xi) \\
&= \frac{1}{2\pi}\left(\log\frac{1}{|P_i+\epsilon z-\xi|} - \log\frac{1}{|P_i-\xi|}\right)
\end{aligned}
$$

$$-H_0(P_i+\epsilon z,\xi)+H_0(P_i,\xi) \quad \text{(by (1.19))}$$
$$=\frac{1}{2\pi}\left(\log\frac{1}{\epsilon|\rho-z|}-\log\frac{1}{\epsilon|\rho|}\right)$$
$$-H_0(P_i+\epsilon z,P_i+\epsilon\rho)+H_0(P_i,P_i+\epsilon\rho)$$
$$=\frac{1}{2\pi}\log\frac{|\rho|}{|\rho-z|}-\epsilon\nabla_P H_0(P,Q)|_{P=Q=P_i}\cdot z+O(\epsilon^2),$$
$$=\frac{1}{2\pi}\log\frac{|\rho|}{|\rho-z|}-\frac{1}{2}\epsilon\nabla_P H_0(P,P)|_{P=P_i}\cdot z+O(\epsilon^2),$$

where $\epsilon\rho=\xi-P_i$, $|\epsilon\rho|<\delta$, and

$$G_0(P_i+\epsilon z,\xi)-G_0(P_i,\xi)$$
$$=G_0(P_i+\epsilon z,P_j+\epsilon\rho)-G_0(P_i,P_j+\epsilon\rho)$$
$$=\epsilon\nabla_P G_0(P,P_j)|_{P=P_i}\cdot z+O(\epsilon^2),$$
$$=\frac{1}{2}\epsilon\nabla_P(G_0(P,P_j)|_{P=P_i}+G_0(P_j,P)|_{P=P_i})\cdot z+O(\epsilon^2),$$

where $\epsilon\rho=\xi-P_j$, $|\epsilon\rho|<\delta$, and $i\neq j$. Substituting (5.11) into (5.10), we have the following key estimate

Lemma 5.1. *For $x=P_j+\epsilon z$, $|\epsilon z|<\delta$, we have the decomposition*

$$S_1(A_{\epsilon,\mathbf{P}},H_{\epsilon,\mathbf{P}})=S_{1,1}+S_{1,2}, \tag{5.12}$$

where

$$S_{1,1}(z)=\beta^2\epsilon^2\xi_\epsilon(H_{\epsilon,P_j}(P_j))^{-2}(\int_{R^2}w^2)w^2(z)\left(\frac{\epsilon}{2}\nabla_{P_j}F(\mathbf{P})\cdot z+O(\beta^2\epsilon|z|)\right) \tag{5.13}$$

and

$$S_{1,2}(z)=\beta^2\epsilon^2\xi_\epsilon w^2(z)R_j(|z|)+O(\beta^4\epsilon^3\xi_\epsilon|z|), \tag{5.14}$$

where $S_{1,2}(|z|)$ is a radially symmetric function with the property that $R_j(|z|)=O(\log(1+|z|))$.

Furthermore, $S_1(A_{\epsilon,\mathbf{P}},H_{\epsilon,\mathbf{P}})=e.s.t.$ for $|x-P_j|\geq\delta$, $j=1,2,...,K$.

Now we study the linearized operator defined by

$$\tilde{L}_{\epsilon,\mathbf{P}}:=S'_\epsilon\begin{pmatrix}A_{\epsilon,\mathbf{P}}\\ H_{\epsilon,\mathbf{P}}\end{pmatrix},$$
$$\tilde{L}_{\epsilon,\mathbf{P}}:H^2_N(\Omega_\epsilon)\times H^2_N(\Omega)\to L^2(\Omega_\epsilon)\times L^2(\Omega),$$

where $\epsilon>0$ is small and $\mathbf{P}\in\overline{\Lambda_\delta}$.

Set

$$K_{\epsilon,\mathbf{P}} := \text{span } \left\{ \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,l}} | j = 1, \dots, K, \, l = 1, 2 \right\} \subset H^2_N(\Omega_\epsilon)$$

and

$$C_{\epsilon,\mathbf{P}} := \text{span } \left\{ \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,l}} | j = 1, \dots, K, \, l = 1, \dots, N \right\} \subset L^2(\Omega_\epsilon).$$

Note that $\tilde{L}_{\epsilon,\mathbf{P}}$ is not uniformly invertible in ϵ due to the approximate kernel

$$\mathcal{K}_{\epsilon,\mathbf{P}} := K_{\epsilon,\mathbf{P}} \oplus \{0\} \subset H^2_N(\Omega_\epsilon) \times H^2_N(\Omega).$$

We choose the approximate cokernel as follows:

$$\mathcal{C}_{\epsilon,\mathbf{P}} := C_{\epsilon,\mathbf{P}} \oplus \{0\} \subset L^2(\Omega_\epsilon) \times L^2(\Omega).$$

We then define

$$\mathcal{K}^\perp_{\epsilon,\mathbf{P}} := K^\perp_{\epsilon,\mathbf{P}} \oplus H^2_N(\Omega) \subset H^2_N(\Omega_\epsilon) \times H^2_N(\Omega),$$

$$\mathcal{C}^\perp_{\epsilon,\mathbf{P}} := C^\perp_{\epsilon,\mathbf{P}} \oplus L^2(\Omega) \subset L^2(\Omega_\epsilon) \times L^2(\Omega),$$

where $C^\perp_{\epsilon,\mathbf{P}}$ and $K^\perp_{\epsilon,\mathbf{P}}$ denote the orthogonal complement with the scalar product of $L^2(\Omega_\epsilon)$ in $H^2_N(\Omega_\epsilon)$ and $L^2(\Omega_\epsilon)$, respectively.

Let $\pi_{\epsilon,\mathbf{P}}$ denote the projection in $L^2(\Omega_\epsilon) \times L^2(\Omega)$ onto $\mathcal{C}^\perp_{\epsilon,\mathbf{P}}$. (Here the second component of the projection is the identity map.) We are going to show that the equation

$$\pi_{\epsilon,\mathbf{P}} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} = 0$$

has the unique solution $\Sigma_{\epsilon,\mathbf{P}} = \begin{pmatrix} \Phi_{\epsilon,\mathbf{P}}(y) \\ \Psi_{\epsilon,\mathbf{P}}(x) \end{pmatrix} \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$ if ϵ is small enough.

Set

$$\mathcal{L}_{\epsilon,\mathbf{P}} = \pi_{\epsilon,\mathbf{P}} \circ \tilde{L}_{\epsilon,\mathbf{P}} : \mathcal{K}^\perp_{\epsilon,\mathbf{P}} \to \mathcal{C}^\perp_{\epsilon,\mathbf{P}}. \tag{5.15}$$

As a preparation, in the following two propositions we show the invertibility of the corresponding linearized operator $\mathcal{L}_{\epsilon,\mathbf{P}}$.

Proposition 5.2. *Assume that (T1) of Theorem 1.1 holds. Let $\mathcal{L}_{\epsilon,\mathbf{P}}$ be given by (5.15). There exist positive constants $\bar{\epsilon}$, C with C indendent of ϵ such that* *for all* $\epsilon \in (0, \bar{\epsilon})$

$$\|\mathcal{L}_{\epsilon,\mathbf{P}}\Sigma\|_{L^2(\Omega_\epsilon)\times L^2(\Omega)} \geq C\|\Sigma\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)} \tag{5.16}$$

for arbitrary $\mathbf{P} \in \overline{\Lambda_\delta}$, $\Sigma \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$.

Proposition 5.3. *Assume that (T1) of Theorem 1.1 holds. There exists a positive constant* $\bar{\bar{\epsilon}}$ *such that for all* $\epsilon \in (0, \bar{\bar{\epsilon}})$ *the map*

$$\mathcal{L}_{\epsilon,\mathbf{P}} = \pi_{\epsilon,\mathbf{P}} \circ \tilde{L}_{\epsilon,\mathbf{P}} : \mathcal{K}^\perp_{\epsilon,\mathbf{P}} \to \mathcal{C}^\perp_{\epsilon,\mathbf{P}}$$

is surjective for arbitrary $\mathbf{P} \in \overline{\Lambda_\delta}$.

The proofs of Propositions 5.2 and 5.3 are similar to that of Appendix A in [42]. A key point is to show that the operator $\tilde{L}_{\epsilon,\mathbf{P}}$ has exactly a $2K$-dimensional kernel. The condition (T1) of Theorem 1.1 is vital since it implies that the limiting operator $\mathcal{L}$ has exactly a $2K$-dimensional kernel (see Lemma 7.2 below). Then by Liapunov-Schmidt reduction the same holds for $\tilde{L}_{\epsilon,\mathbf{P}}$. For the sake of limited space we omit the details.

□

If condition (T1) does not hold, then either Liapunov-Schmidt reduction fails or we have to change the dimension of the kernel and cokernel, respectively, to make it work. It seems that further conditions are needed to distinguish what happens.

Now we are in a position to solve the equation

$$\pi_{\epsilon,\mathbf{P}} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} + \phi \\ H_{\epsilon,\mathbf{P}} + \psi \end{pmatrix} = 0. \tag{5.17}$$

Since $\mathcal{L}_{\epsilon,\mathbf{P}}|_{\mathcal{K}^\perp_{\epsilon,\mathbf{P}}}$ is invertible (call the inverse $\mathcal{L}^{-1}_{\epsilon,\mathbf{P}}$), we can rewrite

$$\Sigma = -(\mathcal{L}^{-1}_{\epsilon,\mathbf{P}} \circ \pi_{\epsilon,\mathbf{P}}) \left(S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} \end{pmatrix} \right) - (\mathcal{L}^{-1}_{\epsilon,\mathbf{P}} \circ \pi_{\epsilon,\mathbf{P}})(N_{\epsilon,\mathbf{P}}(\Sigma)) \equiv M_{\epsilon,\mathbf{P}}(\Sigma), \tag{5.18}$$

where

$$N_{\epsilon,\mathbf{P}}(\Sigma) = S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} + \phi \\ H_{\epsilon,\mathbf{P}} + \psi \end{pmatrix} - S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} \end{pmatrix} - S'_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} \end{pmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix},$$

and the operator $M_{\epsilon,\mathbf{P}}$ is defined by (5.18) for $\Sigma = (\phi,\psi) \in H^2_N(\Omega_\epsilon) \times H^2(\Omega)$. We are going to show that the operator $M_{\epsilon,\mathbf{P}}$ is a contraction on

$$B_{\epsilon,\delta} \equiv \{\Sigma \in H^2(\Omega_\epsilon) \times H^2(\Omega) | \|\Sigma\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)} < \delta\}$$

if δ is small enough. By Lemma 5.1 we have

$$\|S_1(A_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}})\|_{H^2(\Omega_\epsilon)} \le C \frac{1}{\log\frac{1}{\epsilon}}. \tag{5.19}$$

Using (5.19) and the Propositions 5.2 and 5.3 we get

$$\begin{aligned} \|M_{\epsilon,\mathbf{P}}(\Sigma)\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)} &\le \lambda^{-1}(\|\pi_{\epsilon,\mathbf{P}} \circ N_{\epsilon,\mathbf{P}}(\Sigma)\|_{L^2(\Omega_\epsilon)\times L^2(\Omega)} \\ &\quad + \left\| \pi_{\epsilon,\mathbf{P}} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} \end{pmatrix} \right\|_{L^2(\Omega_\epsilon)\times L^2(\Omega)}) \\ &\le \lambda^{-1} C (c(\delta)\delta + \frac{1}{\log\frac{1}{\epsilon}}), \end{aligned}$$

where $\lambda > 0$ is independent of $\delta > 0$ and $c(\delta) \to 0$ as $\delta \to 0$. Similarly we show

$$\|M_{\epsilon,\mathbf{P}}(\Sigma) - M_{\epsilon,\mathbf{P}}(\Sigma')\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)} \le \lambda^{-1} c(\delta) \|\Sigma - \Sigma'\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)},$$

where $c(\delta) \to 0$ as $\delta \to 0$. If we choose δ small enough, then $M_{\epsilon,\mathbf{P}}$ is a contraction mapping on $B_{\epsilon,\delta}$. The existence of a fixed point $\Sigma_{\epsilon,\mathbf{P}}$ for $M_{\epsilon,\mathbf{P}}$ plus an error estimate now follows from the Contraction Mapping Principle and $\Sigma_{\epsilon,\mathbf{P}}$ is a solution of (5.18).

We have thus proved

Lemma 5.4. *There exist an $\bar{\epsilon} > 0$ such that for every pair ϵ, β, $\mathbf{P}$ with $0 < \epsilon < \bar{\epsilon}$, $\mathbf{P} \in \overline{\Lambda_\delta}$ there exists a unique $(\Phi_{\epsilon,\mathbf{P}}, \Psi_{\epsilon,\mathbf{P}}) \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$ satisfying $S_\epsilon\left(\begin{pmatrix} A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix}\right) \in \mathcal{C}_{\epsilon,\mathbf{P}}$ and*

$$\|(\Phi_{\epsilon,\mathbf{P}}, \Psi_{\epsilon,\mathbf{P}})\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)} \le C \frac{1}{\log\frac{1}{\epsilon}}. \tag{5.20}$$

More refined estimates for $\Phi_{\epsilon,\mathbf{P}}$ are needed. We recall from Lemma 5.1 that S_1 can be decomposed into the two parts $S_{1,1}$ and $S_{1,2}$, where $S_{1,1}$ is in leading order an odd function and $S_{1,2}$ is in leading order a radially symmetric function. Similarly, we can decompose $\Phi_{\epsilon,\mathbf{P}}$:

Lemma 5.5. *Let* $\Phi_{\epsilon,\mathbf{P}}$ *be defined in Lemma 5.4. Then for* $x = P_i + \epsilon z$, $|\epsilon z| < \delta$, *we have the decomposition*

$$\Phi_{\epsilon,\mathbf{P}} = \Phi_{\epsilon,\mathbf{P},1} + \Phi_{\epsilon,\mathbf{P},2}, \tag{5.21}$$

where $\Phi_{\epsilon,\mathbf{P},2}$ *is a radially symmetric function in* z *which satisfies*

$$\Phi_{\epsilon,\mathbf{P},2} = O\left(\frac{1}{\log\frac{1}{\epsilon}}\right) \quad \text{in } H^2_N(\Omega_\epsilon). \tag{5.22}$$

and

$$\Phi_{\epsilon,\mathbf{P},1} = O\left(\frac{\epsilon}{\log\frac{1}{\epsilon}}\right) \quad \text{in } H^2_N(\Omega_\epsilon). \tag{5.23}$$

Proof: Let $S[v] := S_1(v, T[v])$. We first solve

$$S[A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P},2}] - S[A_{\epsilon,\mathbf{P}}] + \sum_{j=1}^{K} S_{1,2}(y - \frac{P_j}{\epsilon}) \in C_{\epsilon,\mathbf{P}}, \tag{5.24}$$

for $\Phi_{\epsilon,\mathbf{P},2} \in K^\perp_{\epsilon,\mathbf{P}}$.

Then we solve

$$S[A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P},2} + \Phi_{\epsilon,\mathbf{P},1}] - S[A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P},2}] + \sum_{j=1}^{K} S_{1,1}(y - \frac{P_j}{\epsilon}) \in C_{\epsilon,\mathbf{P}}, \tag{5.25}$$

for $\Phi_{\epsilon,\mathbf{P},1} \in K^\perp_{\epsilon,\mathbf{P}}$.

Using the same proof as in Lemma 5.4, both equations (5.24) and (5.25) have unique solutions for $\epsilon << 1$. By uniqueness, $\Phi_{\epsilon,\mathbf{P}} = \Phi_{\epsilon,\mathbf{P},1} + \Phi_{\epsilon,\mathbf{P},2}$. Since $S_{1,1} = S^0_{1,1} + S^\perp_{1,1}$, where $\|S^0_{1,1}\|_{H^2(\Omega_\epsilon)} = O\left(\frac{\epsilon}{\log\frac{1}{\epsilon}}\right)$ and $S^\perp_{1,1} \in C^\perp_{\epsilon,\mathbf{P}}$, it is easy to see that $\Phi_{\epsilon,\mathbf{P},1}$ and $\Phi_{\epsilon,\mathbf{P},2}$ have the required properties.

□

6. Existence II: The reduced problem

In this section, we solve the reduced problem and complete the proof of Theorem 1.1.

Let $\mathbf{P}^0 \in \overline{\Lambda_\delta}$ be a nondegenerate critical point of $F(\mathbf{P})$.

By Lemma 5.4, if we choose δ small enough, for each $\mathbf{P} \in B_\delta(\mathbf{P}^0)$, there exists a unique solution $(\Phi_{\epsilon,\mathbf{P}}, \psi_{\epsilon,\mathbf{P}}) \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$ such that

$$S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} = \begin{pmatrix} v_{\epsilon,\mathbf{P}} \\ 0 \end{pmatrix} \in \mathcal{C}_{\epsilon,\mathbf{P}}.$$

Now we are going to find a $\mathbf{P} = \mathbf{P}^\epsilon \in B_\delta(\mathbf{P}^0)$ such that

$$S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} \perp \mathcal{C}_{\epsilon,\mathbf{P}}. \tag{6.26}$$

For $\mathbf{P} \in \overline{\Lambda_\delta}$ let

$$W_{\epsilon,j,i}(\mathbf{P}) := \log \frac{1}{\epsilon} \int_{\Omega_\epsilon} (S_1(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}}) \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}), \quad j = 1, ..., K, i = 1, 2, \tag{6.27}$$

$$W_\epsilon(\mathbf{P}) := (W_{\epsilon,1,1}(\mathbf{P}), ..., W_{\epsilon,K,2}(\mathbf{P})). \tag{6.28}$$

Note that $W_\epsilon(\mathbf{P})$ is a map which is continuous in $\mathbf{P}$ and our problem is reduced to finding a zero of the vector field $W_\epsilon(\mathbf{P})$.

Let

$$\Omega_{\epsilon,P_j} = \{y | \epsilon y + P_j \in \Omega\}. \tag{6.29}$$

We calculate the asymptotic expansion of $W_{\epsilon,j,i}(\mathbf{P})$:

$$\begin{aligned}
&\log \frac{1}{\epsilon} \int_{\Omega_\epsilon} S_1(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}}) \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} \\
&= \log \frac{1}{\epsilon} \int_{\Omega_\epsilon} \left[\Delta(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) - (A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) + \frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}}} \right] \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} \\
&= \log \frac{1}{\epsilon} \int_{\Omega_\epsilon} \left[\Delta(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) - (A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) + \frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}}} \right] \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} \\
&\quad + \log \frac{1}{\epsilon} \int_{\Omega_\epsilon} \left[\frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}}} - \frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}}} \right] \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} \\
&= I_1 + I_2,
\end{aligned}$$

where I_1 and I_2 are defined by the last equality.

For I_1, we have by Lemma 5.5,

$$\begin{aligned}
I_1 &= \log \frac{1}{\epsilon} \Big(\int_{\Omega_\epsilon} \left[\Delta(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) - (A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) + \frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}}(P_j)} \right] \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} \\
&\quad - \int_{\Omega_\epsilon} \frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{(H_{\epsilon,\mathbf{P}}(P_j))^2} (H_{\epsilon,\mathbf{P}} - H_{\epsilon,\mathbf{P}}(P_j)) \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} \Big) + o(1) \\
&= \epsilon^{-1} \log \frac{1}{\epsilon} \Big(- \int_{\Omega_{\epsilon,P_j}} [\Delta(\hat{\xi}_{\epsilon,j} w_{\epsilon,j} + \Phi_{\epsilon,\mathbf{P}}) - (\hat{\xi}_{\epsilon,j} w_{\epsilon,j} + \Phi_{\epsilon,\mathbf{P}}) + \frac{(\hat{\xi}_{\epsilon,j} w_{\epsilon,j} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}}(P_j)}] \frac{\hat{\xi}_{\epsilon,j} \partial w_{\epsilon,j}}{\partial y_i}
\end{aligned}$$

$$+\int_{\Omega_{\epsilon,P_j}} \frac{(\hat{\xi}_{\epsilon,j} w_{\epsilon,j} + \Phi_{\epsilon,\mathbf{P},2})^2(y)}{(H_{\epsilon,\mathbf{P}}(P_j))^2} (H_{\epsilon,\mathbf{P}}(P_j+\epsilon y) - H_{\epsilon,\mathbf{P}}(P_j)) \frac{\hat{\xi}_{\epsilon,j} \partial w_{\epsilon,j}(y)}{\partial y_i}\, dy \Big) + o(1).$$

Note that, by Lemma 5.5,

$$\int_{\Omega_{\epsilon,P_j}} [\Delta \Phi_{\epsilon,\mathbf{P}} - \Phi_{\epsilon,\mathbf{P}} + 2 w_{\epsilon,j} \Phi_{\epsilon,\mathbf{P}}] \frac{\partial w_{\epsilon,j}}{\partial y_i}$$

$$= \int_{\Omega_{\epsilon,P_j}} \Phi_{\epsilon,\mathbf{P},1} \frac{\partial}{\partial y_i} [\Delta w - w + w^2] + o\left(\frac{\epsilon}{\log \frac{1}{\epsilon}}\right) = o\left(\frac{\epsilon}{\log \frac{1}{\epsilon}}\right), \tag{6.30}$$

$$\int_{\Omega_{\epsilon,P_j}} (\Phi_{\epsilon,\mathbf{P}})^2 \frac{\partial w_{\epsilon,j}}{\partial y_i} = \int_{\Omega_{\epsilon,P_j}} (\Phi_{\epsilon,\mathbf{P},1})^2 \frac{\partial w_{\epsilon,j}}{\partial y_i} + o\left(\frac{\epsilon}{\log \frac{1}{\epsilon}}\right) = o\left(\frac{\epsilon}{\log \frac{1}{\epsilon}}\right). \tag{6.31}$$

Now, by (5.11), (6.30) and (6.31),

$$I_1 = o(1) - \epsilon^{-1} \log \frac{1}{\epsilon} (\hat{\xi}_{\epsilon,j})^3 (H_{\epsilon,\mathbf{P}}(P_j))^{-2} \int_{\Omega_{\epsilon,P_j}} w_{\epsilon,j}^2(y) (H_{\epsilon,\mathbf{P}}(P_j+\epsilon y) - H_{\epsilon,\mathbf{P}}(P_j)) \frac{\partial w_{\epsilon,j}(y)}{\partial y_i}\, dy$$

$$= o(1) + \pi \eta_0 \hat{\xi}_{\epsilon,j} (H_{\epsilon,\mathbf{P}}(P_j))^{-2} \sum_{k=1}^{2} \frac{\partial F(\mathbf{P})}{\partial P_{j,k}} \int_{R^2} w^2 y_k \frac{\partial w}{\partial y_i}$$

$$= o(1) + \pi \eta_0 \hat{\xi}_{\epsilon,j} (H_{\epsilon,\mathbf{P}}(P_j))^{-2} \frac{\partial F(\mathbf{P})}{\partial P_{j,i}} \int_{R^2} w^2 y_i \frac{\partial w}{\partial y_i}$$

$$= o(1) - \frac{\pi \eta_0}{3} \hat{\xi}_{\epsilon,j} (H_{\epsilon,\mathbf{P}}(P_j))^{-2} \int_{R^2} w^3 \frac{\partial F(\mathbf{P})}{\partial P_{j,i}}, \frac{\partial F(\mathbf{P})}{\partial P_{j,i}}$$

$$= o(1) - \frac{\pi \eta_0}{3} (\hat{\xi}_{\epsilon,j})^{-1} \int_{R^2} w^3 \frac{\partial F(\mathbf{P})}{\partial P_{j,i}} \quad \text{(by (2.4))}, \tag{6.32}$$

where η_0 and ξ have been defined in (1.2) and (1.22), respectively.

Similarly, we compute for I_2:

$$I_2 = \log \frac{1}{\epsilon} \int_{\Omega_\epsilon} \left[\frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}}} - \frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}}} \right] \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}$$

$$= -\log \frac{1}{\epsilon} \int_{\Omega_\epsilon} \frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}}^2} \Psi_{\epsilon,\mathbf{P}} \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} + o(1)$$

$$= -\epsilon^{-1} \log \frac{1}{\epsilon} \hat{\xi}_{\epsilon,j}^3 (H_{\epsilon,\mathbf{P}}(P_j))^{-2} \int_{\Omega_{\epsilon,P_j}} \frac{1}{3} \frac{\partial w_{\epsilon,j}^3}{\partial y_i} (\Psi_{\epsilon,\mathbf{P}} - \Psi_{\epsilon,\mathbf{P}}(P_j)) + o(1). \tag{6.33}$$

Recall that $\Psi_{\epsilon,\mathbf{P}}$ satisfies

$$\Delta \Psi_{\epsilon,\mathbf{P}} - \beta^2 \Psi_{\epsilon,\mathbf{P}} + 2\beta^2 \xi_\epsilon A_{\epsilon,\mathbf{P}} \Phi_{\epsilon,\mathbf{P}} + \beta^2 \xi_\epsilon \Phi_{\epsilon,\mathbf{P}}^2 = 0.$$

Using Lemma 5.5, similar computations as those leading to (5.11) show that

$$\Psi_{\epsilon,\mathbf{P}}(P_j+\epsilon y)-\Psi_{\epsilon,\mathbf{P}}(P_j)$$
$$=\int_\Omega (G_\beta(P_j+\epsilon y,\xi)-G_\beta(P_j,\xi))\beta^2\xi_\epsilon(2A_{\epsilon,\mathbf{P}}(\frac{\xi}{\epsilon})\Phi_{\epsilon,\mathbf{P}}(\frac{\xi}{\epsilon})+\Phi^2_{\epsilon,\mathbf{P}}(\frac{\xi}{\epsilon}))d\xi$$
$$=o\left(\frac{\epsilon}{\log\frac{1}{\epsilon}}|\nabla_{P_j}F(\mathbf{P})|\,|y|\right)+\frac{1}{\log\frac{1}{\epsilon}}R_a(|y|), \tag{6.34}$$

where $R_a(|y|)$ is a radially symmetric function.

Substituting (6.34) into (6.33), we obtain that

$$I_2=o(1) \quad \text{uniformly in } \overline{\Lambda_\delta}. \tag{6.35}$$

Combining the estimates for I_1 and I_2, we obtain

$$W_\epsilon(\mathbf{P})=-\frac{\pi\eta_0}{6}\mathcal{D}^{-1}\nabla_{\mathbf{P}}F(\mathbf{P})+E_\epsilon(\mathbf{P}), \tag{6.36}$$

where the matrix $\mathcal{D}$ is defined by

$$(\mathcal{D})_{ij}=\hat{\xi}_{\epsilon,j}\delta_{ij}, \tag{6.37}$$

δ_{ij} the Kronecker symbol, and $E_\epsilon(\mathbf{P})=o(1)$ is a continuous function of $\mathbf{P}$ which goes to 0 as $\epsilon\to 0$ uniformly in $\overline{\Lambda_\delta}$. Note that the matrix $\mathcal{D}$ is strictly positive definite.

At $\mathbf{P}^0$, we have $\nabla_{\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^0}F(\mathbf{P}^0)=0, \det(\nabla^2_{\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^0}(F(\mathbf{P}))\neq 0$. Therefore, for ϵ small enough and $\delta=\delta(\epsilon)\to 0$ as $\epsilon\to 0$ but so slowly that W_ϵ has exactly one zero in $B_\delta(\mathbf{P}^0)$ (which is possible by (6.36)), we compute the mapping degree of $W_\epsilon(\mathbf{P})$ for the set B_δ and the value 0 as follows:

$$\deg(W_\epsilon,0,B_\delta)=\text{sign}\ \det(-\mathcal{D}^{-1}\nabla^2_{\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^0}(F(\mathbf{P}))$$
$$=\text{sign}\ \det(-\mathcal{D}^{-1}M(\mathbf{P^0}))\neq 0$$

by condition (T2) in Theorem 1.1. Therefore, standard degree theory implies that for ϵ small enough, there exists a $\mathbf{P}^\epsilon\in B_\delta$ such that $W_\epsilon(\mathbf{P}^\epsilon)=0$ and, by (6.36), we have $\mathbf{P}^\epsilon\to\mathbf{P}^0$.

Thus we have proved the following proposition.

Proposition 6.1. *For ϵ sufficiently small, there exist points $\mathbf{P}^\epsilon$ with $\mathbf{P}^\epsilon\to\mathbf{P}^0$ such that $W_\epsilon(\mathbf{P}^\epsilon)=0$.*

Now we complete the proof of Theorem 1.1.

Proof of Theorem 1.1: By Proposition 6.1, there exists $\mathbf{P}^\epsilon \to \mathbf{P}^0$ such that $W_\epsilon(\mathbf{P}^\epsilon) = 0$. In other words, $S_1(A_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}, H_{\epsilon,\mathbf{P}^\epsilon} + \Psi_{\epsilon,\mathbf{P}^\epsilon}) = 0$. Let $A_\epsilon = \xi_\epsilon(A_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}), H_\epsilon = \xi_\epsilon(H_{\epsilon,\mathbf{P}^\epsilon} + \Psi_{\epsilon,\mathbf{P}^\epsilon})$. It is easy to see that $H_\epsilon = \xi_\epsilon T[A_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}] > 0$. Hence $A_\epsilon \geq 0$. By the Maximum Principle, $A_\epsilon > 0$. Therefore (A_ϵ, H_ϵ) satisfies Theorem 1.1.

□

7. Stability Analysis I: Large Eigenvalues

We consider the stability of the steady state (A_ϵ, H_ϵ) constructed in Theorem 1.1.

Linearizing around the equilibrium states

$$\begin{cases} A = A_\epsilon + \phi_\epsilon(x)e^{\lambda_\epsilon t}, \\ H = H_\epsilon + \psi_\epsilon(x)e^{\lambda_\epsilon t}, \end{cases}$$

and substituting the result into (GM) we deduce the following eigenvalue problem

$$\begin{cases} \Delta_y\phi_\epsilon - \phi_\epsilon + 2\frac{A_\epsilon}{H_\epsilon}\phi_\epsilon - \frac{A_\epsilon^2}{H_\epsilon^2}\psi_\epsilon = \lambda_\epsilon\phi_\epsilon, \\ \frac{1}{\beta^2}\Delta\psi_\epsilon - \psi_\epsilon + 2A_\epsilon\phi_\epsilon = \tau\lambda_\epsilon\psi_\epsilon. \end{cases} \tag{7.1}$$

Here $D = \frac{1}{\beta^2}$, λ_ϵ is some complex number and

$$\phi_\epsilon \in H_N^2(\Omega_\epsilon),\ \psi_\epsilon \in H_N^2(\Omega). \tag{7.2}$$

In this section, we study the large eigenvalues, i.e., we assume that $|\lambda_\epsilon| \geq c > 0$ for ϵ small. Furthermore, we may assume that $(1+\tau)c < \frac{1}{2}$. If $\mathrm{Re}(\lambda_\epsilon) \leq -c$, we are done. (Then λ_ϵ is a stable large eigenvalue.) Therefore we may assume that $\mathrm{Re}(\lambda_\epsilon) \geq -c$ and for a subsequence $\epsilon \to 0$, $\lambda_\epsilon \to \lambda_0 \neq 0$. We shall derive the limiting eigenvalue problem which are NLEPs.

The key references are Theorem 4.2, Theorem 4.4, and Theorem 4.5.

The second equation in (7.1) is equivalent to

$$\Delta\psi_\epsilon - \beta^2(1+\tau\lambda_\epsilon)\psi_\epsilon + 2\beta^2 A_\epsilon\phi_\epsilon = 0. \tag{7.3}$$

We introduce the following:

$$\beta_{\lambda_\epsilon} = \beta\sqrt{1+\tau\lambda_\epsilon}, \tag{7.4}$$

where in $\sqrt{1+\tau\lambda_\epsilon}$ we take the principal part of the square root. (This means that the real part of $\sqrt{1+\tau\lambda_\epsilon}$ is positive, which is possible since Re $(1+\tau\lambda_\epsilon)\geq\frac{1}{2}$.)

Let us assume that

$$\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)}=1. \tag{7.5}$$

We cut off ϕ_ϵ as follows: Introduce

$$\phi_{\epsilon,j}(y)=\phi_\epsilon(y)\chi_{\epsilon,P_j^\epsilon}(\epsilon y), \tag{7.6}$$

where $\chi_{\epsilon,P_j^\epsilon}(x)$ is given by (2.3).

From (7.1) using Lemma 5.4, the fact that $\mathrm{Re}(\lambda_\epsilon)\geq -c$, the asymptotic expansion of A_ϵ, given in Theorem 1.1, and the exponential decay of w (see (1.8)), it follows that

$$\phi_\epsilon=\sum_{j=1}^K\phi_{\epsilon,j}+e.s.t.\quad \text{in } H^2(\Omega_\epsilon). \tag{7.7}$$

Then by a standard procedure we extend $\phi_{\epsilon,j}$ to a function defined on R^2 such that

$$\|\phi_{\epsilon,j}\|_{H^2(R^2)}\leq C\|\phi_{\epsilon,j}\|_{H^2(\Omega_\epsilon)},\qquad j=1,\dots,K.$$

Since $\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)}=1$, $\|\phi_{\epsilon,j}\|_{H^2(R^2)}\leq C$. By taking a subsequence of ϵ, we may also assume that $\phi_{\epsilon,j}\to\phi_j$ as $\epsilon\to 0$ in H^1 for $j=1,\dots,K$, strongly on compact subsets of R^2. Therefore, we also have

$$w\phi_{\epsilon,j}\to w\phi_j\quad \text{as } \epsilon\to 0, \text{ strongly in } L^\infty(R^2). \tag{7.8}$$

We have by (7.3)

$$\psi_\epsilon(x)=2\beta^2\int_\Omega G_{\beta_{\lambda_\epsilon}}(x,\xi)A_\epsilon(\frac{\xi}{\epsilon})\phi_\epsilon(\frac{\xi}{\epsilon})\,d\xi. \tag{7.9}$$

Now we use the expansion of A_ϵ and the definitions of ξ_ϵ and $\hat{\xi}_{\epsilon,i}$ which are given in Theorem 1.1. At $x=P_i^\epsilon$, $i=1,\dots,K$, we calculate

$$\psi_\epsilon(P_i^\epsilon)=2\beta^2\int_\Omega G_{\beta_{\lambda_\epsilon}}(P_i^\epsilon,\xi)\sum_{j=1}^K\xi_\epsilon\hat{\xi}_{\epsilon,j}w(\frac{\xi-P_j^\epsilon}{\epsilon})\phi_{\epsilon,j}(\frac{\xi}{\epsilon})\,d\xi+e.s.t.$$

$$=2\beta^2\int_\Omega(\frac{(\beta_{\lambda_\epsilon})^{-2}}{|\Omega|}+G_0(P_i^\epsilon,\xi)+O(|\beta_{\lambda_\epsilon}|^2))\sum_{j=1}^K\xi_\epsilon\hat{\xi}_{\epsilon,j}w(\frac{\xi-P_j^\epsilon}{\epsilon})\phi_{\epsilon,j}(\frac{\xi}{\epsilon})\,d\xi+e.s.t.$$

$$= 2\int_\Omega \Big(\frac{1}{|\Omega|(1+\tau\lambda_\epsilon)} + \beta^2 G_0(P_i^\epsilon,\xi) + O(|\beta_{\lambda_\epsilon}|^4)\Big)\xi_\epsilon\hat{\xi}_{\epsilon,i} w\Big(\frac{x-P_i^\epsilon}{\epsilon}\Big)\phi_{\epsilon,i}\Big(\frac{\xi}{\epsilon}\Big)\,d\xi$$

$$+2\sum_{j\neq i}\int_\Omega \Big(\frac{1}{|\Omega|(1+\tau\lambda_\epsilon)} + \beta^2 G_0(P_i^\epsilon,P_j^\epsilon) + O(|\beta_{\lambda_\epsilon}|^4)\Big)\xi_\epsilon\hat{\xi}_{\epsilon,j} w\Big(\frac{\xi-P_j^\epsilon}{\epsilon}\Big)\phi_{\epsilon,j}\Big(\frac{\xi}{\epsilon}\Big)\,d\xi$$

$$= \Bigg(2\sum_{j=1}^K \frac{1}{|\Omega|(1+\tau\lambda_\epsilon)}\xi_\epsilon\epsilon^2\hat{\xi}_{\epsilon,j}\int_{R^2} w(y)\phi_{\epsilon,j}(y)\,dy$$

$$+ 2\xi_\epsilon\hat{\xi}_{\epsilon,i}\frac{\beta^2}{2\pi}\epsilon^2\log\frac{1}{\epsilon}\int_{R^2} w(y)\phi_{\epsilon,i}(y)\,dy + O(|\beta_{\lambda_\epsilon}|^2\xi_\epsilon\epsilon^2)\Bigg) \quad \text{(by (1.18))}. \tag{7.10}$$

We get from (7.10) together with (1.1) and (1.2), (7.8), and since $\xi_{\epsilon,i} \to \xi_i$, $i = 1, ..., K$ by Theorem 1.1,

$$\psi_\epsilon(P_i^\epsilon) = \Bigg(2\sum_{j=1}^K \frac{1}{|\Omega|(1+\tau\lambda_0)}\xi_\epsilon\hat{\xi}_{\epsilon,j}\epsilon^2\int_{R^2} w\phi_{\epsilon,j} + 2\xi_\epsilon\hat{\xi}_{\epsilon,i}\frac{\eta_0}{|\Omega|}\epsilon^2\int_{R^2} w\phi_{\epsilon,i}\Bigg)(1+o(1)). \tag{7.11}$$

Substituting (7.11) into the first equation of (7.1) and letting $\epsilon \to 0$, we obtain the following nonlocal eigenvalue problem (NLEP):

$$\Delta\phi_i - \phi_i + 2w\phi_i - \frac{2}{1+\tau\lambda_0}\sum_{j=1}^K \hat{\xi}_j\frac{\int w\phi_j}{\int w^2} - 2\eta_0\hat{\xi}_i\frac{\int w\phi_i}{\int w^2}w^2 = \lambda_0\phi_i, \quad i = 1, ..., K. \tag{7.12}$$

Let

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_K \end{pmatrix}.$$

Then we can rewrite (7.12) in matrix form:

$$\Delta\Phi - \Phi + 2w\Phi - \frac{2\int_{R^2} w\mathcal{B}\Phi}{\int_{R^2} w^2}w^2 = \lambda_0\Phi, \tag{7.13}$$

where

$$\mathcal{B} = \begin{pmatrix} \eta_0\hat{\xi}_1 & & \\ & \ddots & \\ & & \eta_0\hat{\xi}_K \end{pmatrix} + \frac{1}{1+\tau\lambda_0}\begin{pmatrix} \hat{\xi}_1 & \dots & \hat{\xi}_K \\ \vdots & & \vdots \\ \hat{\xi}_1 & \dots & \hat{\xi}_K \end{pmatrix} \tag{7.14}$$

Note that in general $\mathcal{B}$ is **not** self-adjoint since $\lambda_0 \in \mathcal{C}$.

Let us now compute the eigenvalues of $\mathcal{B}$ in two special cases. We claim that

Lemma 7.1. *Let $(\hat{\xi}_1, ..., \hat{\xi}_K)$ be given by Lemma 3.1. Then the eigenvalues of $\mathcal{B}$ are solutions of the following quadratic equation*

$$\frac{k_1\rho}{\eta_0\rho - \lambda} + \frac{k_2\eta}{\eta_0\eta - \lambda} + 1 + \tau\lambda_0 = 0, \tag{7.15}$$

where ρ and η are given by (1.16). In particular, if $\tau = 0$, then the eigenvalues of $\mathcal{B}$ are given by

$$\lambda_1 = 1, \quad \lambda_2 = k_1\rho + k_2\eta. \tag{7.16}$$

If $\tau = +\infty$, then the eigenvalues of $\mathcal{B}$ are given by

$$\lambda_1 = \eta_0\rho, \quad \lambda_2 = \eta_0\eta. \tag{7.17}$$

Proof: Let $\mathbf{q} = (q_1, \dots, q_K)^T$ be an eigenvector of $\mathcal{B}$ with eigenvalue λ. Then we have

$$\mathcal{B}\mathbf{q} = \lambda\mathbf{q}. \tag{7.18}$$

Writing (7.18) in components, we have

$$\eta_0\hat{\xi}_i q_i + \frac{1}{1 + \tau\lambda_0}\sum_{j=1}^{N} q_j\hat{\xi}_j = \lambda q_i, \quad i = 1, \dots, K.$$

Hence, we have

$$(\eta_0\hat{\xi}_i - \lambda)q_i = -\frac{1}{1 + \tau\lambda_0}\sum_{j=1}^{N} q_j\hat{\xi}_j = c, \tag{7.19}$$

$$q_i = \frac{c}{\eta_0\hat{\xi}_i - \lambda}. \tag{7.20}$$

Substituting (7.20) into (7.19), we obtain that

$$\sum_{j=1}^{K} \frac{\hat{\xi}_j}{\eta_0\hat{\xi}_j - \lambda} + 1 + \tau\lambda_0 = 0. \tag{7.21}$$

Using (1.16), this can be re-written as

$$\frac{k_1\rho}{\eta_0\rho - \lambda} + \frac{k_2\eta}{\eta_0\eta - \lambda} + 1 + \tau\lambda_0 = 0,$$

which is exactly (7.15).

When $\tau = 0$, using the fact that $\rho + \eta = \frac{1}{\eta_0}$, we obtain the following

$$\lambda^2 - \lambda(k_1\rho + k_2\eta + 1) + \eta_0(K + \eta_0)\rho\eta = 0 \tag{7.22}$$

The two roots of (7.22) are given by (7.16).

Next, for $\tau = +\infty$, $\mathcal{B}$ is diagonal and the result is trivial.

□

By choosing a basis for R^K so $\mathcal{B}$ is diagonal, we see that the eigenvalue problem (7.13) can be reduced to the study of the following two nonlocal eigenvalue problems

$$\Delta\Phi_i - \Phi_i + 2w\Phi_i - \frac{2\lambda_i \int_{R^2} w\Phi_i}{\int_{R^2} w^2} w^2 = \lambda_0\Phi_i,\ i = 1, 2,\ \Phi_i \in H^2(R^2), \tag{7.23}$$

where λ_i are the two eigenvalues of $\mathcal{B}$ satisfying (7.15). We can study these by using the results of Section 3.

When $\tau = 0$, we have $\lambda_1 = 1, \lambda_2 = k_1\rho + k_2\eta$. The first eigenvalue causes no difficulty in the stability of (7.23) by Theorem 4.5. For the second eigenvalue, it is easy to compute that for $(\rho, \eta) = (\rho_\pm, \eta_\pm)$,

$$2\lambda_2 - 1 = \frac{4k_1k_2 - \eta_2^2 \pm (k_1 - k_2)\sqrt{\eta_0^2 - 4k_1k_2}}{2\eta_0(\eta_0 + K)}. \tag{7.24}$$

If $\eta_0 > K$, we have

$$\eta_0^2 > (k_1 + k_2)^2$$

and therefore

$$\eta_0^2 - 4k_1k_2 > (k_1 - k_2)^2.$$

Thus

$$\lambda_2 < \frac{1}{2} \quad \text{if} \quad \eta_0 > K.$$

By Theorem 4.2, there exists a positive real eigenvalue $\lambda_0 > 0$ of (7.23) for all $\tau > 0$. This, together with a perturbation argument of [5], implies instability of (7.1) with respect to the $O(1)$ eigenvalues.

However, in the case when $2\sqrt{k_1k_2} < \eta_0 \le K$, if we choose $k_1 > k_2, (\rho, \eta) = (\rho_+, \eta_+)$, then $\lambda_2 > 1/2$. Thus we have stability of (7.1) with respect to the large eigenvalues, for τ small, by Theorem 4.5.

Finally, when $\tau = +\infty$, we have $\lambda_1 = \eta_0\rho,\ \lambda_2 = \eta_0\eta$. Then, since $\rho + \eta = \frac{1}{\eta_0}$,

$$\lambda_1 + \lambda_2 = 1,$$

which implies that one of the λ_i must be less than $\frac{1}{2}$ unless $\lambda_1 = \lambda_2 = \frac{1}{2}$. In the latter case, $\rho = \eta$ and $\hat{\xi}_1 = ... = \hat{\xi}_K$, which implies that (A_ϵ, H_ϵ) is a symmetric K–peaked solution. Since our solution is asymmetric, the latter case can not happen.

Thus by Theorem 4.4, if τ is large enough, we must have the instability of (7.23) and hence instability of (7.1) with respect to $O(1)$ eigenvalues.

This finishes the proof of Theorem 1.2 in the large eigenvalue case.

In the next section we shall study the eigenvalues λ_ϵ which tend to zero as $\epsilon \to 0$.

Finally, we state a lemma which is vital for the Liapunov-Schmidt reduction process.

Lemma 7.2. *Suppose that (T1) of Theorem 1.1 holds. Let*

$$\mathcal{L}\Phi := \Delta\Phi - \Phi + 2w\Phi - \frac{2\int_{R^2} w\mathcal{B}\Phi}{\int_{R^2} w^2} w^2, \Phi \in (H^2(R^2))^K, \tag{7.25}$$

where $\mathcal{B}$ is given by (7.14). Set

$$X_0 := span \left\{ \frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2} \right\}. \tag{7.26}$$

Then

$$Ker(\mathcal{L}) = X_0 \oplus X_0 \oplus \cdots \oplus X_0 \tag{7.27}$$

and

$$Ker(\mathcal{L}^*) = X_0 \oplus X_0 \oplus \cdots \oplus X_0. \tag{7.28}$$

As a result, the operator

$$\mathcal{L} : (H^2(R^2))^K \to (L^2(R^2))^K$$

is invertible if it is restricted as follows

$$\mathcal{L} : (X_0 \oplus \cdots \oplus X_0)^\perp \cap (H^2(R^2))^K \to (X_0 \oplus \cdots \oplus X_0)^\perp \cap (L^2(R^2))^K.$$

Moreover, $\mathcal{L}^{-1}$ is bounded.

Proof: This follows from choosing a basis in R^K so $\mathcal{B}$ is diagonal and using (3) of Lemma 4.3. □

8. Stability Analysis II: Small Eigenvalues

We now study (7.1) for small eigenvalues. Namely, we assume that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. The analysis follows along the lines of [40] and [42]. We will show that the small eigenvalues are related to the matrix $M(\mathbf{P}^0)$ given in (1.21).

Let us assume that condition (*) of Theorem 1.2 holds true. That is, all eigenvalues of the matrix $M(\mathbf{P}^0)$ are negative. Our main result in this section says that if $\lambda_\epsilon \to 0$, then

$$\lambda_\epsilon \sim \frac{\epsilon^2}{\log \frac{1}{\epsilon}} 2\pi\eta_0 \frac{1}{\int_{R^2} w^2} \sigma_0, \tag{8.1}$$

where σ_0 is an eigenvalue of $\mathcal{D}^{-1}M(\mathbf{P}^0)\mathcal{D}^{-2}$ and $\mathcal{D}$ is the diagonal, positive definite matrix defined in (6.37). From (8.1), we see that all small eigenvalues of $\mathcal{L}_\epsilon$ are stable, provided that condition (*) of Theorem 1.2 holds.

Again let (A_ϵ, H_ϵ) be the equilibrium state of (1.5). which has been rigorously constructed in Theorem 1.1 and let $(\hat{A}_\epsilon, \hat{H}_\epsilon)$ be the rescaled solution given by

$$\hat{A}_\epsilon = \xi_\epsilon^{-1} A_\epsilon, \quad \hat{H}_\epsilon = \xi_\epsilon^{-1} H_\epsilon, \tag{8.2}$$

where ξ_ϵ is defined in (1.22).

We cut off $\hat{A}_\epsilon$ as follows:

$$\hat{A}_{\epsilon,j}(y) = \chi_{\epsilon,P_j^\epsilon}(\epsilon y)\hat{A}_\epsilon(y), \quad j = 1, ..., K, \tag{8.3}$$

where $\chi_{\epsilon,P_j^\epsilon}$ is defined in (2.3).

Then it is easy to see that

$$\hat{A}_\epsilon(y) = \sum_{j=1}^K \hat{A}_{\epsilon,j}(y) + e.s.t. \quad \text{in } H^2(\Omega_\epsilon). \tag{8.4}$$

We now give a formal argument which should motivate to the reader our choice of decomposition of ϕ_ϵ which will be given in (8.6) below. Later, in Step 1 of the proof it will be shown that this choice gives the correct answer in leading order.

Note that $\tilde{A}_{\epsilon,j}(y) \sim \hat{\xi}_j w(y - \frac{P_j^\epsilon}{\epsilon})$ in $H^2(\Omega_\epsilon)$ and $\hat{A}_{\epsilon,j}$ satisfies

$$\Delta_y \hat{A}_{\epsilon,j} - \hat{A}_{\epsilon,j} + \frac{(\hat{A}_{\epsilon,j})^2}{\hat{H}_\epsilon} + e.s.t. = 0.$$

Thus $\frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k}$ satisfies

$$\Delta_y \frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} - \frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} + \frac{2\hat{A}_{\epsilon,j}}{\hat{H}_\epsilon}\frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} - \epsilon \frac{(\hat{A}_{\epsilon,j})^2}{\hat{H}_\epsilon^2}\frac{\partial \hat{H}_\epsilon}{\partial x_k} + e.s.t. = 0, \tag{8.5}$$

and we have $\frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} = \hat{\xi}_j(1+o(1))\frac{\partial w}{\partial y_k}(y - \frac{P_j^\epsilon}{\epsilon})$. We now decompose

$$\phi_\epsilon = \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} + \phi_\epsilon^\perp \tag{8.6}$$

with complex numbers $a_{j,k}^\epsilon$, where

$$\phi_\epsilon^\perp \perp \tilde{\mathcal{K}}_\epsilon := \text{span}\,\{\frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} | j = 1, \dots, K,\ k = 1,2\} \subset H_N^2(\Omega_\epsilon). \tag{8.7}$$

Our main idea is to show that this is a good choice since the error $\phi_\epsilon^\perp$ is small and thus can be neglected (This is done in Step 1.) Then we obtain algebraic equations for $a_{j,k}^\epsilon$ which are related to the matrix $M(\mathbf{P}^0)$. (This is done in Step 2.)

Accordingly, we decompose ψ_ϵ

$$\psi_\epsilon(x) = \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \psi_{\epsilon,j,k} + \psi_\epsilon^\perp, \tag{8.8}$$

where $\psi_{\epsilon,j,k}$ is the unique solution of the problem

$$\begin{cases} \frac{1}{\beta^2}\Delta_x \psi_{\epsilon,j,k} - (1+\tau\lambda_\epsilon)\psi_{\epsilon,j,k} + 2\xi_\epsilon \hat{A}_{\epsilon,j}\frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} = 0 & \text{in } \Omega, \\ \frac{\partial \psi_{\epsilon,j,k}}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.9}$$

and $\psi_\epsilon^\perp$ satisfies

$$\begin{cases} \frac{1}{\beta^2}\Delta_x \psi_\epsilon^\perp - (1+\tau\lambda_\epsilon)\psi_\epsilon^\perp + 2\xi_\epsilon \hat{A}_\epsilon \phi_\epsilon^\perp = 0 & \text{in } \Omega, \\ \frac{\partial \psi_\epsilon^\perp}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.10}$$

Suppose that $\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)} = 1$. Then $|a_{j,k}^\epsilon| \le C$ since

$$a_{j,k}^\epsilon = \frac{\int_{\Omega_\epsilon} \phi_\epsilon \frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k}}{(\hat{\xi}_{\epsilon,j})^2 \int_{R^2} (\frac{\partial w}{\partial y_1})^2} + o(1).$$

Substituting the decompositions of ϕ_ϵ and ψ_ϵ into the eigenvalue problem (7.1) and using (8.5), we have

$$\epsilon\sum_{j=1}^{K}\sum_{k=1}^{2} a_{j,k}^{\epsilon}\frac{(\hat{A}_{\epsilon,j})^2}{(\hat{H}_\epsilon)^2}\left[-\frac{1}{\epsilon}\psi_{\epsilon,j,k}+\frac{\partial \hat{H}_\epsilon}{\partial x_k}\right]+e.s.t.$$

$$+\Delta_y\phi_\epsilon^\perp-\phi_\epsilon^\perp+2\frac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi_\epsilon^\perp-\frac{(\hat{A}_\epsilon)^2}{(\hat{H}_\epsilon)^2}\psi_\epsilon^\perp-\lambda_\epsilon\phi_\epsilon^\perp$$

$$=\lambda_\epsilon\sum_{j=1}^{K}\sum_{k=1}^{2} a_{j,k}^{\epsilon}\frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k}\quad \text{in}\quad \Omega_\epsilon. \tag{8.11}$$

Set

$$I_3:=\epsilon\sum_{j=1}^{K}\sum_{k=1}^{2} a_{j,k}^{\epsilon}\frac{(\hat{A}_{\epsilon,j})^2}{(\hat{H}_\epsilon)^2}\left[-\frac{1}{\epsilon}\psi_{\epsilon,j,k}+\frac{\partial \hat{H}_\epsilon}{\partial x_k}\right] \tag{8.12}$$

and

$$I_4:=\Delta_y\phi_\epsilon^\perp-\phi_\epsilon^\perp+2\frac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi_\epsilon^\perp-\frac{(\hat{A}_\epsilon)^2}{(\hat{H}_\epsilon)^2}\psi_\epsilon^\perp-\lambda_\epsilon\phi_\epsilon^\perp. \tag{8.13}$$

We divide our proof into two steps.

Step 1: Estimates for $\phi_\epsilon^\perp$.

The main contribution of this step is to obtain good error bounds for $\phi_\epsilon^\perp$.

We use equation (8.11). Since $\phi_\epsilon^\perp \perp \tilde{\mathcal{K}}_\epsilon$, then similar to the proof of Proposition 5.2 it follows that

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)}\le C\|I_3\|_{L^2(\Omega_\epsilon)}. \tag{8.14}$$

Let us now compute I_3.

Let ξ_ϵ be the same as in Theorem 1.1. Then we calculate, using (2.2), that for $x\in B_\delta(P_l^\epsilon)$:

$$\frac{\partial \hat{H}_\epsilon}{\partial x_k}(x)=\xi_\epsilon\beta^2\int_\Omega \frac{\partial}{\partial x_k}G_\beta(x,\xi)(\hat{A}_\epsilon(\frac{\xi}{\epsilon}))^2\,d\xi$$

$$=\xi_\epsilon\beta^2\left[\int_\Omega\frac{\partial}{\partial x_k}\left(\frac{1}{2\pi}\log\frac{1}{|x-\xi|}-H_0(x,\xi)\right)(\hat{A}_{\epsilon,l}(\frac{\xi}{\epsilon}))^2\,d\xi\right.$$

$$\left.+\int_\Omega\sum_{s\neq l}\frac{\partial}{\partial x_k}G_0(x,\xi)(\hat{A}_{\epsilon,s}(\frac{\xi}{\epsilon}))^2\,d\xi+O(\beta^2\epsilon^2)\right]$$

and

$$\psi_{\epsilon,l,k}(x) = 2\beta^2\xi_\epsilon \int_\Omega G_{\beta_{\lambda_\epsilon}}(x,z)\hat{A}_{\epsilon,l}\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_k}dz$$

$$= \epsilon\xi_\epsilon\beta^2 \int_\Omega \left(\frac{1}{2\pi}\log\frac{1}{|x-\xi|} - H_0(x,\xi) + O(|\beta_{\lambda_\epsilon}|^2)\right)\frac{\partial}{\partial \xi_k}(\hat{A}_{\epsilon,l})^2\,d\xi.$$

Thus, for $x \in B_\delta(P_l^\epsilon)$, we have

$$\frac{\partial \hat{H}_\epsilon}{\partial x_k}(x) - \frac{1}{\epsilon}\psi_{\epsilon,l,k}(x)$$

$$= \xi_\epsilon\beta^2\Bigg[\left(\int_\Omega[\frac{\partial}{\partial x_k}\frac{1}{2\pi}\log\frac{1}{|x-\xi|}(\hat{A}_{\epsilon,l}(\frac{\xi}{\epsilon}))^2 - \frac{1}{2\pi}\log\frac{1}{|x-\xi|}\frac{\partial}{\partial \xi_k}(\hat{A}_{\epsilon,l}(\frac{\xi}{\epsilon}))^2]d\xi\right)$$

$$- \int_\Omega[\frac{\partial}{\partial x_k}H_0(x,\xi))(\hat{A}_{\epsilon,l}(\frac{\xi}{\epsilon}))^2 - H_0(x,\xi)\frac{\partial}{\partial \xi_k}(\hat{A}_{\epsilon,l}(\frac{\xi}{\epsilon}))^2]d\xi$$

$$+ \int_\Omega \sum_{s\neq l}\frac{\partial}{\partial x_k}G_0(x,\xi)(\hat{A}_{\epsilon,s}(\frac{\xi}{\epsilon}))^2\,d\xi + O(\epsilon^2\beta^2)\Bigg].$$

Using the radial symmetry of $\frac{1}{2\pi}\log\frac{1}{|x|}$ and integrating by parts, we get

$$\frac{\partial \hat{H}_\epsilon}{\partial x_k}(x) - \frac{1}{\epsilon}\psi_{\epsilon,l,k}(x)$$

$$= \epsilon^2\xi_\epsilon\beta^2(\hat{\xi}_{\epsilon,l})^{-2}\int_{R^2} w^2(-\frac{\partial}{\partial x_k}F_l(x) + o(\epsilon)), \tag{8.15}$$

where

$$F_l(x) = H_0(x,P_l^\epsilon)\hat{\xi}_{\epsilon,l}^4 - \sum_{j\neq l}G_0(x,P_j^\epsilon)\hat{\xi}_{\epsilon,j}^2\hat{\xi}_{\epsilon,l}^2. \tag{8.16}$$

Observe that

$$\frac{\partial}{\partial x_m}F_l(x)|_{x=P_l^\epsilon} = o(1)$$

since $\mathbf{P}^\epsilon \to \mathbf{P}^0$ and $\mathbf{P}^0$ is a critical point of $F(\mathbf{P})$.

Hence, we have

$$\|I_3\|_{L^2(\Omega_\epsilon)} = o\left(\frac{\epsilon}{\log\frac{1}{\epsilon}}\sum_{j=1}^K\sum_{k=1}^2|a_{j,k}^\epsilon|\right) \tag{8.17}$$

and

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} \le C\|I_3\|_{L^2(\Omega_\epsilon)} = o\left(\frac{\epsilon}{\log\frac{1}{\epsilon}}\sum_{j=1}^K\sum_{k=1}^2|a_{j,k}^\epsilon|\right). \tag{8.18}$$

Using the equation (8.10) for $\psi_\epsilon^\perp$ and (8.18), we obtain that

$$\psi_\epsilon^\perp(x) = o\left(\frac{\epsilon}{\log\frac{1}{\epsilon}}\sum_{j=1}^K\sum_{k=1}^2 |a_{j,k}^\epsilon|\right). \tag{8.19}$$

We calculate, using (8.5) and (8.13),

$$\int_{\Omega_\epsilon}(I_4\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m})d\xi = \int_{\Omega_\epsilon}(\frac{\hat{A}^2_{\epsilon,l}}{H^2_\epsilon}(\epsilon\frac{\partial \hat{H}_\epsilon}{\partial x_m}\phi_\epsilon^\perp - \frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}\psi_\epsilon^\perp))d\xi - \lambda_\epsilon\int_{\Omega_\epsilon}\phi_\epsilon^\perp\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}$$

$$= \int_{\Omega_{\epsilon,P_l^\epsilon}}\frac{\hat{A}^2_{\epsilon,l}}{\hat{H}^2_\epsilon}(\epsilon\frac{\partial \hat{H}_\epsilon}{\partial x_m}(P_l^\epsilon+\epsilon y) - \epsilon\frac{\partial \hat{H}_\epsilon}{\partial x_m}(P_l^\epsilon))\phi_\epsilon^\perp$$

$$+\int_{\Omega_{\epsilon,P_l^\epsilon}}\frac{\hat{A}^2_{\epsilon,l}}{\hat{H}^2_\epsilon}(\epsilon\frac{\partial \hat{H}_\epsilon}{\partial x_m}(P_l^\epsilon))\phi_\epsilon^\perp$$

$$-\int_{\Omega_{\epsilon,P_l^\epsilon}}\frac{\hat{A}^2_{\epsilon,l}}{\hat{H}^2_\epsilon}\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}(\psi_\epsilon^\perp(P_l^\epsilon+\epsilon y) - \psi_\epsilon^\perp(P_l^\epsilon)))d\xi$$

$$-\lambda_\epsilon\int_{\Omega_\epsilon}\phi_\epsilon^\perp\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}.$$

This implies, using (8.7), (8.18), (8.10), and the estimate

$$\frac{\partial \hat{H}_\epsilon}{\partial x_m} = O(\frac{1}{\log\frac{1}{\epsilon}}) \quad \text{in } \Omega,$$

that

$$\int_{\Omega_\epsilon}(I_4\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m})d\xi = o\left(\frac{\epsilon^2}{\log\frac{1}{\epsilon}}\sum_{j=1}^K\sum_{k=1}^2 |a_{j,k}^\epsilon|\right). \tag{8.20}$$

Step 2: Algebraic equations for $a_{j,k}^\epsilon$.

This step gives us algebraic equations for $a_{j,k}^\epsilon$.

Multiplying both sides of (8.11) by $\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}$ and integrating over Ω_ϵ, we obtain

$$r.h.s. = \lambda_\epsilon\sum_{j=1}^K\sum_{k=1}^2 a_{j,k}^\epsilon\int_{\Omega_\epsilon}\frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k}\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}$$

$$= \lambda_\epsilon\sum_{j=1}^K\sum_{k=1}^2 a_{j,k}^\epsilon\delta_{jl}\delta_{km}\hat{\xi}_{\epsilon,l}\hat{\xi}_{\epsilon,j}\int_{R^2}\left(\frac{\partial w}{\partial y_1}\right)^2 dy\,(1+o(1))$$

$$= \lambda_\epsilon a_{l,m}^\epsilon\hat{\xi}^2_{\epsilon,l}\int_{R^2}\left(\frac{\partial w}{\partial y_1}\right)^2 dy(1+o(1)).$$

Now (8.20) gives

$$l.h.s. = \epsilon \sum_{j=1}^{K}\sum_{k=1}^{2} a_{j,k}^{\epsilon} \int_{\Omega_{\epsilon,P_l^\epsilon}} \frac{(\hat{A}_{\epsilon,j})^2}{(\hat{H}_\epsilon)^2}\left[-\frac{1}{\epsilon}\psi_{\epsilon,j,k} + \frac{\partial \hat{H}_\epsilon}{\partial x_k}\right]\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}$$

$$+ \int_{\Omega_\epsilon} (I_4 \frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}) d\xi$$

$$= \epsilon \sum_{j=1}^{K}\sum_{k=1}^{2} a_{j,k}^{\epsilon} \int_{\Omega_{\epsilon,P_l^\epsilon}} \frac{(\hat{A}_{\epsilon,j})^2}{(\hat{H}_\epsilon)^2}\left[-\frac{1}{\epsilon}\psi_{\epsilon,j,k} + \frac{\partial \hat{H}_\epsilon}{\partial x_k}\right]\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}$$

$$+ o\left(\frac{\epsilon^2}{\log\frac{1}{\epsilon}}\sum_{j=1}^{K}\sum_{k=1}^{2}|a_{j,k}^{\epsilon}|\right). \tag{8.21}$$

Using (8.15), we obtain

$$l.h.s. = \epsilon^3 \xi_\epsilon \beta^2 \sum_{j=1}^{K}\sum_{k=1}^{2} a_{j,k}^{\epsilon} (\hat{\xi}_{\epsilon,j})^{-2}$$

$$\times \int_{\Omega_\epsilon} \frac{(\hat{A}_{\epsilon,j})^2}{(\hat{H}_\epsilon)^2}(-\frac{\partial}{\partial x_k}F_j(x))\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}\int w^2$$

$$+ o\left(\frac{\epsilon^2}{\log\frac{1}{\epsilon}}\sum_{j=1}^{K}\sum_{k=1}^{2}|a_{j,k}^{\epsilon}|\right)$$

$$= \epsilon^4 \xi_\epsilon \beta^2 \hat{\xi}_{\epsilon,l}(\hat{\xi}_{\epsilon,j})^{-2}\int_{R^2} w^2 \frac{\partial w}{\partial y_m} y_m \int w^2 \sum_{j=1}^{K}\sum_{k=1}^{2} a_{j,k}^{\epsilon}\left(-\frac{\partial}{\partial P_{l,m}^\epsilon}\frac{\partial}{\partial P_{j,k}^\epsilon}\frac{1}{2}F(\mathbf{P}^\epsilon)\right)$$

$$+ o\left(\frac{\epsilon^2}{\log\frac{1}{\epsilon}}\sum_{j=1}^{K}\sum_{k=1}^{2}|a_{j,k}^{\epsilon}|\right). \tag{8.22}$$

Note that

$$\int_{R^2} w^2 \frac{\partial w}{\partial y_m} y_m = -\frac{1}{3}\int_{R^2} w^3$$

Thus we have

$$l.h.s. = \frac{\epsilon^4 \xi_\epsilon \beta^2}{6}\hat{\xi}_{\epsilon,l}(\hat{\xi}_{\epsilon,j})^{-2}(\int_{R^2} w^3)(\int_{R^2} w^2)\sum_{j=1}^{K}\sum_{k=1}^{2} a_{j,k}^{\epsilon}\left(\frac{\partial}{\partial P_{l,m}^\epsilon}\frac{\partial}{\partial P_{j,k}^\epsilon}F(\mathbf{P}^\epsilon)\right) \tag{8.23}$$

$$+ o\left(\frac{\epsilon^2}{\log\frac{1}{\epsilon}}\sum_{j=1}^{K}\sum_{k=1}^{2}|a_{j,k}^{\epsilon}|\right).$$

Combining the $l.h.s.$ and $r.h.s$, we have

$$\frac{\epsilon^4 \xi_\epsilon \beta^2}{6}\hat{\xi}_{\epsilon,l}(\hat{\xi}_{\epsilon,j})^{-2}(\int_{R^2} w^3)(\int_{R^2} w^2)\sum_{j=1}^{K}\sum_{k=1}^{2} a_{j,k}^{\epsilon}\left(\frac{\partial}{\partial P_{l,m}^\epsilon}\frac{\partial}{\partial P_{j,k}^\epsilon}F(\mathbf{P}^\epsilon)\right)$$

$$+o\left(\frac{\epsilon^2}{\log\frac{1}{\epsilon}}\sum_{j=1}^{K}\sum_{k=1}^{2}|a^\epsilon_{j,k}|\right)$$

$$=\lambda_\epsilon a^\epsilon_{l,m}\hat{\xi}^2_{\epsilon,l}\int_{R^2}\left(\frac{\partial w}{\partial y_1}\right)^2 dy(1+o(1)). \tag{8.24}$$

Letting $\epsilon \to 0$ in (8.24), we see that the small eigenvalues with $\lambda_\epsilon \to 0$ satisfy $|\lambda_\epsilon| \sim \epsilon^4\xi_\epsilon\beta^2$. Furthermore,

$$\frac{\lambda_\epsilon}{\epsilon^4\xi_\epsilon\beta} \to \frac{\int_{R^2} w^3 \int_{R^2} w^2}{6\int_{R^2}(\frac{\partial w}{\partial y_1})^2 dy}\sigma_0$$

as $\epsilon \to 0$, where σ_0 is an eigenvalue of the matrix $\mathcal{D}^{-1}M(\mathbf{P}^0)\mathcal{D}^{-2}$, $\mathcal{D}$ is given by (6.37), and $\mathbf{P}^\epsilon \to \mathbf{P}^0$ as $\epsilon \to 0$. (The vector $\vec{a}^\epsilon = (a^\epsilon_{1,1}, a^\epsilon_{1,2}, ..., a^\epsilon_{K,2})^T$ approaches an eigenvector of $M(\mathbf{P}^0)$ corresponding to σ_0.) By condition (*) of Theorem 1.2, the matrix $M(\mathbf{P}^0)$ is negative definite. Therefore, we have Re $(\sigma_0) < 0$ and it follows that $\mathrm{Re}(\lambda_\epsilon) < 0$ if ϵ is small enough. Therefore the small eigenvalues λ_ϵ are stable for (7.1) if ϵ is small enough.

Completion of the proof of Theorem 1.2:

Theorem 1.2 now follows from Section 7 and Section 8.

□

Remark 8.1:

We have shown that the small eigenvalues with $\lambda \to 0$ satisfy $\lambda_\epsilon \sim C\frac{\epsilon^2}{\log\frac{1}{\epsilon}}$ with some $C > 0$. Furthermore, asymptotically, they are eigenvalues of the matrix $\mathcal{D}^{-1}M(\mathbf{P}^0)\mathcal{D}^{-2}$ and the coefficients $a^\epsilon_{j,k}$ are the corresponding eigenvectors. If the matrix $M(\mathbf{P}^0) = \frac{\partial^2}{\partial \mathbf{P}^2}F(\mathbf{P})|_{\mathbf{P}=\mathbf{P}^0}$ is strictly negative definite, it follows that $\mathrm{Re}(\lambda_\epsilon) < 0$ if ϵ is small enough.

An open question is whether or not a positive real eigenvalue of $M(\mathbf{P}^0)$ gives rise to a positive (small) eigenvalue λ_ϵ for the system. Similar questions for singularly perturbed Neumann problem, where the role of $M(\mathbf{P}^0)$ is replaced by the mean curvature function, have been studied in [3] and [36]. The main difficulty for the full Gierer-Meinhardt system is that we do not have a variational structure.

□

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Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: wei@math.cuhk.edu.hk

Fachbereich Mathematik, Universität Stuttgart, D-70511 Stuttgart, Germany
E-mail address: winter@mathematik.uni-stuttgart.de