UNIFORM ASYMPTOTIC BEHAVIOR OF INTEGRALS OF BESSEL FUNCTIONS WITH A LARGE PARAMETER IN THE ARGUMENT

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Summary

In this paper we deal with integrals whose integrand has a rapidly oscillating zero-order Bessel function of the first kind with real parameters in its argument which can become large. We introduce and tabulate model integrals which depend on a single parameter, which can determine the behavior of the original integral near the zeros of the argument of the Bessel function. As an example of the uniform asymptotic analysis, we evaluate the multi-parameter integral which arises in the solution of the transition problem for an accelerating moving load on an elastically supported infinite string. Asymptotic predictions are compared with the results obtained by direct numerical integration.

1. Introduction

The method of stationary phase is of particular importance for numerous applications in physics and engineering. It is studied in great detail and described in a number of the classical textbooks in the field of asymptotic analysis (e.g. see (**1**),(**2**)). At the same time uniform generalizations of this method are usually restricted to the integration of rapidly oscillating sinusoidal functions (see the monograph (**3**) and the more recent journal publications (**4**),(**5**)). The most famous example of a uniform stationary phase expansion is the situation when two stationary points merge. As a result, a more complicated asymptotic behavior which occurs is expressed in terms of the Airy function.

When integrating an oscillating Bessel function with a large parameter in its argument, the vicinities of the zeros of the argument require a special treatment (see (**3**), (**6**)). The point is that the well-known sinusoidal-type asymptotics of Bessel functions fail in this case and, therefore, the aforementioned techniques are not applicable. Thus, the peculiarity of uniform analysis of the integrals involving Bessel functions derives from the effect of various parameters on the behavior of the zeros of the argument.

In this paper we study uniform asymptotic behaviors of integrals with a rapidly oscillating zero-order Bessel function of the first kind J_0. The problem is specified in section 2, where we also define three model integrals (2.8), (2.11) and (2.14) with the argument depending on the real non-negative parameter α as $s\sqrt{\alpha+s}$, $\sqrt{s}(\alpha+s)$ and $s\sqrt{\alpha-s}$ (s is the integration parameter), respectively. It is shown how these model integrals arise from uniform asymptotic analysis. In section 3 for each of them we establish local asymptotics both for large and small values of the parameter; in doing so, a more sophisticated limit of large α is studied using the steepest descent method (e.g. see (**6**), (**7**)) implemented for the Hankel function.

In section 4 of the paper we apply, as an example, the model integrals investigated in section 3 for the evaluation of the multi-parameter integral of the Bessel function J_0 arising from the problem of an accelerating moving load on an elastically supported infinite string, previously investigated in (**8**, **9**). In this case the passage through the sound wave speed with a small acceleration is associated with the presence of a large parameter in the argument of the Bessel function. Two other problem parameters correspond to the speed of the load and its moving co-ordinate. Three specific combinations of the last two parameters are tackled by utilizing the model integrals.

Two sets of numerical computations are presented in the paper. The first of these is to tabulate the model integrals and test the accuracy of the local asymptotics in section 3. The second one deals with the application of these integrals to the derivation of uniform asymptotics in the problem for a string.

2. Statement of the problem

We begin with the integral

$$\mathcal{F}(\nu) = \int_a^b J_0(\nu f(p))dp, \ b > a, \tag{2.1}$$

where ν is a large real parameter and J_0 denotes the zero-order Bessel function of the first kind.

Away from the zeros of the argument $f(p)$, the Bessel function of the integrand (2.1) behaves as (**10**)

$$J_0(\nu f(p)) \sim \sqrt{\frac{2}{\pi\nu f(p)}} \cos\left(\nu f(p) - \frac{\pi}{4}\right), \ \nu \gg 1. \tag{2.2}$$

As a result, the integral (2.1) can be evaluated using the standard method of stationary phase.

Let us now assume that $f(0) = 0$, $f'(a) > 0$ and $f(p) = f'(a)(p-a) + ...$ ($|p-a| \ll 1$). Assuming for the sake of simplicity that the function $f(p)$ has no stationary points and zeros over the domain of integration in (2.1) (see (**3**), (**6**) for further details) we have after the substitution $s = \nu f'(a)(p-a)$

$$\mathcal{F}(\nu) \sim \frac{1}{\nu f'(a)} \int_0^{\nu f'(a)(b-a)} J_0(s)ds. \tag{2.3}$$

Finally, we get for $\nu \gg 1$ (**11**)

$$\mathcal{F}(\nu) \sim \frac{1}{\nu f'(a)} \int_0^{\infty} J_0(s)ds = \frac{1}{\nu f'(a)}. \tag{2.4}$$

Thus, the contribution of the zeros of the J_0 argument is of the same asymptotic order

$O(\nu^{-1})$ as that of ordinary stationary phase points (the additional factor $\nu^{-1/2}$ comes from the asymptotic formula (2.2)).

In this paper we investigate integrals of the type

$$\mathcal{F}(\nu, \beta) = \int_a^b J_0(\nu f(p, \beta))dp, \tag{2.5}$$

with an extra real parameter β.

Our main focus is the uniform asymptotic analysis in terms of the parameters ν and β, dealing in particular with the dominant contributions of the J_0 zeros, which cannot be reduced to well known uniform generalizations of the stationary phase method including, for example, the Airy function (e.g. see (**3**) and reference therein).

To this end, we introduce canonical integrals that play, in our considerations, the same role as the above mentioned Airy function (and some others) do in the well established case of oscillating sinusoidal functions.

If, for example, in (2.5)

$$f(p, \beta) = p\sqrt{p + \beta}, \ \beta \geq 0, \tag{2.6}$$

and the limits of integration are $a = 0$ and $b = \infty$, we have after the substitution $p = \nu^{-2/3}s$

$$\mathcal{F}(\nu, \beta) = \nu^{-2/3}\mathcal{F}_1(\alpha), \tag{2.7}$$

where

$$\mathcal{F}_1(\alpha) = \int_0^\infty J_0(y\sqrt{\alpha + s})ds. \tag{2.8}$$

Here and below $\alpha = \beta\nu^{2/3}$ is a real non-negative parameter.

For the same limits of integration in (2.5) and with

$$f(p, \beta) = (p + \beta)\sqrt{p}, \ \beta \geq 0, \tag{2.9}$$

we get

$$\mathcal{F}(\nu, \beta) = \nu^{-2/3}\mathcal{F}_2(\alpha), \tag{2.10}$$

where

$$\mathcal{F}_2(\alpha) = \int_0^\infty J_0(\sqrt{s}(\alpha + s))ds. \tag{2.11}$$

The last canonical integral considered in the paper arises from letting

$$f(p, \beta) = p\sqrt{\beta - p}, \ \beta \geq 0 \tag{2.12}$$

with the limits $a = 0$ and $b = \beta$.

In this case

$$\mathcal{F}(\nu,\beta)=\nu^{-2/3}\mathcal{F}_3(\alpha), \tag{2.13}$$

where

$$\mathcal{F}_3(\alpha)=\int\limits_0^{\alpha} J_0(s\sqrt{\alpha-s})ds. \tag{2.14}$$

In more general situations when the formula (2.6), (2.9) and (2.12) correspond to the local approximations of the Bessel function argument near its zeros and for arbitrary limits of integration we may expect that the canonical integrals (2.8), (2.11) and (2.14) will appear as the leading order terms in related asymptotic expansions. In section 4 of the paper all of these integrals naturally arise in the moving load problem for a string.

3. Canonical integrals

The behavior of the argument of the Bessel function in all the canonical integrals (2.8), (2.11) and (2.14) is strongly affected by the parameter α. In particular, in (2.8) and (2.11) it has, respectively, the limiting forms $\alpha^{1/2}y$ and $\alpha y^{1/2}$ for $\alpha \gg 1$ and tends to $y^{3/2}$ for $\alpha \ll 1$ in both integrals. In (2.14) the argument of J_0 is uniformly small for $\alpha \ll 1$, whereas it takes large values, outside the vicinities of the end points, in this integral for $\alpha \gg 1$.

Let us study in greater detail the asymptotic behavior of the functions $\mathcal{F}_i$ $(i=1,2,3)$ in the domain of small and large values of the parameter α. It is clear that (see e.g. (**11**))

$$\lim_{\alpha\to 0}\mathcal{F}_j(\alpha)=\int\limits_0^{\infty} J_0(s^{3/2})ds=\frac{2\sqrt{\pi}}{3\Gamma\left(5/6\right)},\ j=1,2. \tag{3.1}$$

It is also evident that

$$\mathcal{F}_3(\alpha)\sim\alpha \text{ as } \alpha\ll 1, \tag{3.2}$$

since $J_0(s\sqrt{\alpha-s})\sim 1$.

Asymptotic analysis for $\alpha \gg 1$ requires more delicate calculations. Let us express first the functions $\mathcal{F}_i$ $(i=1,2,3)$ in terms of integrals of the Hankel function $H_0^{(1)}$. Changing variables in (2.8), (2.11) and (2.14) by the formulae $y=-\alpha(z^2+1)$, $y=\alpha z^2$ and $y=\alpha(z^2+1)$, respectively, we have

$$\mathcal{F}_1(\nu)=-2\nu^{2/3}\mathrm{Re}\int\limits_i^{i\infty} H_0^{(1)}(i\nu h)zdz \tag{3.3}$$

$$\mathcal{F}_2(\nu)=2\nu^{2/3}\mathrm{Re}\int\limits_0^{\infty} H_0^{(1)}(\nu h)zdz \tag{3.4}$$

and

$$\mathcal{F}_3(\nu) = 2\nu^{2/3}\mathrm{Re}\int_{-i}^{0} H_0^{(1)}(i\nu h)zdz \tag{3.5}$$

where $\nu = \alpha^{3/2} \gg 1$ and $h(z) = z(z^2+1)$

The asymptotic behavior of the Hankel function in (3.3) - (3.5) for $\nu|h| \gg 1$ is given by (see (**10**))

$$H_0^{(1)}(i\nu h) \sim -i\sqrt{\frac{2}{\pi\nu h}}e^{-\nu h} \tag{3.6}$$

and

$$H_0^{(1)}(\nu h) \sim e^{-\frac{\pi i}{4}}\sqrt{\frac{2}{\pi\nu h}}e^{i\nu h}. \tag{3.7}$$

The exponentials in the right-hand sides of the formulae (3.6) - (3.7) motivate making use of the steepest descent method (e.g. see (**6**), (**7**)) when evaluating the original integrals (3.3) - (3.5). To this end we introduce a complex variable $z = x + iy$ having

$$h(x+iy) = h_r(x,y) + ih_i(x,y) \tag{3.8}$$

where

$$\begin{aligned} h_r(x,y) &= \mathrm{Re}h(x+iy) = x(x^2-3y^2+1), \\ h_i(x,y) &= \mathrm{Im}h(x+iy) = y(3x^2-y^2+1). \end{aligned} \tag{3.9}$$

In case of the function $\mathcal{F}_1$, we present the integral (3.3) as (see Fig. 1)

$$\int\limits_{C_1} = \int\limits_{C_{11}} + \int\limits_{C_{12}} . \tag{3.10}$$

where C_1 is the original path of integration in (3.3), C_{11} is the steepest descent path through the point $z = i$ corresponding to the exponential in (3.6) and C_{12} is the path along the circle of an infinitely large radius. Here and below we omit integrands in all symbolic formulae.

Along the steepest descent path C_{11} we have $\mathrm{Im}h(z) = \mathrm{Im}h(i) = 0$ (see (3.6) and (3.9)). Therefore

$$y = \sqrt{3x^2+1}. \tag{3.11}$$

We start from the first integral in (3.10). Near the end point $z = i$ we get $h(z) \approx -2x,\ z \approx i$ and $dz \approx dx$. Thus,

$$\int\limits_{C_{11}} \sim i\int_{0}^{-\infty} H_0^{(1)}(-2i\nu x)dx. \tag{3.12}$$

It is clear that the contribution of the integral along C_{12} vanishes. Then, by substituting $x_1 = -2\nu x$ in (3.12) we obtain from (3.10)

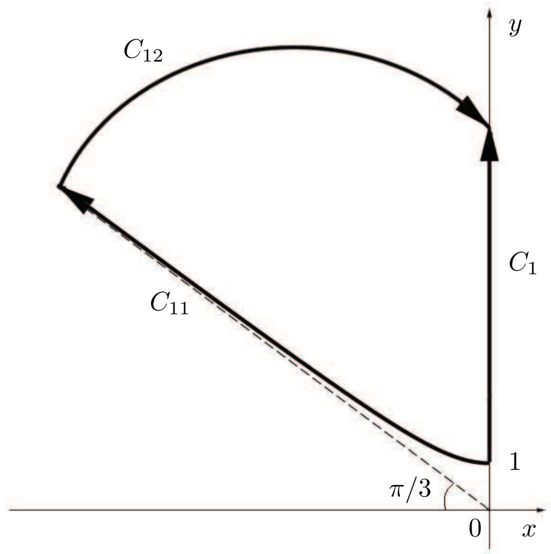

Fig. 1 Contour integration in (3.3)

$$\int_{C_1} \sim -\frac{i}{2\nu}\int_0^\infty H_0^{(1)}(ix_1)dx_1 = -\frac{1}{\pi\nu}\int_0^\infty K_0(x_1)dx_1 = -\frac{1}{2\nu} \tag{3.13}$$

where K_0 - denotes the Macdonald function. Finally, we have from (3.3)

$$\mathcal{F}_1(\alpha) \sim \frac{1}{\sqrt{\alpha}} \tag{3.14}$$

To establish the asymptotic behavior of the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ we need to calculate the saddle points of the function $h(z)$. By setting $h'(z) = 0$ we obtain $3z^2 + 1 = 0$. The saddle points become $z_{1,2} = \pm\frac{i}{\sqrt{3}}$.

Next, consider the integral (3.4). The steepest descent path through the saddle point $z_1 = \frac{i}{\sqrt{3}}$ is determined by the condition $\mathrm{Re}h(z) = \mathrm{Re}h(\frac{i}{\sqrt{3}}) = 0$ (see (3.7) and (3.9)) resulting in

$$y = \frac{1}{\sqrt{3}}\sqrt{x^2+1} \tag{3.15}$$

Similarly to (3.10) we present the integral (3.4) as (see Fig. ...)

$$\int_{C_2} = \int_{C_{21}} + \int_{C_{22}} + \int_{C_{23}} \tag{3.16}$$

where C_{21} is the part of the imaginary axis between the points $z = 0$ and $z = i/\sqrt{3}$, C_{22} is the steepest descent path and C_{23} is the path along the circle of an infinitely large radius.

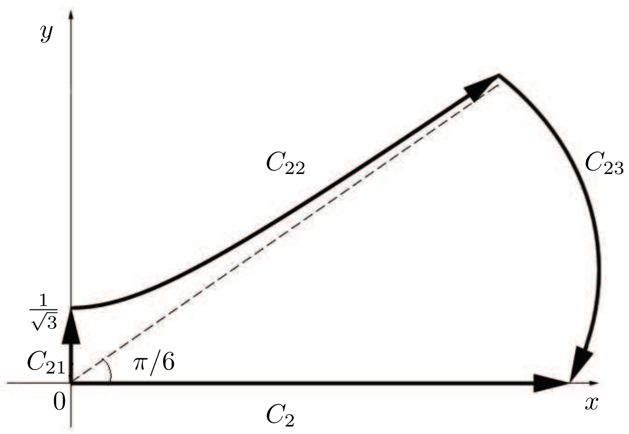

Fig. 2 Contour integration in (3.4)

Along the path C_{21} we get $z = iy$ and $h(z) = i(1 - y)$. Then

$$\int\limits_{C_{21}} = -\int\limits_{0}^{\frac{1}{\sqrt{3}}} H_0^{(1)}\left[i(1-y^2)\right] dy. \tag{3.17}$$

The real part of the last integral is equal to zero and it does not affect the asymptotic behavior of $\mathcal{F}_2$ (see (3.4)). By using the formula (3.7) we obtain

$$\int\limits_{C_{22}} \sim -i\sqrt{\frac{2}{\pi\nu}} \int\limits_{0}^{\infty} \frac{1}{h_i} \exp\left(-\nu h_i\right)(x+iy)\left(1+i\frac{dy}{dx}\right) dx. \tag{3.18}$$

where the steepest descent path $y(x)$ is given by (3.15) whereas

$$\frac{dy}{dx} = \frac{x}{\sqrt{3(x^2+1)}}, \; h_i = \frac{2}{3\sqrt{3}}\sqrt{x^2+1}(4x^2+1).$$

By applying the Laplace method (e.g. see (**6**), (**7**)) in (3.18) we arrive at the sought for asymptotic formulae. It is

$$\int\limits_{C_2} \sim \exp\left(-\frac{2\nu}{3\sqrt{3}}\right) \frac{1}{\sqrt{\pi\nu}} \int\limits_{0}^{\infty} \left(-\sqrt{3}\nu x^2\right) dx = \frac{1}{2\nu} \exp\left(-\frac{2\nu}{3\sqrt{3}}\right). \tag{3.19}$$

Now, by inserting (3.19) into (3.4) we get

$$\mathcal{F}_2(\alpha) = \frac{1}{\alpha^{1/2}} \exp\left(-\frac{2\alpha^{3/2}}{3\sqrt{3}}\right). \tag{3.20}$$

The path of integration for the function $\mathcal{F}_3$ is shown in Fig. 3 . Here the path C_{31} goes along the real axis, the path C_{32} goes along the steepest descent pathes associated with the saddle point $z = -\frac{i}{\sqrt{3}}$ and C_{33} is the steepest descent path through the point $z = -i$.

Along the path C_{32} we have $\text{Im}h(z) = \text{Im}h(-\frac{i}{\sqrt{3}}) = -\frac{2}{3\sqrt{3}}$ and

$$x = \left(y + \frac{1}{\sqrt{3}}\right)\sqrt{\frac{y - 2/\sqrt{3}}{3y}} \tag{3.21}$$

The equation of the path C_{33} follows from the condition $\text{Im}h(z) = \text{Im}h(-i) = 0$. The result is

3.22)

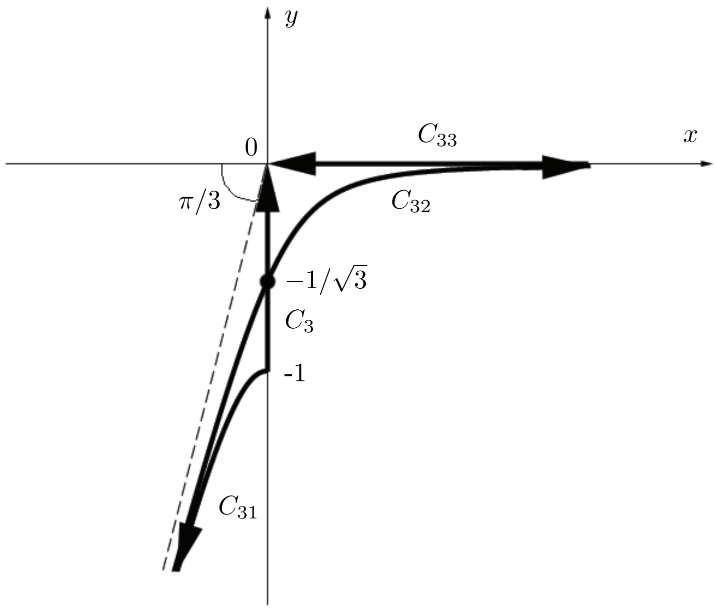

Fig. 3 Contour integration in (3.5)

As above, we present the studied integral as a sum, i.e.

$$\int_{C_3} = \int_{C_{31}} + \int_{C_{32}} + \int_{C_{33}} \tag{3.23}$$

Along the path C_{31}, we get $y = 0$ and $h(z) = x(x^2 + 1)$. The integral

$$\int\limits_{C_{31}} = \int\limits_{0}^{\infty} H_0^{(1)} \left[ix(x^2+1)\right] x dx \tag{3.24}$$

takes an imaginary value and does not contribute to the function $\mathcal{F}_3$.

The integral along the path C_{33} is similar to that along C_{11} (see (3.12)). In this case

$$\int\limits_{C_{31}} \sim -i \int\limits_{0}^{-\infty} H_0^{(1)}(-2i\nu x) dx = \frac{1}{2\nu}. \tag{3.25}$$

Near the saddle point $z = -\frac{i}{\sqrt{3}}$, we derive from (3.21) $y \approx -\frac{1}{\sqrt{3}} + x$, $z \approx -\frac{i}{\sqrt{3}}$, $dz \approx (1+i)dx$ and $h(z) \approx 2\sqrt{3}x^2 - \frac{2}{3\sqrt{3}}i$. Then, we obtain

$$\int\limits_{C_{32}} \sim i\sqrt{\frac{2\sqrt{3}}{\pi\nu}} \exp\left(-i\frac{2\nu}{3\sqrt{3}}\right) \int\limits_{-\infty}^{+\infty} e^{-2\sqrt{3}\nu x^2} dx \tag{3.26}$$

and

$$\int\limits_{C_3} \sim \frac{1}{\nu^{1/2}} \left[\frac{1}{2} + i\exp\left(-i\frac{2\nu}{3\sqrt{3}}\right)\right]. \tag{3.27}$$

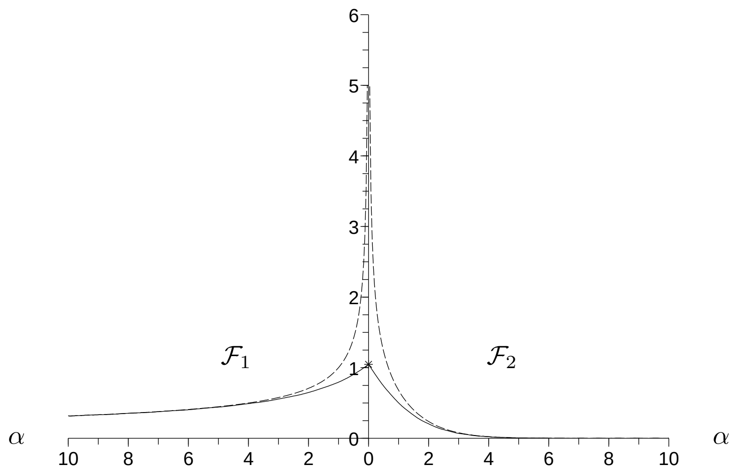

Fig. 4 The function $\mathcal{F}_1$ and $\mathcal{F}_2$. Asymptotics (dashed line and asterisk) and numerics (solid line).

By substituting (3.27) into (3.5) we arrive at the formula

$$\mathcal{F}_3(\alpha) \sim \frac{1}{\alpha^{1/2}} \left(1 + 2\sin\left(\frac{2\alpha^{3/2}}{3\sqrt{3}}\right)\right) \tag{3.28}$$

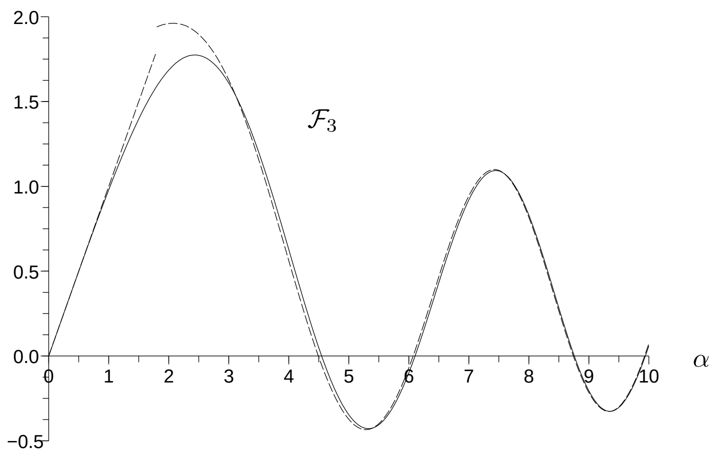

Fig. 5 The function $\mathcal{F}_3$. Asymptotics (dashed line) and numerics (solid line).

Further, we may expect that the canonical integrals (2.8), (2.11) and (2.14) describe uniform asymptotic behavior of more complicated integrals of this type for the case in which the intermediate range $\alpha \sim 1$ is also of interest.

A comparison of the asymptotics of the functions $\mathcal{F}_i (i = 1, 2, 3)$ with the results of numerical computations for the integrals (2.8), (2.11) and (2.14) is presented in Figs 4 and 5. The solid line in the first and second quadrants of Fig 4 corresponds to the computed values of the integrals (2.11) and (2.8), respectively. The asymptotics (3.14) and (3.20) are plotted in this figure by the dashed line. In addition, the limiting value (3.1) is denoted by an asterisk. In Fig 5 the results of the numerical evaluation of the integral (2.14) (solid line) are shown with its asymptotic forms (3.2) and (3.28) (dashed line). The tabulated values of the functions $\mathcal{F}_i$ are also displayed in Table 1.

Here and below we use MatLab for calculating integrals.

4. An elastically supported string under accelerating load.

As an example, consider an infinite string lying on an elastic support and subject to a point force uniformly accelerating from rest (see Fig 6). An appropriately nondimesionalised equation of motion can be written as (e.g. see (**8**), (**9**) for more details)

$$-\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \tau^2} + w = \delta(\xi - \frac{1}{2} a \tau^2) \qquad (4.1)$$

where τ is time, ξ is coordinate, a is acceleration, w is transverse displacement and δ denotes Dirac's delta; in doing so, dimensionless sound wave speed, stiffness of the elastic support and magnitude of the moving load all take unit values.

The solution of the equation (4.1) with homogeneous initial data can be expressed as

α	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	α	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$
0.0	1.04713	1.04713	0.00000	2.6	0.58912	0.11115	1.76045
0.1	1.01559	0.98280	0.10000	2.7	0.58088	0.09995	1.73827
0.2	0.98573	0.91931	0.19997	2.8	0.57263	0.08843	1.70562
0.3	0.95731	0.85697	0.29983	2.9	0.56431	0.07774	1.66250
0.4	0.93014	0.79721	0.39947	3.0	0.55592	0.06911	1.60906
0.5	0.90419	0.74135	0.49870	3.1	0.54759	0.06270	1.54554
0.6	0.87948	0.68948	0.59730	3.2	0.53947	0.05738	1.47232
0.7	0.85611	0.64039	0.69501	3.3	0.53174	0.05167	1.38993
0.8	0.83418	0.59273	0.79150	3.4	0.52454	0.04511	1.29901
0.9	0.81374	0.54625	0.88640	3.5	0.51796	0.03857	1.20037
1.0	0.79476	0.50200	0.97932	3.6	0.51197	0.03343	1.09492
1.1	0.77713	0.46135	1.06979	3.7	0.50648	0.03026	0.98372
1.2	0.76065	0.42469	1.15733	3.8	0.50131	0.02824	0.86796
1.3	0.74507	0.39101	1.24142	3.9	0.49626	0.02589	0.74891
1.4	0.73014	0.35877	1.32152	4.0	0.49116	0.02241	0.62798
1.5	0.71564	0.32725	1.39704	4.1	0.48587	0.01832	0.50663
1.6	0.70144	0.29713	1.46740	4.2	0.48036	0.01503	0.38642
1.7	0.68751	0.26977	1.53200	4.3	0.47467	0.01345	0.26893
1.8	0.67391	0.24590	1.59023	4.4	0.46893	0.01316	0.15576
1.9	0.66077	0.22483	1.64149	4.5	0.46332	0.01276	0.04852
2.0	0.64826	0.20502	1.68519	4.6	0.45802	0.01116	-0.05122
2.1	0.63651	0.18539	1.72079	4.7	0.45315	0.00852	-0.14195
2.2	0.62562	0.16623	1.74774	4.8	0.44880	0.00612	-0.22225
2.3	0.61556	0.14884	1.76558	4.9	0.44492	0.00515	-0.29082
2.4	0.60625	0.13423	1.77387	5.0	0.44142	0.00561	-0.34651
2.5	0.59751	0.12214	1.77225				

Table 1 Tabulated values of canonical integrals

$$w(\nu,\ \lambda,\ u) = \nu \int_0^u J_0(\nu\phi(u,\lambda,z))H(\phi^2(u,\lambda,z))dz \tag{4.2}$$

with

$$\phi(u,\lambda,z) = \sqrt{z^2 - (\lambda + uz - z^2/2)^2} \tag{4.3}$$

where H denotes Heaviside step function. The last integral depends on three problem parameters including the load speed $u = a\tau$, the moving coordinate $\lambda = \xi - \frac{1}{2}a\tau^2$, and the parameter $\nu = 1/a$. The large values of the parameter ν ($\nu \gg 1$) are associated with dynamic phenomena observing when the load speed passes through the critical value $u = 1$, i.e. the sound wave speed in a string, with a small acceleration a. Non-uniform asymptotics of the function (4.2) were derived in (**8**).

Below we investigate the string behavior at the moving point $\lambda = 0$, where the force is applied and also at the moving singularity $\lambda = -\frac{(u-1)^2}{2}$. The latter arises when passing through the sound speed and coincides with the point $\lambda = 0$ at $u = 1$. The aforementioned

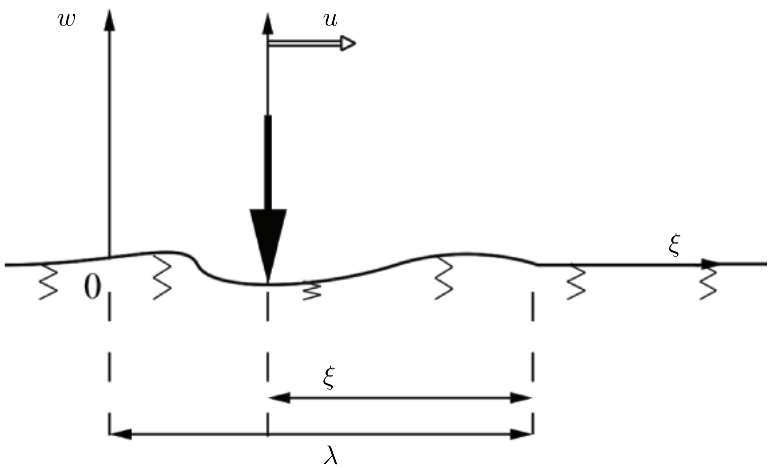

Fig. 6 Elastically supported string under moving load

moving points are of the main interest when investigating the passage through the sound wave barrier. Below we study three combinations of the problem parameters.

(i) *The displacement under the load before the passage* ($u \le 1$ *and* $\lambda = 0$).
In this case the original integral in (4.2) becomes

$$I = \int_0^u J_0(\nu\phi(u,0,t))dt, \tag{4.4}$$

with

$$\phi(u,0,t) = t\sqrt{1 - u^2 + t - t\left[(1-u) + \frac{t}{4}\right]}. \tag{4.5}$$

In the vicinity of the end point $t = 0$ in the integral (4.4) we have $\phi(u,0,t) \approx t\sqrt{1-u^2}$, if $t \ll 1 - u$. Otherwise, for $1 - u \ll t \ll 1$ we get $\phi(u,0,t) \approx t^{3/2}$. The formula

$$\phi(u,0,t) \approx t\sqrt{1 - u^2 + t}, \ t \ll 1 \tag{4.6}$$

contains both of the limiting forms.
As above, to the leading order we may substitute infinity at the upper limit of the last integral. Finally, we obtain a simpler integral, which is

$$I \sim \int_0^\infty J_0(\nu t\sqrt{1-u^2+t})dz. \tag{4.7}$$

Next, changing the independent variable by $t_1 = t\nu^{2/3}$ we establish the sought for uniform asymptotic behavior in the parameters ν and u

$$I \sim \nu^{-2/3} \int_0^\infty J_0(t_1\sqrt{\nu^{2/3}(1-u^2)+t_1})dt_1, \tag{4.8}$$

or

$$I \sim \nu^{-2/3}\mathcal{F}_1(\eta). \tag{4.9}$$

where the fundamental parameter η determines the scaling of the problem. It is given by

$$\eta = \nu^{2/3}(1-u^2) \tag{4.10}$$

Outside the characteristic zone $\eta \sim 1$ $(1-u^2 \sim \nu^{-2/3})$ the function $\mathcal{F}_1$ in (4.9) can be reduced to the local forms (3.1) for $\eta \ll 1$ $(1-u^2 \ll \nu^{-2/3})$ and (3.14) for $\eta \gg 1$ $(1-u^2 \gg \nu^{-2/3})$. Such an observation is relevant for other integrals considered below in this section.

(ii) *The displacement under the load after the passage* ($u \geq 1$ *and* $\lambda = 0$).
Now, we get

$$I = \int_{2(u-1)}^{u} J_0(\nu\phi(u,0,t))dt, \tag{4.11}$$

with

$$\phi(u,0,t) = \frac{t}{2}\sqrt{(t-2(u-1))(2(u+1)-t)}. \tag{4.12}$$

In this case we present the function $\phi(u,0,t)$ near the end point $t = 2(u-1)$ as

$$\phi(u,0,t) \approx t\sqrt{t-2(u-1)}. \tag{4.13}$$

After changing the independent variable by

$$t_1 = \nu^{2/3}(t-2(u-1)), \tag{4.14}$$

we obtain

$$I \sim \nu^{-2/3} \int_0^\infty J_0((t_1+2\nu^{2/3}(u-1))\sqrt{t_1})dt_1 = \nu^{-2/3}\mathcal{F}_2(2\nu^{2/3}(u-1)). \tag{4.15}$$

This asymptotics is of interest only over the narrow vicinity of the sound wave speed $(u-1 \sim \nu^{-2/3})$ due to exponential decay of the function $\mathcal{F}_2$. In this case, we may introduce the parameter η in the last formula setting $2-u \approx u^2-1$.

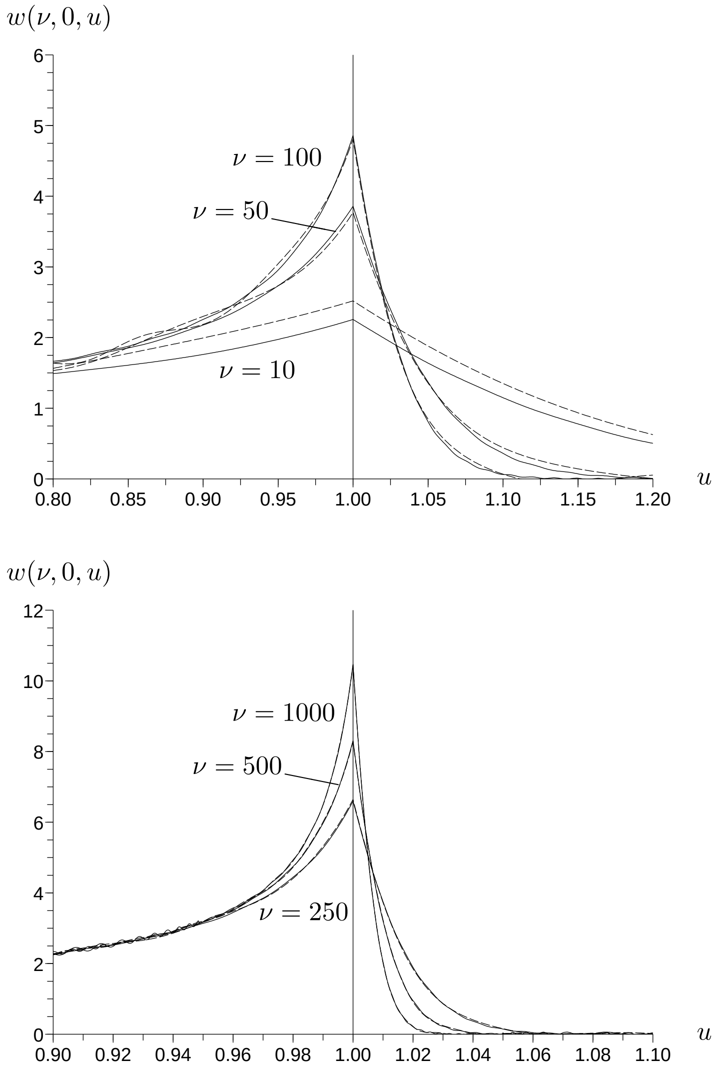

Fig. 7 Uniform asymptotics (solid line) and numerics (dashed line) of the function (4.2) using integrals (4.4) ($u \leq 1$) and (4.9) ($u \geq 1$).

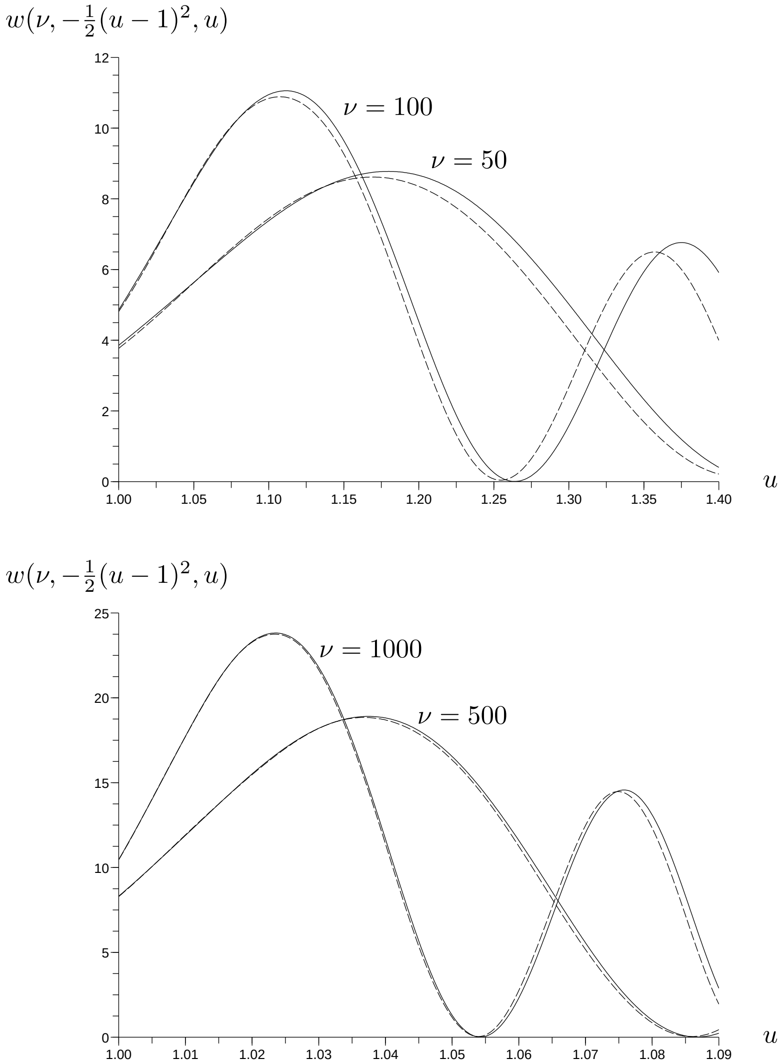

Fig. 8 Uniform asymptotics (dashed line) and numerics (solid line) of the function (4.2) using integrals (4.16).

(iii) *The displacement at the moving singularity* ($u \geq 1$ *and* $\lambda = -\frac{1}{2}(u-1)^2$).
We have in (4.2)

$$I = I_1 + I_2, \tag{4.16}$$

with

$$I_1 = \int\limits_{u-1}^{u} J_0(\nu\phi(u, -\frac{1}{2}(u-1)^2), t))dt, \tag{4.17}$$

and

$$I_2 = \int\limits_{(\sqrt{u}-1)^2}^{u-1} J_0(\nu\phi(u, -\frac{1}{2}(u-1)^2), t))dt, \tag{4.18}$$

where

$$\phi(u, -\frac{1}{2}(u-1)^2, t) = \pm(t-(u-1))\sqrt{-\frac{1}{4}(t-(\sqrt{u}-1)^2)(t-(\sqrt{u}+1)^2)}. \tag{4.19}$$

Here and below the signs "+" and "−" correspond to the integrals I_1 and I_2, respectively.
Let us restrict ourselves to the parameter range in which the values of u are close enough to the value $u = 1$. In this case we may set $\sqrt{u}+1 \approx 2$. Next, we expand the function (4.19) near the left end point in the integral (4.17) and over the whole integration domain in the integral (4.18). It becomes

$$\phi(u, -\frac{1}{2}(u-1)^2, t) \approx \pm(t-(u-1))\sqrt{t-(\sqrt{u}-1)^2}.$$

Now, we change the variables in the integrals (4.17) and (4.18) by the formula

$$t_1 = \pm\nu^{2/3}(t-(u-1)). \tag{4.20}$$

The result is

$$I_1 \sim \nu^{-2/3}\int\limits_0^{\infty} J_0(t_1\sqrt{t_1+2(\sqrt{u}-1)\nu^{2/3}})dt_1 = \nu^{-2/3}\mathcal{F}_1(2(\sqrt{u}-1)\nu^{2/3}), \tag{4.21}$$

and

$$I_2 \sim \nu^{-2/3}\int\limits_0^{2(\sqrt{u}-1)\nu^{2/3}} J_0(t_1\sqrt{2(\sqrt{u}-1)\nu^{2/3}-t_1})dt_1 = \nu^{-2/3}\mathcal{F}_3(2(\sqrt{u}-1)\nu^{2/3}). \tag{4.22}$$

Finally,

$$I \sim \nu^{-2/3}[\mathcal{F}_1(2\nu^{2/3}(\sqrt{u}-1)) + \mathcal{F}_3(2\nu^{2/3}(\sqrt{u}-1))]. \tag{4.23}$$

Similar to the previous case, we may operate with the functions $\mathcal{F}_j(-\frac{1}{2}\eta)$ $(j = 1,3)$ in the vicinity of the sound wave speed where $\sqrt{u}-1 \approx \frac{1}{4}(u^2-1)$.

Numerical examples are presented in Figs 7 and 8. In Fig 7 we display the computed values of the function (4.2) using the formulae (4.4) and (4.11) (dashed line) along with its uniform asymptotic behaviors given by the formulae (4.9) and (4.15) (solid line). Graphs of the function (4.2) in case of integral (4.16) (dashed line) and uniform asymptotic formula (4.23) (solid line) are plotted in Fig 8.

These figures illustrate uniform validity of the derived asymptotic formulae in case of the moving load problem for a string. The striking difference between asymptotic behaviors of the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ for $\alpha \gg 1$ leads to a strong asymmetry of the resonant curves in Fig. 7. In Fig. 8 the function $\mathcal{F}_3$ reproduces oscillatory patterns associated with the passage through sound wave barrier.

5. Concluding remarks

In this paper we define canonical integrals (2.8), (2.11) and (2.14) involving the Bessel function with a real parameter in its argument. They demonstrate a potential for constructing uniform asymptotic expansions due to their distinct limiting behavior at large and small values of the parameter.

These integrals are successfully applied to the evaluation of the three-parameter integral (4.2). The large values of one of the parameters in that integral correspond to the case of a slowly accelerating point load on an infinite string resting on an elastic foundation.

The derived asymptotic formulae are useful not only for a numerical investigation of the passage through the sound wave speed, but also allow a better physical insight. In particular they provide an important information on the scaling law. In fact, all the uniform asymptotic formulae in Section 4 involve the parameter η (see (4.10)) expressing the scaling of the analyzed phenomenon.

The developed methodology may be easily extended to deal with other types of oscillating Bessel functions with arguments of different forms, that will result in making use of other canonical integrals. In addition, the applications of the uniform asymptotics of the integrals of Bessel functions are expected to have a wider application than to the here considered problem for a string.

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