SOME SCATTERING

AND

SLOSHING PROBLEMS IN

LINEAR WATER WAVE

THEORY

A THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

BY

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TABLE OF THE NUMERICAL TECHNIQUES USED, AND THE GEOMETRIES CONSIDERED

	Matched asymptotic expansions	Eigen-function expansion	Wide-spacing approximation	Small-body approximation	Boundary element method
Surface piercing barrier	Sections 2.3 & 4.3	Section 3.2	Section 3.4(a)	Section 3.4(b)	
Bottom mounted barrier	Section 2.2	Section 3.2			
Circular cylinder	Section 4.2		Section 3.4(a)	Section 3.4(b)	Section 3.3
Elliptic cylinder	Section 4.2				Section 3.3
Two circular cylinders with different radii	Sections 6.2 & 6.3				
One circular and one elliptic cylinder	Section 6.3				

ABSTRACT

Using the method of matched asymptotic expansions the reflection and transmission coefficients are calculated for scattering of oblique water waves by a vertical barrier. Here an assumption is made that the barrier is small compared to the wavelength and the depth of water.

A number of sloshing problems are considered. The eigenfrequencies are calculated when a body is placed in a rectangular tank. Here the bodies considered are a vertical surface-piercing or bottom-mounted barrier, and circular and elliptic cylinders.

When the body is a vertical barrier, the eigenfunction expansion method is applied. When the body is either a circular or elliptic cylinder, and the motion is two-dimensional, the boundary element method is applied to calculate the eigenfrequencies. For comparison, two approximations, "a wide-spacing", and "a small-body" are used for a vertical barrier and circular cylinder. In the wide-spacing approximation, the assumption is made that the wavelength is small compared with the distance between the body and walls. The small-body approximation means that a typical dimension of the body is much larger than the cross-sectional length scale of the fluid motion.

For an elliptic cylinder, the method of matched asymptotic expansions is used and compared with the result of the boundary- element method. Also a higher-order solution is obtained using the method of matched asymptotic expansions, and it is compared with the exact solution for a surface-piercing barrier. Again the assumption is made that the length scale of the motion is much larger than a typical body dimension.

Finally, the drift force on multiple bodies is considered. The ratio of horizontal drift force in the direction of wave advance on two cylinders to that on an isolated cylinder is calculated. The method of matched asymptotic expansions is used under the assumption that the wavelength is much greater than the cylinder spacing.

CONTENTS

CHAPTER 1

INTRODUCTION

The behaviour of water waves is familiar from everyday experience. For example, when oil or liquid natural gas is carried by ships it will slosh within its container, and this may affect the ship's stability. Sloshing of liquid propellents in space craft and missiles can also cause stability problems. It is therefore desirable to avoid external excitation at the sloshing frequencies of the container by suitable choice of its dimensions. Here I will calculate the natural small-amplitude frequencies, and not the amplitude of the sloshing motions for which viscosity and non-linearity are both important. Hence the theory of linear water waves is applied, as described by several authors, for example Milne-Thomson (1968, pp.426-428).

In linear theory, we neglect viscosity, compressibility and surface tension. The effects of air movement in the atmosphere, too, will be neglected, being replaced by a uniform pressure p_0 on the upper surface of the water. With these assumptions, the velocity vector $\underline{V}(x, y, z, t)$ satisfies

$$\text{curl } \underline{V} = 0 ,$$

and so

$$\underline{V} = \nabla\Phi ,$$

where Cartesian co-ordinates (x , y , z) are chosen with x, z horizontal in the undisturbed free surface and y vertically upwards, and Φ is a velocity potential. The equation of continuity

$$\nabla . \underline{V} = 0 ,$$

implies that

$$\nabla^2\Phi = 0 , \tag{1.1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

The pressure on the surface derived from Bernoulli's equation for unsteady potential flow is given by

$$\frac{\partial\Phi}{\partial t} + \frac{p_0}{\rho} + \frac{1}{2}|\underline{V}^2| + gy = F(t) \tag{1.2}$$

on the surface $y = \eta$. Here ρ is a fluid density and g the acceleration due to gravity. For small amplitude waves, we neglect $\frac{1}{2}\underline{V}^2$ as a small term; and we may take $F(t)$ and $\frac{p_0}{\rho}$ into the potential Φ, where they do not affect any velocities. This leaves us with

$$\left(\frac{\partial\Phi}{\partial t}\right)_{y=\eta} + g\eta = 0 .$$

Since η is small we use Taylor's theorem to approximate the first term by its value at $y =$ 0. The boundary condition is thus taken to be

$$\left(\frac{\partial\Phi}{\partial t}\right)_{y=0} + g\eta = 0 .$$

Also there is a relation between surface motion and fluid velocity which states that free surface particles remain in the free-surface. Thus the full non-linear condition is

$$v - \frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} - w\frac{\partial \eta}{\partial z} = 0 \,,$$

where $\underline{V} = (u, v, w)$. Now We linearise this equation to give

$$\frac{\partial \eta}{\partial t} = \left(\frac{\partial \Phi}{\partial y}\right)_{y=0}$$

again evaluated at $y = 0$. From both of conditions, η can be eliminated to give

$$\frac{\partial^2 \Phi}{\partial t^2} + g\,\frac{\partial \Phi}{\partial y} = 0 \,, \text{ on } y = 0 \,. \tag{1.3}$$

For time-harmonic motions of angular frequency ω we write

$$\Phi(x\,,\,y\,,\,z\,,\,t) = \text{Re}\left(\,\phi(x\,,\,y\,,\,z)\,e^{-i\omega t}\,\right), \tag{1.4}$$

and

$$\eta(x\,,\,z\,,\,t) = \text{Re}\left(\,\eta(x\,,\,z)\,e^{-i\omega t}\,\right), \tag{1.5}$$

where Re stands for the real part with respect to i. We then have the boundary-value problem

$$\nabla^2 \phi = 0 \,, \tag{1.6}$$

in the fluid, and the free-surface condition is given by substituting equation (1.4) into (1.3) resulting in

$$\frac{\partial \phi}{\partial y} = K \phi \quad \text{on} \quad y = 0 , \tag{1.7}$$

where $K = \frac{\omega^2}{g}$. For water of constant depth h, the condition of no vertical motion at the bottom is

$$\frac{\partial \phi}{\partial y} = 0 \text{ on } y = -h . \tag{1.8}$$

If we consider simple two-dimensional solutions of these equations describing propagating waves of amplitude A such that

$$\eta = A\, e^{ikx} , \tag{1.9}$$

the corresponding velocity potential is

$$\phi = -\frac{igA}{\omega} \frac{\cosh k(y+h)}{\cosh kh} e^{ikx} , \tag{1.10}$$

provided the real positive wavenumber k satisfies

$$\frac{\omega^2}{g} = K = k \tanh kh , \tag{1.11}$$

i.e. the dispersion relation. Depending on the fluid geometry considered, the potential may have to satisfy a radiation condition in addition to the above.

The thesis is divided into five further chapters. In chapter 2 the scattering of oblique waves by a vertical barrier is considered, the barrier may be either surface-piercing or bottom-mounted. Under the assumption that the barrier is small compared to the wavelength and the depth of the water, the method of matched asymptotic expansions is applied to obtain approximations to the reflection and transmission coefficients.

In chapters 3 and 4, a number of sloshing problems are considered involving a vertical rectangular tank containing various bodies. The bodies considered are vertical surface-piercing and bottom-mounted barriers, and circular and elliptic cylinders.

In chapter 3, the eigenfunction expansion method is applied for the case of a vertical barrier. For a circular or elliptic cylinder, the boundary element is applied to obtain the eigenfrequencies. For comparison, wide-spacing and small body approximations are applied for a circular cylinder or for a vertical surface-piercing barrier. The wide-spacing approximation assumes that the wavelength is small compared with the distance between the body and walls. The small body approximation means that a typical dimension of the body is much smaller than the cross-sectional length scale of the fluid motion. In the wide-spacing approximation, the results for the eigenfrequencies involve the reflection coefficient. In the case of the surface-piercing barrier, this reflection coefficient was obtained in chapter 2.

In chapter 4, using the method of matched asymptotic expansions, the lowest-order solution for the eigenfrequencies is obtained in terms of the cross-sectional area and dipole strengths of an arbitrary shaped body. Here the assumption is made that the length scale of the motion is much larger than a typical body dimension. A higher-order solution is difficult for an arbitrary shaped body.

Therefore only a circular cylinder or a vertical surface-piercing barrier is considered to obtain a higher-order solution. When a body is either a circular or elliptic cylinder, the lowest-order solution is compared with the solution of the boundary-element method in chapter 3. When a body is a vertical surface-piercing barrier, the highest-order solution is compared with the solution by the eigenfunction expansion method of chapter 3.

Chapter 5 contains results for the theories of chapters 3 and 4. Comparisons are made for different body geometries, and water depths.

Chapter 6 is concerned with the drift force on multiple bodies. Maruo (1960) and Newman (1967) obtained a formula for the mean horizontal drift force in the direction of wave advance. Under the assumptions that the waves are long relative to body separation, and the bodies are widely spaced relative to body size, McIver (1987) calculated the mean drift force on a group of N vertical circular cylinders by using the method of matched asymptotic expansions. He obtained an additional higher-order term, beside the N^2 term found previously. In the present work, the work of of McIver (1987) is extended in two ways. Firstly the assumption that the cylinders are widely spaced relative to body size is relaxed, while retaining the assumption on the wave length. This is done in section 6.2, considering the case of two different sizes of circular cylinders. In section 6.3, the effects of body geometry are investigated by allowing the N cylinders to be of arbitrary cross-section. The basic procedure of this method is the same as that used by McIver (1987). The result is derived in terms of cross-sectional area and dipole strengths of bodies.

CHAPTER 2

SCATTERING OF OBLIQUE WAVES BY A VERTICAL BARRIER

2.1 INTRODUCTION

The method of matched asymptotic expansions is used to develop a theoretical solution for the diffraction of oblique incident waves by a submerged vertical barrier in water of constant depth. Dean (1945) considered the effect of a normally incident train of waves on a fixed vertical barrier, in deep water, extending down from a point below the free surface, and obtained the reflection and transmission coefficients in integral forms. The effect of a normally incident train of surface waves on a fixed surface-piercing vertical barrier, immersed to a given depth beneath the surface in deep water, was considered by Ursell (1947). He showed that an explicit solution was possible for the velocity potential everywhere in the fluid, and that the reflection and transmission coefficients may be expressed as combinations of modified Bessel functions. The reflection coefficient, R, and the transmission coefficient, T, are defined as the ratios of modulus of the amplitude of the reflected and transmitted waves to the amplitude of the incident wave. If this problem is generalised by considering an obliquely incident wave-train, an explicit solution is no longer possible and only short-wavelength asymptotic and numerical results have been obtained, see particularly Evans and Morris (1972).

The matched asymptotic expansions method is appropriate for the study of the diffraction of water waves by a small object or a small gap. Tuck (1971) used this method to obtain analytical solutions for the diffraction of normally incident waves through a small horizontal slit in a vertical barrier of zero thickness in deep water. Guiney et al (1972) extended Tuck's theory to include the effects of the thickness of the barrier and their theory was verified by experimental data. Liu and Wu (1986) considered the diffraction of an obliquely incident wave by a slit in a barrier of finite width using the same method; Tuck's (1971) solution can be considered as a special case of Liu and Wu's solution. Liu and Wu (1987) also obtained expressions for the reflection and transmission coefficients when the barrier is in finite depth water. Although their theory was developed for the wide barrier case, it can be used for the case of a thin barrier. In this previous work it was assumed that the wave length is much greater than the gap size. Solutions were obtained for reflection and transmission coefficients as far as the first-order term in $\varepsilon = a/h$, where a and h are the length of the barrier and the depth of the water respectively.

Here the assumption is made that the wave length is much greater than the barrier length. The barrier is either surface-piercing or bottom-mounted and is assumed to be uniform in the z-direction. The solutions for reflection and transmission coefficients are obtained to orders ε^2, $\varepsilon^4 \ln \varepsilon$, and ε^4. In the surface-piercing barrier case the solution is more difficult because the free-surface appears within the inner region.

Cartesian coordinates(x,y,z) are employed with the origin in the free surface and y vertically downwards. The barrier occupies the interval, which is

$$x = 0\,, \quad 0 \leq y \leq a\,, \quad -\infty < z < +\infty$$

for the surface-piercing barrier, and

$$x = 0\,, \quad h - a < y < h\,, \quad -\infty < z < \infty.$$

for the bottom-mounted barrier, in each case. A time-harmonic factor $e^{-i\omega t}$ is removed. With the usual assumptions of the linearized water wave theory, a velocity potential $\Phi\,(x,y,z)$ exists which satisfies Laplace's equation,

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \tag{2.1.1}$$

Since the barriers are assumed to have infinite extent in the z –direction and the motion to be periodic in z, the potential $\Phi\ (x,y,z)$ can be expressed as

$$\Phi(x,\ y\ ,\ z)=\phi_T(x,\ y)\ e^{ipz} \tag{2.1.2}$$

where p is the wave number component in the z direction. The total potential ϕ_T is written in the form

$$\phi_T=\phi+\phi_I \tag{2.1.3}$$

where the ϕ and ϕ_I are scattered and incident wave potentials respectively. From equations (2.1.1-3) ϕ satisfies

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}-p^2\phi=0 \tag{2.1.4}$$

in the fluid. The conditions of no flow through the bed are

$$\frac{\partial\phi}{\partial y}=0\ ,\ y=h\ ,\ |x|>0, \tag{2.1.5}$$

and

$$\frac{\partial\phi}{\partial y}=0\ ,\ \ y=h\ ,\ \text{for all } x, \tag{2.1.6}$$

for a bottom-mounted and surface-piercing barrier respectively. The linearized free-surface boundary condition is

$$K\phi + \frac{\partial \phi}{\partial y} = 0\ ,\ y = 0,\ |x| > 0, \tag{2.1.7}$$

where

$$K = \frac{\omega^2}{g} = k \tanh kh \tag{2.1.8}$$

and ω is the radian frequency, g is the acceleration due to gravity and k is the wave number of the incident wave. There is no flow across the barrier, i.e.

$$\frac{\partial \phi_T}{\partial x} = 0\ , \tag{2.1.9}$$

on the barrier.

In sections 2.2 and 2.3 we assume that the body is small compared to all other length scales so that a solution may be constructed by the method of matched asymptotic expansions. Two solution domains are considered here as inner and outer regions, as shown in the figure 2.1.1.

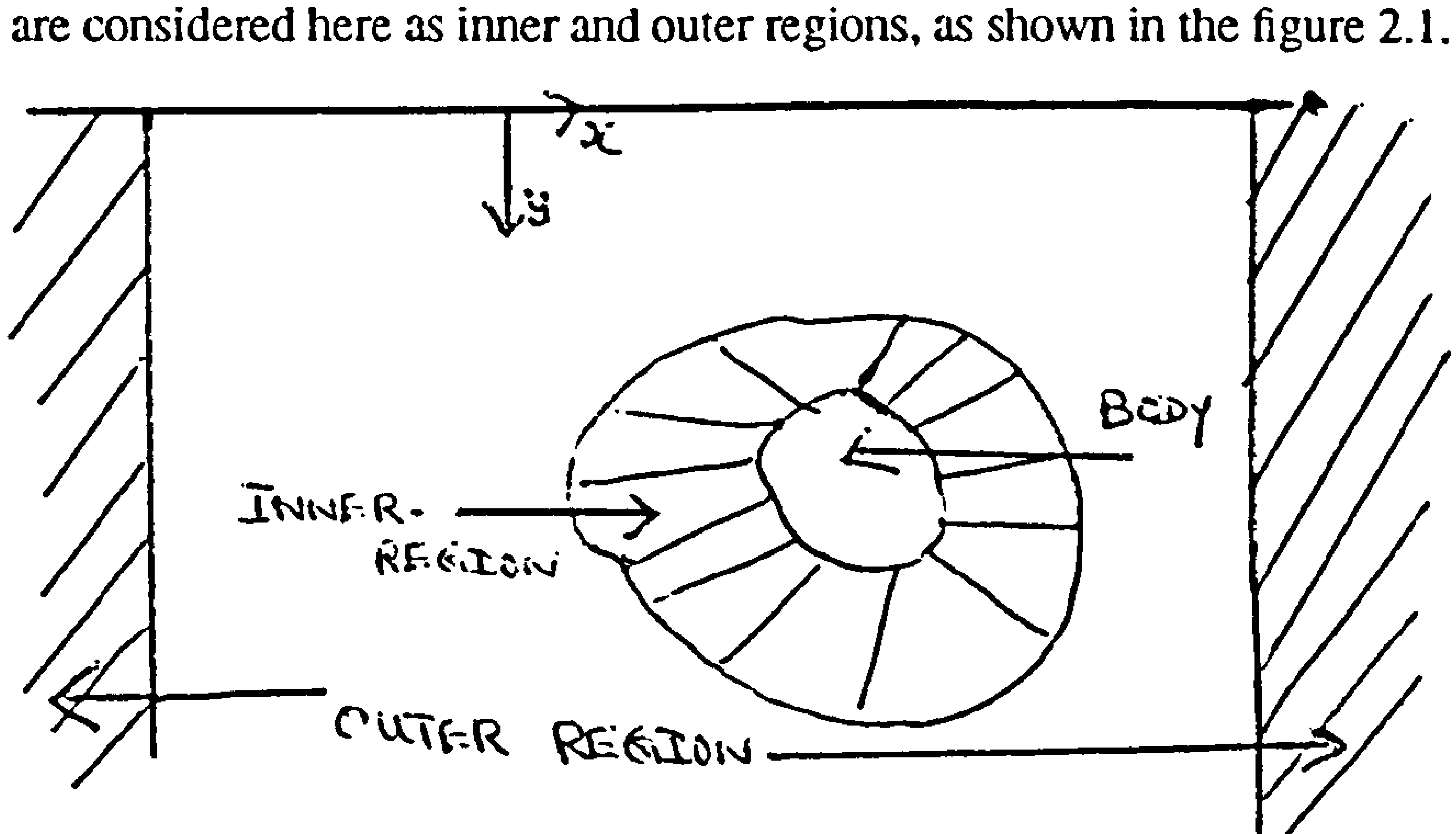


FIGURE 2.1.1

If a is a typical dimension, k is the wavenumber and r a polar coordinate measured from the body, then the inner region is close to the body at distances $r \ll 1/k$ and the outer region is far from the body at distances $r \gg a$. These definitions allow the existence of an overlap region so that matching may be carried out. The full solution is found by using the method of matching principle. Let $\psi^{(m,n)}$ denotes the m^{th} inner approximation, rewritten in terms of the outer variables and expanded to order n. Similarly $\Psi^{(n,m)}$ denotes the n^{th} outer approximation, rewritten in terms of inner variable and expanded to order m. The matching principle requires that, in the overlap region, each term in the inner approximation of the outer solution and outer approximation of the inner solution are identical. That is, $\Psi^{(n,m)} \equiv \psi^{(m,n)}$ for any integer n and m. More details of the matching principle are given by Crighton and Leppington (1973). I consider here two cases, a bottom-mounted and surface-piercing barrier in the sections 2.2 and 2.3 respectively.

2.2 Bottom-mounted barrier

A thin barrier of length a stands in water of depth h so that the barrier occupies $x = 0$, $h - a < y < h$, $-\infty < z < \infty$. The boundary-value problem is defined by the modified Helmholtz equation (2.1.4), the bed condition (2.1.5), the free-surface condition (2.1.7), and the body condition (2.1.9). The incident wave potential ϕ_I is written as,

$$\phi_I = e^{i\alpha x} \cosh k(h - y) \tag{2.2.1}$$

where $\alpha^2 = k^2 - p^2$. From the body condition (2.1.9),

$$\frac{\partial}{\partial x}(\phi_I + \phi) = 0\,,\quad x = 0\,,\quad h - a < y < h. \tag{2.2.2}$$

Therefore from equations (2.2.1) and (2.2.2), the condition on the barrier to be satisfied by the scattered potential is

$$\frac{\partial \phi}{\partial x} = -i\alpha \cosh k(h - y)\,,\ x = 0\,,\ h - a < y < h. \tag{2.2.3}$$

OUTER REGION

Outer coordinates are defined by

$$X = x/h \text{ and } Y = (h - y)/h. \tag{2.2.4}$$

In terms of the outer coordinates $(X\,,Y)$, equations (2.1.4-5) and (2.1.7) take the form

$$\frac{\partial^2\Psi}{\partial X^2} + \frac{\partial^2\Psi}{\partial Y^2} - \delta^2\Psi = 0 \qquad (2.2.5)$$

$$\frac{\partial\Psi}{\partial Y} = 0, \; Y = 0 \, , \; |X| > 0 \qquad (2.2.6)$$

and

$$\mu\Psi - \frac{\partial\Psi}{\partial Y} = 0 \, , \; Y = 1 \qquad (2.2.7)$$

where Ψ is the outer potential in terms of (X,Y) coordinates, $\mu = \mathrm{K\,h}$, and $\delta = \mathrm{ph}$.

At infinity on both sides of the barrier, the scattered waves must be outgoing. That is, on the left side of the barrier, the waves comprise the incident and reflected wave, and on the right side of the barrier there is only the transmitted wave. This radiation condition is written as

$$\Psi \approx (e^{i\lambda X} + R\, e^{-i\lambda X}) \cosh \tau_0 Y \, , \qquad X -> -\infty \qquad (2.2.8)$$

and

$$\Psi \approx T\, e^{i\lambda X} \cosh \tau_0 Y \, , \qquad X -> \infty \qquad (2.2.9)$$

where $\lambda = \alpha h$, $\tau_0 = kh$, and R and T are the reflection and transmission coefficients respectively.

INNER REGION

For the inner region, scaled coordinates are defined by

$$\xi = \frac{x}{a} \, , \, \eta = \frac{(h-y)}{a} \tag{2.2.10}$$

so from (2.2.4), and (2.2.10) the inner and outer coordinates are related by

$$X = \varepsilon\xi \, , \, Y = \varepsilon\eta \tag{2.2.11}$$

where $\varepsilon = \frac{a}{h}$ is a small parameter. In terms of (ξ, η) equations (2.1.4-5) and (2.2.3) become

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} - \delta^2 \varepsilon^2 \psi = 0 \tag{2.2.12}$$

$$\frac{\partial \psi}{\partial \eta} = 0 \, , \, \eta = 0 \, , \, |\xi| > 0 \tag{2.2.13}$$

and

$$\frac{\partial \psi}{\partial \xi} = -i \, \lambda \, \varepsilon \cosh \tau_0 \varepsilon \eta = -i \, \lambda \, \varepsilon \left(1 + \frac{(\tau_0 \, \varepsilon \eta)^2}{2} + \frac{(\tau_0 \, \varepsilon \, \eta)^4}{4!} + O(\varepsilon^6) \right) , \; \xi = 0 \, , \; 0 < \eta < 1 \tag{2.2.14}$$

where ψ is the inner potential in terms of (ξ, η). The boundary condition (2.2.14) suggests that the first-order inner potential must have the form

$$\psi^{(1)} = \varepsilon \, \psi_1 \tag{2.2.15}$$

where from the equations (2.2.12-14), ψ_1 is a harmonic function satisfying the boundary conditions,

$$\frac{\partial \psi_1}{\partial \eta} = 0\ ,\ \ \eta = 0\ ,\ \ \xi > 0 \tag{2.2.16}$$

and

$$\frac{\partial \psi_1}{\partial \xi} = -i\lambda\ ,\ \ \xi = 0\ ,\ \ 0 < \eta < 1 \tag{2.2.17}$$

The solutions are easily found with the aid of a conformal mapping. We define a complex variable $z = \xi + j\eta$ and map the flow region onto the upper half of the $\zeta = \mathrm{u} + j\mathrm{v}$ plane by

$$\zeta = (z^2 + 1)^{\frac{1}{2}}. \tag{2.2.18}$$

This solution ζ has zero normal derivative on $v = 0$ (see appendix 2.1). Therefore from the appendix 2.1 (equation A2.1.3) the boundary conditions for ψ_1 in terms of ζ - plane variables, equations (2.2.16) and (2.2.17) become ,

$$\frac{\partial \psi_1}{\partial v} = 0,\ \ |u| > 1 \tag{2.2.19}$$

and

$$\frac{\partial \psi_1}{\partial v} = -i\lambda \frac{u}{(1-u^2)^{\frac{1}{2}}}\ ,\ \ |u| < 1 \tag{2.2.20}$$

Now to find ψ_1, we define the complex potential

$$W_1(\zeta) = i\,\lambda(\zeta - (\zeta^2 - 1)^{\frac{1}{2}}) \tag{2.2.21}$$

and differentiate W_1 with respect to ζ , giving

$$\frac{dW_1}{d\zeta} = i\lambda\left(1 - \frac{\zeta}{(\zeta^2 - 1)^{\frac{1}{2}}} \right) \tag{2.2.22}$$

with real part satisfying (2.2.19) and (2.2.20) . Therefore,

$$\psi_1 = \mathrm{Re}_j\left(i\,\lambda\,(\zeta - (\zeta^2 - 1)^{\frac{1}{2}}) \right). \tag{2.2.23}$$

Here the notation Re_j means that the real part with respect to j. By expanding, ψ_1, in terms of outer variable up to $O(\varepsilon^2)$,

$$\psi^{(1,2)} = i\,\lambda\,\varepsilon^2\,\frac{X}{2R^2} \tag{2.2.24}$$

where $R^2 = X^2 + Y^2$.

Equation (2.2.24) suggests that the leading-order outer potential is of the form

$$\Psi^{(2)} = \varepsilon^2 A G_2(X, Y) \tag{2.2.25}$$

where A is a constant to be determined, G_2 is a dipole potential, singular at the origin, and is given in appendix 2.3 (equation A2.3.19), and

$$G_2 \approx \frac{X}{\pi R^2} \quad \text{as} \quad R -> 0.$$

Using the matching principle, $\psi^{(1,2)} \equiv \Psi^{(2,1)}$ gives

$$A = i\,\lambda\,\frac{\pi}{2}. \tag{2.2.26}$$

Looking at the inner expansion of this outer potential and the boundary condition (2. 2. 14) suggests that

$$\psi^{(2)} = \varepsilon\,\psi_1 + \varepsilon^2\,\psi_2. \tag{2.2.27}$$

Where from the equations (2.2.12-14), ψ_2 is a harmonic function satisfying the boundary conditions,

$$\frac{\partial \psi_2}{\partial \eta} = 0\,,\;\; \eta = 0\,,\;\; |\xi| > 0 \tag{2.2.28}$$

and

$$\frac{\partial \psi_2}{\partial \xi} = 0\,,\;\; \xi = 0\,,\;\; 0 < \eta < 1 \tag{2.2.29}$$

As before, the solutions can be easily found by writing the boundary conditions in transformed plane variables. That is,

$$\frac{\partial \psi_2}{\partial v} = 0\,,\; v = 0. \tag{2.2.30}$$

We define the complex potential

$$W_2(\zeta) = \sum_{n=0}^{\infty} (b_n \zeta^n + c_n \zeta^{-n}).$$

Here b_n and c_n are constants. Differentiating this with respect to ζ shows that the real part satisfies (2.2.30). However terms of the form ζ^{-n} give a non-physical singularity at $\zeta = 0$. Therefore,

$$\psi_2 = \mathrm{Re}_j \sum_{n=0}^{\infty} b_n \zeta^n \tag{2.2.31}$$

From (2.2.24-25),

$$\Psi^{(2,2)} = \psi^{(1,2)} = i\lambda\varepsilon^2 \frac{X}{2R^2} \tag{2.2.32}$$

Therefore, using the matching principle,

$$\Psi^{(2,2)} \equiv \psi^{(2,2)} \tag{2.2.33}$$

shows that all the constants b_n are zero. Therefore from equations (2.2.23) and (2.2.27), the outer expansion of the inner potential,

$$\psi^{(2,4)} = i\,\lambda\,\varepsilon^2 \frac{X}{2R^2} - i\,\lambda\,\varepsilon^4 \frac{(X^3 - 3\,X\,Y^2)}{8\,R^6} \tag{2.2.34}$$

which will be used later.

HIGHER-ORDER SOLUTION:

Looking at the inner expansion of the outer solution, equation (2.2.25), and the boundary condition (2.2.14) suggests that,

$$\psi^{(3)} = \varepsilon\,\psi_1 + \varepsilon^2\,\psi_2 + \varepsilon^3\,\psi_3 \tag{2.2.35}$$

where, from equations (2.2.12-14), ψ_3 satisfies,

$$\frac{\partial^2 \psi_3}{\partial \xi^2} + \frac{\partial^2 \psi_3}{\partial \eta^2} = \delta^2\,\psi_1 \tag{2.2.36}$$

$$\frac{\partial \psi_3}{\partial \eta} = 0\,,\ \eta = 0\,,\ |\xi| > 0 \tag{2.2.37}$$

and

$$\frac{\partial \psi_3}{\partial \xi} = -\,i\,\lambda\,\tau_0^{\,2}\,\frac{\eta^2}{2}\,,\ \xi = 0\,,\ 0 < \eta < 1 \tag{2.2.38}$$

where ψ_1 is defined in (2.2.23). The solution of ψ_3 may be broken down into a number of stages. A particular solution of the field equation (2.2.36) may be found by writing $\psi_3 = \mathrm{Re}_j\ W$ where W is the solution of

$$4\,\frac{\partial^2 W}{\partial z \partial \overline{z}} = i\,\lambda\,\delta^2 \left[\ (z^2 + 1)^{\frac{1}{2}} - z\ \right] \tag{2.2.39}$$

and bar denotes complex conjugate. Integrating twice yields the particular solution

$$W = i\,\lambda\,\delta^2 \left[\ \frac{1}{8}\,\overline{z}\left(\ \sinh^{-1} z + z\,(1 + z^2)^{\frac{1}{2}}\ \right) - \frac{1}{8}\,z^2\,\overline{z}\ \right] + f_1(z) + \int f_2\,(\,\overline{z}\,)\,dz \tag{2.2.40}$$

where $f_1(z)$ and $f_2(\bar{z})$ are arbitrary functions of z and $\bar{z}$ respectively.

If we choose $f_2(\bar{z}) = 0$ and $f_1(z) = \frac{1}{8}\, i\,\lambda\,\delta^2\, z \sinh^{-1} z$, in order to satisfy the equation (2.2.36), then

$$\psi_{3_p} = \frac{1}{8}\, i\,\lambda\,\delta^2\, \mathrm{Re}_j \left[\sinh^{-1} z\,(z+\bar{z}) + z\,\bar{z}\,(1+z^2)^{\frac{1}{2}} - z^2\,\bar{z} \right] \tag{2.2.41}$$

Differentiating equation (2.2.41) with respect to η and ξ gives

$$\frac{\partial \psi_{3p}}{\partial \eta} = 0\,,\ \ \eta = 0\,,\ \ \xi \neq 0 \tag{2.2.42}$$

and

$$\frac{\partial \psi_{3p}}{\partial \xi} = -\frac{i\,\lambda\,\delta^2}{8}\,\eta^2\,,\ \ \xi = 0\,,\ \ 0 < \eta < 1. \tag{2.2.43}$$

We can now write

$$\psi_3 = \psi_{3_p} + \psi_{31}$$

where ψ_{31} is a harmonic function. By writing

$$\psi_{31} = \frac{1}{24}\, i\,\lambda\,\ (4\,\tau_0^{\,2} - \delta^2)\,\mathrm{Re}_j\, z^3 \tag{2.2.44}$$

we see that $\psi_{3_p} + \psi_{31}$ satisfy the boundary conditions (2.2.37-38). Therefore from equations (2.2.41) and (2.2.44),

$$\psi_3 = i\,\lambda\,\frac{\delta^2}{4}\,\mathrm{Re}_j\left[\,\xi\,\sinh^{-1} z + \frac{1}{2}\rho^2\,(1+z^2)^{\frac{1}{2}} - \frac{1}{2}\rho^2\,z\,\right] + \frac{1}{24}\,i\,\lambda\,(4\,\tau_0^{\;2} - \delta^2\,)\,(\,\xi^3 - 3\xi\eta^2\,).$$

(2.2.45)

Using the expansions,

$$\sinh^{-1} z = \ln z + \ln 2 + \frac{1}{4z^2} + O\left(\frac{1}{z^4}\right),$$

$$\mathrm{Re}_j(\xi\,\sinh^{-1} z\,) = \xi\,\ln 2\,\rho + \frac{\xi}{4\rho^4}\,(\xi^2 - \eta^2) + O\left(\frac{1}{\rho^6}\right),$$

and

$$\mathrm{Re}_j\left(\,\rho^2\,(1+z^2)^{\frac{1}{2}}\,\right) = \rho^2\,\xi + \frac{\xi}{2} - \frac{\xi}{8\rho^4}\,(\xi^2 - 3\,\eta^2) + O\left(\frac{1}{\rho^6}\right)$$

equation (2.2.45) becomes,

$$\psi_3 = \frac{1}{4}\,i\,\lambda\,\delta^2\left(\,\xi\,\ln 2\rho + \frac{\xi}{4} - \frac{\xi}{16\rho^2} + \frac{\xi^3}{4\rho^4}\,\right)$$

$$+\frac{1}{6}\,i\,\lambda\,(\tau_0^{\;2} - \frac{1}{4}\,\delta^4\,)\,(\,\xi^3 - 3\,\xi\,\eta^2) + O\left(\frac{1}{\rho^6}\right). \qquad (2.2.46)$$

In order to match with the outer solution, we must add the homogeneous solutions $\psi_{31,h}$ and $\psi_{32,h}$ of order O ($\varepsilon^3\,\ln\varepsilon$) to ψ_3

where

$$\psi_{31,h} = b_1 \operatorname{Re}_j (z^2 + 1)^{\frac{1}{2}} = b_1 \left(\xi + \frac{\xi}{2\rho^2} + O(\frac{1}{\rho^3}) \right) = b_1 \left(\frac{X}{\varepsilon} + \frac{\varepsilon X}{2R^2} + O(\varepsilon^3) \right) \quad (2.2.47)$$

where b_1 is a constant, and

$$\psi_{32,h} = \operatorname{Re}_j \left(a_0 + a_1\zeta + a_2\zeta^2 + a_3 \zeta^3 + O(\zeta^4) \right)$$

$$= \operatorname{Re}_j \left(a_3 z^3 + a_2 z^2 + (\frac{3 a_3}{2} + a_1) z + a_2 + a_0 + (\frac{3 a_3}{8} + \frac{a_1}{2}) \frac{1}{z} + O(\frac{1}{z^2}) \right) \quad (2.2.48)$$

Here a_0, a_1, a_2, and a_3 are constants. Therefore from equations (2.2.34), and (2.2.46)-(2.2.48), the outer expansion of $\psi^{(3)}$ can be calculated as,

$$\psi^{(3,4)} = i \lambda \varepsilon \frac{\xi}{2 \rho^2} - i \lambda \varepsilon \frac{(\xi^3 - 3 \xi \eta^2)}{8\rho^6} + \varepsilon^3 \ln \varepsilon \, b_1 \left(\frac{\xi}{2\rho^2} + \xi \right)$$

$$+ \varepsilon^3 \left(i \lambda \frac{\delta^2}{4} \left(\xi \ln 2\rho + \frac{\xi}{4} + \frac{\xi^3}{4\rho^4} - \frac{\xi}{16 \rho^2} \right) + \frac{i \lambda}{6} (\tau_0^2 - \frac{\delta^2}{4}) (\xi^3 - 3 \xi \eta^2) \right.$$

$$+ a_3 (\xi^3 - 3 \xi \eta^2) + a_2(\xi^2 - \eta^2) + (\frac{3a_3}{2} + a_1) \xi +$$

$$\left. + a_2 + a_0 + (3 \frac{a_3}{8} + \frac{a_1}{2}) \frac{\xi}{\rho^2} \right) \quad (2.2.49)$$

Equations (2. 2. 34) and (2. 2. 46) suggest that the continuing outer solution is

$$\Psi^{(4)} = \varepsilon^2 \, i \lambda \frac{\pi}{2} G_2 + \varepsilon^4 \ln \varepsilon \, D \, G_2 + \varepsilon^4 \, (B \, G_2 + C \, G_4) \tag{2.2.50}$$

Where B,C and D are constants, and $G_4 = \dfrac{\partial^2 G_2}{\partial X^2}$ is defined in appendix 2.3 (equation 2.3.20).

Therefore from equation(2.2.50), the inner expansion of $\Psi^{(4)}$ to order ε^3 is

$$\Psi^{(4,3)} = \varepsilon \left(2i\lambda \, \xi \, \rho^2 + 2 \, C \, \xi \, \frac{(\xi^2 - 3 \, \eta^2)}{\pi \, \rho^6} \right) + \frac{1}{\pi} \varepsilon^3 \ln \varepsilon \left(D \frac{\xi}{\rho^2} + i \, \lambda \, \delta^2 \, \frac{\xi}{4} \right)$$

$$+ \varepsilon^3 \left(\frac{i\lambda}{2} \left(\delta^2 \, \frac{\xi}{2} \left(-\frac{1}{2} + \ln \rho + \ln \frac{\delta}{2} + \gamma \right) - \delta^2 \, \xi \left(\frac{\pi i}{2 \, \lambda \, N_0^2} - T_0 \right) \right. \right.$$

$$\left. \left. - \xi \left(- \frac{\pi i \, \tau_0^2}{2 \, \lambda \, N_0^2} + \frac{\delta^2}{4} \, (T_2 + T_0) \right) \right) + \frac{B\xi}{\pi \, \rho^2} + \frac{C\delta^2}{\pi} \left(\frac{3 \, \xi}{2 \, \rho^2} - \frac{\xi^3}{\rho^4} \right) \right) \tag{2.2.51}$$

where

$$T_0 = \int_{-1}^{1} \frac{M(\tau)}{\beta} \, d\tau, \tag{2.2.52}$$

$$M(\tau) = \frac{(K + \beta) \cosh^2 \dfrac{\beta h}{2} \, e^{-\beta h}}{(K \cosh \beta h - \beta \sinh \beta h)},$$

$$\beta^2 = p^2 + \tau^2,$$

and

$$T_2 = \frac{1}{p} \int_{-1}^{1} M(\tau) \tanh \frac{\beta h}{2} \, d\tau, \tag{2.2.53}$$

are defined in the appendix 2.3 where the source solution is obtained.

Now matching (2.2.49) with (2.2.51), gives

$$a_2 = a_0 = 0,$$

$$a_3 = -\frac{i\lambda}{6} (\tau_0^2 - \frac{\delta^2}{4}), \tag{2.2.54}$$

$$C = -\frac{i\lambda\pi}{16}, \tag{2.2.55}$$

$$b_1 = \frac{i\lambda\delta^2}{4\pi}, \tag{2.2.56}$$

$$D = \frac{\pi b_1}{2}, \tag{2.2.57}$$

$$\frac{1}{\pi}\left(B + \frac{3C\delta^2}{2} \right) = -\frac{i\lambda\delta^2}{16} + \frac{3a_3}{8} + \frac{a_1}{2}, \tag{2.2.58}$$

and

$$\frac{i\lambda}{2}\left(\frac{\delta^2}{2}\left(-\frac{1}{2}+\ln\frac{\delta}{2}+\gamma\right)-\delta^2\left\{\frac{i\pi}{2\lambda N_0^2}-T_0\right\}+i\tau_0^2\frac{\pi}{2\lambda N_0^2}-\delta^2\frac{(T_2+T_0)}{4}\right)$$

$$=\frac{i\lambda\delta^2}{8}\left(\frac{1}{2}+2\ln 2\right)+\frac{3a_3}{2}+a_1. \tag{2.2.59}$$

Substituting equation (2.2.54) into (2.2.59), we get

$$a_1=\frac{i\lambda}{2}\left(\frac{\delta^2}{2}\left(-\frac{1}{2}+\ln\frac{\delta}{2}+\gamma\right)-\delta^2\left\{\frac{\pi i}{2\lambda N_0^2}-T_0\right\}+\frac{\pi i\tau_0^2}{2\lambda N_0^2}\right.$$

$$\left.-\frac{\delta^2}{4}(T_2+T_0)-\frac{\delta^2}{4}\left(\frac{1}{2}+2\ln 2\right)\right)+\frac{i\lambda}{4}\left(\tau_0^2-\frac{\delta^2}{4}\right). \tag{2.2.60}$$

Substituting equations (2.2.54 – 55) and (2.2.60) into (2.2.58), after little algebra gives

$$B=\frac{\pi i\lambda}{4}\left(-\frac{\delta^2}{8}+\frac{\tau_0^2}{4}+\frac{\delta^2}{2}\left(\ln\frac{\delta}{4}+\gamma\right)-\delta^2\left\{\frac{i\pi}{2\lambda N_0^2}-T_0\right\}\right.$$

$$\left.+\frac{\pi i\tau_0^2}{2\lambda N_0^2}-\frac{\delta^2}{4}(T_2+T_0)\right). \tag{2.2.61}$$

From equations (2.2.8), (2.2.50) and far-field potential which is calculated in appendix 2. 3 we obtain

$$R = -\frac{\varepsilon^2}{2N_0^2}\left(i\lambda\frac{\pi}{2} + \varepsilon^2\left\{D\ln\varepsilon + B - C\lambda^2\right\}\right). \tag{2.2.62}$$

Substituting the values of B, C and D into the equation (2. 2. 62) gives

$$R = -i\lambda\pi\frac{\varepsilon^2}{4N_0^2}\left(1 + \frac{\varepsilon^2}{2}\left\{\frac{\delta^2}{2\pi}\ln\varepsilon + \frac{\lambda^2}{4} + \frac{\tau_0^2}{4} - \frac{\delta^2}{8} + \frac{\delta^2}{2}(\ln\frac{\delta}{4} + \gamma)\right.\right.$$

$$\left.\left. - \delta^2(\frac{\pi i}{2\lambda N_0^2} - T_0) + i\pi\frac{\tau_0^2}{2\lambda N_0^2} - \frac{\delta^2}{4}(T_2 + T_0)\right\}\right) \tag{2.2.63}$$

and transmission coefficient T is

$$T = 1 - R. \tag{2.2.64}$$

By using the definitions of variables in the equation (2.2.63), and after a little simplification, equation (2.2.63) can be rewritten in the simplest form as

$$R = -i\pi\frac{\varepsilon^2((kh)^2 - (ph)^2)^{\frac{1}{2}}}{4N_0^2}\left\{1 + \frac{\varepsilon^2}{4}\left[\frac{(pa)^2}{\pi}\ln\varepsilon + (kh)^2 - \frac{3}{4}(ph)^2 + (ph)^2(\ln\frac{ph}{4} + \gamma)\right.\right.$$

$$\left.\left. + \frac{\pi i((kh)^2 - (ph)^2)^{\frac{1}{2}}}{N_0^2} - \frac{(ph)^2}{2}(T_2 - 3T_0)\right]\right\}, \tag{2.2.65}$$

where

$$N_0^{\,2} = \frac{1}{2}\left(1 + \frac{\sinh 2kh}{2kh}\right)$$

and integrals T_0 and T_2 are defined in the equations (2.2.52) and (2.2.53) respectively.

As $p -> 0$, equation (2.2.65) becomes,

$$R -> -ik\pi \frac{a^2}{4N_0^{\,2}h}\left(1 + \frac{a^2}{2}\left(\frac{k^2}{2} + i\pi k \frac{h}{2N_0^{\,2}} - \frac{1}{2}\int_{-1}^{1} M(\tau)\beta d\tau\right)\right) \tag{2.2.66}$$

which is the two-dimensional reflection coefficient.

2.3 Surface-piercing barrier

A thin barrier of length a stands in water depth h, such that the barrier occupies $x = 0, 0 < y < a$, $-\infty < z < \infty$. The boundary-value problem is defined as for the bottom-mounted barrier, the modified Helmholtz equation (2.1.4), the bed condition (2.1.6), the linearized free-surface condition (2.1.7), and the body condition (2.1.9). The incident wave potential ϕ_I is written in the form,

$$\phi_I = e^{i\alpha x} \cosh k(h-y) \tag{2.3.1}$$

where $\alpha^2 = k^2 - p^2$. From the body condition (2.1.9),

$$\frac{\partial}{\partial x}(\phi_I + \phi) = 0 \,, \quad x = 0 \,, \quad 0 < y < a. \tag{2.3.2}$$

Therefore from equations (2.3.1) and (2.3.2), the condition on the barrier to be satisfied by the scattered potential is

$$\frac{\partial \phi}{\partial x} = -i\,\alpha \cosh k(h-y) \quad , \quad x = 0 \,, \quad 0 < y < a. \tag{2.3.3}$$

OUTER REGION

Outer coordinates are defined by,

$$X = \frac{x}{h} \,, \text{ and } Y = \frac{y}{h} \,. \tag{2.3.4}$$

Therefore in terms of (X,Y), equations (2.1.4), (2.1.6), and (2.1.7) take the form

$$\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} - \delta^2 \Psi = 0 \tag{2.3.5}$$

$$\mu \Psi + \frac{\partial \Psi}{\partial Y} = 0 \, , \ Y = 0 \, , \ |X| > 0 \tag{2.3.6}$$

and

$$\frac{\partial \Psi}{\partial Y} = 0 \, , \ Y = 1 \, . \tag{2.3.7}$$

Here $\mu = Kh$. As for the bottom-mounted barrier, the radiation condition is written as

$$\Psi -> (e^{i\lambda X} + Re^{-i\lambda X}) \cosh \tau_0 (1 - Y) \, , \qquad X -> -\infty \tag{2.3.8}$$

and

$$\Psi -> Te^{i\lambda X} \cosh \tau_0 (1 - Y) \, , \qquad X -> \infty \tag{2.3.9}$$

where $\lambda = \alpha h$, $\tau_0 = kh$, and R and T are reflection and transmission coefficients respectively.

INNER REGION

For the inner region, scaled coordinates are defined by,

$$\xi = \frac{x}{a} \, , \text{ and } \eta = \frac{y}{a} \, . \tag{2.3.10}$$

Therefore in terms of (ξ, η), equations (2.1.4), (2.1.6), and (2.3.3) become

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} - \delta^2 \varepsilon^2 \psi = 0 \tag{2.3.11}$$

$$\mu\psi + \frac{\partial \psi}{\partial \eta} = 0 \,,\ \eta = 0 \,,\ \xi > 0 \tag{2.3.12}$$

and

$$\frac{\partial \psi}{\partial \xi} = -i\,\lambda\,\varepsilon \cosh\left(\tau_0(-\varepsilon\eta + 1)\right) = -i\,\lambda\,\varepsilon \cosh\tau_0 \left(1 - \tau_0 \varepsilon\eta + \frac{(\tau_0 \varepsilon \eta)^2}{2} + O(\varepsilon^3) \right),$$

$$0 < \eta < 1 \,,\ \xi = 0 \tag{2.3.13}$$

where ψ is the inner potential in terms of (ξ, η). The boundary condition (2.3.13) suggests that the first-order inner potential must have the form

$$\psi^{(1)} = \varepsilon\, \psi_1 \tag{2.3.14}$$

where from equations (2.3.11-13), ψ_1 is a harmonic function satisfying the boundary conditions,

$$\frac{\partial \psi_1}{\partial \eta} = 0 \,,\ \eta = 0 \,,\ \xi > 0 \tag{2.3.15}$$

and

$$\frac{\partial \psi_1}{\partial \xi} = -i\,\lambda \cosh\tau_0 = -iv_0 \,,\ \xi = 0 \,,\ 0 < \eta < 1 \tag{2.3.16}$$

where $v_0 = \lambda \cosh \tau_0$.

The solutions are easily found with the aid of a conformal mapping. Define a complex variable $z = \xi + j\,\eta$ and map the flow region onto the lower half of the $\zeta = u + j\,v$ plane by

$$\zeta = (\, z^2 + 1\,)^{\frac{1}{2}}. \tag{2.3.17}$$

As for the bottom-mounted barrier, this solution has zero normal derivative on $v = 0$ (see appendix 2.1). Therefore, from appendix 2.1 (equation A2.1.4), the boundary condition for ψ_1 in terms of ζ -plane variables, equations (2.3.15) and (2.3.16) become,

$$\frac{\partial \psi_1}{\partial v} = 0\,,\ |u| > 1 \tag{2.3.18}$$

and

$$\frac{\partial \psi_1}{\partial v} = -\frac{i\,v_0 u}{(1-u^2)^{\frac{1}{2}}}\,,\ |u| < 1\,. \tag{2.3.19}$$

Now to find ψ_1 , define the complex potential,

$$W_1(\zeta) = i v_0 (\zeta - (\zeta^2 - 1)^{\frac{1}{2}}) \tag{2.3.20}$$

and differentiate W_1 with respect to ζ ,

$$\frac{dW_1}{d\zeta} = i v_0 \left(1 - \frac{\zeta}{(\zeta^2-1)^{\frac{1}{2}}} \right). \tag{2.3.21}$$

The real part of (2.3.21) satisfies (2.3.18) and (2.3.19), that is

$$\psi_1 = \mathrm{Re}_j\left\{ iv_0(\zeta - (\zeta^2 - 1)^{\frac{1}{2}}) \right\} \tag{2.3.22}$$

By expanding ψ_1 in terms of outer variables up to $O(\varepsilon^2)$,

$$\psi^{(1,2)} = iv_0\varepsilon^2 \frac{X}{2R^2} \tag{2.3.23}$$

where $R^2 = X^2 + Y^2$.

Equation (2.3.23) suggests that the leading-order outer potential

$$\Psi^{(2)} = \varepsilon^2 AG_2(X,Y) \tag{2.3.24}$$

where A is a constant to be determined, G_2 is a dipole potential at the origin and is given in appendix 2.4 (equation A2.4.12) and

$$G_2 \approx \frac{X}{\pi R^2} \text{ as} R -> 0.$$

Using the matching principle, $\psi^{(1,2)} \equiv \Psi^{(2,1)}$

$$A = \frac{iv_0\pi}{2}. \tag{2.3.25}$$

By looking at the inner expansion of this outer potential and from the boundary condition

(2.3.13)

$$\psi^{(2)} = \varepsilon\,\psi_1 + \varepsilon^2\,\psi_2 \qquad (2.3.26)$$

where from (2.3.13-15), ψ_2 is a harmonic function satisfying the boundary conditions,

$$\frac{\partial \psi_2}{\partial \eta} = -\,\mu\psi_1 \;,\; \eta = 0\,,\, |\xi| > 0 \qquad (2.3.27)$$

and

$$\frac{\partial \psi_2}{\partial \xi} = i v_0 \mu \eta \,,\; \xi = 0\,,\; 0 < \eta < 1\,. \qquad (2.3.28)$$

As before, the solution can be easily found by writing the boundary conditions in ζ-plane variables. That is

$$\frac{\partial \psi_2}{\partial v} = -\,i v_0 \mu \left(\frac{u^2}{(u^2-1)^{\frac{1}{2}}} - u \right), |u| > 1 \qquad (2.3.29)$$

and

$$\frac{\partial \psi_2}{\partial v} = i v_0 \mu\, u \,,\; |u| < 1. \qquad (2.3.30)$$

In order to find ψ_2, define

$$W_2(\zeta) = \frac{i\,v_0\,\mu}{2}\, j\left[\ln(\zeta + (\zeta^2-1)^{\frac{1}{2}}) + \zeta\,(\zeta^2-1)^{\frac{1}{2}} - \zeta^2 \right] + C$$

where C is a complex constant and $\mathrm{Re}_j\left(\dfrac{dW_2}{d\zeta}\right)$ satisfies equations (2.3.29) and (2.3.30). Thus,

$$\psi_2 = \frac{iv_0\mu}{2}\,\mathrm{Re}_j\left\{j\left(\ln(\zeta + (\zeta^2 - 1)^{\frac{1}{2}}) + \zeta\,(\zeta^2 - 1)^{\frac{1}{2}} - \zeta^2\right)\right\} + C_R \tag{2.3.31}$$

where C_R is a real constant.

By expanding ψ_2 in terms of outer variables up to the order of ε^2, we obtain

$$\psi^{(2,2)} = \frac{1}{2}\varepsilon^2 iv_0\left(\frac{X}{R^2} - \mu\left(\frac{\pi}{2} - \theta\right)\right) + C_R \tag{2.3.32}$$

where $X = R\sin\theta$ and $Y = R\cos\theta$.

From equations (2.3.24) and (2.3.25), the inner expansion of outer potential becomes,

$$\Psi^{(2,2)} = \frac{1}{2}\,iv_0\varepsilon^2\left(\frac{X}{R^2} + \mu\,\theta\right) \tag{2.3.33}$$

Matching equations (2.3.32) with (2.3.33) gives,

$$C_R = \frac{\pi}{4}\,iv_0\mu\,. \tag{2.3.34}$$

By substituting C_R into equation (2.3.31),

$$\psi_2 = \frac{iv_0\mu}{2}\,\mathrm{Re}_j\left\{j\left(\ln(z + (z^2 - 1)^{\frac{1}{2}}) + z(z^2 - 1)^{\frac{1}{2}} - (z^2 + 1)\right)\right\} + \frac{iv_0\mu\pi}{4} \tag{2.3.35}$$

and from equations (2.3.22), (2.3.26), and (2.3.35), the outer expansion of the inner potential is

$$\psi^{(2,4)} = \frac{iv_0}{2}\left\{ \varepsilon^2\left(\frac{X}{R^2} + \mu\,\theta\right) - \frac{1}{4}\varepsilon^4\left(\frac{X^3 - 3XY^2}{R^6} - \mu\frac{XY}{R^4}\right)\right\} \tag{2.3.36}$$

which will be used later.

HIGHER-ORDER SOLUTION:

From the inner expansion of the outer solution and the boundary condition (2.3.13)

$$\psi^{(3)} = \varepsilon\,\psi_1 + \varepsilon^2\,\psi_2 + \varepsilon^3\,\psi_3 \tag{2.3.37}$$

where from (2.3.11-13), ψ_3 satisfies,

$$\frac{\partial^2\psi_3}{\partial\xi^2} + \frac{\partial^2\psi_3}{\partial\eta^2} = \delta^2\psi_1, \tag{2.3.38}$$

$$\frac{\partial\psi_3}{\partial\eta} = -\,\mu\psi_2\ ,\ \eta = 0\,,\ |\xi| > 0 \tag{2.3.39}$$

and

$$\frac{\partial\psi_3}{\partial\xi} = -\frac{iv_0\tau_0^2\,\eta^2}{2}\ ,\ \xi = 0\ ,\ 0 < \eta < 1 \tag{2.3.40}$$

where ψ_1 and ψ_2 are defined in equations (2.3.22) and (2.3.35) respectively.

Since equation (2.3.38) is very similar to equation (2.2.36), the particular solution of the field equation may be found as for the bottom-mounted barrier. Therefore the particular solution is

$$\psi_{3_p} = \frac{i v_0 \delta^2}{8} \mathrm{Re}_j \left\{ \sinh^{-1} z\,(z+\bar{z}) + z\,\bar{z}\,(1+z^2)^{\frac{1}{2}} - z^2\,\bar{z} \right\}. \tag{2.3.41}$$

Differentiate with respect to η and ξ to give,

$$\frac{\partial \psi_{3p}}{\partial \eta} = 0 \ , \ \eta = 0 \, , \ \xi \neq 0 \tag{2.3.42}$$

and

$$\frac{\partial \psi_{3p}}{\partial \xi} = -\frac{i v_0 \delta^2}{8} \eta^2 \ , \ \xi = 0 \, , \ 0 < \eta < 1 \, . \tag{2.3.43}$$

Now write

$$\psi_3 = \psi_{3_p} + \psi_{31} + \psi_{32} + \psi_{33} \tag{2.3.44}$$

where $\psi_{3_i}(i = 1, 2, 3)$'s are harmonic functions and $\psi_{3_p} + \psi_{31}$ and ψ_{32} satisfy equations (2.3.40) and (2.3.39) respectively.

By writing,

$$\psi_{31} = \frac{i v_0}{6} (\tau_0^2 - \frac{\delta^2}{4}) \mathrm{Re}_j\, z^3 \tag{2.3.45}$$

$\psi_{3_p} + \psi_{31}$ satisfies the boundary condition (2.3.40). To calculate ψ_{32} write the boundary condition

(2.3.39) in terms of ζ -plane variables,

$$\frac{\partial \psi_{32}}{\partial v} = \frac{-1}{4} \pi i v_0 \mu^2 \frac{u \; sgn \; u}{(u^2 - 1)^{\frac{1}{2}}} \; , \; |u| > 1 \tag{2.3.46}$$

To satisfy the boundary condition (2.3.46), choose

$$\psi_{32} = -\frac{1}{2} i v_0 \mu^2 \operatorname{Re}_j \left\{ (\zeta^2 - 1)^{\frac{1}{2}} \left(\ln(\zeta^2 - 1)^{\frac{1}{2}} - \frac{j\pi}{2} \right) \right\} . \tag{2.3.47}$$

For $\zeta = u$, real and $u < -1$,

$$\frac{\partial \psi_{32}}{\partial v} = + \frac{1}{4} i v_0 \pi \mu^2 \frac{u}{(u^2 - 1)^{\frac{1}{2}}} \tag{2.3.48}$$

and for $u > 1$,

$$\frac{\partial \psi_{32}}{\partial v} = \frac{1}{4} i v_0 \pi \mu^2 \frac{u}{(u^2 - 1)^{\frac{1}{2}}} . \tag{2.3.49}$$

That is ψ_{32} satisfies the boundary condition (2.3.46).

However for $\zeta = u$, real and $|u| < 1$,

$$\frac{\partial \psi_{32}}{\partial v} = \frac{1}{2} i v_0 \mu^2 \frac{u}{(1 - u^2)^{\frac{1}{2}}} \left\{ \ln(1 - u^2)^{\frac{1}{2}} + 1 \right\} . \tag{2.3.50}$$

Therefore from equations (2.3.41), and (2.3.44-50), ψ_{33} must satisfy the boundary conditions,

$$\frac{\partial \psi_{33}}{\partial v} = 0 \, , \, |u| > 1 \tag{2.3.51}$$

and

$$\frac{\partial \psi_{33}}{\partial v} = -\frac{1}{2} i v_0 \mu^2 \frac{u}{(1-u^2)^{\frac{1}{2}}} \left(\ln(1-u^2)^{\frac{1}{2}} + 1 \right) , \, |u| < 1 \, . \tag{2.3.52}$$

To calculate ψ_{33}, Green's theorem is applied in appendix 2.2, which gives

$$\psi_{33} = -\frac{1}{8} i v_0 \mu^2 (1 - 2 \ln 2) \frac{\xi}{\rho^2} + O(\frac{1}{\rho^3}) \tag{2.3.53}$$

As in section 2.2, in order to match with the outer solution we must add the homogeneous solutions of O($\varepsilon^3 \ln \varepsilon$) and O(ε^3) which are $\psi_{31,h}$ and $\psi_{32,h}$ respectively, and are defined in equations (2.2.47-48). Therefore from equations (2.3.36-37), (2.3.41), (2.3.44-45), (2.3.47), and (2.2.47-48), the outer expansion of $\psi^{(3)}$ may now be calculated as,

$$\psi^{(3,4)} = i v_0 \varepsilon \frac{\xi}{2\rho^2} + \frac{\varepsilon^2}{2} i v_0 \mu \theta - i v_0 \varepsilon \frac{(\xi^3 - 3\xi\eta^2)}{8\rho^6}$$

$$+ \frac{1}{8} i v_0 \mu \varepsilon^2 \frac{\xi\eta}{\rho^4} + \varepsilon^3 \ln \varepsilon \left(\frac{\xi}{2\rho^2} + \xi \right) b_1$$

$$+ \varepsilon^3 \left(i v_0 \frac{\delta^2}{8} \left[\frac{\xi}{2} + 2\xi \left(\ln 2 + \ln \rho \right) + \frac{\xi^3}{2\rho^4} - \frac{\xi}{8\rho^2} \right] \right.$$

$$+\frac{iv_0}{6}(\tau_0^2-\frac{\delta^2}{4})(\xi^3-3\xi\eta^2)-\frac{1}{2}iv_0\mu^2(\xi\ln\rho+\eta\theta)-\frac{1}{8}iv_0\mu^2(1-2\ln 2)\frac{\xi}{\rho^2}$$

$$+a_3(\xi^3-3\xi\eta^2)+a_2(\xi^2-\eta^2)+(\frac{3a_3}{2}+a_1)\xi+$$

$$\left. a_2+a_0+(\frac{3}{8}a_3+\frac{a_1}{2})\frac{\xi}{\rho^2}\right). \tag{2.3.54}$$

where b_1 is a constant and has to be determined.

Equations (2. 3. 36) and (2. 3. 54) suggest that the continuing outer solution is

$$\Psi^{(4)}=\varepsilon^2 iv_0\frac{\pi}{2}G_2+\varepsilon^4\ln\varepsilon\, DG_2++\varepsilon^4(BG_2+CG_4) \tag{2.3.55}$$

where B,C and D are constants, and $G_4=\dfrac{\partial^2 G_2}{\partial X^2}$ is defined in appendix 2.4 (equation 2.4.14).

Therefore the inner expansion of (2.3.55) is,

$$\Psi^{(4,3)}=\varepsilon\left(\frac{iv_0\xi}{2\rho^2}+2C\xi\frac{(\xi^2-3\eta^2)}{\pi\rho^6}\right)+\varepsilon^2\left(\frac{iv_0}{2}\mu\theta-2C\mu\frac{\xi\eta}{\pi\rho^4}\right)$$

$$+\varepsilon^3\ln\varepsilon\left(-\frac{i}{2}v_0\mu^2\xi+i\frac{v_0}{4}\xi\delta^2+D\frac{\xi}{\pi\rho^2}\right)+\varepsilon^3\left[\frac{iv_0}{2}\left(\xi\alpha_7-\mu^2\xi\ln\mu\rho\right.\right.$$

$$\left.\left.+\frac{\delta^2}{2}\xi\ln\mu\rho-\mu^2\eta\,\theta\right)+\frac{\xi}{\pi\rho^2}\left\{B-C\left(\mu^2-\frac{\delta^2}{2}\right)\right\}+C\,\delta^2\xi\frac{\eta^2}{\pi\rho^4}\right] \tag{2.3.56}$$

where α_7 is defined in appendix 2.4 (equation A2.4.13).

Matching equation (2. 3. 54) with equation (2. 3. 56),

$$a_2 = a_0 = 0 \, , \, C = -\frac{iv_0 \pi}{16} \, , \tag{2.3.57}$$

$$a_3 = -\frac{iv_0}{6} \, (\tau_0^{\,2} - \frac{\delta^2}{4}) \, , \tag{2.3.58}$$

$$b_1 = \frac{iv_0}{2} (\frac{\delta^2}{2} - \mu^2), \tag{2.3.59}$$

$$D = \frac{\pi \, b_1}{2} \, , \tag{2.3.60}$$

$$\frac{iv_0}{2} \, (\, \alpha_7 - \mu^2 \ln \mu + \frac{\delta^2}{2} \ln \mu \,) = \frac{3}{2} \, a_3 + a_1 + i \, \frac{v_0}{8} \, \delta^2 (\, \frac{1}{2} + 2 \ln 2 \,), \tag{2.3.61}$$

and

$$-\frac{1}{64} iv_0 \delta^2 - iv_0 \frac{\mu^2}{8} (1 - 2 \ln 2) + 3 \frac{a_3}{8} + \frac{a_1}{2} = \frac{1}{\pi} \left(B - C(\mu^2 - \frac{\delta^2}{2}) + C\delta^2 \right). \tag{2.3.62}$$

From equations (2.3.57-62) and after little algebra,

$$B = \pi i \frac{v_0}{4} \left(\frac{\delta^2}{8} - \frac{3 \, \mu^2}{4} + \frac{\tau_0^{\,2}}{4} + \alpha_7 + (\frac{\delta^2}{2} - \mu^2) \ln \frac{\mu}{2} \right) \tag{2.3.63}$$

and

$$D = \frac{iv_0\pi}{8}\,(\,\delta^2 - 2\,\mu^2\,). \tag{2.3.64}$$

From equations (2.3.8), (2.3.55), and far-field potentials which are calculated in appendix 2.4 (equations A2.4.17-18). we obtain

$$R = -\frac{\cosh kh}{2N_0^{\;2}}\left(\varepsilon^2 iv_0\frac{\pi}{2} + \varepsilon^4(D\ln\varepsilon + B - \lambda^2 C)\right). \tag{2.3.65}$$

Substituting the values of B, C and D into equation (2. 3. 65), gives

$$R = -\pi i v_0 \varepsilon^2 \frac{\cosh kh}{4N_0^{\;2}}\left(1 + \frac{1}{4}\,\varepsilon^2 \ln\varepsilon\,(\delta^2 - 2\,\mu^2)\right.$$

$$\left. + \frac{\varepsilon^2}{2}\left(\frac{\delta^2}{8} - \frac{3\mu^2}{4} + \frac{\tau_0^{\;2}}{4} + \alpha_7 + (\frac{\delta^2}{2} - \mu^2)\ln\frac{\mu}{2} - \frac{\lambda^2}{4}\right)\right). \tag{2.3.66}$$

and transmission coefficient is

$$T = 1 - R\,. \tag{2.3.67}$$

By using the definitions of variables in equation (2.3.66), and after little simplification, equation (2.3.66) can be rewritten in the simplest form as

$$R = -\pi\, i\,((kh)^2 - (ph)^2)^{\frac{1}{2}}\,\frac{\varepsilon^2\cosh^2 kh}{4\,N_0^{\;2}}\left\{1 + \frac{\varepsilon^2}{2}\left[\frac{1}{2}\ln\varepsilon\,((ph)^2 - 2(Kh)^2) + \frac{5\,(ph)^2}{8} + \frac{(Kh)^2}{4}\right.\right.$$

$$+ \pi\, \delta_0\, ((kh)^2 - (ph)^2) - (Kh)^2\, \Delta_6 + (ph)^2\, \Delta_4 + \pi\, iKh\, ((Kh)^2 - (ph)^2)^{\frac{1}{2}}$$

$$+ \left(\frac{(ph)^2}{2} - (Kh)^2 \right) t_1 + \left((Kh)^2 - (ph)^2 \right) t_2 \Bigg] \Bigg\} \tag{2.3.68}$$

where

$$K = k \tanh kh\ ,$$

$$\delta_0 = \frac{i \cosh^2 kh}{2\,(k^2 - p^2)\, hN_0^{\ 2}}\, ,$$

$$\alpha_1 = \cosh^{-1} \frac{k}{p}\, ,$$

$$t_1 = \gamma - \ln\,(2 \cosh \alpha_1) + \ln \frac{Kh}{2}\, ,$$

$$t_2 = \pi\, i\, coth\ \alpha_1 - \alpha_1\, coth\ \alpha_1 ,$$

$$\Delta_4 = \int_0^\infty \left(\frac{1}{K - \beta \tanh \beta h} - \frac{1}{K - \beta} \right) dt,$$

$$\beta^2 = p^2 + t^2,$$

and

$$\Delta_6 = \frac{1}{K^2} \int_0^\infty \left\{ \frac{1}{K - \beta \tanh \beta h} - \frac{1}{K - \beta} \right\} \beta^2 \, dt.$$

Now we define asymptotic expressions. When $kh \to 0$, the reflection coefficient in equation (2.3.68) tends to zero and when $kh \to \infty$, the reflection coefficient in equation (2.3.68) becomes

$$R_\infty = -\frac{\pi i}{2} Ka \left((Ka)^2 - (pa)^2 \right)^{\frac{1}{2}} \left(1 + \frac{(pa)^2}{2} \left(-\frac{1}{4} - \ln 2 + \frac{\gamma}{2} - \pi i \coth \alpha_2 + \alpha_2 \coth \alpha_2 + \frac{1}{2} \ln pa \right.\right.$$

$$\left.\left. + \frac{(Ka)^2}{(pa)^2} \left(\frac{3}{4} - \gamma + 2 \ln 2 - \ln pa + \pi i \coth \alpha_2 - \alpha_2 \coth \alpha_2 \right) \right) \right),$$

(2.3.69)

where

$$\alpha_2 = \cosh^{-1} \left(\frac{K}{p} \right).$$

As p- > 0, equation (2.3.68) becomes,

$$R -> -i\pi\varepsilon^2 kh \frac{\cosh^2 kh}{4 N_0^2} \left(1 + \frac{\varepsilon^2}{2} \left(\frac{(kh)^2}{2} - (Kh)^2 \left[-\frac{1}{4} + \ln \frac{Ka}{2} + \Delta_6 + \gamma \right] + \pi (kh)^2 \delta_0 \right) \right)$$

(2.3.70)

which is the two-dimensional reflection coefficient. In deep water equation (2.3.70) becomes

$$R_{2D,\infty} = -\pi i \, \frac{(Ka)^2}{2} \left(1 + \frac{(Ka)^2}{2} \left(\ln 2 - \ln Ka - \gamma + \frac{3}{4} \right) \right). \tag{2.3.71}$$

which is the expansion of Ursell's result.

2.4 Results

The reflection coefficient can be calculated from the solutions given in the section 2.2, equation (2.2.65), and section 2.3, equation (2.3.68) by using the asymptotic form for the dipole potential. My main aim is to find how the reflection coefficient changes with different barrier length, ka and depth, kh and for different angles $\alpha_0 = \pi/6$, $\pi/4$, and $\pi/3$ of incident wave, measured from the x-axis. Here α_0 is defined by

$$\alpha_0 = \sin^{-1} \frac{p}{k}. \tag{2.4.1}$$

In the case of a bottom-mounted barrier, the reflection coefficient, $|R|$, is plotted against different barrier lengths, ka, and depths, kh. Figure 2.1 shows that the reflection coefficient increases monotonically with kh for the range of kh plotted. However as $kh \to \infty$ it tends to zero since high frequency waves will not feel the barrier. Also a comparison is made with the first-order solution in figure 2.1 which agrees well for small values of $kh \leq 0.2$. In Figure 2.2 the reflection coefficient increases with barrier length, $ka \leq 0.5$, and decreases with angle of incident wave for fixed $kh = 0.5$. The reason is that amplitude of the disturbances increases with increasing barrier length. Since the maximum allowable value of a/l is one, the barrier length is therefore considered up to 0.5 for fixed $kh =$ 0.5 in figure 2.2. As for figure 2.1, a comparison is made with the first-order solution and they agree for small values of $ka \leq 0.2$. Figure 2.3 shows $|R|$ plotted against $kh > 0.5$ for a fixed barrier length $ka = 0.5$. When $kh \leq 0.5$ the method is not valid, since $h/a > 1$ and so $ka = 0.5$ implies that $kh > 0.5$. From figure 2.3, we see that for all values of kh, when α_0 increases the reflection coefficient decreases. When kh increases, the reflection coefficient tends to zero, as the wave motion is not much affected by the barrier.

Dean (1945) obtained reflection coefficients in terms of the depth of the barrier below the free surface in the deep water. In the present method the reflection coefficient would tend to zero as water depth tends to infinity since we have a fixed barrier length. Since the amplitude of the disturbance decreases rapidly with increasing depth, the wave motion is not much affected by the barrier. So comparison with Dean's result for different finite depths of barrier submergence below the free surface is not possible.

In the case of a surface-piercing barrier, the reflection coefficient $|R|$ is plotted against different barrier lengths ka and depths kh in figures 2.4-2.6. Figure 2.4 shows $|R|$ plotted against different depths, kh, for fixed $h/a = 2.0$. As for bottom-mounted barrier the reflection coefficient increases with kh and decreasing angles of incident wave. But when $kh = 0.51$ and $\alpha_0 = \frac{\pi}{6}$, the reflection coefficient is 0.9941, and in this case there is almost zero wave amplitude on the right of the barrier and the standing wave on the left. When $kh > 0.51$ for angle of incidence $\pi/6$, the reflection coefficient keeps on increasing. Clearly the solution is not valid near $kh = 0.51$. A comparison is made with the first-order solution in figure 2.4 and the two solutions agree well for small values of $kh \leq 0.1$. Figure 2.5 shows $|R|$ plotted against different barrier lengths for fixed $kh = 1.0$. From figure 2.5, we can see that the reflection coefficient increases with ka, but is not monotonic in α_0. When $\alpha_0 = \pi/3$, the reflection coefficient is much higher than the other values of α_0 shown and when $ka = 0.46$, the reflection coefficient is 0.9851. Again the solution is not valid near $kh = 0.46$. A comparison is made with the first-order solution in figure 2.5 and the two solutions agree well for $\alpha_0 = \pi/6$, and $\pi/4$ when $ka \leq 0.25$. But when $\alpha_0 = \pi/3$ those results do not agree even for small values of ka. Figure 2.6 shows $|R|$ plotted against different values of $kh > 0.5$ for fixed barrier length 0.5. By using the same argument as for the bottom-mounted barrier case, when $kh < 0.5$, the method is not valid. From the figure 2.6, the reflection coefficient is increasing with increasing kh up to $kh = 4.0$ and when $kh \geq 4.0$, the reflection coefficient is efficiently constant as we would expect for high frequency waves.

A similar problem in deep water was treated by Evans and Morris (1972), who obtained upper and lower bounds for reflection and transmission coefficients for all wavelengths and angles of incidence; their method was based on complementary variational approximations to an integral equation. Their results are compared with the results from equation (2.3.69). In figure 2.7, $|R|$ is plotted against angle α_0. Comparison for different values of angle α_0 shows the agreement between the present and Evans and Morris result is very good for $0 < \mu_0 = Ka \leq 0.4$. Here K has been used instead of k, as $k \to K$ in deep water. When μ_0, increases up to one, there is a difference in the reflection coefficient. For increasing values of μ_0, the results do not agree in the range $0 < \alpha_0 < \frac{\pi}{3}$, but in the range $\frac{\pi}{3} \leq \alpha_0 \leq \frac{\pi}{2}$ they agree quite well. The reason is that the assumption made here is that the barrier length is small compared to wave length and depth of water.

Finally we note that as $p \to 0$, that is in two-dimensional case, in deep water the reflection coefficient in the equation (2.3.63) is obtained and comparison is made with Ursell's (1947) result in figure 2.8, where $|R|$ is plotted against Ka. From this figure, we notice that the agreement is good for small values of $Ka \leq 0.4$, since the solution derived here is valid for small values of Ka.

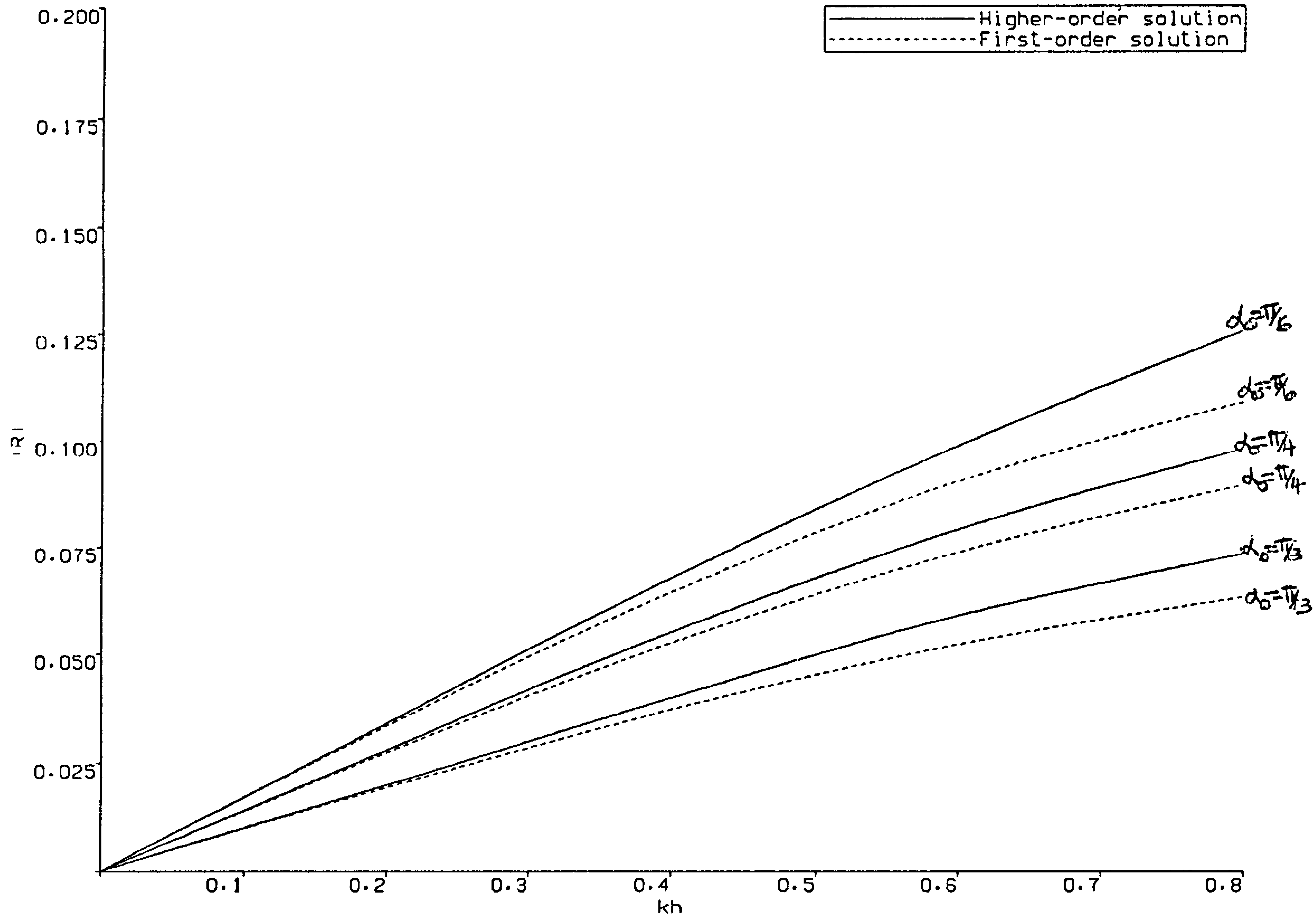


Figure 2.1 Bottom-mounted barrier for fixed h/a=2.0

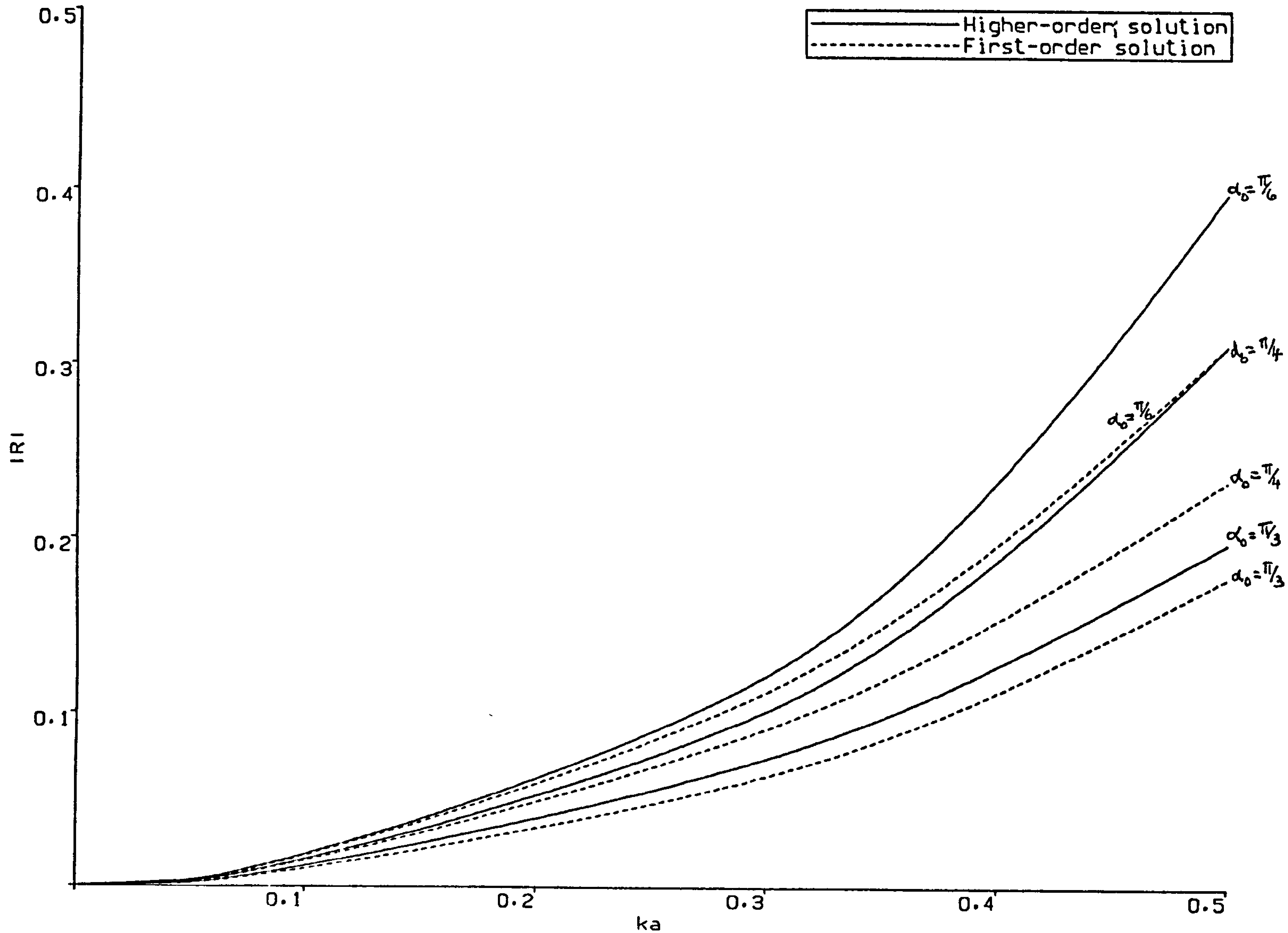


Figure 2.2 Bottom-mounted barrier for fixed kh=0.5

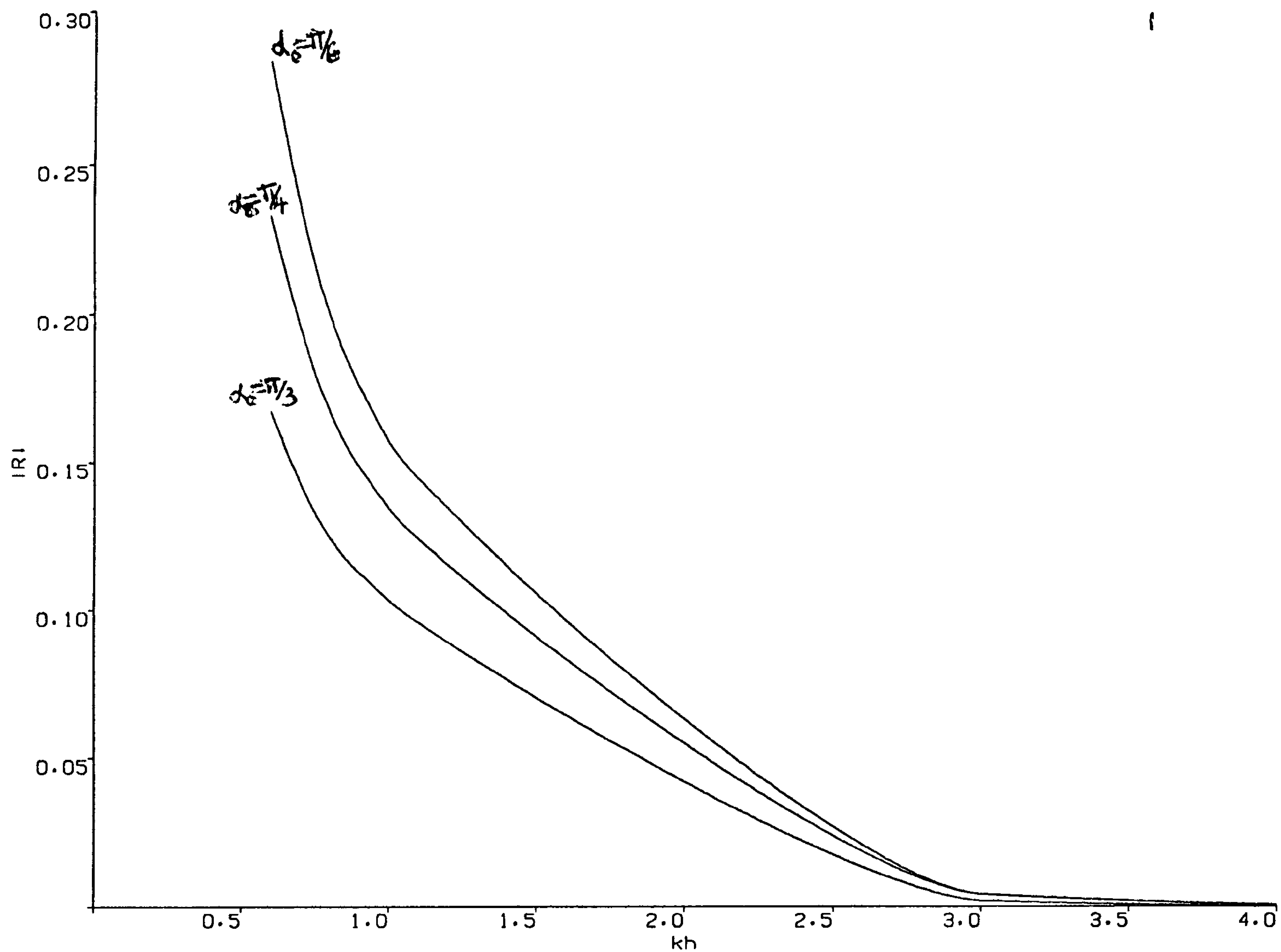


Figure 2.3 Bottom-mounted barrier for fixed ka=0.5

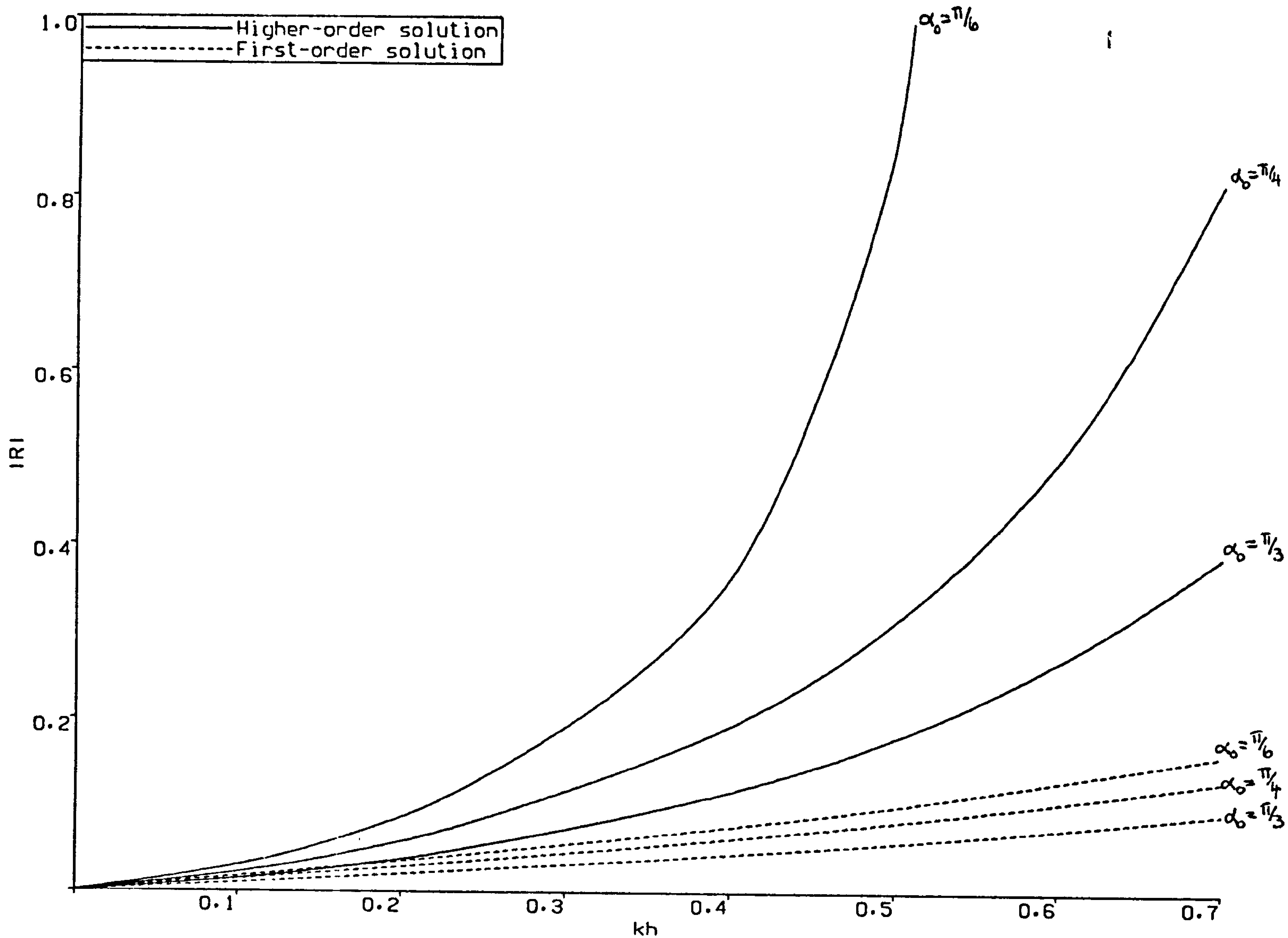


Figure 2.4 Surface-piercing barrier for fixed h/a=2.0

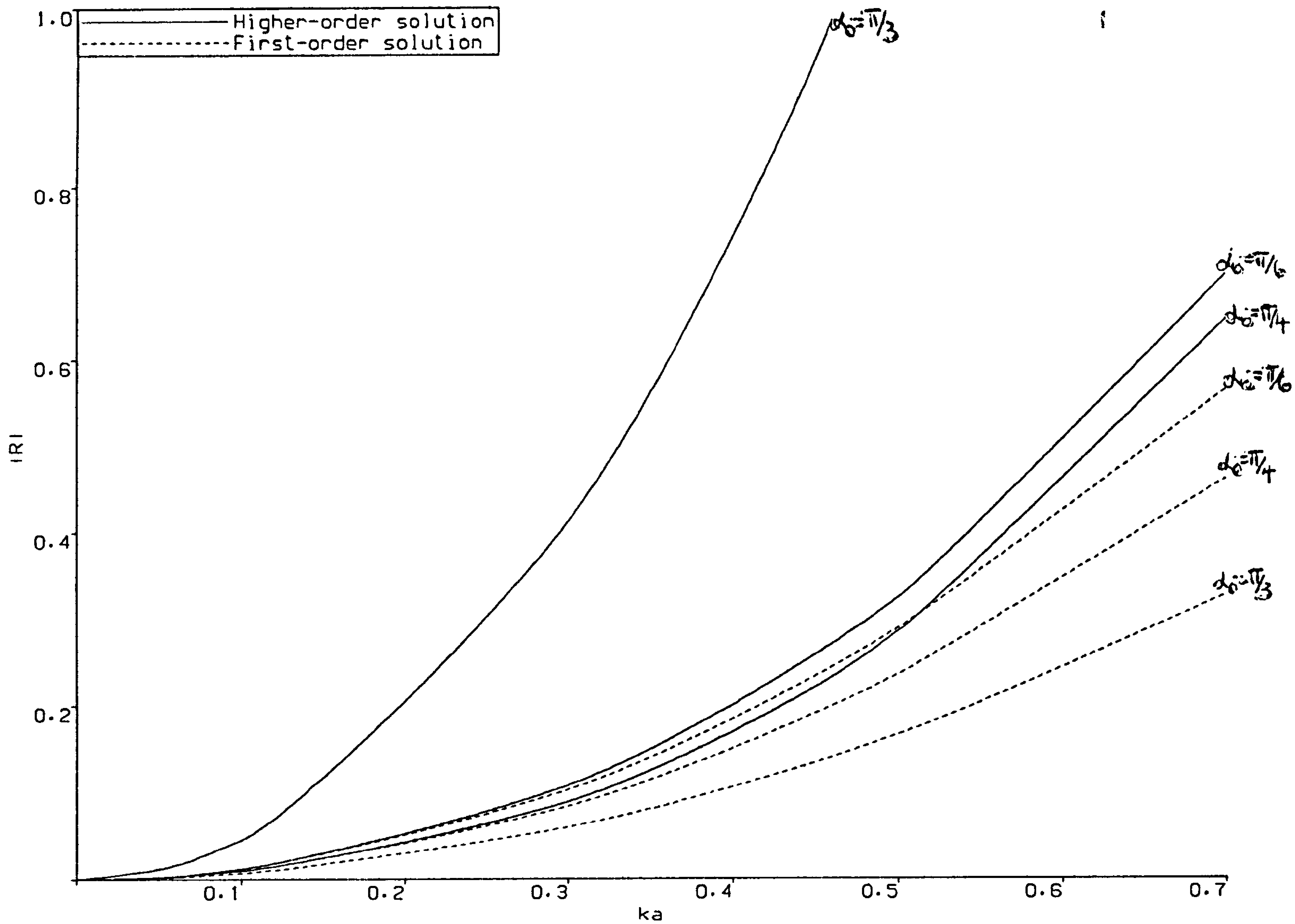


Figure 2.5 Surface-piercing barrier for fixed kh=1.0

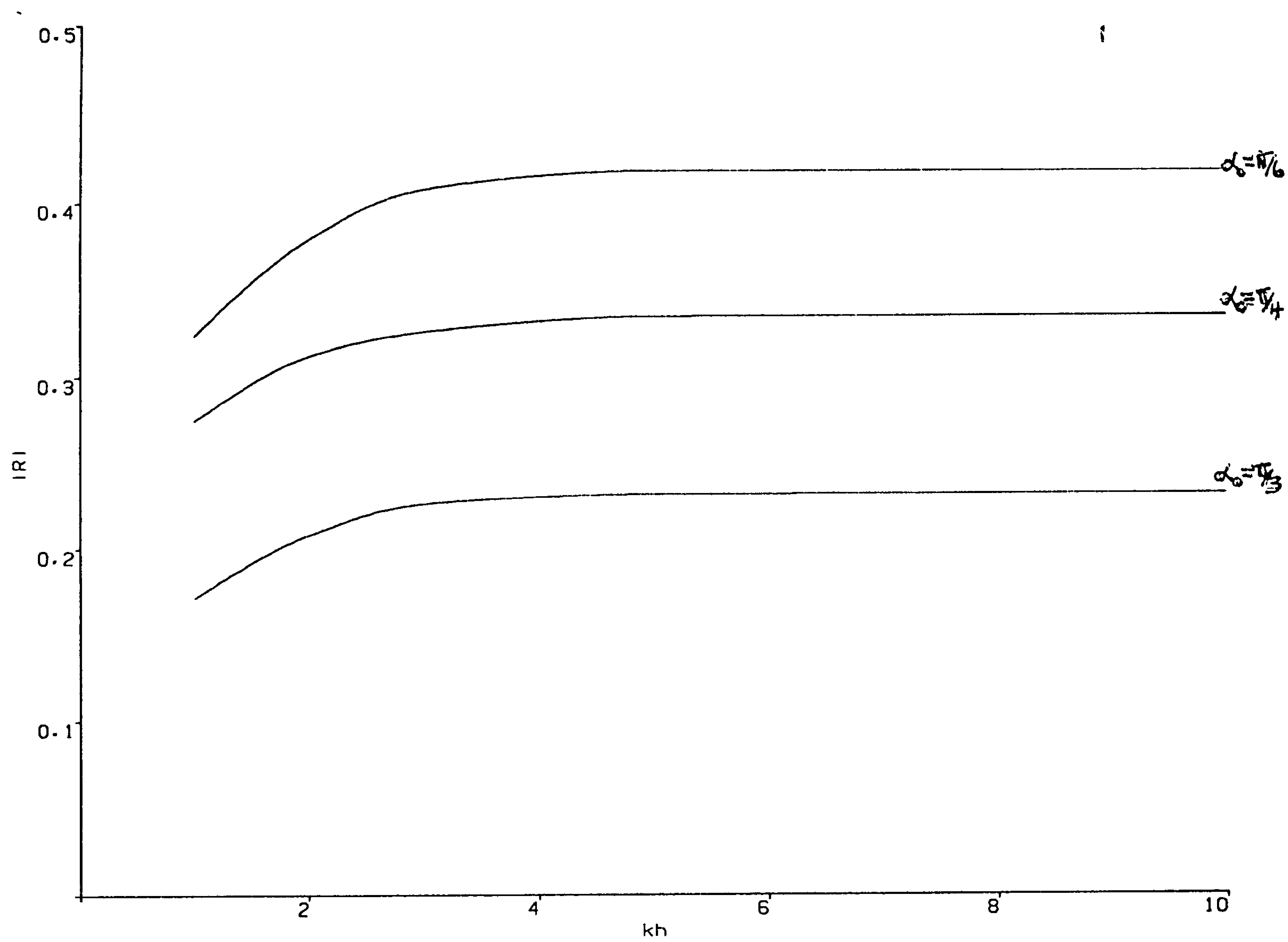


Figure 2.6 Surface-piercing barrier for fixed ka=0.5

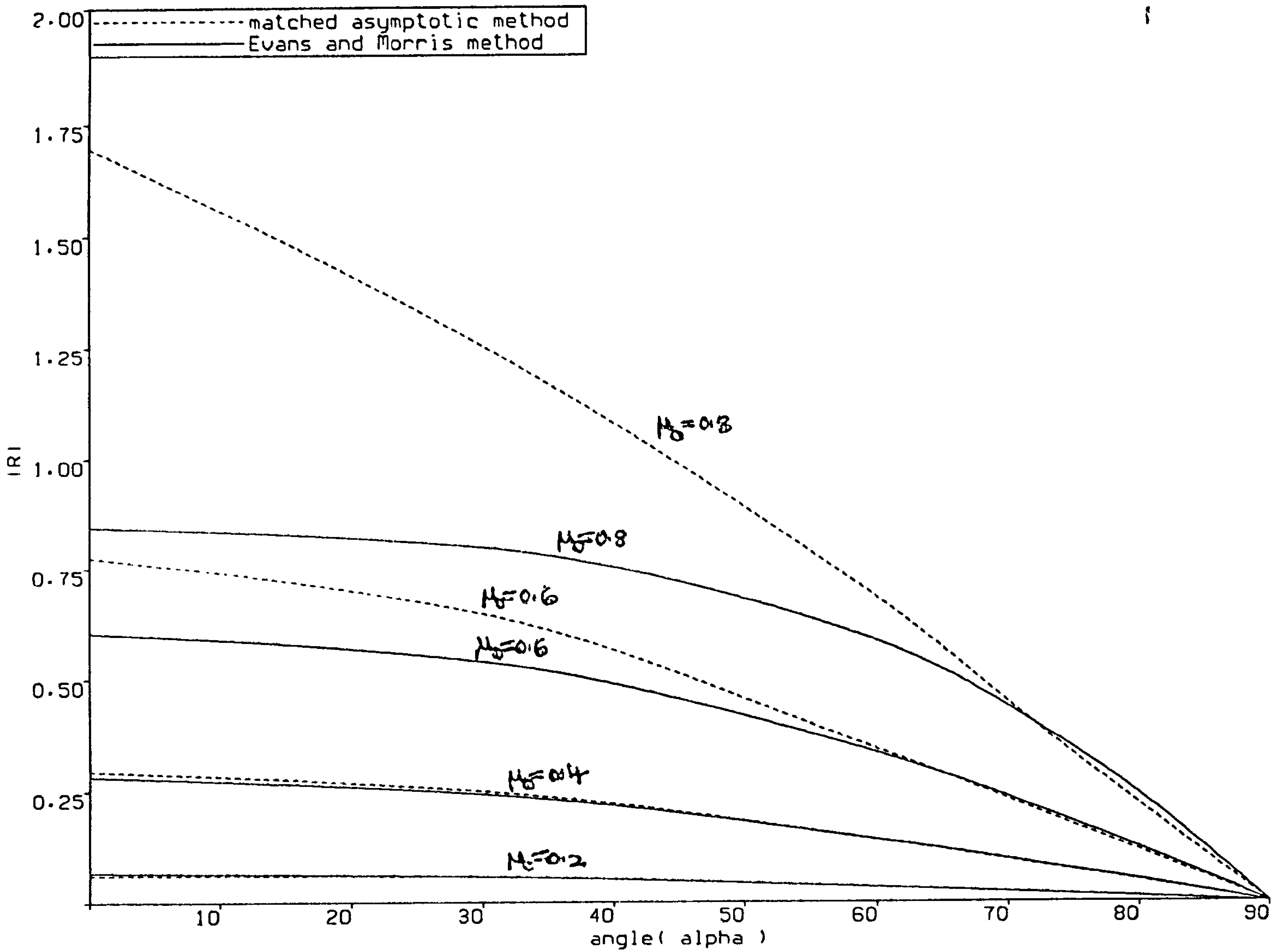


Figure 2.7 surface piercing barrier in a deep water

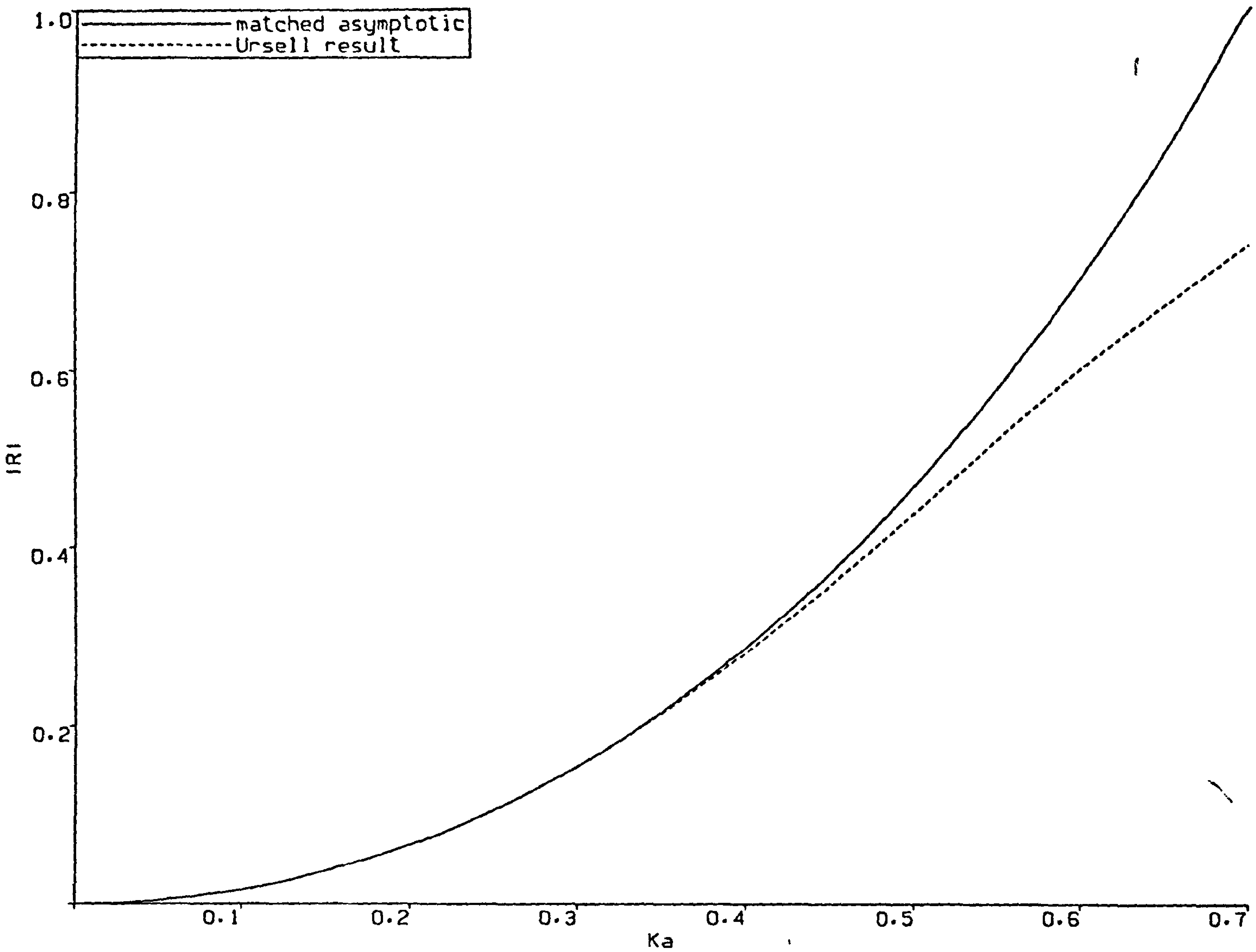


Figure 2.8 surface-piercing barrier in a deep water

APPENDIX 2.1: CONFORMAL MAPPING

(a) Bottom-mounted barrier

Using conformal mapping, the barrier condition in the physical plane is mapped into a transformed plane and the solution in the transformed plane is used in sections 2.1 and 2.2 to obtain the inner solution. Here the barrier is considered as a bottom-mounted or surface-piercing barrier. The transformation

$$z = (\zeta^2 - 1)^{\frac{1}{2}} \tag{A2.1.1}$$

where $z = \xi + j\eta$ and $\zeta = u + jv$ and maps the flow region onto the upper half of the ζ - plane as shown in figure 2.1.1. The barrier is mapped onto the interval (-1,1) and the real part of the z-axis AB and DE are mapped onto the interval $A'B'$ and $D'E'$ respectively.

FIGURE 2.1.1.

Define the complex potential,

$$W = \varphi + j\psi$$

and so

$$\frac{dW}{d\zeta} = \frac{\partial\psi}{\partial v} - j\frac{\partial\phi}{\partial v} = \frac{dW}{dz}\frac{dz}{d\zeta}$$

$$= \left(\frac{\partial \phi}{\partial \xi} + j \frac{\partial \psi}{\partial \xi} \right) \frac{\zeta}{(\zeta^2 - 1)^{\frac{1}{2}}} . \qquad \text{(A2.1.2)}$$

On BCD, $\zeta = u$, so

$$\arg(\zeta^2 - 1) = \arg(\zeta + 1) + \arg(\zeta - 1) = \pi.$$

Therefore

$$(\zeta^2 - 1)^{\frac{1}{2}} = j\,(1 - u^2)^{\frac{1}{2}} (= j\eta)$$

and so from (A2.1.2),

$$\frac{\partial \psi}{\partial v} = \frac{\partial \psi}{\partial \xi} \frac{u}{(1 - u^2)^{\frac{1}{2}}} \qquad \text{(A2.1.3)}$$

(b) Surface-piercing barrier

The above transformation (A2.1.1) maps the flow region onto the lower half of the ζ- plane as shown in the figure A2.1.2. That is the barrier is mapped onto the interval $(-1\,,\,1)$ and the real part of the z-axis AB and DE are mapped onto the interval A'B' and D'E' respectively, see figure 2.1.2. As for (a), the complex potential, W can be defined as in the equation (A2.1.2).

On BCD, $\zeta =$ u, so

$$\arg(\zeta^2 - 1) = - \pi$$

therefore

$$(\zeta^2 - 1)^{\frac{1}{2}} = -j\,(1 - u^2)^{\frac{1}{2}}\ (= j\eta\,)$$

and so from (A2.1.2)

$$\frac{\partial\psi}{\partial v} = -\frac{\partial\psi}{\partial\xi}\,\frac{u}{(1-u^2)^{\frac{1}{2}}}\,. \qquad \text{(A2.1.4)}$$

FIGURE 2.1.2.

APPENDIX 2.2: APPLICATION OF GREEN'S THEOREM FOR THE SOURCE POTENTIAL

Applying Green's theorem to inner and source potentials, and using the boundary condition for the barrier, the inner potential, ψ, is calculated in finite integral form. Now applying the above theorem to the potential, ψ , and source potential

$$G(s,t;\xi,\eta) = \ln \rho_0 + \ln \rho_1 \qquad \text{(A2.2.1)}$$

where

$$\rho_0^2 = (s-\xi)^2 + (t-\eta)^2$$

and

$$\rho_1^{\,2} = (s-\xi)^2 + (t+\eta)^2$$

gives

$$\int_{S_\infty} \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \right) ds = 0 \qquad \text{(A2.2.2)}$$

Where $S_\infty = s_\infty + s_{x\infty}$, and n is a normal in the outward direction as shown in the figure 2.2.1.

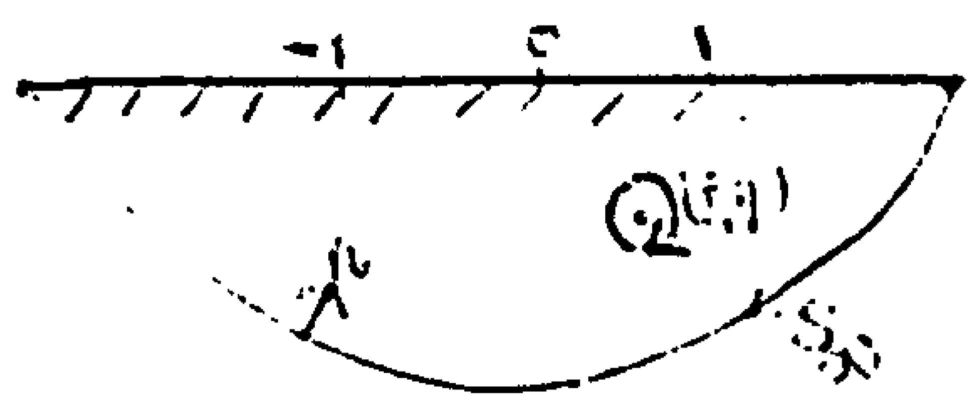


FIGURE 2.2.1

Here s_∞ and $s_{x\infty}$ are an infinite surface and x- plane. Assuming ϕ decays sufficiently rapidly on s_∞, then equation $A(2.2.2)$ gives,

$$2\pi\ \phi(\xi,\eta) + \int_{-\infty}^{\infty} (\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n})\ ds = 0 \tag{A2.2.3}$$

Using the conditions,

$$\frac{\partial \phi}{\partial n} = f(s)\ , |s| < 1$$

and

$$\frac{\partial \phi}{\partial n} = 0\ , |s| > 1$$

Then $A(2.2.3)$ gives,

$$\phi(\xi,\eta) = \frac{1}{2\pi} \int_{-1}^{1} f(s)\ G(s,0;\xi,\eta)\ ds \tag{A2.2.4}$$

Now write

$$G(s,0;\xi,\eta) = \ln\left((s-\xi)^2 + \eta^2\right) = \ln\left(\left(\xi^2+\eta^2\right)\left(1 + \frac{s^2 - 2s\xi}{\xi^2+\eta^2}\right)\right)$$

so for $\rho = (\xi^2 + \eta^2)^{\frac{1}{2}} >> 1$,

$$G(s,0;\xi,\eta) = \ln\rho^2 + \frac{s^2 - 2s\xi}{\rho^2} - \frac{2s^2\xi^2}{\rho^4} + O(\frac{1}{\rho^3}) . \tag{A2.2.5}$$

Substituting this into $A(2.2.4)$, and noting that $f(s)$ is odd, gives

$$\phi(\xi,\eta) = \frac{1}{2\pi}\int_{-1}^{1} f(s)\left(-\frac{2s\xi}{\rho^2} + O(\frac{1}{\rho^3})\right)ds$$

$$= -\frac{1}{\pi}\int_{-1}^{1} \frac{\xi}{\rho^2} f(s)\, s\, ds + O(\frac{1}{\rho^3}) \tag{A2.2.6}$$

APPENDIX 2.3: SUBMERGED SOURCE POTENTIAL

Here the fundamental singular solution of the modified Helmholtz equation that satisfies the water wave boundary conditions is considered. Expansions about the singular point are derived when the singularity is on the bed, and hence far-field potentials for dipole and higher-order singularities are derived. Throughout, the position of the origin is on the free surface with the y axis directed vertically downwards and water depth h.

(a) THE SINGULAR SOLUTION OF THE MODIFIED HELMHOLTZ EQUATION

The line source solution of the modified Helmholtz equation is given by, MacCamy (1957)

$$G = \frac{1}{\pi} \int_0^\infty \frac{\cosh \beta(h-\eta)\,(-K \sinh \beta y + \beta \cosh \beta y)}{\beta\,(K \cosh \beta h - \beta \sinh \beta h)} \cos \tau x \, d\tau \,, \text{ for } y > \eta \qquad \text{(A2.3.1)}$$

where $\int_0^\infty$ indicates that the contour runs below the pole at $\beta = k$, here k is the real root of the equation

$$(K \cosh \beta h - \beta \sinh \beta h) = 0,$$

and

$$\beta^2 = p^2 + \tau^2 \qquad \text{(A2.3.2)}$$

For y < η , the expansion for G is similar to $(A2.3.1)$ but y and η are interchanged.

The function

$$G \approx \frac{1}{2\pi} \ln r$$

close to the singular point, where

$$r^2 = x^2 + (y-\eta)^2 . \qquad \text{(A2.3.3)}$$

Using the identity (Gradshtegn and Ryzhik, 1980, $pp.$ 498)

$$K_0(pr) = \int_0^\infty \frac{e^{-\beta(y-\eta)}}{\beta} \cos \tau x d\tau, \text{ for} y > \eta \qquad \text{(A2.3.4)}$$

gives, after a little algebra,

$$\frac{1}{2\pi} (K_0(pr) + K_0(pr_1)) = \frac{1}{\pi} \int_0^\infty \frac{e^{-\beta(h-y)}}{\beta} \cosh \beta (h-\eta) \cos \tau x \, d\tau, \; y > \eta \qquad \text{(A2.3.5)}$$

where K_0 is the modified Bessel function of the first kind and order zero , and

$$r_1^{\,2} = x^2 + (2h - y - \eta)^2 \qquad \text{(A2.3.6)}$$

Adding equation $(A2.3.1)$ and $(A2.3.5)$ yields

$$G = -\frac{1}{2\pi} (K_0(pr) + K_0(pr_1)) + \frac{1}{\pi} \int_0^\infty \frac{(K+\beta)\, e^{-\beta h} \cosh \beta(h-y) \cosh \beta(h-\eta) \cos \tau x}{\beta(K \cosh \beta h - \beta \sinh \beta h)} d\tau$$

$$\text{(A2.3.7)}$$

By considering the residue from the pole at $\beta = k$ and shifting the contour $\int_0^\infty$ into a new contour $\int_{c^1}$ as shown in the figure. 2.3.1,

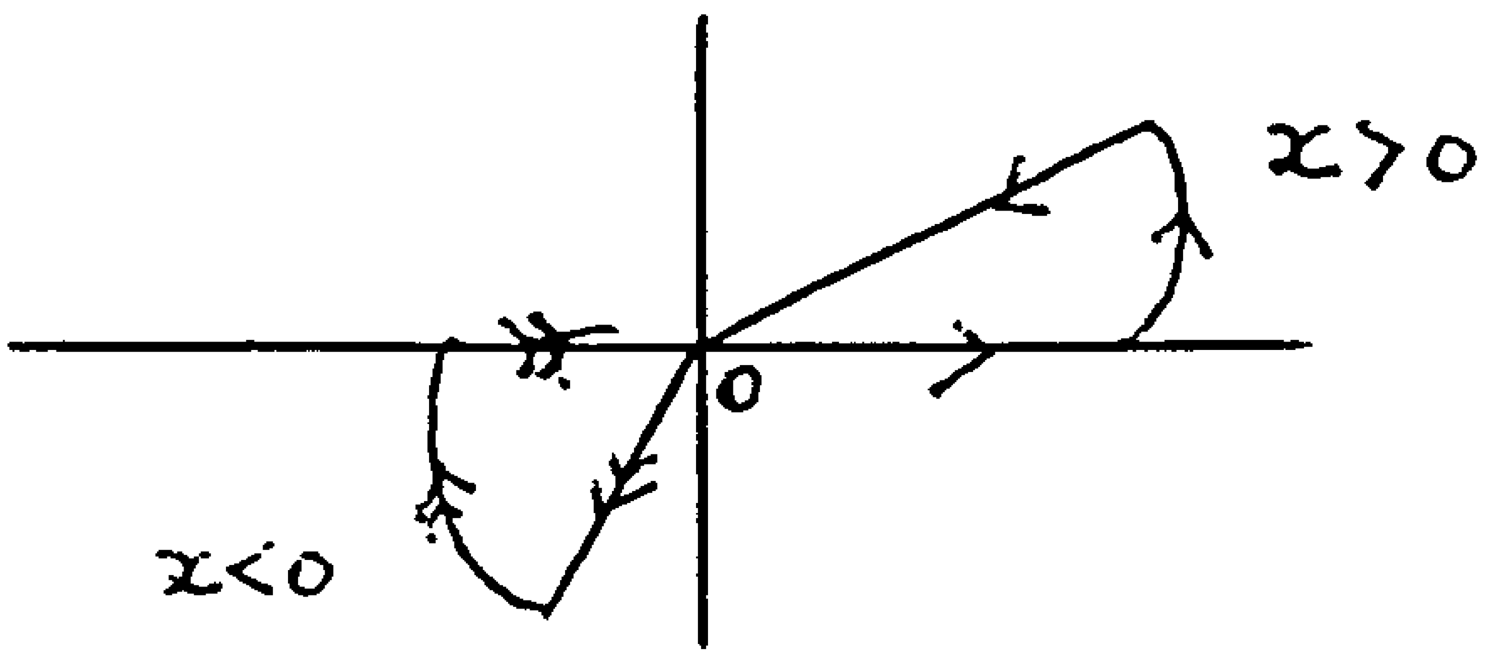


FIGURE 2.3.1.

$$G = -\frac{1}{2\pi}(K_0(pr) + K_0(pr_1)) - \frac{i \cosh k(h-y) \cosh k(h-\eta)\, e^{i\alpha|x|}}{2\alpha h N_0^{\,2}}$$

$$-\frac{1}{\pi} \operatorname{Re} \int_{c^1} \frac{(K+\beta)\, e^{-\beta h} \cosh \beta(h-y)\, e^{i\tau|x|} \cosh \beta(h-\eta)}{\beta(K \cosh \beta h - \beta \sinh \beta h)}\, d\tau \qquad \text{(A2.3.8)}$$

where

$$\alpha^2 = k^2 - p^2,$$

and

$$\text{(A2.3.9)}$$

$$N_0^{\,2} = \frac{1}{2}\left(1 + \frac{\sinh 2kh}{2kh}\right)$$

It follows that

$$G \approx -\frac{i \cosh k(h-y) \cosh k(h-\eta)\, e^{i\alpha|x|}}{2\alpha h N_0^{\,2}}, \qquad \text{as } |x| -> \infty \qquad \text{(A2.3.10)}$$

which will be used later. By considering the residue from the pole at $\beta = k$ in equation $(A2.3.7)$, G may be written as

$$G = -\frac{1}{2\pi}(K_0(pr) + K_0(pr_1)) - \frac{i}{2\alpha h N_0^{\,2}} \cosh k(h-y) \cosh k(h-\eta) \cos \alpha x$$

$$+ \frac{1}{\pi} \int_0^\infty \frac{(K+\beta)\, e^{-\beta h} \cosh \beta(h-y) \cosh \beta(h-\eta) \cos \tau x}{\beta(K \cosh \beta h - \beta \sinh \beta h)}\, d\tau \qquad \text{(A2.3.11)}$$

where $\int_0^\infty$ indicates that the principal value of the integral is to be taken.

(b) THE EXPANSION OF THE SOURCE POTENTIAL:

From $(A2.3.11)$ the expansion of the source potential about the singular point may be obtained using the result (Abramowitz and Stegun, 1965)

$$\exp\left(\frac{1}{2}\omega\,(t + \frac{1}{t})\right) = \sum_{n=-\infty}^{\infty} \omega^n I_n(\omega) \quad (t \neq 0) \qquad \text{(A2.3.12)}$$

where I_n is the modified Bessel function. Define μ by

$$\beta = p \cosh \mu \qquad \text{(A2.3.13)}$$

and, in equation ($A2.3.11$), put

$$\omega = -pr\ ,\ t = \exp(\mu \pm i\theta)$$

to obtain

$$\exp(\ \beta(y-\eta) -+ i\tau x\ ) = \sum_{n=0}^{\infty} \varepsilon_n\ (\cosh n\mu \cos n\theta \pm i \sinh n\mu \sin n\theta)\ (-1)^n\ I_n(pr) \quad \text{(A2.3.14)}$$

where,

$$\varepsilon_n = 2,\ \text{if } n \neq 0$$

$$= 1\ ,\ \text{if } n = 0\ . \quad \text{(A2.3.15)}$$

A similar result for $\exp(-\beta\ (y-\eta) +- i\tau x)$ may be obtained by letting $\omega = pr$. By substituting these results into equation ($A2.3.11$) gives

$$G = -\frac{1}{2\pi}\left(\ K_0(pr) + K_0(pr_1)\ \right) + \sum_{n=0}^{\infty} \varepsilon_n\ Q_n\ I_n(pr) \cos n\theta \quad \text{(A2.3.16)}$$

which is general expression for when the source is on x = 0, and y $= \eta$ and is not on the free-surface, where

$$x = r \sin\theta,$$

$$y - \eta = r \cos\theta,$$

$$N = \frac{\cosh^2 \frac{kh}{2}}{2\alpha h N_0^{\,2}},$$

$$M(\beta) = \frac{(K+\beta)\cosh^2 \frac{\beta h}{2}\, e^{-\beta h}}{(K\cosh\beta h - \beta\sinh\beta h)},$$

$$Q_{2n} = -iN\cosh 2n\mu_0 + \frac{1}{\pi}\,T_{2n}, \text{ for all } n$$

$$T_{2n} = \int_{C^1} \frac{M(\beta)}{\beta}\cosh 2n\mu\, d\tau, \text{ for all } n$$

$$Q_{(2n-1)} = iN\tanh\frac{kh}{2}\cosh(2n-1)\mu_0 - \frac{1}{\pi}\,T_{2n-1}, \text{ for } n > 0$$

$$T_{2n-1} = \int_{C^1} \frac{M(\beta)}{\beta}\tanh\frac{\beta h}{2}\cosh(2n-1)\mu\, d\tau,$$

and

$$k = p\cosh\mu_0.$$

(c) SOURCE IS ON THE BED

When the source is on the bed $(\eta = h)$, the equation $(A2.3.11)$ becomes as in the general case and equation (A2.3.16) gives

$$G = -\frac{1}{\pi} K_0(pr) + \sum_{n=0}^{\infty} \epsilon_n Q_{2n} I_{2n}(pr) \cos 2n\theta \tag{A2.3.17}$$

(i) EXPANSION ABOUT SINGULAR POINT $x = 0$, AND $y = h$

By expanding $(A2.3.17)$ about the singular point, G becomes

$$G = \frac{1}{\pi} \left(\ln \frac{pr}{2} + \gamma \right) + T_0 - \frac{\pi i}{2\alpha h N_0^{\,2}}$$

$$+ \frac{p^2 r^2}{4} \left(\ln \frac{pr}{2} + \gamma - 1 \right) - \frac{p^2 x^2}{2} \left(\frac{\pi i}{2\alpha h N_0^{\,2}} - T_0 \right)$$

$$+ \frac{1}{2} \left((h-y)^2 - x^2 \right) \left(- \frac{\pi i k^2}{2\alpha h N_0^{\,2}} + \frac{p^2}{4} (T_2 + T_0) \right) + O(r^3 \ln r) \tag{A2.3.18}$$

where γ is a Euler's constant. Using non-dimensional form,

$$X = \frac{x}{h}, \; Y = \frac{(h-y)}{h},$$

and

$$R = \frac{r}{h}.$$

Differentiating $(A2.3.18)$ with respect to X, gives

$$\frac{\partial G}{\partial X} = \frac{1}{\pi}\left(\frac{X}{R^2} + \frac{\delta^2 X}{2}\,(\ln\frac{\delta R}{2} + \gamma) - \frac{\delta^2 X}{4} - \delta^2 X\,(\frac{\pi i}{2\lambda N_0^{\;2}} - T_0) \right.$$

$$\left. - X\left(- \frac{\pi i \tau_0^{\;2}}{2\lambda N_0^{\;2}} + \frac{\delta^2}{4}(T_2 + T_0) \right)\right) + O(R^2 \ln R) \qquad \text{(A2.3.19)}$$

which represents a horizontal dipole. Higher-order singularities can be obtained by differentiating $(A2.3.19)$ twice with respect to X, giving

$$\frac{\partial^3 G}{\partial X^3} = \frac{1}{\pi}\left(\frac{2X(X^2 - 3Y^2)}{R^6} + \delta^2\,(\frac{3X}{2R^2} - \frac{X^3}{R^4}) \right) + O(R \ln R) \qquad \text{(A2.3.20)}$$

(ii) FAR-FIELD POTENTIALS

In non-dimensional form equation (A2.3.10) becomes,

$$G \approx - \frac{i \cosh \tau_0 Y\, e^{i\lambda|X|}}{2\lambda N_0^{\;2}}, \qquad \text{as } |X| -> \infty \qquad \text{(A2.3.21)}$$

$$G_2 = \frac{\partial G}{\partial X} \approx \frac{\cosh \tau_0 Y\, e^{i\lambda|X|}\, sgn(X)}{2N_0^{\;2}}. \qquad \text{as } |X| -> \infty \qquad \text{(A2.3.22)}$$

and

$$G_4 = \frac{\partial^3 G}{\partial X^3} \approx - \frac{\lambda^2 \cosh \tau_0 Y\, e^{i\lambda|X|}\, sgn(X)}{2N_0^{\;2}}, \qquad \text{as } |x| -> \infty. \qquad \text{(A2.3.23)}$$

Equations (A2.3.21-23) represent far-field potentials when the source is on the bed.

APPENDIX 2.4: SOURCE ON FREE SURFACE

As for appendix 2.3, the fundamental singular solution of the modified Helmholtz equation that satisfies the water wave boundary conditions is considered. Expansions about the singular point are derived when the singularity is on free-surface, and hence far-field potentials for dipole and higher-order singularities are derived. Throughout, the position of the origin is on the free surface with the y axis directed vertically downwards and water depth h.

(a) SINGULAR SOLUTION OF MODIFIED HELMHOLTZ EQUATION

When the source is on the free surface, MacCamy (1957) shows that

$$G = \frac{1}{\pi} \int_p^{\infty} \frac{\cosh \beta(h-y) \cos (\beta^2 - p^2)^{\frac{1}{2}} x \; \beta \, d\beta}{(K \cosh \beta h - \beta \sinh \beta h)(\beta^2 - p^2)^{\frac{1}{2}}} \qquad \text{(A2.4.1)}$$

By considering the residue of the pole at $\beta = k$, we obtain

$$G = -\frac{i \cosh kh \cosh k(h-y) \cos \alpha x}{2\alpha \, h \, N_0^{\,2}} + \frac{1}{2\pi} \mathrm{Re} \left(\int_p^{\infty} \frac{e^{(\beta(h-y) + i(\beta^2 - p^2)^{\frac{1}{2}} x)} \, \beta \, d\beta}{(K \cosh \beta h - \beta \sinh \beta h)(\beta^2 - p^2)^{\frac{1}{2}}} \right.$$

$$\left. + \int_p^{\infty} \frac{e^{-(\beta(h-y) - i(\beta^2 - p^2)^{\frac{1}{2}} x)} \, \beta \, d\beta}{(K \cosh \beta h - \beta \sinh \beta h)(\beta^2 - p^2)^{\frac{1}{2}}} \right) \qquad \text{(A2.4.2)}$$

that is,

$$G = -\frac{i \cosh kh \cosh k(h-y) \cos \alpha x}{2 \, \alpha h \, N_0^{\,2}} + \frac{1}{2\pi} (T_1 + T_2)$$

where

$$T_1 = \mathrm{Re} \int_p^\infty \frac{e^{(\beta(h-y) + i(\beta^2 - p^2)^{\frac{1}{2}} x)} \beta \, d\beta}{(K \cosh \beta h - \beta \sinh \beta h)(\beta^2 - p^2)^{\frac{1}{2}}}, \text{ and}$$

and

$$T_2 = \mathrm{Re} \int_p^\infty \frac{e^{-(\beta(h-y) - i(\beta^2 - p^2)^{\frac{1}{2}} x)} \beta \, d\beta}{(K \cosh \beta h - \beta \sinh \beta h)(\beta^2 - p^2)^{\frac{1}{2}}}$$

T_2 is a convergent integral for all x and y. However T_1 is not a convergent integral for $x - y = 0$ and neither are its derivatives with respect to x and y. Consider

$$T_1 = \mathrm{Re} \left(\int_p^\infty \left(\frac{e^{(\beta(h-y) + i(\beta^2 - p^2)^{\frac{1}{2}} x)} \beta \, d\beta}{(K \cosh \beta h - \beta \sinh \beta h)(\beta^2 - p^2)^{\frac{1}{2}}} - \frac{2e^{(\beta y + i(\beta^2 - p^2)^{\frac{1}{2}} x)} \beta \, d\beta}{(K - \beta)(\beta^2 - p^2)^{\frac{1}{2}}} \right) \right.$$

$$\left. + 2 \int_p^\infty \frac{e^{(\beta y + i(\beta^2 - p^2)^{\frac{1}{2}} x)} \beta \, d\beta}{(K - \beta)(\beta^2 - p^2)^{\frac{1}{2}}} \right) \qquad \text{(A2.4.3)}$$

and

$$2 \, \mathrm{Re} \int_p^\infty \frac{e^{(\beta y + i(\beta^2 - p^2)^{\frac{1}{2}} x)} \beta \, d\beta}{(K - \beta)(\beta^2 - p^2)^{\frac{1}{2}}} = 2 \int_p^\infty \frac{e^{\beta y} \cos(\beta^2 - p^2)^{\frac{1}{2}} x \, \beta \, d\beta}{(K - \beta)(\beta^2 - p^2)^{\frac{1}{2}}}$$

$$= -2 \frac{\pi i K e^{Ky} \cos \alpha_0 x}{\alpha_0} + 2 \int_p^\infty \frac{\beta e^{\beta y} \cos(\beta^2 - p^2)^{\frac{1}{2}} x \, \beta \, d\beta}{(K - \beta)(\beta^2 - p^2)^{\frac{1}{2}}} \qquad \text{(A2.4.4)}$$

where

$$\alpha_0^2 = K^2 - p^2 . \qquad \text{(A2.4.5)}$$

From (A2.4.4), and using $\beta = p \cosh \mu_0$, equation (A2.4.4) becomes,

$$2 \operatorname{Re} \int_p^\infty \frac{e^{(\beta y + i(\beta^2 - p^2)^{\frac{1}{2}} x)} \beta \, d\beta}{(K - \beta)(\beta^2 - p^2)^{\frac{1}{2}}} = - \frac{2 \pi i K e^{Ky} \cos \alpha_0 x}{\alpha_0}$$

$$- 2 p \int_0^\infty \frac{\cosh \mu_0 \, e^{py \cosh \mu_0} \cos (px \sinh \mu_0) \, d\mu_0}{(p \cosh \mu_0 - K)} . \qquad \text{(A2.4.6)}$$

The integral on the right side was considered by Ursell (1962). Therefore from (A2.4.1-4) and $(A2.4.6)$, G becomes

$$G = - \frac{i \cosh kh \cosh k(h-y) \cos \alpha x}{2 \alpha h N_0^2} - \frac{iK}{\alpha_0} e^{Ky} \cos \alpha_0 x + \frac{1}{2\pi} (T_2 - U_0)$$

$$+ \frac{1}{2\pi} \operatorname{Re} \int_p^\infty \left(\frac{e^{(\beta(h-y) + i(\beta^2 - p^2)^{\frac{1}{2}} x)}}{(K \cosh \beta h - \beta \sinh \beta h)(\beta^2 - p^2)^{\frac{1}{2}}} \right.$$

$$\left. - \frac{2 e^{(\beta y + i(\beta^2 - p^2)^{\frac{1}{2}} x)}}{(K - \beta)(K \cosh \beta h - \beta \sinh \beta h)(\beta^2 - p^2)^{\frac{1}{2}}} \right) \beta \, d\beta \qquad \text{(A2.4.7)}$$

where

$$x = r \sin\theta,$$

$$y = r \cos\theta,$$

$$U_0 = (\pi i - \alpha_1) \coth\alpha_1 \left(2 I_0(pr) + 4 \sum_{m=1}^{\infty} (-1)^m I_m(pr) \cos m\theta \cos m\alpha_1 \right)$$

$$+ 2K_0(pr) + 4 \sum_{m=1}^{\infty} (-1)^{m-1} \left[\frac{\partial}{\partial v} (I_v(pr) \cos v\theta) \right]_{v=m} \sinh m\alpha_1 \coth\alpha_1, \qquad \text{(A2.4.8)}$$

$$\alpha_1 = \cosh^{-1}\left(\frac{K}{p}\right), \qquad \text{(A2.4.9)}$$

and here K_0 and I_m denote modified Bessel functions and γ is a Euler's constant.

(b) EXPANSION ABOUT THE SINGULAR POINT

Using the expansions of K_0, I_0 and I_m about the singular point (0,0) (Erde'lyi 1953), equation (A2.4.7) becomes

$$G = \frac{1}{\pi}\left[\ln Kr - \pi\delta_0 + \Delta_4 + \gamma - F_1 - F_2 - \frac{\pi K i}{\alpha_0} \right.$$

$$\left. - Ky\left(-\pi\delta_0 + \Delta_5 - \frac{\pi K i}{\alpha_0} + \gamma - 1 - F_1 - F_2 \right) - Ky \ln Kr \right.$$

$$+ Kx\,\theta + \frac{p^2 r^2}{4}\ln Kr + \frac{K^2}{2}(y^2 - x^2)\left(-\frac{\pi\delta_0 k^2}{K^2} + \Delta_6 - \frac{\pi i K}{\alpha_0} - F_1 - F_2 \right.$$

$$\left. + \frac{p^2}{2K^2}(\pi i - \alpha_1)\coth\alpha_1 + \gamma - \frac{3}{2} + \ln Kr \right) + \frac{p^2 x^2}{2}\left(-\pi\delta_0 + \Delta_4 - \frac{\pi i K}{\alpha_0} \right)$$

$$-\frac{p^2 r^2}{4}\left(F_1 + F_2 + 1 - \gamma \right) + K^2\, xy\theta \Big] + O(r^3 \ln r) \qquad \text{(A2.4.10)}$$

where,

$$\delta_0 = \frac{i\cosh^2 kh}{2\alpha h N_0^{\,2}},$$

$$F_1 = \pi\, i \coth\alpha_1,$$

$$F_2 = \ln(2\cosh\alpha_1) - \alpha_1 \coth\alpha_1,$$

$$\Delta_{n+4} = \frac{1}{K^n}\int_p^\infty \left(\frac{1}{(K - \beta\tanh\beta h)\,(1 - \frac{p^2}{\beta^2})^{\frac{1}{2}}} - \frac{1}{(K-\beta)\,(1 - \frac{p^2}{\beta^2})^{\frac{1}{2}}} \right)\beta^n\, d\beta\,, \text{ for } n = 0, 2$$

and

$$\Delta_5 = \frac{1}{K}\int_p^\infty \left(\frac{\tanh\beta h}{(K - \beta\tanh\beta h)\,(1 - \frac{p^2}{\beta^2})^{\frac{1}{2}}} - \frac{1}{(K-\beta)\,(1 - \frac{p^2}{\beta^2})^{\frac{1}{2}}} \right)\beta\, d\beta.$$

In non-dimensional form,

$$X = \frac{x}{h}, \; Y = \frac{y}{h},$$

and

$$\delta = ph, \; \mu = Kh,$$

equation (A2.4.10) becomes ,

$$G = \frac{1}{\pi}\left(\ln \mu R + \alpha_3 - \mu Y \alpha_4 - \mu Y \ln \mu R + \mu X \, \theta + \frac{\delta^2 R^2}{4} \ln \mu R \right.$$

$$\left. + \frac{\mu^2}{2}(Y^2 - X^2) \ln \mu R + \frac{\mu^2}{2}(Y^2 - X^2)\, \alpha_5 + \frac{\delta^2 X^2}{2} \alpha_6 - \frac{\delta^2 R^2 \alpha_2}{4} + \mu^2 XY\theta \right) + O(R^3 \ln R)$$

(A2.4.11)

where,

$$\alpha_2 = F_1 + F_2 - \gamma,$$

$$\alpha_3 = -\pi \delta_0 + \Delta_4 - \alpha_2 - \frac{\pi i K}{\alpha_0},$$

$$\alpha_4 = -\pi \delta_0 + \Delta_5 - \frac{\pi i K}{\alpha_0} - 1 - \alpha_2,$$

$$\alpha_5 = -\frac{\pi\delta_0 k^2}{K^2} + \Delta_6 - \frac{\pi i K}{\alpha_0} - \alpha_2 + \frac{\delta^2}{2\mu^2}(\pi i - \alpha_1)\coth\alpha_1 - \frac{3}{2},$$

and

$$\alpha_6 = -\pi\delta_0 + \Delta_4 - \frac{\pi i K}{\alpha_0}.$$

The horizonal dipole can be defined by differentiating equation (A2.4.11) with respect to X,

$$\frac{\partial G}{\partial X} = \frac{1}{\pi}\left(\frac{X}{R^2} + \mu\,\theta + X\alpha_7 - \mu^2 X \ln \mu R \right.$$

$$\left. + \frac{\delta^2 X}{2}\ln \delta R - \mu^2 Y\,\theta \right) + O(R^2 \ln R) \qquad \text{(A2.4.12)}$$

where

$$\alpha_7 = -\mu^2\alpha_5 + \delta^2\alpha_6 - \frac{\delta^2}{2}\alpha_2 + \frac{\delta^2}{4} - \frac{\mu^2}{2} \qquad \text{(2.4.13)}$$

Higher-order singularities can obviously be generated by further differentiation; thus differentiating (A2.4.12) twice with respect to X gives,

$$\frac{\partial^3 G}{\partial X^3} = \frac{1}{\pi}\left(\frac{2X(X^2 - 3Y^2)}{R^6} - \frac{2\mu XY}{R^4} - \frac{\mu^2 X}{R^2} + \frac{\delta^2 X}{2R^2} + \frac{\delta^2 XY^2}{R^4} \right) + O(1) \qquad \text{(A2.4.14)}$$

(c) FAR-FIELD POTENTIAL

From equation (A2.3.10),

$$G \approx -\frac{i \cosh k(-y+h) \cosh kh \, e^{i\alpha |x|}}{2\alpha h N_0^{\,2}}, \qquad \text{as } |x| -> \infty \qquad \text{(A2.4.15)}$$

In non- dimensional form, we obtain

$$G \approx -\frac{i \cosh \tau_0(-Y+1) \cosh \tau_0 \, e^{i\lambda |x|}}{2\lambda N_0^{\,2}}, \qquad \text{as } |X| -> \infty \qquad \text{(A2.4.16)}$$

$$G_2 = \frac{\partial G}{\partial X} \approx \frac{\cosh \tau_0(Y+1) \cosh \tau_0 \, e^{i\lambda |X|} \, sgn|X|}{2N_0^{\,2}}, \qquad \text{as } |X| -> \infty \qquad \text{(A2.4.17)}$$

and

$$G_4 = \frac{\partial^3 G}{\partial X^3} \approx -\frac{\lambda^2 \cosh \tau_0(-Y+1) \cosh \tau_0 \, e^{i\lambda |X|} \, sgn|X|}{2N_0^{\,2}}, \qquad \text{as } |X| -> \infty . \qquad \text{(A2.4.18)}$$

Equations (A2.4.16-18) represent far field potentials when the source is on the free surface.

CHAPTER 3

SLOSHING IN A RECTANGULAR TANK WITH INTERNAL BODIES

3.1 INTRODUCTION

In a number of circumstances it is important to know the natural frequencies of fluid with a free surface in a container. For example, when liquid natural gas carried by ships sloshes within its container this could seriously effect the ship's stablity, and so it is desirable for the ship's design to avoid external excitation at the natural oscillation frequencies of the liquid by suitable choice of container dimensions. These oscillation periods each correspond to different sloshing modes of the liquid, which are functions of the tank geometry and size. Another important application of the results below concerns the stablity of spacecraft and missiles with liquid-fuel tanks.

Here the main aim is to calculate the natural frequencies, and not the amplitudes of the sloshing motions for which viscosity and non-linearity are both important. We suppose that such effects will only slightly de-tune the natural frequencies or eigenfrequencies of the container from their small amplitude oscillation values. Hence the linearised, inviscid equations of motion will be used in the following.

The Cartesian co-ordinates (x, y, z) are employed with the origin in the undisturbed free surface and y measured vertically downwards. The fluid occupies the container which has a flat bottom at $y = h$, and the walls correspond to

$$x = b\,;\, 0 \le z \le l\,;\, 0 \le y \le h$$

$$x = -b\,;\, 0 \le z \le l\,;\, 0 \le y \le h$$

$$z = 0\,;\, -b \le x \le b\,;\, 0 \le y \le h$$

$$z = l\,;\, -b \le x \le b\,;\, 0 \le y \le h. \tag{3.1.1}$$

See also figure 3.1.1.

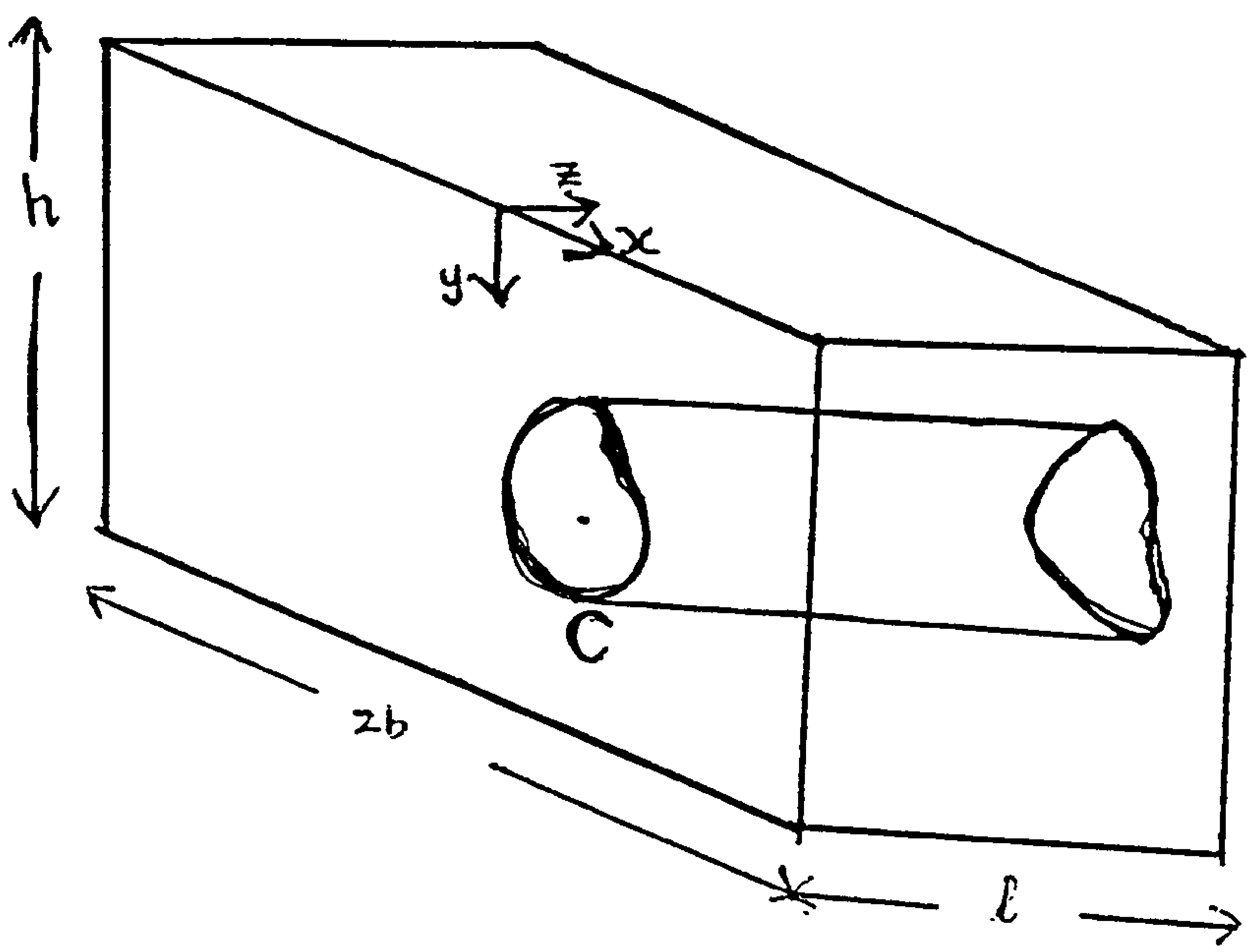


FIGURE 3.1.1.

Bodies with generators parallel to the z direction will be introduced into the tank and a cross-section is illustrated in figure 3.1.1. In this cross-section, a reference point within the body has coordinates $(x, y) = (x_0, y_0)$ and the surface of the body cross-section is denoted by C. A time-harmonic factor $e^{-i\omega t}$ is removed. Under the usual assumptions of linearized water wave theory, a velocity potential $\Phi(x, y, z)$ exists which satisfies,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{3.1.2}$$

in the fluid. No flow through the walls at $z = 0, l$ requires

$$\frac{\partial \Phi}{\partial z} = 0 \text{ on } z = 0 , l \tag{3.1.3}$$

which may be satisfied by taking

$$\Phi(x , y , z) = \phi(x , y) \cos pz \tag{3.1.4}$$

with $p = \dfrac{N \pi}{l}$ and N is any integer. From (3.1.2), $\phi(x , y)$ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - p^2 \phi = 0 \tag{3.1.5}$$

in the fluid. The problem is now two dimensional because the factor cos pz has been removed. The potential ϕ must also satisfy the linearised free-surface condition

$$K \phi + \frac{\partial \phi}{\partial y} = 0 , y = 0 , |x| < b \tag{3.1.6}$$

where $K = \dfrac{\omega^2}{g}$. There is no flow through the solid boundaries and so for the bed

$$\frac{\partial \phi}{\partial y} = 0 \text{ on } y = h , |x| \leq b \tag{3.1.7}$$

for the walls

$$\frac{\partial \phi}{\partial x} = 0 \text{ on } x = \pm b , 0 < y < h \tag{3.1.8}$$

and for the body

$$\frac{\partial \phi}{\partial n} = 0 \text{ on } C \tag{3.1.9}$$

where n is a normal coordinate measured into the fluid. This boundary-value problem will be solved by a number of different methods which are described in sections 3.2 and 3.3 .

In the absence of the body, the potential $\Phi(x , y , z)$ can easily be found by using the separation of variables method and the boundary conditions (3.1.6-7). Thus

$$\Phi(x , y , z) = \cos [\alpha (x - b)] \cosh [k (h - y)] \cos pz \tag{3.1.10}$$

where

$$\alpha = \alpha_M = \frac{M\pi}{2b} , \ k = (\alpha^2 + p^2)^{\frac{1}{2}} \tag{3.1.11}$$

and M is any integer. The eigenfrequencies of the tank may then be found from the finite depth dispersion relation

$$\frac{\omega^2}{g} = K = k \tanh kh. \tag{3.1.12}$$

These values are shown on the presented graphs of eigenfrequencies with bodies for easy comparison.

Firstly, I consider a thin vertical barrier. When the barrier penetrates the entire depth of the fluid so as to form two separate containers, it will make two sets of eigenfrequencies appropriate to the dimensions of the respective containers (see equations 3.1.11 and 3.1.12). When the barrier is introduced which extends from the free surface but does not reach the bottom, it can be shown from standard eigenvalue theory (Courant and Hilbert, 1953, chapter 6) that the eigenfrequencies are decreased in general. In fact, as the length of the barrier increases from 0 to h with p fixed, Courant and Hilbert show that the nth eigenfrequency decreases continuously from its value in the absence of the barrier, to the closest eigenfrequency not greater than it corresponding to the two separate containers obtained when the barrier length is h. When the position of the barrier coincides with an antinode of the oscillation at which the horizonal velocity is zero throughout the depth, introducing the barrier has no effect. Another deduction from Courant and Hilbert is that when the geometry is fixed p and K may only change in the same sense. Thus, for example, if p is increased then every eigenfrequency will also increase. My aim is to examine in detail how these frequencies are influenced by the presence of the body in detail, but it is worth noting that the calculated eigenfrequencies are consistent with these more general results.

In section 3.2 the problem is formulated by using the eigenfunction expansion method. By using the condition of continuity of pressure and horizontal velocity, the eigenfunction expansions valid either side of the barrier are matched across the gap in the fluid not occupied by the barrier. An integral equation is obtained for the unknown velocity across the gap, and an explicit condition is obtained for the wavenumbers in terms of a quantity A which is related to this velocity. This method has been applied by other authors, for example Evans and McIver (1987) in a related 2-D problem (when $p = 0$). The expansion of the unknown velocity in a series of orthogonal functions enables, after truncating the series, a determinant form to be obtained in terms of A and known matrices B and C.

In section 3.3 a more general approach is described applicable to bodies of any shape. Green's second identity is applied to two different potentials, and the original boundary-value problem becomes an integral equation eigenvalue problem for the eigenfrequencies. By using constant values of ϕ along the boundary elements, the integral equation eigenvalue problem is converted to a matrix eigenvalue problem.

In section 3.4, two approximate solutions based on the "wide-spacing approximation" and a "small body approximation" are obtained. The wide-spacing approximation assumes that the wave-length is small compared with the distance between the body and walls. The small body approximation means that a typical dimension of the body is much smaller than the cross-sectional length scale of the fluid motion. Both methods are used to consider two cases: a vertical surface-piercing barrier, and submerged circular cylinder on the line $x = 0$. By using the wide-spacing approximation, we obtain a general expression to determine the eigenfrequencies in terms of the reflection and transmission coefficients for that body. Approximations to the reflection and transmission coefficients for the barrier are known from Chapter 2 (equation (2.3.66)). The same coefficients for the cylinder case have been calculated by Davis and Leppington (1977) who used the method of matched asymptotic expansions. In the small-body approximation, Green's theorem is applied to the potential with and without the body being present. We obtain the explicit form for the eigenfrequencies in terms of free-surface and body integrals, which are estimated for both body geometries above to obtain analytical results for various body dimensions within a fixed container geometry.

3.2 EIGENFUNCTION METHOD FOR THE VERTICAL BARRIER

The contour C is here taken to be a thin vertical barrier which will be denoted by L_1. That is L_1 is $x = x_0$, $0 \le y \le a$ for a surface piercing barrier, and L_1 is $x = x_0$, $a \le y \le h$ for a bottom-mounted barrier, with $0 \le a \le h$ in each case. Here the boundary value problem is defined by the modified Helmholtz equation (3.1.5), the linearized free-surface condition (3.1.6), the bed, wall conditions (3.1.7-8) and the body condition (3.1.9). The body boundary condition is

$$\frac{\partial \phi}{\partial x} = 0 \text{ on } L_1. \tag{3.2.1}$$

As in equation (3.1.12) we wish to find the wavenumber k which is related to the frequency ω through

$$\frac{\omega^2}{g} = K = k \tanh kh. \tag{3.2.2}$$

We introduce the orthonormal eigenfunctions chosen to satisfy the bottom boundary condition (3.1.7)

$$\psi_n(y) = N_n^{-1} \cos k_n(h-y) \tag{3.2.3}$$

where k_n ($n = 1,2...$) are the real positive roots of the equation

$$K + k_n \tan k_n h = 0 \tag{3.2.4}$$

with $k_0 = ik$, and

$$2\, N_n^2 = (\, h + \frac{\sin 2k_n h}{2k_n}\,) \tag{3.2.5}$$

Let ϕ_1 and ϕ_2 be the solutions of the equation (3.1.5) for the regions on either side of the barrier.

Using the separation of variables method

$$\phi_1(x, y) = \sum_{n=0}^{\infty} (A_n e^{m_n x} + B_n e^{-m_n x}) \psi_n(y), \text{ for } x_0 < x < b \tag{3.2.6}$$

and

$$\phi_2(x, y) = \sum_{n=0}^{\infty} (C_n e^{m_n x} + D_n e^{-m_n x}) \psi_n(y), \text{ for } -b < x < x_0 \tag{3.2.7}$$

where A_n , B_n , C_n , and D_n are constants, and $\psi_n(y)$ is given by the equation (3.2.3),

$$m_n^2 = p^2 + k_n^2 \tag{3.2.8}$$

and

$$m_0 = i (k^2 - p^2)^{\frac{1}{2}} = i \alpha, \tag{3.2.9}$$

say. Let L be the interval on $x = x_0$ not occupied by the barrier. Then continuity of pressure and horizonal velocity requires,

$$\phi_1 = \phi_2 \text{ on } L;$$

$$\frac{\partial \phi_1}{\partial x} = \frac{\partial \phi_2}{\partial x} \text{ on } L, \tag{3.2.10}$$

and the wall condition (3.1.8) , ϕ_1 and ϕ_2 may be conveniently written in the form

$$\phi_1(x\,,\,y) = -\sum_{n=0}^{\infty} u_n \frac{\cosh m_n(b-x)}{m_n \sinh m_n b}\, \psi_n(y)\,,\ \text{for}\ \ x_0 \le x \le b \qquad (3.2.11)$$

$$\phi_2(x\,,\,y) = \sum_{n=0}^{\infty} u_n \frac{\cosh m_n(b+x)}{m_n \sinh m_n b}\, \psi_n(\,y\,)\,,\ \text{for}\ \ -b \le x \le x_0 \qquad (3.2.12)$$

where the u_n are the Fourier coefficients in the expansion of the horizonal velocity $U(y)$ across $x = x_0\,, 0 \le y \le h$. Using the condition $U(y) = 0$ on L', we have

$$U(y) = \sum_{n=0}^{\infty} u_n\, \psi_n(y) \text{ with } u_n = \langle U\,,\, \psi_n \rangle \equiv \int_L U(y)\, \psi_n(y)\, dy\,. \qquad (3.2.13)$$

The continuity of ϕ across L gives

$$2\sum_{n=0}^{\infty} u_n\, m_n^{-1}\left(\coth m_n(b-x_0) + \coth m_n(b+x_0)\right)\psi_n(y) = 0\,,\ (\,on\ L\,). \qquad (3.2.14)$$

It follows on substitution from (3. 2. 13) that

$$\int_L U\,(t)\, K\,(\,y\,,t\,)\, dt = 0 \quad a < y < h\,; 0 < y < h - a \qquad (3.2.15)$$

where

$$K\,(y\,,\,t) = \sum_{n=0}^{\infty} s_n\, \psi_n(y)\, \psi_n(t) \qquad (3.2.16)$$

and

$$s_n = \alpha\, m_n^{-1} \left(\coth m_n(b - x_0) + \coth m_n(b + x_0) \right) \qquad (3.2.17)$$

Here a factor α has been included for later convenience. By introducing

$$K(y, t) = -A^{-1}\, \psi_0(y)\, \psi_0(t) + K_1(y, t) \qquad (3.2.18)$$

where

$$K_1(y, t) = \sum_{n=1}^{\infty} s_n\, \psi_n(y)\, \psi_n(t) \qquad (3.2.19)$$

and

$$A = -\left(\tanh[\alpha(b - x_0)] + \tanh[\alpha(b + x_0)] \right) \qquad (3.2.20)$$

and defining $u(y)$ by

$$U(y) = U_0\, A^{-1}\, u(y) \qquad (3.2.21)$$

equation (3.2.15) becomes,

$$\int_L u(t)\, K_1(y, t)\, dt = \psi_0(y), \quad (y \,\varepsilon\, L). \qquad (3.2.22)$$

Multiplication of (3.2.21) by $\psi_0(y)$ and integration over L gives

$$< u , \psi_0 > = \int_L u(t)\, \psi_0(t)\, dt = A . \tag{3.2.23}$$

It follows from (3.2.20) and (3.2.22) that

$$A = \frac{< u , \psi_0 >^2}{\sum_{n=1}^{\infty} < u , \psi_n >^2 s_n} . \tag{3.2.24}$$

Now U(y) can be expanded in an infinite series of terms of the orthonormal set { $\psi_m(y)$ }, $m =$ 0,1,2.... If we substitute, as a trial function, the truncated expansion

$$U(y) = \sum_{m=0}^{M} u_m\, \psi_m(y) \tag{3.2.25}$$

into (3.2.24), we obtain,

$$A = \frac{U^T\, C\, U}{U^T\, B\, U} \tag{3.2.26}$$

where,

$$U^T = (u_0 , u_1 , \ldots , u_m) , C = cc^T , c^T = (c_{00} , c_{10} , c_{20} , \ldots , c_{m0}),$$

$$c_{mn} = < \psi_m , \psi_n > , \; B = (B_{mn}) , \tag{3.2.27}$$

and

$$B_{mp} = \sum_{n=1}^{\infty} c_{mn} \, c_{pn} \, s_n .$$

The best possible approximation of the form (3.2.25) is now obtained by requiring u_m to be chosen so that equation (3.2.24) is stationary. By differentiation of (3.2.26) with respect to the elements of U, we obtain

$$\det (C - AB) = 0. \tag{3.2.28}$$

In this equation C and B are known matrices (given by (3.2.27)) and A is related to the unknown eigenvalues k via equations (3.2.8-9) and (3.2.20).

3.3 BOUNDARY ELEMENT METHOD

The following is based on the work of Brebbia and Walker (1980, pp.35-38). Let the potential ϕ and source potential G (except at the source point (ξ,η)) satisfy the modified Helmholtz equation (3.1.5), and the condition of no flow through the solid boundaries of the tank, equations (3.1.7-8). We consider the case $p=0$, i.e. the problem is two-dimensional. By the application of Green's second identity to G and the potential ϕ of equations (3.1.6)-(3.1.9), we have that

$$\int_{\Omega} (\phi\, \nabla^2 G - G\, \nabla^2 \phi\,)dV = \int_{S} (\phi\, \frac{\partial G}{\partial n} - G\, \frac{\partial \phi}{\partial n}\,)ds \tag{3.3.1}$$

where n is a normal directed out of the fluid, and $S = S_W + S_B + S_F + S_\varepsilon$. Here S_W denotes the tank walls, S_B the boundary of the internal body, S_F the free surface and S_ε is a small semicircle of radius ε taken about the source point (ξ,η) which is on the boundary. By the application of equation (3.1.5) into equation (3.3.1), we have

$$\int_{S_W+S_B+S_F+S_\varepsilon} (\phi\, \frac{\partial G}{\partial n} - G\, \frac{\partial \phi}{\partial n}\,)ds = 0 \tag{3.3.2}$$

Applying the wall conditions, equation (3.1.8) to equation (3.3.2), gives

$$\int_{S_W} (\phi\, \frac{\partial G}{\partial n} - G\, \frac{\partial \phi}{\partial n}\,)ds = 0 \tag{3.3.3}$$

and hence

$$\int_{S_B+S_F+S_\varepsilon} (\phi\, \frac{\partial G}{\partial n} - G\, \frac{\partial \phi}{\partial n}\,)ds = 0. \tag{3.3.4}$$

Since S_ε is a circle of radius ε centred at the source point (ξ, η),

$$\int_{S_\varepsilon} (\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n})ds = \lim_{\varepsilon \to 0} \int_0^\pi [\phi \frac{\partial G}{\partial R} \frac{\partial R}{\partial n} - G \frac{\partial \phi}{\partial R} \frac{\partial R}{\partial n}] \varepsilon \, d\theta$$

where R is measured from the source point, so

$$\frac{\partial R}{\partial n} = -1$$

Also near to the source point,

$$G \approx \ln R$$

Therefore,

$$\int_{S_\varepsilon} [\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}]ds = \lim_{\varepsilon \to 0} \int_0^\pi [- \frac{\phi}{\varepsilon} + \frac{\partial \phi}{\partial R} \ln \varepsilon] \varepsilon \, d\theta.$$

Since $\lim_{\varepsilon \to 0} \varepsilon \ln \varepsilon = 0$ we have

$$\int_{S_\varepsilon} [\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}] \, ds = - \pi \, \phi \tag{3.3.5}$$

Thus for any boundary point (ξ, η)

$$- \pi \, \phi + \int_{S_B + S_F} (\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n})ds = 0 \tag{3.3.6}$$

Applying the boundary condition (3.1.9) and

$$\frac{\partial \phi}{\partial n} = 0 \text{ on } S_B \tag{3.3.7}$$

gives

$$-\pi\,\phi + \int_{S_B} \phi\,\frac{\partial G}{\partial n}\,ds + \int_{S_F} (\,\phi\,\frac{\partial G}{\partial n} - K\,\phi\,G\,)\,ds = 0.$$

That is,

$$-\pi\,\phi + \int_{S_B} \phi\,\frac{\partial G}{\partial n}\,ds + \int_{S_F} \phi\,\frac{\partial G}{\partial n}\,ds = K \int_{S_F} \phi\,G\,ds \tag{3.3.8}$$

Now by using the boundary-element method with constant values of ϕ over each element, this integral equation eigenvalue problem for K can be solved.

The constant boundary element can be described as follows:

(a) The boundary S_B is divided into N straight-line elements

(b) Assume that ϕ is constant over each individual element, and takes the value at the node (the mid-point).

If the nodes are numbered in an anticlockwise direction such that the nodes numbered $1, 2, 3, \ldots, N_B$ lie on the boundary S_B , and the nodes numbered $N_B + 1, N_B + 2$,...,N lie on the boundary S_F , then equation(3.3.8) can be expressed as

$$-\pi\,\phi_i + \sum_{j=1}^{N_B} \phi_j \int_{S_{B_j}} \frac{\partial G_{ij}}{\partial n}\,ds + \sum_{j=N_B+1}^{N} \phi_j \int_{S_{F_j}} \frac{\partial G_{ij}}{\partial n}\,ds = K \sum_{j=N_B+1}^{N} \phi_j \int_{S_{F_j}} G_{ij}\,ds \tag{3.3.9}$$

(c) Take the source point (ξ, η) at each node in turn. Then by using (3.3.9) , we have

$$-\pi\,\phi_i + \sum_{j=1}^{N_B} \phi_j \int_{S_{B_j}} \frac{\partial G_{ij}}{\partial n}\,ds + \sum_{j=N_B+1}^{N} \phi_j \int_{S_{F_j}} \frac{\partial G_{ij}}{\partial n}\,ds$$

$$= K \sum_{j=N_B+1}^{N} \phi_j \int_{S_{F_j}} G_{ij}\,ds \;\;, i = 1, 2, 3, \ldots, N \tag{3.3.10}$$

Now the equation (3.3.10) can be written in matrix notation as

$$(-\pi\, I + B + C\,)\,\phi_i = K\; D\;\phi_i \tag{3.3.11}$$

where ϕ_i are unknowns,

$$B_{ij} = \int_{S_{B_j}} \frac{\partial G_{ij}}{\partial n}\,ds,$$

(3.3.12)

$$C_{ij} = \int_{S_F} \frac{\partial G_{ij}}{\partial n}\,ds,$$

$$D_{ij} = \int_{S_F} G_{ij}\, ds \,,$$

and I is an identical matrix of order $N \times N$. Here B_{ij}, C_{ij} and D_{ij} are evaluated in appendix 3.

3.4 APPROXIMATE METHODS

(a) Wide-spacing approximation

This approximation is applied by several authors, for example Evans and McIver (1987), and assumes that the interaction of the wave field with the body is governed by the appropriate reflection and transmission coefficients for waves incident upon the body in a fluid having a free surface extending to infinity in either direction.

Near $x = b$ we require the potential to satisfy the boundary condition in the container walls. We can thus write $\phi(x\ ,\ y)$ as

$$\phi(x\ ,\ y) = 2B \cos \alpha(x - b)\, \psi_0(y)\ ,$$

$$= B\, (\, e^{i\alpha(x-b)} + e^{-i\alpha(x-b)}\,)\, \psi_0(y)\ , \tag{3.4.1}$$

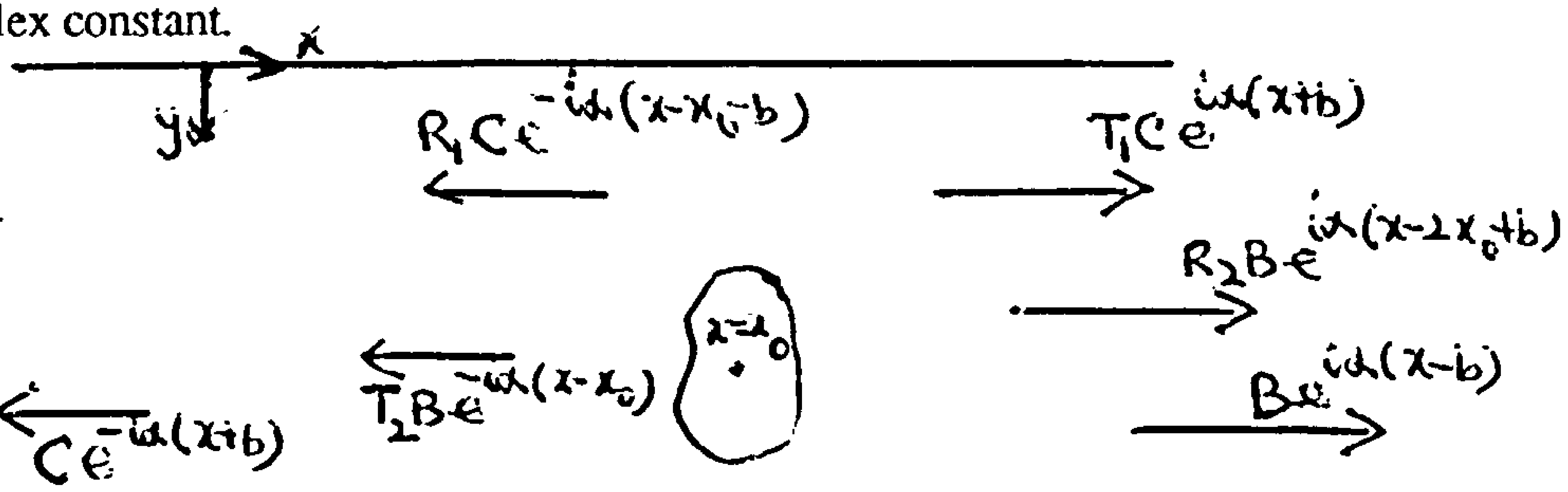
where B is a complex constant and $\psi_0(y)$ is defined by the equation (3.2.3), with $n = 0$, whilst near $x = -b$,

$$\phi(x\ ,\ y) = 2C \cos \alpha(x + b)\, \psi_0(y)\ ,$$

$$= C\, (\, e^{i\alpha(x+b)} + e^{-i\alpha(x+b)}\,)\, \psi_0(y). \tag{3.4.2}$$

Here C is a complex constant.

FIGURE 3.4.1:



Let R_1 and T_1 be the reflection and transmission coefficients for waves incident from the left of the body and R_2 and T_2 be the reflection and transmission coefficients for waves incident from the right of the body (as shown in the figure 3.4.1).

Newman (1976) has shown that $T_1 = T_2 = T$, i.e. the transmission coefficient is independent of direction of the incident wave, for any arbitary two dimensional body. Using his idea we can prove that $T_1 = T_2 = T$, for any arbitrary three-dimensional body in the wave tank. Using conservation of energy

$$|R_1|^2 + |T_1|^2 = |R_2|^2 + |T_2|^2 = 1$$

gives, $|R_1| = |R_2| = |R|$.

Consider the source of waves travelling away from the body($x = x_0$), as shown in the figure, and assume C = 1

$$B\, e^{i\alpha(x_0 - b)} = R\, B\, e^{-i\alpha(x_0 - b)} + T\, e^{i\alpha(x_0 + b)} \tag{3.4.3}$$

and

$$e^{-i\alpha(x_0 + b)} = R\, e^{i\alpha(x_0 + b)} + T\, B\, e^{-i\alpha(x_0 - b)} \tag{3.4.4}$$

where α is defined by equation (3.2.9). From (3.4.3) and (3.4.4) we have

$$T^2\, e^{2i\alpha b} = e^{-2i\alpha b} - 2\, R \cos(2\alpha x_0) + R^2\, e^{2i\alpha b}. \tag{3.4.5}$$

This is a general expression derived under the wide-spacing approximation for the determination of the frequencies in a tank containing any body whatsoever, in terms of the reflection and transmission coefficients for that body when in open water. Define

$$\alpha = (k^2 - p^2)^{\frac{1}{2}} = \alpha_M - \frac{1}{2b} f(\varepsilon)\, \sigma_2 \tag{3.4.6}$$

where $\alpha_M = \dfrac{M\pi}{2b}$, M is an integer and the form of f(ε) is to be determined. My aim is to determine σ_2 . From equation (3.4.6), we have

$$k = k_M - \frac{M\pi}{4b^2 k_M} f(\varepsilon)\, \sigma_2 + \ldots , \tag{3.4.7}$$

where

$$k_M = (\alpha_M{}^2 + p^2)^{\frac{1}{2}} \tag{3.4.8}$$

and

$$\cosh^2 kh = \frac{1}{2}\left(\cosh 2k_M h + 1 - \frac{M\pi h}{4k_M b^2} f(\varepsilon)\, \sigma_2 \sinh 2k_M h + \cdots \right). \tag{3.4.9}$$

Now I am going to consider two cases, the surface-piercing barrier and the submerged cylinder.

(i) Surface-piercing barrier:

From the section 2.3 (equation 2.3.66) , R can be written as

$$R = -\frac{\alpha\,\pi\, ih}{4N_0^{\,2}} \cosh^2 kh\, \varepsilon^2 + O(\varepsilon^4) \tag{3.4.10}$$

where $\varepsilon = \frac{a}{h}$. For the barrier,

$$R + T = 1. \tag{3.4.11}$$

(To prove this, take an incident wave coming from the left, that is it is moving in the direction of x increasing. Applying Green's theorem twice to the unknown potential ϕ and $e^{(i\alpha x)}$ and ϕ and $e^{(-i\alpha x)}$ gives and expressions for R and T which yield (3.4.11).)

From equations (3.4.5-6), and (3.4.9-11) and after little simplification, we obtain

$$f(\varepsilon)\,\sigma_2 = -\frac{\alpha_M\,\pi h}{4\,N_{0,M}^{\,2}} \cosh^2 k_M h\, \varepsilon^2\, (\,1 - (-1)^M \cos 2\alpha_M x_0\,) \tag{3.4.12}$$

and

$$N_{0,M}^{\,2} = \frac{1}{4\,k_M\, h}\,(\,\sinh\,[2k_M h] + 2\,k_M\, h\,).$$

In order to find σ_2, from (3.4.12) we define $f(\varepsilon) = \varepsilon^2$. Therefore

$$\sigma_2 = \frac{\alpha_M\,\pi\, h \cosh^2 k_M h}{4N_{0,M}^{\,2}} \left(\,1 - (-1)^M \cos 2\alpha_M x_0\,\right). \tag{3.4.13}$$

Substituting this into equation (3.4.6), we have

$$\alpha = \frac{M\pi}{2b} - \varepsilon^2 \frac{M\pi^2 h}{16 b^2 N_{0.M}^2} \cosh^2 k_M h \left(1 - (-1)^M \cos 2\alpha_M x_0\right) \tag{3.4.14}$$

(ii) Circular cylinder, centred on (x_0, y_0) with radius a and $y_0 > a$.

From Davis and Leppington (1977), we have

$$R = -\frac{\pi kh i}{N_0^2} \varepsilon^2 + O(\varepsilon^4) \tag{3.4.15}$$

and

$$T = 1 + \frac{\pi kh i}{N_0^2} \varepsilon^2 \cosh\left[2 kh \left(\frac{y_0}{h} + 1\right)\right] \tag{3.4.16}$$

By substituing equations (3.4.15-16) into equation (3.4.5) and using the equations (3.4.7-9), we get

$$1 + \frac{2\pi i k_M h}{N_{0,M}^2} \varepsilon^2 \cos 2\alpha_M x_0 \, e^{2i\alpha_M b} = e^{-2 i f(\varepsilon) \sigma_2} \left(1 + \frac{2\pi k_M h i}{N_{0,M}^2} \varepsilon^2 \cosh\left[2k_M h \left(\frac{y_0}{h} + 1\right)\right]\right).$$

As before, in order to find σ_2 we observe $f(\varepsilon) = \varepsilon^2$. Therefore,

$$\sigma_2 = \frac{\pi k_M h}{N_{0,M}^2} \left(\cosh\left[2 k_M h \left(\frac{y_0}{h} + 1\right)\right] - (-1)^M \cos 2\alpha_M x_0\right) \tag{3.4.17}$$

By substituting σ_2 into equation(3.4.6), we obtain

$$\alpha = \frac{M\pi}{2b} - \frac{\pi k_M h}{2b N_{0,M}^2} \varepsilon^2 \left(\cosh\left[2 k_M h \left(\frac{y_0}{h} + 1\right)\right] - (-1)^M \cos 2\alpha_M x_0\right) \tag{3.4.18}$$

(b) A small body approximation

It has been pointed out in section 3.1 that solutions of the boundary-value problem (3.1.5-9) when there is no body within the tank are of the form

$$\phi_M = \cos \alpha_M (x - b) \cosh k_M (y - h), \tag{3.4.19}$$

where

$$\alpha_M = M\pi/2b \, , \, k_M = (\alpha_M^2 + p^2)^{\frac{1}{2}} \tag{3.4.20}$$

and M is any integer. From the free-surface condition (3.1.6) the corresponding value of the frequency parameter K is

$$K_M = k_M \tanh k_M h. \tag{3.4.21}$$

If M is odd, these modes of oscillation are antisymmetric in x, while if M is even the modes are symmetric in x. For a given tank geometry there are a doubly infinite set of modes which can be identified by the integers N and M in the definition of p and α_M .

As before, the aim is to calculate the change in K when a body of uniform cross-section spans the tank in the z direction. Since ϕ and ϕ_M, the potentials with and without the body respectively, both satisfy the modified Helmholtz equation we may apply the Green's theorem to the potentials over the fluid domain exterior to C giving,

$$\int_F (\phi \frac{\partial \phi_M}{\partial n} - \phi_M \frac{\partial \phi}{\partial n}) \, ds + \int_C \phi \frac{\partial \phi_M}{\partial n} \, ds = 0 \, . \tag{3.4.22}$$

Here F denotes the free-surface contour and C denotes the body surface contour. The body boundary condition (3.1.9) has been used to eliminate a number of terms. Replacing the normal derivatives on F by using the free-surface condition (3.1.6) and rearranging yields

$$K = K_M - \int_C \phi \frac{\partial \phi_M}{\partial n} \, ds \Big/ \int_F \phi \, \phi_M \, ds \, . \tag{3.4.23}$$

When a typical dimension C is much smaller than the cross-sectional length scale of the fluid motion, equation (3.4.23) may be used, with care, to estimate K by using the fact that over most of the fluid domain the solution will differ little from that given in equations (3.4.19-21). A typical length scale of the fluid motion is k_M^{-1}, so an approximation is sought under the assumption $k_M a \ll 1$. Now I am going to consider two cases, a circular cylinder and barrier as before.

(i) Circular cylinder: Suppose C is a circle of radius a centred on $(x_0 \, , y_0)$ and the motion is two-dimensional, that is $p = 0$. We define the polar coordinates $(r \, , \theta)$ by

$$(x - x_0) = r \sin \theta \, , (y - y_0) = r \cos \theta \, . \tag{3.4.24}$$

Near $(x_0 \, , y_0)$, the potential ϕ_M may be expanded as,

$$\phi_M = \cos\,[k_M(x_0 - b)] \cosh\,[k_M(y_0 - h)] - k_M r \sin\theta \sin\,[k_M(x_0 - b)] \cosh\,[k_M(y_0 - h)]$$

$$+ k_M r \cos\theta \cos\,[k_M(x_0 - b)] \sinh\,[k_M(y_0 - h)] + O((k_M \, r)^2) \, . \tag{3.4.25}$$

All terms in equation (3.4.25) are solutions of Laplace's equation and represent a uniform flow. We are here using the small-body approximation, namely that the undisturbed potential ϕ_M does not vary significantly over a typical body dimension scale. Providing $k_M a$ is small, the perturbation to the motion from introducing C will be neglible except in the immediate vicinity of the body. Near the body , ϕ is written

$$\phi = \text{constant} - k_M \left(r + \frac{a^2}{r}\right) \sin\theta \sin\left[k_M(x_0 - b)\right] \cosh\left[k_M(y_0 - h)\right]$$

$$+ k_M \left(r + \frac{a^2}{r}\right) \cos\theta \cos\left[k_M(x_0 - b)\right] \sinh\left[k_M(y_0 - h)\right] + O((k_M r)^2) \qquad (3.4.26)$$

where terms $r\sin\theta$ and $r\cos\theta$ represent the uniform flow and the term $\frac{a^2}{r}$ represents the dipole. Together these have zero normal derivative on C. The value of the constant in (3.4.26) is not needed for the present calculation. The integral over C in (3.4.23) can be calculated approximately by using the forms of ϕ and ϕ_M given by the terms displayed explicitly in equations (3.4.25) and (3.4.26). The influence of the body is small in the free-surface integral and ϕ is approximated by ϕ_M over F as seen from equation (3.1.9). With these approximations to the potentials equation (3.4.23) yields

$$K = K_M - \frac{2\pi k_M^2 a^2}{b \cosh^2 k_M h} \left(\sin^2 k_M(x_0 - b) \cosh^2 k_M(y_0 - h)\right.$$

$$\left. + \cos^2 k_M(x_0 - b) \sinh^2 k_M(y_0 - h)\right) \qquad (3.4.27)$$

with $k_M = M\,\pi/2b$

(ii) Thin vertical barrier: Here the body is a thin vertical barrier piercing the free surface and extending downwards a distance a. The motion is three-dimensional, but its variation is taken to be small on the other length scales of the problem so that the plate is far from the walls and base of the container.

The reference point in C is now chosen as $(x, y) = (x_0, 0)$ and the polar coordinates (r, θ) are defined by

$$x - x_0 = r \sin\theta \, , \; y = r \cos\theta \, . \tag{3.4.28}$$

As in (i), near $(x_0, 0)$ the potential may be expanded as

$$\phi_M = \cos\left[\alpha_M (x_0 - b)\right] \cosh k_M h - \alpha_M r \sin\theta \sin\left[\alpha_M (x_0 - b)\right] \cosh k_M h$$

$$- \alpha_M r \cos\theta \cos\left[\alpha_M (x_0 - b)\right] \sinh k_M h + O((k_M r)^2) \tag{3.4.29}$$

Near the barrier, ϕ is written as,

$$\phi \approx \varepsilon \cosh k_M h \sin\left[\alpha_M (x_0 - b)\right] \mathrm{Re}_j (z^2 + 1)^{\frac{1}{2}} \tag{3.4.30}$$

where

$$z = \frac{x - x_0}{a} + j \frac{y}{a} \quad \text{and} \quad \varepsilon = \frac{a}{h} \tag{3.4.31}$$

which has zero normal derivative on the barrier and for large r the uniform flow terms correspond with the potential in equation(3.4.29).

To calculate the integral over the barrier in equation (3.4.21) we approximate using equation (3.4.29) and (3.4.30). This gives

$$\int_C \phi \frac{\partial \phi_M}{\partial n} \, ds = \int_0^a \varepsilon \, \alpha_M \cosh^2 k_M h \sin^2 \alpha_M (x_0 - b) \left(1 - \frac{y^2}{a^2}\right)^{\frac{1}{2}} dy$$

$$-\int_a^0 \varepsilon\, \alpha_M \cosh^2 k_M h \sin^2 \alpha_M(x_0 - b)\, (1 - \frac{y^2}{a^2})^{\frac{1}{2}}\, dy$$

$$= \frac{\pi}{2}\, \varepsilon\, \alpha_M \cosh^2 k_M h \sin^2 \alpha_M(x_0 - b) \qquad (3.4.32)$$

As in (i), the influence of the body is small for the free-surface integral and

$$\phi \approx \phi_M \quad \text{on } F \qquad (3.4.33)$$

Therefore,

$$\int_F \phi\, \phi_M ds = \int_{-b}^{b} \cos^2 \alpha_M(x - b) \cosh^2 k_M h\, dx = b \cosh^2 k_M h \qquad (3.4.34)$$

By substituting (3.4.32) and (3.4.34) into (3.4.23), we obtain

$$K = K_M - \frac{\pi\, \alpha_M\, a}{2bh} \sin^2 \alpha_M(x_0 - b)\,. \qquad (3.4.35)$$

APPENDIX 3: SOURCE POTENTIAL

Here the source potential which satisfies the two-dimensional Laplace equation, and the condition of no flow through the solid boundaries, equations (3.1.7-8), is calculated using a conformal mapping. From the source potential the matrix coefficients B_{ij}, C_{ij}, and D_{ij} which are defined in (3.3.12) are calculated.

(a) Conformal mapping

Firstly consider containing two walls a distance 2b apart and without the free surface nor bottom. Define a complex variable $z = \xi + j\eta$, and map the flow region onto the upper half of the $\zeta = u + jv$ plane (see figure . 3.1) by

$$\zeta = e^{\frac{-i\pi}{2b}(z-b)} \tag{A3.1}$$

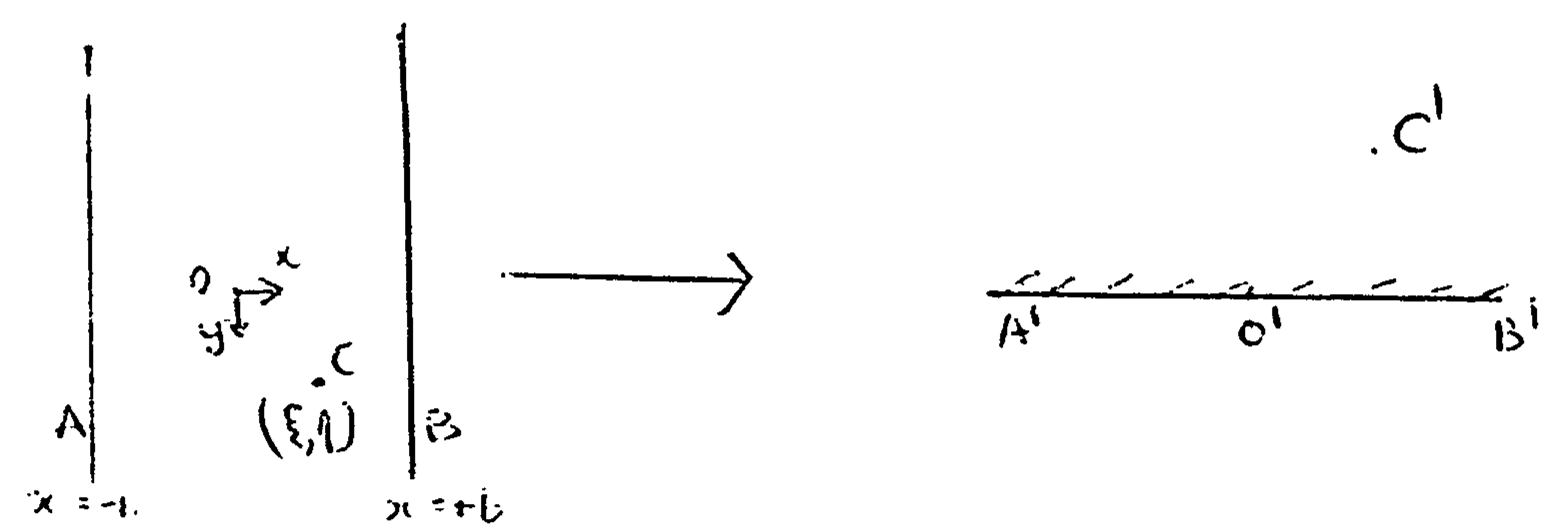


FIGURE 3.1

Therefore the complex source potential in the ζ plane can be written as,

$$W = \ln\left(\zeta - e^{\frac{-i\pi}{2b}(z_0 - b)}\right) + \ln\left(\zeta - e^{\frac{-i\pi}{2b}(-\overline{z_0} + b)}\right) - \ln\zeta \tag{A3.2}$$

where $z_0 = \xi + j\eta$ and $\bar{z}_0$ is a complex conjugate. After simplification, W can be written as,

$$W = \frac{\pi\eta}{2b} + 2\ln 2 + + \ln\left[\sin\frac{\pi}{4b}(z - z_0)\right] + \ln\left[\cos\frac{\pi}{4b}(z + \bar{z}_0)\right] + i\pi .$$

By neglecting the constant and taking real part, the potential, ϕ becomes

$$\phi = \frac{1}{2}\ln\left[\cosh^2\frac{\pi}{4b}(y - \eta) - \cos^2\frac{\pi}{4b}(x - \xi)\right] + \frac{1}{2}\ln\left[\cosh^2\frac{\pi}{4b}(y - \eta) - \sin^2\frac{\pi}{4b}(x + \xi)\right].$$

(A3.3)

By the method of images, when the tank contains the bottom the source potential becomes,

$$G = \frac{1}{2}\ln\left[\cosh^2\frac{\pi}{4b}(y - \eta) - \cos^2\frac{\pi}{4b}(x - \xi)\right] + \frac{1}{2}\ln\left[\cosh^2\frac{\pi}{4b}(y - \eta) - \sin^2\frac{\pi}{4b}(x + \xi)\right]$$

$$+ \frac{1}{2}\ln\left[\cosh^2\frac{\pi}{4b}(y - 2h + \eta) - \cos^2\frac{\pi}{4b}(x - \xi)\right]$$

$$+ \frac{1}{2}\ln\left[\cosh^2\frac{\pi}{4b}(y - 2h + \eta) - \sin^2\frac{\pi}{4b}(x + \xi)\right]. \quad \text{(A3.4)}$$

This is a required source potential which satisfies the wall and bed conditions.

(b) To obtain B_{ij}:

Now

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial x}\sin\alpha_i + \frac{\partial G}{\partial y}\cos\alpha_i \quad \text{(A3.5)}$$

where each line element makes an angle α_i with horizontal. Differentiating (A3.4) with respect to x and y and substituting into (A3.5), and using

$$x = x_i + s\cos\alpha_i$$

and

$$y = y_i + s\sin\alpha_i$$

we get,

$$\frac{\partial G}{\partial n} = \frac{\pi}{4b}\,(e_{ij} + f_{ij} + g_{ij} + h_{ij})\,, \tag{A3.6}$$

where,

$$e_{ij} = \frac{\sinh\frac{\pi}{2b}(y_j + s\sin\alpha_j - \eta_i)\cos\alpha_j - \sin\frac{\pi}{2b}(x_j + s\cos\alpha_j - \xi_i)\sin\alpha_j}{\cosh\frac{\pi}{2b}(y_j + s\sin\alpha_j - \eta_i) - \cos\frac{\pi}{2b}(x_j + s\cos\alpha_j - \xi_i)}\,, \tag{A3.7}$$

$$f_{ij} = \frac{\sinh\frac{\pi}{2b}(y_j + s\sin\alpha_j - \eta_i)\cos\alpha_j + \sin\frac{\pi}{2b}(x_j + s\cos\alpha_j + \xi_i)\sin\alpha_j}{\cosh\frac{\pi}{2b}(y_j + s\sin\alpha_j - \eta_i) + \cos\frac{\pi}{2b}(x_j + s\cos\alpha_j + \xi_i)}\,, \tag{A3.8}$$

$$g_{ij} = \frac{\sinh\frac{\pi}{2b}(y_j + s\sin\alpha_j - 2h + \eta_i)\cos\alpha_j - \sin\frac{\pi}{2b}(x_j + s\cos\alpha_j - \xi_i)\sin\alpha_j}{\cosh\frac{\pi}{2b}(y_j + s\sin\alpha_j - 2h + \eta_i) - \cos\frac{\pi}{2b}(x_j + s\cos\alpha_j - \xi_i)}\,, \tag{A3.9}$$

and

$$h_{ij} = \frac{\sinh \frac{\pi}{2b}(y_j + s \sin \alpha_j - 2h + \eta_i) \cos \alpha_j + \sin \frac{\pi}{2b}(x_j + s \cos \alpha_j + \xi_i) \sin \alpha_j}{\cosh \frac{\pi}{2b}(y_j + s \sin \alpha_j - 2h + \eta_i) + \cos \frac{\pi}{2b}(x_j + s \cos \alpha_j + \xi_i)} . \quad \text{(A3.10)}$$

Here (x_j , y_j) are coordinates of elements and (ξ_i , η_i) are coordinates of singularities. Express the variable s in terms of v, i.e.

$$s = (v + 1) \frac{l_j}{2}, \quad \text{(A3.11)}$$

where,

$$l_j = \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2}.$$

From the definition of B_{ij} and $(A3.6)$

$$B_{ij} = \frac{1}{2} \int_{-1}^{1} \frac{\partial G}{\partial n} l_j \, dv \, , \; j = 1, 2, .., N_B; i \neq j \text{ and } i = j \quad \text{(A3.12)}$$

can be calculated. Here N_B represents number of elements on the body.

(c) To obtain C_{ij}:

Differentiating (A3.4) with respect to y at free surface gives

$$\frac{\partial G_{ii}}{\partial y} = -\frac{\pi}{4b} \left[\frac{\sinh \frac{\pi h}{b}}{\cosh \frac{\pi h}{b} - \cos \frac{\pi}{2b}(x - \xi)} - \frac{\sinh \frac{\pi h}{b}}{\cosh \frac{\pi h}{b} + \cos \frac{\pi}{2b}(x + \xi)} \right],$$

$$i = N_{B+1}, N_{B+2}, \ldots, N. \tag{A3.13}$$

where N is a total number of elements on the body and free-surface. Writing $x = x_i + s$ and, since

$\xi = \dfrac{x_i + x_{i+1}}{2}$, we get from (A3.11) and (A3.13)

$$\frac{\partial G_{ii}}{\partial y} = -\frac{\pi}{4b}\left[\frac{\sinh\frac{\pi h}{b}}{\cosh\frac{\pi h}{b} - \cos\frac{\pi v d}{4}} + \frac{\sinh\frac{\pi h}{b}}{\cosh\frac{\pi h}{b} + \cos\frac{\pi}{2}\left(\frac{vd}{2} + x_i + x_{i+1}\right)}\right], i = j \tag{A3.14}$$

where d is the distance between two nodes and from (A3.4),

$$\frac{\partial G_{ij}}{\partial y} = \frac{\pi}{4b}\left(-\frac{\sinh\frac{\pi\eta_j}{2b}}{\cosh\frac{\pi\eta_j}{2b} - \cos\frac{\pi}{2b}(x_i + s - \xi_j)} - \frac{\sinh\frac{\pi\eta_j}{2b}}{\cosh\frac{\pi\eta_j}{2b} + \cos\frac{\pi}{2b}(x_i + s + \xi_j)}\right.$$

$$\left. + \frac{\sinh\frac{\pi}{2b}(-2h + \eta_j)}{\cosh\frac{\pi}{2b}(-2h + \eta_j) - \cos\frac{\pi}{2b}(x_i + s - \xi_j)} + \frac{\sinh\frac{\pi}{2b}(-2h + \eta_j)}{\cosh\frac{\pi}{2b}(-2h + \eta_j) + \cos\frac{\pi}{2b}(x_i + s - \xi_j)}\right),$$

$$j = N_B + 1, N_B + 2, \ldots, N \text{ and } j \neq i., i = ! \; j \tag{A3.15}$$

where

$$\xi_j = \frac{x_i + x_{i+1}}{2}, \quad \text{and} \quad \eta_j = \frac{y_i + y_{i+1}}{2}.$$

Therefore from the definition of C_{ij} and from equations (A3.13) and (A3.15)

$$C_{ij} = \frac{d}{2} \int_{-1}^{1} \frac{\partial G_{ij}}{\partial y} \, dv \ , \ j = N_B + 1, N_B + 2, \ldots, N; \ i \neq j \text{ and } i = j \tag{A3.16}$$

can be calculated.

(d) To obtain D_{ij}:

Near source point $G_{ii} \approx \ln r$, where $r = | x - \xi |$, we have from (A3.4), when $(x, y) \rightarrow (\xi, \eta) \rightarrow (\xi, 0)$,

$$G_{ii} - \ln r \rightarrow t_{\xi} \ , i = N_B + 1, N_B + 2, \ldots N \tag{A3.17}$$

where

$$t_{\xi} = \ln \left(\frac{\pi}{4b} \cos \frac{\pi\xi}{2b}\right) + \ln \sinh \frac{\pi h}{2b} + \frac{1}{2} \ln \left(\sinh^2 \frac{\pi h}{2b} + \cos^2 \frac{\pi\xi}{2b}\right). \tag{A3.18}$$

Integrating this with respect to x gives

$$\int_0^d (G_{ii} - \ln r) \, dx = d \, t_{\xi} \tag{A3.19}$$

and

$$\int_0^d \ln r \, dr = d \left(\ln \frac{d}{2} - 1\right) \tag{A3.20}$$

Therefore from the definition of D_{ij} for $i = j$, and from equations (A3.19-20), we have

$$D_{ii} = d\left(\frac{t_\xi}{2} - 1 + \ln\frac{d}{2}\right),\ i = N_B + 1, N_B + 2, .. N \tag{A3.21}$$

and from (A3.4) and (A3.18), we have

$$G_{ij} = \frac{1}{2}\left(p_{ij} + q_{ij} + r_{ij} + u_{ij} - 4\ln 2\right),\ i \neq j \tag{A3.22}$$

where,

$$p_{ij} = \ln\left(\cosh\frac{\pi\eta_j}{2b} - \cos\frac{\pi}{2b}(x_i + s - \xi_j)\right), \tag{A3.23}$$

$$q_{ij} = \ln\left(\cosh\frac{\pi\eta_j}{2b} + \cos\frac{\pi}{2b}(x_i + s + \xi_j)\right), \tag{A3.24}$$

$$r_{ij} = \ln\left(\cosh\frac{\pi}{2b}(-2h + \eta_j) - \cos\frac{\pi}{2b}(x_i + s - \xi_j)\right), \tag{A3.25}$$

and

$$u_{ij} = \ln\left(\cosh\frac{\pi}{2b}(-2h + \eta_j) + \cos\frac{\pi}{2b}(x_i + s + \xi_j)\right),\ i \neq j \tag{A3.26}$$

From the definition of D_{ij} and using (A3.22), we have

$$D_{ij} = \frac{1}{2}\int_{-1}^{1} G_{ij}\, l_i dv,\ j = N_B + 1, N_B + 2, \dots, N;\ j \neq i,\ j = i. \tag{A3.27}$$

CHAPTER 4

SLOSHING IN A RECTANGULAR TANK WITH INTERNAL BODIES - SOLUTION BY MATCHED ASYMPTOTIC EXPANSIONS

4.1 INTRODUCTION

Here solutions to the problems discussed in chapter 3 are obtained by formal application of the method of matched asymptotic expansions. The boundary-value problem is defined by the modified Helmholtz equation (3.1.5), the linearized free-surface condition (3.1.6), and the condition of no flow through the solid boundaries (3.1.7-9). We assume that the length scale of the motion is much larger than a typical body dimension. The solutions obtained will not be valid for higher modes as the fundamental length scale decreases as mode number increases.

In section 3.4, a solution was obtained on the above assumption and for the submerged cylinder case but it is not clear how to find a solution for non-zero p. Here $p = n\pi/2l$ and n is any integer and $2l$ is tank walls is in z -direction. To calculate the higher-order solution for non-zero p we use the method of matched asymptotic expansions. This method is a powerful tool which has been used by several authors, for example Davis and Leppington (1978) who obtained a higher-order solution for wave scattering by various obstacles.

From the previous work on scattering by submerged bodies, for example Davis and Leppington (1978), and from the related work of McIver (1991) it is clear that the outer solution will contain only sources and dipoles at leading order. The form of solution may be justified by retaining a full expansion of the multipole potentials defined in Appendix 4.1. These are singular solutions of the modified Helmholtz equation satisfying all the conditions of the problem except that on the body. It follows that the leading-order inner solution corresponds to the potential flow for a uniform stream past the body cross-section. The disturbance to the stream is dipole-like at infinity and the only way this can be reconciled with the inner expansion of a multipole representation in the outer region is if the dipole in that representation has a certain form. In particular, the terms in the inner expansion of the dipole potential in the outer region are to be matched onto the uniform stream and dipole terms of the inner solution, at the same order in $\varepsilon = a/h \ll 1$. Here a and h are the cross-sectional length of the body and depth of the water respectively. This determines the order of eigenfrequency in ε and matching gives an explicit expression for the eigenfrequency in terms of the body geometry and the dipole strength resulting from a uniform flow past the body.

In section 4.2, the lowest-order solution is obtained for a submerged cylinder of arbitrary cross section. By matching inner and outer expansions a standard matrix eigen-value problem is obtained and the lowest-order solution is formed in terms of the cross-sectional area and dipole strengths of the body. The higher-order solution is very difficult for a body of arbitrary cross section. Therefore a circular cylinder is considered to obtain the higher-order solution up to ε^4. The calculation is different from the first-order solution. The fourth-order outer and inner solution are required to find this solution. Using the matching principle gives a non-standard eigenvalue problem. Writing these equations in matrix form gives the same matrix as in the lowest-order solution which is of order three by three and rank two. So using matrix theory, any rows or columns in the matrix can be written as a linear combination of any two rows or columns of the same matrix and then by simplifying the higher-order solution can be obtained.

In section 4.3, a surface-piercing vertical barrier is considered. The solution procedure is very similar to section 4.2, but here the leading order outer solution contains only dipoles. Since the body is a vertical barrier of zero cross-sectional area. The calculation for a higher-order solution is much easier than in section 4.2.

The lowest-order solution for an arbitrary cross-sectional body with coordinates (x_0, y_0) is given by

$$\sigma_2 = \frac{\pi h}{2\alpha_M {N_{0,M}}^2} \left(\frac{Sp^2}{\pi a^2} \left(1 + (-1)^M \cos 2\alpha_M x_0 \right) \right) \cosh^2 k_M(h - y_0)$$

$$+ 2\, \nu {k_M}^2 \left(1 + (-1)^M \cos 2\alpha_M x_0 \right) \sinh^2 k_M(y - y_0)$$

$$- \left(\Upsilon - \lambda \right) \alpha_M k_M (-1)^M \sin 2\alpha_M x_0 \sinh 2k_M(h - y_0)$$

$$+ 2\Lambda\, {\alpha_M}^2 \left(1 - (-1)^M \cos 2\alpha_M x_0 \right) \cosh^2 k_M(h - y_0) \Bigg) . \tag{4.1.1}$$

Here S is a cross-sectional area of the body, ν, Υ, λ, and Λ are dipole strengths of the body and σ_2 is related to the frequency parameter K by the equation

$$K = K_M \left(1 - \varepsilon^2 V \right), \tag{4.1.2}$$

where

$$\alpha_M = \frac{M\pi}{2b}, \quad k_M = (\alpha_M^2 + p^2)^{\frac{1}{2}},$$

$$N_{0,M}^2 = \frac{1}{4k_M h} \left(\sinh 2k_M h + 2k_M h \right),$$

$$K_M = k_M \tanh k_M h \, ,$$

$$V = \frac{M\pi\sigma_2}{4k_M{}^2 b^2}\left(1 + \frac{k_M{}^2 h}{K_M} - K_M h \right), \tag{4.1.3}$$

and M is any integer. For a circular cylinder the higher-order solution, σ_4 is given by

$$\frac{\sigma_4}{\delta}\left(c_{10,0}\,\frac{c_{11,0}}{d_{11,0}} + (\,c_{11,0} + f_{11,0} - f_{11,0}\,\frac{c_{00,0}}{d_{11,0}}\,)\right) = c_{10,0}\left(\frac{c_{00,0}}{d_{01,0}}\,b_{11} + \frac{c_{11,0}}{d_{11,0}}\,B_2 + b_{13} \right)$$

$$+\left(c_{11,0} + f_{11,0} - f_{11,0}\,\frac{c_{00,0}}{d_{11,0}} \right)\left(b_{21}\,\frac{c_{00,0}}{d_{01,0}} + b_{22}\,\frac{c_{11,0}}{d_{11,0}} + B_3 \right) + c_{00,0}\left(b_{31}\,\frac{c_{00,0}}{d_{01,0}} + b_{32}\,\frac{c_{11,0}}{d_{11,0}} + b_{33} \right) \tag{4.1.4}$$

where

$$B_2 = \frac{7\delta^3}{16}\,c_{11,0} + \frac{\delta}{2}\left(c_{11,2} + \sigma_2\,(\,\ln\frac{\delta}{2} + \gamma - \frac{\delta}{2}\,)\right) + \frac{\delta^5}{16\sigma_2}\,(\,e_{21,0}\,d_{12,0} + c_{21,0}\,c_{12,0}\,)\, ,$$

$$B_3 = \frac{7\delta^3}{16}\,f_{11,0} + \frac{\delta}{2}\left(f_{11,2} + \sigma_2\,(\,\ln\frac{\delta}{2} + \gamma - \frac{\delta}{2}\,)\right) + \frac{\delta^5}{16\sigma_2}\,(\,f_{21,0}\,f_{12,0} + d_{21,0}\,e_{12,0}\,)\, ,$$

$$b_{11} = \frac{7\delta^3}{16}\,c_{01,0} + \frac{\delta}{2}\,c_{01,2} + \frac{\delta^5}{16\sigma_2}\,(\,e_{21,0}\,d_{02,0} + c_{21,0}\,c_{02,0}\,)\, ,$$

$$b_{13} = \frac{7\delta^3}{16} e_{11,0} + \frac{\delta}{2} e_{11,2} + \frac{\delta^5}{16\sigma_2} (e_{21,0} f_{12,0} + c_{21,0} e_{12,0}) ,$$

$$b_{21} = \frac{7\delta^3}{16} d_{01,0} + \frac{\delta}{2} d_{01,2} + \frac{\delta^5}{16\sigma_2} (f_{21,0} d_{02,0} + d_{21,0} c_{02,0}) ,$$

$$b_{22} = \frac{7\delta^3}{16} d_{11,0} + \frac{\delta}{2} d_{11,2} + \frac{\delta^5}{16\sigma_2} (f_{21,0} d_{12,0} + d_{21,0} c_{12,0}) ,$$

$$b_{31} = \sigma_2 - \frac{3\delta^4}{16} c_{00,0} - \frac{\delta^2}{2} \left(c_{00,2} - \sigma_2 (\gamma + \ln \frac{\delta}{2}) \right) - \frac{\delta^5}{16\sigma_2} (d_{02,0} e_{20,0} + c_{20,0} c_{02,0}) ,$$

$$b_{32} = - \frac{3\delta^4}{16} c_{10,0} - \frac{\delta^2}{2} c_{10,2} - \frac{\delta^5}{16\sigma_2} (c_{20,0} c_{12,0} + e_{20,0} d_{12,0}) ,$$

$$b_{33} = - \frac{3\delta^4}{16} e_{10,0} - \frac{\delta^2}{2} e_{10,2} - \frac{\delta^5}{16\sigma_2} (c_{20,0} e_{12,0} + e_{20,0} f_{12,0}) ,$$

gamma is Euler's constant, and $\delta = ph$. The coefficients $c_{ij,k}, d_{ij,k}, e_{ij,k}$, and $f_{ij,k}(i, j, k = 0, 1, 2)$ are known from the appendices 4.1-2.

When the body is a surface-piercing vertical barrier and the higher-order solution is obtained up to ε^4, that is

$$\sigma^{(4)} = \varepsilon^2 \sigma_2 \Big[1 + \varepsilon^2 \ln \varepsilon \, \sigma_{41}$$

$$+\varepsilon^2\left\{-\frac{\sigma_2\alpha_M}{4k_M{}^2 bN_{0,M}{}^2}\left(2K_M h+\frac{\sinh 2k_M h}{2k_M h}\right)\right.$$

$$-\frac{\pi x_0\alpha_M h\cosh^2 k_M h}{4bN_{0,M}{}^2}(-1)^M\sin 2\alpha_M x_0$$

$$+\frac{1}{8}\left(S_{1,M}+\mathrm{Re}_j\, c_{1,M}+\frac{k_M{}^2h^2}{\cosh^2 k_M h}+\alpha_M{}^2h^2\right)$$

$$\left.\left.-\frac{p^2h^2}{4}\left((\gamma+\ln\frac{ph}{4}-\frac{1}{2})\cosh 2\hat{v}+\hat{v}\sinh 2\hat{v}+\frac{1}{4}\right)\right\}\right], \tag{4.1.5}$$

where

$$\sigma_2=\frac{\pi\alpha_M h}{4N_{0,M}{}^2}\cosh^2 k_M h\,(1-(-1)^M\cos 2\alpha_M x_0), \tag{4.1.6}$$

$$\sigma_{41}=-\frac{1}{4}\delta^2\cosh 2\hat{v}\,\sigma_2, \tag{4.1.7}$$

$$S_{1,M}=2\pi\sum_{n=1}^{\infty}\frac{\alpha_n h\cos^2 k_n h}{N_{n,M}{}^2\sin 2\alpha_n b}(\cosh 2\alpha_n x_0-e^{-2\alpha_n b}), \tag{4.1.8}$$

$$c_{1,M}=-2h^2\, in4\left\{\left(\frac{e^{\beta h}}{K_M-\beta\sinh\beta h}-\frac{2}{K_M-\beta}\right)+\frac{e^{-\beta h}}{K_M\cosh\beta h-\beta\sinh\beta h}\right\}t^2\,dt$$

$$+2\pi i\left\{\frac{\alpha_M h\cosh^2 kh}{N_{0,M}^{\;2}}-\delta^2\sinh 2\hat{v}\right\}, \tag{4.1.9}$$

$$\beta=(p^2+t^2)^{\frac{1}{2}},$$

$$\cosh\hat{v}=\frac{K_M}{p},$$

and $\mathrm{Re}_i\, c_{1,M}$ is a real part of $c_{1,M}$.

4.2 A submerged cylinder

The contour C is taken to be fully submerged, with the typical dimension a much less than the distance of C from the boundaries (including the free surface). The boundary-value problem to be solved is given by the modified Helmholtz equation (3.1.5), the linearised free-surface condition (3.1.6), the bed, wall conditions (3.1.7-8) and the body condition (3.1.9).

OUTER REGION

At distances $r \gg a$ from C, a non-dimensional outer radial coordinate is defined by

$$R = r/h, \tag{4.2.1}$$

where the polar coordinates (r , θ) are defined by (3.4.24).

OUTER SOLUTION

The complete outer solution $\Psi(R , \theta) \equiv \phi(r , \theta)$ is expressed as

$$\Psi = A_0 \, g_0(R , \theta) + \sum_{n=1}^{\infty} (A_n \, g_n(R , \theta) + B_n \, h_n(R , \theta)) \tag{4.2.2}$$

where

$$g_n = \sin \sigma \, \phi_n^{(b)} \quad \text{and} \quad h_n = \sin \sigma \, \psi_n^{(b)} \tag{4.2.3}$$

and $\phi_n^{(b)}$ and $\psi_n^{(b)}$ are the multipole potentials defined in appendix 4.1. They are singular solutions of the modified Helmholtz equation satisfying all the conditions of the problem except that on C. The additional factor of sin σ has been introduced for convenience. Write

$$g_n = g_{n,1} + \sin\sigma\, g_{n,2} \quad \text{and} \quad h_n = h_{n,1} + \sin\sigma\, h_{n,2} \tag{4.2.4}$$

where

$$g_{n,2} = K_n(\delta R)\cos n\theta \quad \text{and} \quad h_{n,2} = K_n(\delta R)\sin n\theta \tag{4.2.5}$$

are the singular parts of the multipoles. Thus, for example, g_0 is a source while g_1 and h_1 are a horizonal and vertical dipole respectively. From the results in appendix 4.1, part(b), the non-singular parts have expansions of the form

$$g_{n,1} = \sum_{q=0}^{\infty} (c_{nq} \cos q\theta + d_{nq} \sin q\theta) I_q (\delta R) \tag{4.2.6}$$

$$h_{n,1} = \sum_{q=0}^{\infty} (e_{nq} \cos q\theta + f_{nq} \sin q\theta) I_q (\delta R), \tag{4.2.7}$$

where $\delta = ph$. In the above K_n and I_q denote modified Bessel functions. By virtue of (4.1.2) and (4.1.3), the expansion coefficients in equation (4.2.6) have expansions in terms of ε in the form

$$c_{nq} = c_{nq,0} + f(\varepsilon)\, c_{nq,2} + \dots, \tag{4.2.8}$$

where $c_{nq,j} = O(1)$, with similar expansions for the remaining coefficients in (4.2.6-7). Note that the $O(1)$ terms in these coefficient expansions arise from the first terms of the summations over m in equations (A4.1.31-32).

From previous work on scattering by submerged bodies, for example Davis and Leppington (1978), and from the related work of McIver (1991) it is clear that the outer solution will contain only sources and dipoles at leading order. Thus, the leading-order outer solution is written

$$\Psi^{(0)} = A_0\, g_{0,1}{}^{(0)} + A_1\, g_{1,1}{}^{(0)} + B_1\, h_{1,1}{}^{(0)} \tag{4.2.9}$$

where, in a standard notation, a superscript in parentheses is used to denote the order in ε of a quantity. (This form for $\Psi^{(0)}$ may be justified by retaining a full multipole expansion and allowing the matching to eliminate all but the source and dipole terms.) The problem is homogeneous so the order in ε of the solution may be freely chosen. It is natural to take it to be O(1) as in equation (4.2.9).

INNER REGION

Within distances $r \ll h$ of C, a radial inner coordinate is defined by

$$\rho = r/a. \tag{4.2.10}$$

In terms of the inner coordinates the inner solution $\psi(\rho, \theta) \equiv \phi(r, \theta)$ must satisfy the field equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} - \varepsilon^2\delta^2\psi = 0 \tag{4.2.11}$$

and the boundary condition

$$\partial\psi/\partial n = 0 \text{ on } C. \tag{4.2.12}$$

The solution is fully determined by matching with the outer solution.

Using well-known expansions of the modified Bessel functions, the inner expansion of the leading-order outer solution, equation (4.2.9), is

$$\Psi^{(0,1)} = \Pi_0 + \Pi_1 \, \varepsilon \, \rho \cos\theta + \Pi_2 \, \varepsilon \, \rho \sin\theta, \tag{4.2.13}$$

where

$$\Pi_0 = A_0 \, c_{00,0} + A_1 \, c_{10,0} + B_1 \, e_{10,0}, \tag{4.2.14}$$

$$\Pi_1 = \frac{1}{2} \, \delta \, (A_0 \, c_{01,0} + A_1 \, c_{11,0} + B_1 \, e_{11,0}), \tag{4.2.15}$$

$$\Pi_2 = \frac{1}{2} \, \delta \, (A_0 \, d_{01,0} + A_1 \, d_{11,0} + B_1 \, f_{11,0}). \tag{4.2.16}$$

Here, $\Psi^{(p,q)}$ denotes the result of expressing $\Psi^{(p)}$ in inner variables and expanding up to $O(\varepsilon^q)$. A similar notation is used for the inner solution. Thus, $\psi^{(q)}$ is the inner solution up to $O(\varepsilon^q)$ which, when expressed in terms of outer variables and expanded to $O(\varepsilon^p)$, is denoted by $\psi^{(q,p)}$. The matching principle requires that $\psi^{(q,p)} \equiv \Psi^{(p,q)}$ when both are expressed in the same coordinates.

INNER SOLUTION

Equation (4.2.13) suggests an inner development

$$\psi^{(1)} = P_0 + \varepsilon \, (P_1 + P_2 \, (\rho \cos\theta + \tau_1(\rho , \theta)) + P_3 \, (\rho \sin\theta + T_1(\rho , \theta))), \tag{4.2.17}$$

where, from equations (4.2.11) and (4.2.12), τ_1 and T_1 are harmonic functions satisfying

$$\frac{\partial \tau_1}{\partial n} = -\frac{\partial}{\partial n}(\rho\cos\theta) \text{ and } \frac{\partial T_1}{\partial n} = -\frac{\partial}{\partial n}(\rho\sin\theta) \text{ on } C. \tag{4.2.18}$$

The potentials τ_1 and T_1 are respectively the disturbances to a uniform flow past C in the horizontal and vertical directions respectively. From Batchelor (1967, pp.127), as $\rho \to \infty$

$$\tau_1 = \nu\frac{\cos\theta}{\rho} + \lambda\frac{\sin\theta}{\rho} + O(\rho^{-2}) \tag{4.2.19}$$

and

$$T_1 = \Upsilon\frac{\cos\theta}{\rho} + \Lambda\frac{\sin\theta}{\rho} + O(\rho^{-2}) \tag{4.2.20}$$

where the dipole coefficients ν, λ, Υ and Λ are assumed known. The outer expansion of equation (4.2.17) when expressed in inner coordinates, is therefore

$$\psi^{(1,0)} = P_0 + \varepsilon(P_2\,\rho\cos\theta + P_3\sin\theta). \tag{4.2.21}$$

Matching (4.2.13) and (4.2.21) gives

$$P_0 = \Pi_0, \quad P_2 = \Pi_1, \quad P_3 = \Pi_2\,. \tag{4.2.22}$$

Further expansion of the inner solution (4.2.17) using equations (4.2.19-20) gives

$$\psi^{(1,2)} = \Pi_0 + \varepsilon\left\{ P_1 + \Pi_1\left(\frac{R\cos\theta}{\varepsilon} + \frac{\varepsilon}{R}(\nu\cos\theta + \lambda\sin\theta)\right)\right.$$

$$+\Pi_2\left(\frac{R\sin\theta}{\varepsilon}+\frac{\varepsilon}{R}\left(\Upsilon\cos\theta+\Lambda\sin\theta\right)\right)\Bigg\}. \tag{4.2.23}$$

The dipole terms in equation (4.2.23) appear at an $O(\varepsilon^2)$ higher than the uniform flow terms, which can only be reconciled with the outer solution if the same is true in the inner expansions of the dipole potentials. From equation (4.2.4) this requires $\sigma = O(\varepsilon^2)$, and so the choice $f(\varepsilon)=\varepsilon^2$ is made.

Retaining only the multipoles that can possibly match with equation (4.2.23), the outer solution can now be continued as

$$\Psi^{(2)} = A_0\, g_0^{(2)} + A_1\, g_1^{(2)} + B_1\, h_1^{(2)} + \varepsilon\left\{ C_0\, g_0^{(0)} + C_1\, g_1^{(0)} + D_1\, h_1^{(0)} \right\}$$

$$+\varepsilon^2\left\{ E_0\, g_0^{(0)} + \sum_{n=1}^{2}\left(E_n\, g_n^{(0)} + F_n\, h_n^{(0)} \right)\right\}. \tag{4.2.24}$$

Note that

$$g_n^{(2)} = \sum_{q=0}^{\infty}\left[\left(c_{nq,0}+\varepsilon^2 c_{nq,2}\right)\cos q\theta + \left(d_{nq,0}+\varepsilon^2 d_{nq,2}\right)\sin q\theta\right] I_q(\delta R) + \varepsilon^2\sigma_2\, K_n(\delta R)\cos n\theta \tag{4.2.25}$$

with a similar expression for $h_n^{(2)}$. The inner expansion of equation (4.2.24) yields

$$\Psi^{(2,2)} = \Pi_0 + \varepsilon\left\{ \Pi_3 + \Pi_1\,\rho\cos\theta + \Pi_2\rho\sin\theta + \frac{\sigma_2 A_1}{\delta}\frac{\cos\theta}{\rho} + \frac{\sigma_2 B_1}{\delta}\frac{\sin\theta}{\rho}\right\} - \varepsilon^2 \ln\varepsilon\, \sigma_2\, A_0$$

$$+ \varepsilon^2 \left\{ \Pi_4 - \sigma_2 A_0 \ln \rho + \Pi_5 \rho \cos \theta + \Pi_6 \rho \sin \theta + \Pi_7 \rho^2 \cos 2\theta + \Pi_8 \rho^2 \sin 2\theta + \Pi_0 \frac{1}{4} \delta^2 \rho^2 \right\}.$$

(4.2.26)

where

$$\Pi_3 = C_0 c_{00,0} + C_1 c_{10,0} + D_1 e_{10,0} ,$$

$$\Pi_4 = A_0 c_{00,2} + A_1 c_{10,2} + B_1 e_{10,2} - \sigma_2 A_0 \gamma - \sigma_2 A_0 \ln (\delta/2) + E_0 c_{00,0}$$

$$+ \sum_{n=1}^{2} (E_n c_{n0,0} + F_n e_{n0.0}) ,$$

$$\Pi_5 = \frac{\delta}{2} (C_0 c_{01,0} + C_1 c_{11,0} + D_1 e_{11,0}) ,$$

$$\Pi_6 = \frac{\delta}{2} (C_0 d_{01,0} + C_1 d_{11,0} + D_1 f_{11,0}) ,$$

$$\Pi_7 = \frac{\delta^2}{4} (A_0 c_{02,2} + A_1 c_{12,2} + B_1 e_{12,2}) ,$$

and

$$\Pi_8 = \frac{\delta^2}{4} (A_0 d_{02,0} + A_1 d_{12,0} + B_1 f_{12,0}).$$

However the above constants $\Pi_m, m = 3,4,..8$, are not required to calculate the σ_2. The equation (4.2.26) is suggests that the inner solution is continued as

$$\psi^{(2)} = \psi^{(1)} + \varepsilon^2 \ln \varepsilon \, P_4 + \varepsilon^2 \psi_2, \tag{4.2.27}$$

where $\psi^{(1)}$ is given by equations (4.2.17) and (4.2.22). The term at $O(\varepsilon^2 \ln \varepsilon)$ is chosen as a constant since it is the only harmonic function satisfying (4.2.12) that can match with (4.2.26). Substituting (4.2.27) into (4.2.11-12) and equating like terms in ε shows that ψ_2 must satisfy

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi_2}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi_2}{\partial \theta^2} = \delta^2 \Pi_0 \tag{4.2.28}$$

in the fluid and

$$\partial \psi_2 / \partial n = 0 \text{ on } C. \tag{4.2.29}$$

A particular solution of equations (4.2.28-29) is chosen in the form

$$\psi_{2,p} = \frac{1}{4} \delta^2 \Pi_0 \rho^2 + \Omega(\rho, \theta), \tag{4.2.30}$$

where $\Omega(\rho, \theta)$ is a harmonic function satisfying

$$\Omega + \delta^2 \Pi_0 \frac{S}{2\pi a^2} \ln \rho \to 0 \text{ as } \rho \to \infty \tag{4.2.31}$$

and the boundary condition

$$\frac{\partial \Omega}{\partial n} = -\frac{1}{4}\,\delta^2\,\Pi_0\,\frac{\partial \rho^2}{\partial n} \quad \text{on} \quad C. \tag{4.2.32}$$

Here S is the area of the cross section inside C, the logarithmic term in equation (4.2.31) is due to the flux across C indicated by equation (4.2.32). Bearing in mind the inner expansion (4.2.26), the full form for ψ_2 is taken as

$$\psi_2 = \psi_{2,p} + Q_0 + Q_1\,(\,\rho\cos\theta + \tau_1\,(\,\rho\,,\theta\,)\,) + Q_2\,(\,\rho\sin\theta + T_1(\,\rho\,,\theta\,)\,)$$

$$+\,Q_3\,(\,\rho^2\cos 2\theta + \tau_2\,(\rho\,,\theta\,)\,) + Q_4\,(\,\rho^2\sin 2\theta + T_2\,(\,\rho\,,\theta\,)\,) \tag{4.2.33}$$

which leads to an outer expansion for $\psi^{(2)}$ of

$$\psi^{(2,2)} = \Pi_0 + \varepsilon\left\{ P_1 + \Pi_1\left(\rho\cos\theta + \nu\,\frac{\cos\theta}{\rho} + \lambda\,\frac{\sin\theta}{\rho}\right) + \Pi_2\left(\rho\sin\theta + \Upsilon\,\frac{\cos\theta}{\rho} + \Lambda\,\frac{\sin\theta}{\rho}\right)\right\}$$

$$+\,\varepsilon^2\ln\varepsilon\,\Pi_4 + \varepsilon^2\left\{\frac{1}{4}\,\delta^2\,\Pi_0\,\rho^2 - \delta^2\,\Pi_0\,\frac{S}{2\,\pi\,a^2}\ln\rho + Q_0 + Q_1\,\rho\cos\theta + Q_2\,\rho\sin\theta\right.$$

$$\left. +\,Q_3\,\rho^2\cos 2\theta + Q_4\,\rho^2\sin 2\theta\right\}. \tag{4.2.34}$$

Matching (4.2.34) and (4.2.26) gives, in particular,

$$\sigma_2\,A_0 = \delta^2\,\Pi_0\,\frac{S}{2\,\pi\,a^2},\quad \frac{\sigma_2\,A_1}{\delta} = \Pi_1\,\nu + \Pi_2\,\Upsilon,\quad \frac{\sigma_2\,B_1}{\delta} = \Pi_1\,\lambda + \Pi_2\,\Delta. \tag{4.2.35}$$

Substituting for Π_i, $i = 0 , 1 , 2$, from (4.2.14-16) and expressing in matrix form gives

$$\begin{pmatrix} c_{00,0}\,(S/\pi a^2) & c_{10,0}\,(S/\pi a^2) & e_{10,0}\,(S/\pi a^2) \\ \nu\, c_{01,0} + \Upsilon\, d_{01,0} & \nu\, c_{11,0} + \Upsilon\, d_{11,0} & \nu\, e_{11,0} + \Upsilon\, f_{11,0} \\ \lambda\, c_{01,0} + \Lambda\, d_{01,0} & \lambda\, c_{11,0} + \Lambda\, d_{11,0} & \lambda\, e_{11,0} + \Lambda\, f_{11,0} \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ B_1 \end{pmatrix} = \frac{2\,\sigma_2}{\delta^2} \begin{pmatrix} A_0 \\ A_1 \\ B_1 \end{pmatrix},$$

(4.2.36)

which is a standard eigenvalue problem to determine σ_2. The eigenvalues are a repeated value $\sigma_2 = 0$ and

$$\sigma_2 = \frac{\delta^2}{2} \left\{ c_{00,0}\,\frac{S}{\pi a^2} + (\,\nu\, c_{01,0} + \Upsilon\, d_{01,0}\,)\,\frac{c_{11,0}}{c_{01,0}} + (\,\lambda\, c_{01,0} + \Lambda\, d_{01,0}\,)\,\frac{e_{11,0}}{c_{01,0}} \right\}. \qquad (4.2.37)$$

The zero eigenvalue leads to the remaining non-zero parts of the multipole potentials in equation (4.2.9) combining in such a way as to give a zero total potential; therefore the required result is given by equation (4.2.37).

The expansion coefficients appearing in equation (4.2.37) follow by comparison of equations (4.2.6-7) with equations (A4.1.31-32). From (A4.1.17) and (4.1.2)

$$\sinh 2\alpha_0 b = -\,i \sin 2\alpha b = i(-1)^M \sin\sigma \qquad (4.2.38)$$

so that, bearing in mind (4.2.3), we have

$$c_{00,0} = -\,i\,\Gamma_{00}\,(\,(-1)^M \sin 2\alpha_M x_0 + 1\,)\cosh k_M(h - y_0),$$

$$c_{01,0} = \frac{2ik_M}{p}\,\Gamma_{00}\,(\,(-1)^M \sin 2\alpha_M x_0 + 1\,)\sinh k_M(h-y_0),$$

$$c_{11,0} = \frac{2ik_M}{p}\,\Gamma_{10}\,(\,(-1)^M \sin 2\alpha_M x_0 + 1\,)\sinh k_M(h-y_0), \tag{4.2.39}$$

$$d_{01,0} = \frac{2i\alpha_M}{p}\,\Gamma_{00}\,(-1)^M \cos 2\alpha_M x_0 \cosh k_M(h-y_0),$$

$$e_{11,0} = \frac{2k_M}{p}\,\delta_{10}\,(-1)^M \cos 2\alpha_M x_0 \sinh k_M(h-y_0)$$

and so

$$\sigma_2 = \frac{\pi h}{2\,\alpha_M\,N_{0,M}^{\;2}}\left(\frac{S\,p^2}{\pi\,a^2}\,(\,1+(-1)^M \cos 2\alpha_M x_0\,)\cosh^2 k_M(h-y_0)\right.$$

$$+\,2\,\nu\,k_M^{\;2}\left(1+(-1)^M \cos 2\alpha_M x_0\right)\sinh^2 k_M\,(h-y_0)$$

$$-\,(\,\Upsilon-\lambda\,)\alpha_M\,k_M\,(-1)^M \sin 2\alpha_M x_0 \sinh 2k_M(h-y_0)$$

$$\left.+\,2\,\Lambda\,\alpha_M^{\;2}\left(1-(-1)^M \cos 2\alpha_M x_0\right)\cosh^2 k_M(h-y_0)\right). \tag{4.2.40}$$

The above approximation, to order ε^2 is not given for greater accuracy. In order to obtain greater accuracy, we have to consider a higher-order solution. But in the case of arbitrary cross-sectional cylinders higher-order solution is difficult. So a circular cylinder is considered to obtain the higher-order solution.

HIGHER-ORDER SOLUTION FOR THE CIRCULAR CYLINDER

In section 3.4 the solution was obtained to order ε^2 for the submerged circular cylinder case. So my aim is to calculate the higher-order solution up to the order ε^4 for the special case of a circular cylinder. For the circular cylinder, $S = \pi a^2$, $\nu = \Lambda = 1$ and $\lambda = \Upsilon = 0$.

Looking at the inner expansion of $\Psi^{(2)}$ to order ε^3 suggests that the continuing inner solution is,

$$\psi^{(3)} = \Pi_0 + \varepsilon \left\{ P_1 + \Pi_1 \left(\rho \cos\theta + \frac{\cos\theta}{\rho} \right) + \Pi_2 \left(\rho \sin\theta + \frac{\sin\theta}{\rho} \right) \right\}$$

$$+ \varepsilon^2 \ln\varepsilon \, P_4 + \varepsilon^2 \psi_2 + \varepsilon^3 \ln\varepsilon \, \psi_{31} + \varepsilon^3 \psi_3 \qquad (4.2.41)$$

where ψ_2 is given by the equation (4.2.33).

TO COMPUTE ψ_{31}

Since ψ_{31} is a harmonic function and from equation (4.2.12) satisfies,

$$\frac{\partial \psi_{31}}{\partial \rho} = 0 \ , \text{on } \rho = 1 \qquad (4.2.42)$$

then to match with inner expansion of outer solution, ψ_{31} can be written as

$$\psi_{31} = P_5 + P_6 \left(\rho \cos\theta + \frac{\cos\theta}{\rho} \right) + P_7 \left(\rho \sin\theta + \frac{\sin\theta}{\rho} \right) \tag{4.2.43}$$

where P_5, P_6 and P_7 are unknown constants.

TO COMPUTE ψ_3

From the equations (4.2.11-12), ψ_3 satisfies,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho \frac{\partial\psi_3}{\partial\rho} \right) + \frac{1}{\rho^2}\frac{\partial^2\psi_3}{\partial\theta^2} = \delta^2 \left(P_1 + \Pi_1 \left(\rho\cos\theta + \frac{\cos\theta}{\rho}\right) + \Pi_2 \left(\rho\sin\theta + \frac{\sin\theta}{\rho}\right) \right) \tag{4.2.44}$$

and

$$\frac{\partial\psi_3}{\partial\rho} = 0 \quad \text{on} \quad \rho = 1 . \tag{4.2.45}$$

The particular solution of (4.2.44) is easily found,

$$\psi_{3p} = \frac{\delta^2}{4} P_1 \rho^2 + \frac{\delta^2}{8} \left(\Pi_1 \rho^3 \cos\theta + \Pi_2 \rho^3 \sin\theta \right) + \frac{\delta^2}{2} \left(\Pi_1 \rho \ln\rho \cos\theta + \Pi_2 \rho \ln\rho \sin\theta \right) . \tag{4.2.46}$$

Therefore the complete solution for ψ_3 is,

$$\psi_3 = \psi_{3p} + \Omega_3 (\rho , \theta) \tag{4.2.47}$$

where Ω_3 is a harmonic function and satisfies,

$$\frac{\partial \Omega_3}{\partial \rho} = -\frac{\delta^2}{2}\left(P_1 + \frac{7}{4}\left(\Pi_1 \cos\theta + \Pi_2 \sin\theta \right) \text{ on } \rho = 1 \right. \tag{4.2.48}$$

In order to find Ω_3 , we write

$$\Omega_3 = \Omega_{31} + \Omega_{32} \tag{4.2.49}$$

where

$$\frac{\partial \Omega_{31}}{\partial \rho} = -\frac{\delta^2}{2} P_1 \text{ on } \rho = 1 \tag{4.2.50}$$

and

$$\frac{\partial \Omega_{32}}{\partial \rho} = -\frac{7\,\delta^2}{8} (\Pi_1 \cos\theta + \Pi_2 \sin\theta) \text{ on } \rho = 1 \tag{4.2.51}$$

Therefore from (4.2.51),

$$\Omega_{32} = \frac{7\,\delta^2}{8} (\Pi_1 \frac{\cos\theta}{\rho} + \Pi_2 \frac{\sin\theta}{\rho}) \tag{4.2.52}$$

and from (4.2.50)

$$\Omega_{31} + \frac{\delta^2}{2} P_1 \ln\rho \to 0 \text{ as } \rho \to \infty \tag{4.2.53}$$

Therefore the complete solution for ψ_3 is,

$$\psi_3 = \psi_{3p} + \Omega_3 + Q_5 + Q_6\,(\rho\cos\theta + \frac{\cos\theta}{\rho}) + Q_7\,(\rho\sin\theta + \frac{\sin\theta}{\rho})$$

$$+ Q_8\,(\rho^2\cos 2\theta + \frac{\cos 2\theta}{\rho^2}) + Q_9\,(\rho^2\sin 2\theta + \frac{\sin 2\theta}{\rho^2})$$

$$+ Q_{10}\,(\rho^3\cos 3\theta + \frac{\cos 3\theta}{\rho^3}) + Q_{11}\,(\rho^3\sin 3\theta + \frac{\sin 3\theta}{\rho^3}) \qquad (4.2.54)$$

where $Q_5, Q_6, \ldots, Q_{11}$ are unknown constants. Therefore from (4.2.33), (4.2.41), (4.2.43), and (4.2.54), the outer expansion for $\psi^{(3)}$ of

$$\psi^{(3,4)} = \Pi_0 + \varepsilon\left\{ P_1 + \Pi_1\,(\,\rho\cos\theta + \frac{\cos\theta}{\rho}\,) + \Pi_2\,(\,\rho\sin\theta + \frac{\sin\theta}{\rho}\,)\right\} + \varepsilon^2\ln\varepsilon\, P_4$$

$$+ \varepsilon^2\left(\frac{\delta^2}{4}\Pi_0\,\rho^2 - \frac{\delta^2}{2}\Pi_0\ln\rho + Q_0 + Q_1\,(\,\rho\cos\theta + \frac{\cos\theta}{\rho}\,) + Q_2\,(\,\rho\sin\theta + \frac{\sin\theta}{\rho}\,)\right.$$

$$\left. + Q_3\,(\,\rho^2\cos 2\theta + \frac{\cos 2\theta}{\rho^2}\,) + Q_4\,(\,\rho^2\sin 2\theta + \frac{\sin 2\theta}{\rho^2}\,)\right)$$

$$+ \varepsilon^3\ln\varepsilon\left\{ P_5 + P_6\,(\,\rho\cos\theta + \frac{\cos\theta}{\rho}\,) + P_7\,(\,\rho\sin\theta + \frac{\sin\theta}{\rho}\,)\right\}$$

$$+ \varepsilon^3\left(\frac{\delta^2}{4}P_1\,\rho^2 + \frac{\delta^2}{8}(\,\Pi_1\,\rho^3\cos\theta + \Pi_2\,\rho^3\sin\theta\,)\right.$$

$$+\frac{\delta^2}{2}\,(\,\Pi_1\;\rho\ln\rho\cos\theta+\Pi_2\;\rho\ln\rho\sin\theta\,)$$

$$+\frac{\delta^2}{2}\,P_1\ln\rho+\frac{7}{8}\,\delta^2\,(\,\Pi_1\,\frac{\cos\theta}{\rho}+\Pi_2\,\frac{\sin\theta}{\rho}\,)+Q_5+Q_6\,(\,\rho\cos\theta+\frac{\cos\theta}{\rho}\,)$$

$$\left.+\,Q_7\,(\rho\sin\theta+\frac{\sin\theta}{\rho}\,)+Q_8\,\rho^2\cos 2\theta+Q_9\,\rho^2\sin 2\theta+Q_{10}\,\rho^3\cos 3\theta+Q_{11}\,\rho^3\sin 3\theta\right\}.$$

(4.2.55)

This outer expansion of $\psi^{(3)}$ can only match with the inner expansion of $\Psi^{(4)}$ if σ has the expansion,

$$\sigma^{(3)}=\varepsilon^2\,\sigma_2+\varepsilon^3\ln\varepsilon\;\sigma_{31}+\varepsilon^3\,\sigma_3. \tag{4.2.56}$$

Further expansion of the inner solution (4.2.27) using equations (4.2.17) and (4.2.33) gives,

$$\psi^{(2,4)}=\Pi_0+\Pi_1\;R\cos\theta+\Pi_2\;R\sin\theta+\delta^2\,\frac{\Pi_0\,R^2}{4}+Q_3\,R^2\cos 2\theta+Q_4\,R^2\sin 2\theta$$

$$+\,\varepsilon\left\{\,P_1+Q_1\;R\cos\theta+Q_2\;R\sin\theta\,\right\}+\varepsilon^2\ln\varepsilon\left(\Pi_4+\frac{\delta^2}{2}\,\Pi_0\right)$$

$$+\,\varepsilon^2\left(Q_0-\frac{\delta^2}{2}\,\Pi_0\ln R+\Pi_1\,\frac{\cos\theta}{R}+\Pi_2\,\frac{\sin\theta}{R}\right)$$

$$+\varepsilon^3\left(Q_1\frac{\cos\theta}{R}+Q_2\frac{\sin\theta}{R}\right)+\varepsilon^4\left(Q_3\frac{\cos 2\theta}{R^2}+Q_4\frac{\sin 2\theta}{R^2}\right). \tag{4.2.57}$$

FOURTH-ORDER OUTER SOLUTION

Equation (4.2.57) suggests that the continuing outer solution is,

$$\Psi^{(4)}=A_0\,g_0^{(4)}+A_1\,g_1^{(4)}+B_1\,h_1^{(4)}+\varepsilon\left\{C_0\,g_0^{(3)}+C_1\,g_1^{(3)}+D_1\,h_1^{(3)}\right\}$$

$$+\varepsilon^2\left\{E_0\,g_0^{(2)}+\sum_{n=1}^{2}\left(E_n\,g_n^{(2)}+F_n\,h_n^{(2)}\right)\right\}+\varepsilon^3\left(H_0\,g_0^{(1)}+H_1\,g_1^{(1)}+M_1\,h_1^{(1)}\right)$$

$$+\varepsilon^4\left(W_0\,g_0^{(0)}+\sum_{n=1}^{2}\left(W_n\,g_n^{(0)}+U_n\,h_n^{(0)}\right)\right). \tag{4.2.58}$$

By using (4.2.25), the inner expansion of equation (4.2.58) yields,

$$\Psi^{(4,3)}=\Pi_0+\varepsilon\left\{\Pi_3+\Pi_1\,\rho\cos\theta+\Pi_2\,\rho\sin\theta+\frac{\sigma_2\,A_1}{\delta}\frac{\cos\theta}{\rho}+\frac{\sigma_2\,B_1}{\delta}\frac{\sin\theta}{\rho}\right\}$$

$$+\varepsilon^2\ln\varepsilon\left\{-\sigma_2\,A_0+\frac{\sigma_{31}}{\delta}\left(A_1\frac{\cos\theta}{\rho}+B_1\frac{\sin\theta}{\rho}\right)\right\}$$

$$+\varepsilon^2\left\{\Pi_4-\sigma_2\,A_0\,\ln\rho+\Pi_5\,\rho\cos\theta+\Pi_6\,\rho\sin\theta+\frac{\cos\theta}{\rho}\left(\frac{\sigma_2\,C_1}{8}+\frac{\sigma_3\,A_1}{\delta}\right)\right.$$

$$+\frac{\sin\theta}{\rho}\left(\frac{\sigma_2 D_1}{8}+\frac{\sigma_3 B_1}{\delta}\right)+\frac{\Pi_0\,\delta^2\rho^2}{4}+\Pi_7\,\rho^2\cos 2\theta+\Pi_8\,\rho^2\sin 2\theta$$

$$\left.+2\,\frac{\sigma_2}{\delta^2}\left(E_2\,\frac{\cos 2\theta}{\rho^2}+F_2\,\frac{\sin 2\theta}{\rho^2}\right)\right\}-(\varepsilon^3\ln\varepsilon)^2 A_0\,\sigma_{31}+\varepsilon^3\ln\varepsilon\left\{\Pi_9\right.$$

$$\left.-\sigma_{31}A_0\ln\rho+\frac{\delta}{2}\sigma_2(A_1\,\rho\cos\theta+B_1\,\rho\sin\theta)+\Pi_{10}\frac{\cos\theta}{\rho}+\Pi_{11}\frac{\sin\theta}{\rho}\right\}$$

$$+\varepsilon^3\left\{\Pi_{12}-\Pi_{13}\ln\rho+\frac{\delta}{2}\sigma_2\ln\rho\,(A_1\,\rho\cos\theta+B_1\,\rho\sin\theta)\right.$$

$$+\frac{\delta}{2}(\Pi_{14}\,\rho\cos\theta+\Pi_{15}\,\rho\sin\theta)+\Pi_{16}\frac{\cos\theta}{\rho}+\Pi_{17}\frac{\sin\theta}{\rho}+\frac{\delta^2}{4}\Pi_3\,\rho^2$$

$$+\frac{\delta^2}{4}(\Pi_{18}\,\rho^2\cos 2\theta+\Pi_{19}\,\rho^2\sin 2\theta)+\frac{\delta^3}{8}(\Pi_1\,\rho^3\cos\theta+\Pi_2\,\rho^3\sin\theta)$$

$$\left.+\frac{\delta^3}{48}(\Pi_{20}\,\rho^3\cos 3\theta+\Pi_{21}\,\rho^3\sin 3\theta)\right\} \qquad (4.2.59)$$

where,

$$\Pi_9=-\sigma_2 C_0-\sigma_{31}A_0\left(\ln\frac{\delta}{2}+\gamma\right)-\sigma_3 A_0\,,$$

$$\Pi_{10} = \sigma_{31} \frac{C_1}{\delta} + \sigma_{41} \frac{A_1}{\delta} ,$$

$$\Pi_{11} = \sigma_{31} \frac{D_1}{\delta} + \sigma_{41} \frac{B_1}{\delta} ,$$

$$\Pi_{12} = C_0\, c_{00,2} + C_1\, c_{10,2} + D_1\, e_{10,2} + H_0\, c_{00,0} + H_1\, c_{10,0} + M_1\, e_{10,0}$$

$$- \sigma_2\, C_0 \left(\ln \frac{\delta}{2} + \gamma \right) - \sigma_3\, A_0 \left(\ln \frac{\delta}{2} + \gamma \right),$$

$$\Pi_{13} = \sigma_3\, A_0 + \sigma_2\, C_0 ,$$

$$\Pi_{14} = A_0 c_{01,2} + A_1\, c_{11,2} + B_1\, e_{11,2} + E_0\, c_{01,0}$$

$$+ \sum_{n=1}^{2} \left(E_n\, c_{n1,0} + F_n\, e_{n1,0} \right) + \sigma_2\, A_1 \left(\ln \frac{\delta}{2} + \gamma - \frac{\delta}{2} \right),$$

$$\Pi_{15} = A_0\, d_{01,2} + A_1\, d_{11,2} + B_1\, f_{11,2} + E_0\, d_{01,0}$$

$$+ \sum_{n=1}^{2} \left(E_n\, d_{n1,0} + F_n\, f_{n1,0} \right) + \sigma_2\, B_1 \left(\ln \frac{\delta}{2} + \gamma - \frac{\delta}{2} \right),$$

$$\Pi_{16} = \frac{1}{\delta} \left(\sigma_4\, A_1 + \sigma_2\, E_1 + \sigma_{31}\, C_1 \right),$$

$$\Pi_{17} = \frac{1}{\delta} \left(\sigma_4 B_1 + \sigma_2 F_1 + \sigma_{31} D_1 \right),$$

$$\Pi_{18} = C_0 \, d_{02,0} + C_1 \, d_{12,0} + D_1 \, f_{12,0} \, ,$$

$$\Pi_{19} = C_0 \, c_{02,0} + C_1 \, c_{12,0} + D_1 \, e_{12,0} \, ,$$

$$\Pi_{20} = A_0 \, c_{03,0} + A_1 \, c_{13,0} + B_1 \, e_{13,0} \, ,$$

and

$$\Pi_{21} = A_0 \, d_{03,0} + A_1 \, d_{13,0} + B_1 \, f_{13,0}.$$

Matching (4.2.55) with (4.2.59), the coefficient of $(\varepsilon^3 \ln \varepsilon)^2$ gives,

$$\sigma_{31} = 0. \tag{4.2.60}$$

From this matching we do not have enough information to find σ_3. In order to find σ_3 we have to consider the matching principle $\psi^{(4,4)} \equiv \Psi^{(4,4)}$.

FOURTH-ORDER INNER SOLUTION

By looking at inner expansion of $\Psi^{(4)}$ suggests that the continuing inner solution is,

$$\psi^{(4)} = \psi^{(3)} + \varepsilon^4 \ln \varepsilon \, \psi_{41} + \varepsilon^4 \, \psi_4 \tag{4.2.61}$$

where, $\psi^{(3)}$ is given by the equation (4.2.41).

TO COMPUTE ψ_{41}

From the equations (4.2.11-12), ψ_{41} satisfies,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi_{41}}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi_{41}}{\partial\theta^2}=\delta^2 P_4 \qquad (4.2.62)$$

and

$$\frac{\partial\psi_{41}}{\partial\rho}=0 \text{ on } \rho=1\,. \qquad (4.2.63)$$

The particular solution of (4.2.62) is,

$$\psi_{41p}=\frac{\delta^2}{4}P_4\,\rho^2 \qquad (4.2.64)$$

Therefore the complete solution for ψ_{41} is,

$$\psi_{41}=\frac{\delta^2}{4}P_4\,\rho^2+\Omega_{41}(\rho,\theta) \qquad (4.2.65)$$

where Ω_{41} is a harmonic function and

$$\Omega_{41}+\frac{\delta^2}{4}P_4\ln\rho\rightarrow 0 \text{ as } \rho\rightarrow\infty. \qquad (4.2.66)$$

TO COMPUTE ψ_4

From the equations (4.2.11-12), ψ_4 satisfies,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi_4}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi_4}{\partial \theta^2} = \delta^2 \psi_2 \tag{4.2.67}$$

and

$$\frac{\partial \psi_4}{\partial \rho} = 0 \text{ on } \rho = 1 \tag{4.2.68}$$

where

$$\psi_2 = \frac{1}{4}\delta^2 \Pi_0 \rho^2 - \frac{\delta^2}{2}\Pi_0 \ln \rho + Q_0 + Q_1 (\rho \cos\theta + \frac{\cos\theta}{\rho})$$

$$+ Q_2 (\rho \sin\theta + \frac{\sin\theta}{\rho}) + Q_3 (\rho^2 \cos 2\theta + \frac{\cos 2\theta}{\rho^2}) + Q_4 (\rho^2 \sin 2\theta + \frac{\sin 2\theta}{\rho^2}) . \tag{4.2.69}$$

From the equation (4.2.67), the particular solution of (4.2.67) is,

$$\psi_{4_p} = \frac{\delta^2}{2}(Q_1 \ln \rho \, \rho \cos\theta + Q_2 \ln \rho \, \rho \sin\theta) + \frac{\delta^2}{4} Q_0 \rho^2 - \frac{\delta^4}{8}\Pi_0 (\rho^2 \ln \rho - \rho^2)$$

$$+ \frac{\delta^2}{8}(Q_1 \rho^3 \cos\theta + Q_2 \rho^3 \sin\theta) - \frac{Q_3}{4}\delta^2 \cos 2\theta - \frac{Q_4}{4}\delta^2 \sin 2\theta + \frac{\delta^4}{64}\Pi_0 \rho^4$$

$$+ \frac{\delta^2}{12}(Q_3 \rho^4 \cos 2\theta + Q_4 \rho^4 \sin 2\theta) \tag{4.2.70}$$

By differentiating ψ_{4_p} with respect to ρ and on $\rho = 1$, the equation (4.2.70) becomes

$$\frac{\partial \psi_{4p}}{\partial \rho} = \frac{3\delta^4}{16}\,\Pi_0 + \frac{\delta^2}{2}\,Q_0 + \frac{7}{8}\,\delta^2\,(Q_1 \cos\theta + Q_2 \sin\theta)$$

$$+ \frac{\delta^2}{3}\,(Q_3 \cos 2\theta + Q_4 \sin 2\theta) \quad \text{on} \quad \rho = 1 \tag{4.2.71}$$

Therefore the complete solution is,

$$\psi_4 = \psi_{4_p} + \Omega_4 \tag{4.2.72}$$

where Ω_4 is a harmonic function, and from the equations (4.2.68) and (4.2.71), must satisfy

$$\frac{\partial \Omega_4}{\partial \rho} = -\left(\frac{3\delta^4}{16}\,\Pi_0 + \frac{\delta^2}{2}\,Q_0 + \frac{7}{8}\,\delta^2\,(Q_1 \cos\theta + Q_2 \sin\theta) + \frac{\delta^2}{3}\,(Q_3 \cos 2\theta + Q_4 \sin 2\theta)\right)$$

$$\tag{4.2.73}$$

write

$$\Omega_4 = \Omega_{42} + \Omega_{43}$$

where Ω_{42} and Ω_{43} satisfy

$$\frac{\partial \Omega_{42}}{\partial \rho} = -\left(\frac{3\,\delta^4}{16}\,\Pi_0 + \frac{\delta^2}{2}\,Q_0\right)$$

and

$$\frac{\partial \Omega_{43}}{\partial \rho} = - \delta^2 \left(\frac{7}{8} (Q_1 \cos \theta + Q_2 \sin \theta) + \frac{1}{3} (Q_3 \cos 2\theta + Q_4 \sin 2\theta) \right) \text{on } \rho = 1$$

respectively. That is, from the above equations

$$\Omega_{42} + (\frac{3\,\delta^4}{16} \Pi_0 + \frac{\delta^2}{2} Q_0) \ln \rho \to 0 \quad \text{as} \quad \rho \to \infty$$

and

$$\Omega_{43} = \delta^2 \left\{ \frac{7}{8} ((Q_1 \frac{\cos \theta}{\rho} + Q_2 \frac{\sin \theta}{\rho}) + \frac{1}{6} (Q_3 \frac{\cos 2\theta}{\rho^2} + Q_4 \frac{\sin 2\theta}{\rho^2}) \right\} \text{on } \rho = 1 \qquad (4.2.74)$$

From the inner expansion of outer solution the homogeneous solution is required in the inner solution at $O(\varepsilon^4 (\ln \varepsilon)^2)$. That is,

$$\psi_{42} = P_8 \qquad (4.2.75)$$

where P_8 is a constant. Therefore from the equations (4.2.61),(4.2.64-66),(4.2.70),(4.2.72) and (4.2.74-75), the outer expansion of (4.2.61) when expressed in inner coordinates, is therefore

$$\psi^{(4,4)} = \Pi_0 + \varepsilon \left\{ P_1 + \Pi_1 (\rho \cos \theta + \frac{\cos \theta}{\rho}) + \Pi_2 (\rho \sin \theta + \frac{\sin \theta}{\rho}) \right\} + \varepsilon^2 \ln \varepsilon\, P_4$$

$$+ \varepsilon^2 \left\{ \frac{\delta^2}{4} \Pi_0 \rho^2 - \frac{\delta^2}{2} \Pi_0 \ln \rho + Q_0 + Q_1 (\rho \cos \theta + \frac{\cos \theta}{\rho}) \right.$$

$$+ Q_2 \left(\rho \sin \theta + \frac{\sin \theta}{\rho} \right) + Q_3 \left(\rho^2 \cos 2\theta + \frac{\cos 2\theta}{\rho^2} \right) + Q_4 \left(\rho^2 \sin 2\theta + \rho^3 \cos \theta \right) \Bigg\}$$

$$+ \varepsilon^3 \ln \varepsilon \left\{ P_5 + P_6 \left(\rho \cos \theta + \frac{\cos \theta}{\rho} \right) + P_7 \left(\rho \sin \theta + \frac{\sin \theta}{\rho} \right) \right\}$$

$$+ \varepsilon^3 \Bigg\{ Q_5 + \frac{\delta^2}{2} P_1 \ln \rho + \frac{\delta^2}{2} \rho \ln \rho \left(\Pi_1 \cos \theta + \Pi_2 \sin \theta \right) + \frac{7}{8} \delta^2 \left(\Pi_1 \frac{\cos \theta}{\rho} + \Pi_2 \frac{\sin \theta}{\rho} \right)$$

$$+ Q_6 \left(\rho \cos \theta + \frac{\cos \theta}{\rho} \right) + Q_7 \left(\rho \sin \theta + \frac{\sin \theta}{\rho} \right) + \frac{\delta^2}{4} P_1 \rho^2 + Q_8 \rho^2 \cos 2\theta + Q_9 \rho^2 \sin 2\theta$$

$$+ \frac{\delta^2}{8} \left(\Pi_1 \rho^3 \cos \theta + \Pi_2 \rho^3 \sin \theta \right) + Q_{10} \rho^3 \cos 3\theta + Q_{11} \rho^3 \sin 3\theta \Bigg\}$$

$$+ \varepsilon^4 \ln \varepsilon \left\{ Q_{12} - \frac{\delta^2}{2} P_4 \ln \rho + Q_{13} \rho \cos \theta + Q_{14} \rho \sin \theta + \frac{\delta^2}{4} P_4 \rho^2 \right\}$$

$$+ \varepsilon^4 (\ln \varepsilon)^2 P_8 + \varepsilon^4 \Bigg\{ Q_{15} - \left(\frac{3\delta^4}{16} \Pi_0 + \frac{\delta^2}{2} Q_0 \right) \ln \rho + \frac{\delta^2}{2} \rho \ln \rho \left(Q_1 \cos \theta + Q_2 \sin \theta \right)$$

$$+ Q_{16} \rho \cos \theta + Q_{17} \rho \sin \theta - \frac{\delta^4}{8} \Pi_0 \rho^2 \ln \rho + \left(\frac{\delta^2}{4} Q_0 + \frac{\delta^4}{8} \Pi_0 \right) - \frac{\delta^2}{4} Q_3 \cos 2\theta$$

$$- \frac{\delta^2}{4} Q_4 \sin 2\theta + Q_{18} \rho^2 \cos 2\theta + Q_{19} \rho^2 \sin 2\theta + \frac{\delta^2}{8} \left(Q_1 \rho^3 \cos \theta + Q_2 \rho^3 \sin \theta \right)$$

$$+ Q_{20}\,\rho^3 \cos 3\theta + Q_{21}\,\rho^3 \sin 3\theta + \frac{\delta^4}{64}\,\Pi_0\,\rho^4$$

$$+ \frac{\delta^2}{12}\,(Q_3\,\rho^4 \cos 2\theta + Q_4\,\rho^4 \sin 2\theta) + Q_{22}\,\rho^4 \cos 4\theta + Q_{23}\,\rho^4 \sin 4\theta \Big\}.$$

(4.2.76)

This outer expansion of $\psi^{(4)}$ can only match with the inner expansion of $\Psi^{(4)}$ if σ has the expansion

$$\sigma^{(4)} = \sigma^{(3)} + \varepsilon^4 \ln \varepsilon\, \sigma_{41} + \varepsilon^4\, \sigma_4 \tag{4.2.77}$$

where $\sigma^{(3)}$ is given by equation (4.2.56). Therefore from (4.2.25), (4.2.58), (4.2.60), and (4.2.77) the inner expansion of outer solution (4.2.58) yields,

$$\Psi^{(4,4)} = \Pi_0 + \varepsilon \left\{ \Pi_3 + \Pi_1\,\rho \cos \theta + \Pi_2\,\rho \sin \theta + \frac{\sigma_2\,A_1}{\delta}\,\frac{\cos \theta}{\rho} + \frac{\sigma_2\,B_1}{\delta}\,\frac{\sin \theta}{\rho} \right\}$$

$$- \varepsilon^2 \ln \varepsilon\, \sigma_2\, A_0 + \varepsilon^2 \Big\{ \Pi_4 - \sigma_2\, A_0 \ln \rho + \Pi_5\,\rho \cos \theta + \Pi_6\,\rho \sin \theta$$

$$+ \frac{\cos \theta}{\rho}\,(\frac{\sigma_2\,C_1}{8} + \frac{\sigma_3\,A_1}{\delta}) + \frac{\sin \theta}{\rho}\,(\frac{\sigma_2\,D_1}{8} + \frac{\sigma_3\,B_1}{\delta}) + \Pi_0\,\frac{\delta^2\,\rho^2}{4} + \Pi_7\,\rho^2 \cos 2\theta$$

$$+ \Pi_8\,\rho^2 \sin 2\theta + \frac{2\,\sigma_2}{\delta^2}\,(E_2\,\frac{\cos 2\theta}{\rho^2} + F_2\,\frac{\sin 2\theta}{\rho^2}) \Big\}$$

$$+\varepsilon^3 \ln\varepsilon \left\{ \Pi_9 + \frac{\delta}{2}\sigma_2 (A_1 \rho\cos\theta + B_1 \rho\sin\theta) + \Pi_{10}\frac{\cos\theta}{\rho} + \Pi_{11}\frac{\sin\theta}{\rho} \right\}$$

$$+\varepsilon^3 \left\{ \Pi_{12} - \Pi_{13}\ln\rho + \frac{\delta}{2}\sigma_2 \ln\rho (A_1 \rho\cos\theta + B_1 \rho\sin\theta) \right.$$

$$+\frac{\delta}{2}(\Pi_{14}\rho\cos\theta + \Pi_{15}\rho\sin\theta) + \Pi_{16}\frac{\cos\theta}{\rho} + \Pi_{17}\frac{\sin\theta}{\rho} + \frac{\delta^2}{4}\Pi_3\rho^2$$

$$+\frac{\delta^2}{4}(\Pi_{18}\rho^2\cos 2\theta + \Pi_{19}\rho^2\sin 2\theta) + \frac{\delta^3}{8}(\Pi_1\rho^3\cos\theta + \Pi_2\rho^3\sin\theta)$$

$$\left. +\frac{\delta^3}{48}(\Pi_{20}\rho^3\cos 3\theta + \Pi_{21}\rho^3\sin 3\theta) \right\} - \varepsilon^4 (\ln\varepsilon)^2 A_0 \sigma_{41}$$

$$+\varepsilon^4 \ln\varepsilon \left\{ \Pi_{22} + \Pi_{23}\rho\cos\theta + \Pi_{24}\rho\sin\theta - \sigma 41\ A_0\ln\rho - \frac{\delta^2}{4}\sigma_2 A_0\rho^2 \right\}$$

$$+\varepsilon^4 \left\{ \Pi_{25} - \Pi_{26}\ln\rho + \Pi_{27}\rho\cos\theta + \Pi_{28}\rho\sin\theta \right.$$

$$+\Pi_{29}\frac{\delta}{2}\rho\cos\theta\ln\rho + \Pi_{30}\frac{\delta}{2}\rho\sin\theta\ln\rho + \frac{\delta^2}{4}(\Pi_4 - A_0)\rho^2$$

$$-\frac{\delta^2}{4}\sigma_2 A_0\rho^2\ln\rho + \frac{\delta^2}{8}(\Pi_{31}\rho^2\cos 2\theta + \Pi_{32}\rho^2\sin 2\theta)$$

$$+ \frac{\delta^2}{8} \Pi_6 \rho^3 \sin\theta + \frac{\delta^2}{8} \Pi_5 \rho^3 \cos\theta + \Pi_{33} \rho^3 \cos 3\theta$$

$$\left. + \Pi_{34} \rho^3 \sin 3\theta + \frac{\delta^4}{64} \Pi_0 \rho^4 \right\}, \qquad (4.2.78)$$

where

$$\Pi_{22} = -\sigma_4 A_0 - \sigma_2 E_0 - \sigma_{41} A_0 (\ln \frac{\delta}{2} + \gamma),$$

$$\Pi_{23} = \frac{\delta}{2} (\sigma_2 C_1 + A_1 \sigma_3),$$

$$\Pi_{24} = \frac{\delta}{2} (\sigma_2 D_1 + B_1 \sigma_3),$$

$$\Pi_{25} = A_0 c_{00,4} + A_1 c_{10,4} + B_1 e_{10,4} + E_0 c_{00,2} + \sum_{n=1}^{2} (E_n c_{n0,2} + F_n e_{n0,2}),$$

$$- \sigma_2 E_0 (\ln \frac{\delta}{2} + \gamma) - \sigma_4 A_0 (\ln \frac{\delta}{2} - \gamma) + W_0 c_{00,0} + \sum_{n=1}^{2} (W_n c_{n0,0} + U_n e_{n0,0}),$$

$$\Pi_{26} = \sigma_2 (A_0 + E_0),$$

$$\Pi_{27} = \frac{\delta}{2} (C_0 c_{01,2} + C_1 c_{11,2} + D_1 e_{11,2} - \frac{\sigma_2 C_1}{2}$$

$$+ \sigma_2 C_1 (\ln \frac{\delta}{2} + \gamma) + \sigma_3 A_1 \frac{\delta}{2} (\ln \frac{\delta}{2} + \gamma - \frac{1}{2}),$$

$$\Pi_{28} = \frac{\delta}{2} (C_0 d_{01,2} + C_1 d_{11,2} + D_1 f_{11,2} - \frac{\sigma_2 D_1}{2}$$

$$+ \sigma_2 D_1 (\ln \frac{\delta}{2} + \gamma) + \sigma_3 B_1 (\ln \frac{\delta}{2} + \gamma - \frac{1}{2})),$$

$$\Pi_{29} = A_1 \sigma_3 + C_1 \sigma_2,$$

$$\Pi_{30} = B_1 \sigma_3 + D_1 \sigma_2,$$

$$\Pi_{31} = A_0 c_{02,2} + A_1 c_{12,2} + B_1 e_{12,2} + E_0 c_{02,0} + \sum_{n=1}^{2} (E_n c_{n2,0} + F_n e_{n2,0}),$$

$$\Pi_{32} = A_0 d_{02,2} + A_1 d_{12,2} + B_1 f_{12,2} + E_0 d_{02,0} + \sum_{n=1}^{2} (E_n d_{n2,0} + F_n f_{n2,0}),$$

$$\Pi_{33} = \frac{\delta^3}{48} (C_0 c_{03,0} + C_1 c_{13,0} + D_1 e_{13,0}),$$

and

$$\Pi_{34} = \frac{\delta^3}{48} (C_0 d_{03,0} + C_1 d_{13,0} + D_1 f_{13,0}). \qquad (4.2.79)$$

Matching (4.2.76) with (4.2.78), and considering only the equations below in order to calculate the higher-order solution up to σ_4 gives

$$\varepsilon : \ \Pi_3 = P_1, \qquad \varepsilon^2 : \ Q_0 = \Pi_4 \tag{4.2.80}$$

$$\varepsilon^2 \rho \cos\theta : \ Q_1 = \Pi_5, \qquad \varepsilon^2 \rho \sin\theta : \ Q_2 = \Pi_6 \tag{4.2.81}$$

$$\varepsilon^2 \rho^2 \sin 2\theta \ : \ Q_4 = \Pi_8, \qquad \varepsilon^2 \rho^2 \sin 2\theta \ : \ Q_3 = \Pi_7 \tag{4.2.82}$$

$$\varepsilon^2 \frac{\cos\theta}{\rho} : Q_1 = \frac{1}{\delta} (\sigma_2 C_1 + \sigma_3 A_1), \qquad \varepsilon^2 \frac{\sin\theta}{\rho} \ : \ Q_2 = \frac{1}{\delta} (\sigma_2 D_1 + \sigma_3 B_1) \tag{4.2.83}$$

$$\varepsilon^2 \frac{\cos 2\theta}{\rho^2} \ : \ Q_3 = \frac{2 \sigma_2 E_2}{\delta^2}, \qquad \varepsilon^2 \frac{\sin 2\theta}{\rho^2} \ : \ Q_4 = \frac{2 \sigma_2 F_2}{\delta^2} \tag{4.2.84}$$

$$\varepsilon^3 \ln \varepsilon \ \rho \cos\theta \ : \ P_6 = \frac{\delta}{2} \sigma_2 A_1, \qquad \varepsilon^3 \ln \varepsilon \ \frac{\cos\theta}{\rho} \ : \ \Pi_{10} = P_6 \tag{4.2.85}$$

$$\varepsilon^3 \ln \rho \ : \ \Pi_{13} = \frac{\delta^2}{2} P_1, \qquad \varepsilon^3 \rho \cos\theta \ : \ \frac{\delta}{2} \Pi_{14} = Q_6 \tag{4.2.86}$$

$$\varepsilon^3 \rho \sin\theta \ : \ \frac{\delta}{2} \Pi_{15} = Q_7, \qquad \varepsilon^3 \frac{\cos\theta}{\rho} \ : \ \Pi_{16} = Q_6 + \frac{7 \delta^2}{8} \Pi_1 \tag{4.2.87}$$

and

$$\varepsilon^3 \frac{\sin\theta}{\rho} \ : \ \Pi_{17} = Q_7 + \frac{7 \delta^2}{8} \Pi_2, \quad \varepsilon^4 \ln \rho \ : \ \Pi_{26} = \frac{3 \delta^4}{16} \Pi_0 + \frac{\delta^2}{2} Q_0 . \tag{4.2.88}$$

(i) TO CALCULATE σ_{41}

From the equations (4.2.85), σ_{41} is easily find that

$$\Pi_{10} = \frac{\delta}{2} \sigma_2 A_1 \tag{4.2.89}$$

but $\Pi_{10} = \frac{\sigma_{41} A_1}{\delta}$. Substituting into the above equation, we get

$$\sigma_{41} = \frac{\delta^2}{2} \sigma_2 \tag{4.2.90}$$

where σ_2 is known and is given by the equation (4.2.40).

(ii) TO CALCULATE σ_3

From equations (4.2.80), (4.2.86), and definitions of Π_3 and Π_{13} gives

$$\sigma_3 A_0 + \sigma_2 C_0 = \frac{\delta^2}{2} \left\{ C_0 \, c_{00,0} + C_1 \, c_{10,0} + D_1 \, e_{10,0} \right\}. \tag{4.2.91}$$

From (4.2.81), (4.2.83), and definition of Π_5 gives

$$\sigma_2 C_1 + \sigma_3 A_1 = \frac{\delta^2}{2} \left\{ C_0 \, c_{01,0} + C_1 \, c_{11,0} + D_1 \, e_{11,0} \right\}, \tag{4.2.92}$$

and from (4.2.81), (4.2.83) and definition of Π_6 gives

$$\sigma_2 D_1 + \sigma_3 B_1 = \frac{\delta^2}{2}\left\{C_0\, d_{01,0} + C_1\, d_{11,0} + D_1\, f_{11,0}\right\}. \tag{4.2.93}$$

From equation (4.2.35) A_0, and A_1 can be written in terms of B_1 and so there are five unknowns in the equations (4.2.91-93). To solve this we can write above equations (4.2.91-93) in the matrix form

$$\underline{A}\begin{pmatrix} C_0 \\ C_1 \\ D_1 \end{pmatrix} = \underline{B}\ \underline{X} \tag{4.2.94}$$

where,

$$\underline{A} = \begin{pmatrix} c_{01,0} & -f_{11,0} - c_{00,0} & e_{11,0} \\ d_{01,0} & d_{11,0} & -c_{11,0} - c_{00,0} \\ -c_{11,0} - f_{11,0} & c_{10,0} & e_{10,0} \end{pmatrix} = (\,\underline{a}_1\,, \underline{a}_2\,, \underline{a}_3\,), \tag{4.2.95}$$

$$\underline{B} = \begin{pmatrix} 0 & \sigma_3 & 0 \\ 0 & 0 & \sigma_3 \\ \sigma_3 & 0 & 0 \end{pmatrix}, \tag{4.2.96}$$

and

$$\underline{X} = \begin{pmatrix} A_0 \\ A_1 \\ B_1 \end{pmatrix}. \tag{4.2.97}$$

By rewriting equation (4.2.36) in one side gives a matrix which is same as the matrix A. Since matrix A is singular and $r\,(\underline{A}) = 2$ and where r stands for the rank of the matrix. Therefore using the matrix theory, for equation (4.2.94) to have a solution,

$$r(\underline{A}) = r(\underline{A}, \underline{B}\,\underline{X}) = 2.$$

Therefore matrix $\underline{B}\ \underline{X}$ can be written as a linear combination of any two rows or columns of matrix A.

That is

$$\underline{B}\underline{X} = \gamma_1\ \underline{a}_2 + \gamma_2\ \underline{a}_{3,} \tag{4.2.98}$$

where γ_1 and γ_2 are unknown constants.

TO SOLVE MATRIX EQUATION

From equation (4.2.98)

$$\sigma_3 A_1 = \gamma_1 a_{21} + \gamma_2 a_{31}, \tag{4.2.99}$$

$$\sigma_3 B_1 = \gamma_1 a_{22} + \gamma_2 a_{32}, \tag{4.2.100}$$

and

$$\sigma_3 A_0 = \gamma_1 a_{23} + \gamma_2 a_{33}. \tag{4.2.101}$$

However from the matrix equation for σ_2

$$A_0 = B_1\ \frac{c_{00,0}}{d_{01,0}}\ ,\ A_1 = B_1\ \frac{c_{11,0}}{d_{11,0}} \tag{4.2.102}$$

and substituting into equations (4.2.99-101), and by eliminating γ_1 and γ_2 we get,

$$\sigma_3 = 0. \tag{4.2.103}$$

(iii) TO CALCULATE σ_4

This calculation is same as in (ii). Therefore from equation (4.2.86) and (4.2.87) we have

$$\Pi_{16} = \frac{\delta}{2}\,\Pi_{14} + \frac{7}{8}\,\delta^2\,\Pi_1, \tag{4.2.104}$$

and similarly considering equations (4.2.80), (4.2.87), and (4.2.88) we get,

$$\Pi_{17} = \frac{\delta}{2}\,\Pi_{15} + \frac{7}{8}\,\delta^2\Pi_2, \tag{4.2.105}$$

and

$$\Pi_{26} = \frac{3\delta^4}{16}\,\Pi_0 + \frac{\delta^2}{2}\,\Pi_4. \tag{4.2.106}$$

By using the definition of Π's, equations (4.2.104-106) can be written in the matrix form

$$\underline{B}\,\underline{X} = \frac{\delta^2}{2}\,\underline{A}\begin{pmatrix} E_0 \\ E_1 \\ F_1 \end{pmatrix} \tag{4.2.107}$$

where

$$\underline{B} = (b_{ij})_{3\times3}, \quad \underline{A} = (a_{ij})_{3\times3}\,,$$

and elements of matrix B are

$$b_{11} = \frac{7\delta^3}{16} c_{01,0} + \frac{\delta}{2} c_{01,2} + \frac{\delta^5}{16\sigma_2} (e_{21,0} d_{02,0} + c_{21,0} c_{02,0}),$$

$$b_{12} = \frac{7\delta^3}{16} c_{11,0} + \frac{\delta}{2} \left(c_{11,2} + \sigma_2 (\ln \frac{\delta}{2} + \gamma - \frac{\delta}{2}) \right)$$

$$- \frac{\sigma_4}{\delta} + \frac{\delta^5}{16\sigma_2} (e_{21,0} d_{12,0} + c_{21,0} c_{12,0}),$$

$$b_{13} = \frac{7\delta^3}{16} e_{11,0} + \frac{\delta}{2} e_{11,2} + \frac{\delta^5}{16\sigma_2} (e_{21,0} f_{12,0} + c_{21,0} e_{12,0}),$$

$$b_{21} = \frac{7\delta^3}{16} d_{01,0} + \frac{\delta}{2} d_{01,2} + \frac{\delta^5}{16\sigma_2} (f_{21,0} d_{02,0} + d_{21,0} c_{02,0}),$$

(4.2.108)

$$b_{22} = \frac{7\delta^3}{16} d_{11,0} + \frac{\delta}{2} d_{11,2} + \frac{\delta^5}{16\sigma_2} (f_{21,0} d_{12,0} + d_{21,0} c_{12,0}),$$

$$b_{23} = \frac{7\delta^3}{16} f_{11,0} + \frac{\delta}{2} \left(f_{11,2} + \sigma_2 (\ln \frac{\delta}{2} + \gamma - \frac{\delta}{2}) \right)$$

$$- \frac{\sigma_4}{\delta} + \frac{\delta^5}{16\sigma_2} (f_{21,0} f_{12,0} + d_{21,0} e_{12,0}),$$

$$b_{31} = \sigma_2 - \frac{3\delta^4}{16} c_{00,0} - \frac{\delta^2}{2}\left(c_{00,2} - \sigma_2 (\gamma + \ln \frac{\delta}{2}) \right) - \frac{\delta^5}{16\sigma_2} (d_{02,0}\, e_{20,0} + c_{20,0}\, c_{02,0}),$$

$$b_{32} = - \frac{3\delta^4}{16} c_{10,0} - \frac{\delta^2}{2} c_{10,2} - \frac{\delta^5}{16\sigma_2} (c_{20,0}\, c_{12,0} + e_{20,0}\, d_{12,0}),$$

and

$$b_{33} = - \frac{3\delta^4}{16} e_{10,0} - \frac{\delta^2}{2} e_{10,2} - \frac{\delta^5}{16\sigma_2} (c_{20,0}\, e_{12,0} + e_{20,0}\, f_{12,0}).$$

TO SOLVE MATRIX EQUATION

As for (ii), from equation (4.2.107), by writing matrix $\underline{B}\ \underline{X}$ as a linear combination of any two rows or columns of matrix A and after simplification, we have

$$a_{32} (b_{11} A_0 + b_{12} A_1 + b_{13} B_1) - a_{31} (b_{21} A_0 + b_{22} A_1 + b_{23} B_1) = \gamma_1 (a_{21} a_{32} - a_{22} a_{31}) \tag{4.2.109}$$

and

$$a_{33} (b_{21} A_0 + b_{22} A_1 + b_{23} B_1) - a_{32} (b_{31} A_0 + b_{32} A_1 + b_{33} B_1) = \gamma_1 (a_{22} a_{33} - a_{23} a_{32}). \tag{4.2.110}$$

But from the definition of (a_{ij}) $(i, j = 1, 2, 3)$,

$$a_{21} a_{32} - a_{22} a_{31} = \frac{\delta^3}{4} c_{00,0} (f_{11,0} + c_{00,,0} + c_{11,0}), \tag{4.2.110}$$

and

$$a_{22}\, a_{33} - a_{23}\, a_{32} = \frac{\delta^3}{4}\, c_{10,0}\, (f_{11,0} + c_{00,0} + c_{11,0}\,). \qquad (4.2.112)$$

Now from equations (4.2.101) and (4.2.109-112) we get

$$c_{10,0}\, a_{32}\, (\, b_{11}\, \frac{c_{00,0}}{d_{01,0}} + b_{12}\, \frac{c_{11,0}}{d_{11,0}} + b_{13}\,) - (\, a_{31}\, c_{10,0} + a_{33}\, c_{00,0}\,)\, (\, b_{21}\, \frac{c_{00,0}}{d_{01,0}} + b_{22}\, \frac{c_{11,0}}{d_{11,0}} + b_{23}\,)$$

$$= - a_{32}\, c_{00,0}\, (\, b_{31}\, \frac{c_{00,0}}{d_{01,0}} + b_{32}\, \frac{c_{11,0}}{d_{11,0}} + b_{33}\,). \qquad (4.2.113)$$

But from the definition of a_{ij},

$$a_{31}\, c_{10,0} + a_{33}\, c_{00,0} = - a_{32}\, (\, c_{11,0} + f_{11,0} - \frac{f_{11,0}}{d_{11,0}}\, c_{00,0}\,).$$

Substituting this into (4.2.113), and after little simplification, gives,

$$\frac{\sigma_4}{\delta} \left(c_{10,0}\, \frac{c_{11,0}}{d_{11,0}} + (\, c_{11,0} + f_{11,0} - f_{11,0}\, \frac{c_{00,0}}{d_{11,0}}\,) \right) = c_{10,0} \left(\frac{c_{00,0}}{d_{01,0}}\, b_{11} + \frac{c_{11,0}}{d_{11,0}}\, B_2 + b_{13} \right)$$

$$+ \left(c_{11,0} + f_{11,0} - f_{11,0}\, \frac{c_{00,0}}{d_{11,0}} \right) \left(b_{21}\, \frac{c_{00,0}}{d_{01,0}} + b_{22}\, \frac{c_{11,0}}{d_{11,0}} + B_3 \right) + c_{00,0} \left(b_{31}\, \frac{c_{00,0}}{d_{01,0}} + b_{32}\, \frac{c_{11,0}}{d_{11,0}} + b_{33} \right)$$

(4.2.114)

where,

$$B_2 = \frac{7\delta^3}{16}\, c_{11,0} + \frac{\delta}{2}\left(c_{11,2} + \sigma_2\,(\ln\frac{\delta}{2} + \gamma - \frac{\delta}{2}\,)\right) + \frac{\delta^5}{16\sigma_2}\,(\, e_{21,0}\, d_{12,0} + c_{21,0}\, c_{12,0}\,),$$

and

$$B_3 = \frac{7\delta^3}{16}\, f_{11,0} + \frac{\delta}{2}\left(\, f_{11,2} + \sigma_2\,(\ln\frac{\delta}{2} + \gamma - \frac{\delta}{2}\,)\right) + \frac{\delta^5}{16\sigma_2}\,(\, f_{21,0}\, f_{12,0} + d_{21,0}\, e_{12,0}\,).$$

This is an explicit expression for σ_4 and where coefficients $c_{ij,k}, d_{ij,k}, e_{ij,k}(i, j, k = 0, 1, 2)$, and $f_{ij,k}$ are known from the appendix 4.1.

4.3 A surface-piercing vertical barrier

The contour C is now taken to be a thin vertical plate piercing the free surface. Here the boundary-value problem to be solved is given by the modified Helmholtz equation (3.1.5), the linearised free-surface condition (3.1.6), the bed and wall conditions (3.1.7-8) and the body condition (3.1.9). The parameter σ_2 is introduced as in equations (4.1.1-6).

The reference point in C is now chosen as $(x, y) = (x_0, 0)$ and polar coordinates (r, θ) are defined by

$$x - x_0 = r \sin\theta , \; y = r \cos\theta. \tag{4.3.1}$$

INNER REGION

Within distances $r \ll h$ of C, suitable non-dimensional coordinates are

$$\xi = \frac{x - x_0}{a} , \; \eta = \frac{y}{a} , \; \rho = \frac{r}{a}. \tag{4.3.2}$$

In terms of these coordinates the governing equations for the inner potential ψ are the field equation

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} - \varepsilon^2 \delta^2 \psi = 0, \tag{4.3.3}$$

the linearized free-surface condition

$$\frac{\partial \psi}{\partial \eta} + \varepsilon \kappa (1 - \varepsilon^2 V + \ldots) \psi = 0 \text{ on } \eta = 0 , \xi \neq 0 \tag{4.3.4}$$

and the boundary condition

$$\frac{\partial \psi}{\partial \xi} = 0 \ \text{ on } \ \xi = 0\,,\, 0 < \eta < 1. \tag{4.3.5}$$

Here $\kappa = K_0\, h$ and K has been replaced in the free-surface condition using (4.1.5-6).

OUTER SOLUTION

For this problem, the complete outer solution has the form

$$\Psi = \sum_{n=0}^{\infty} B_n\, h_n(\, R\,,\, \theta\,), \tag{4.3.6}$$

where

$$h_n = \sin \sigma\, \psi_n^{(f,b)} \tag{4.3.7}$$

and $\psi_n^{(f,b)}$ are the antisymmetric multipoles defined in appendix 4.2. Each multipole is decomposed as

$$h_n = h_{n,1} + \sin \sigma\, h_{n,2}. \tag{4.3.8}$$

The singular parts are the infinite depth multipoles given in appendix 4.2, thus

$$h_{0,2} = -\frac{2}{p} \int_0^{\infty} \frac{t\, e^{-\beta y} \sin tX}{K - \beta}\, dt \tag{4.3.9}$$

and

$$h_{n,2} = K_{2n+1}(\delta R)\sin(2n+1)\theta + \frac{\kappa}{\delta}(1 - \varepsilon^2 V + \ldots) K_{2n}(\delta R)\sin 2n\theta$$

$$+ K_{2n-1}(\delta R)\sin(2n-1)\theta, \ n = 1, 2, \ldots . \tag{4.3.10}$$

The non-singular parts $h_{n,1}$ have coordinate expansions in the form of equation (4.2.7).

Without loss of generality, dipole coefficient B_0 may be chosen to be unity because the problem is homogeneous. Thus, the leading-order outer solution is written

$$\Psi^{(0)} = h_{0,1}{}^{(0)} \tag{4.3.11}$$

which has an inner expansion

$$\Psi^{(0,1)} = e_{00,0} + (e_{01,0}\cos\theta + f_{01,0}\sin\theta)\,\frac{1}{2}\,\delta\,\varepsilon\,\rho. \tag{4.3.12}$$

LEADING-ORDER INNER SOLUTION

The equation (4.3.12) suggests that the first term in the inner solution must have the form

$$\psi^{(1)} = P_0 + \varepsilon\,\psi_1 \tag{4.3.13}$$

where, from (4.3.3-5), ψ_1 is a harmonic function satisfying the boundary conditions

$$\frac{\partial \psi_1}{\partial \eta} + \kappa P_0 = 0 \ \text{ on } \ \eta = 0, \xi \neq 0 \tag{4.3.14}$$

and

$$\frac{\partial \psi_1}{\partial \xi} = 0 \text{ on } \xi = 0\,, 0 < \eta < 1\,. \tag{4.3.15}$$

TO COMPUTE ψ_1

By inspection, a particular solution is $\psi_{1,p} = -\kappa\, P_0\, \eta$; homogeneous solutions are easily found with the aid of a conformal mapping. Define a complex variable $z = \xi + j\,\eta$ and map the flow region onto the upper half of the $\zeta = u + j\,v$ plane by

$$\zeta = (z^2 + 1)^{\frac{1}{2}}. \tag{4.3.16}$$

Solutions having zero normal derivative on $v = 0$ are of the form $\mathrm{Re}_j\, \zeta^n$ where n is a non-negative integer (negative integers give non-integrable singularities in the velocity. Retaining only those homogeneous solutions needed for matching with (4.3.12) gives

$$\psi_1 = -\kappa\, P_0\, \eta + P_1 + P_2\, \mathrm{Re}_j\, (\, z^2 + 1\,)^{\frac{1}{2}}. \tag{4.3.17}$$

This may now be used in (4.3.13) and expanded to obtain

$$\psi^{(1,0)} = P_0 + \varepsilon\, (\, -\kappa\, P_0\, \rho \cos\theta + P_2\, \rho \sin\theta\,) \tag{4.3.18}$$

which when matched with (4.3.12) gives

$$P_0 = e_{00,0},\;\; P_2 = \frac{1}{2}\,\delta\, f_{01,0}. \tag{4.3.19}$$

An examination of the outer expansion of $\psi^{(1)}$ yields,

$$\psi^{(1,2)} = P_0 - \kappa\, P_0\, R \cos\theta + P_2\, R \sin\theta + \varepsilon\, P_1 + \varepsilon^2\, P_2\, \frac{\sin\theta}{2R}\,. \tag{4.3.20}$$

SECOND-ORDER INNER SOLUTION

Equation(4.3.20) suggests that the continuing outer solution is,

$$\Psi^{(2)} = h_0^{(2)} + \varepsilon^2\, B_1\, h_1^{(0)} \tag{4.3.21}$$

which has an inner expansion

$$\Psi^{(2,3)} = e_{00,0} + \varepsilon \left\{ (e_{01,0} \cos\theta + f_{01,0} \sin\theta)\, \frac{1}{2}\, \delta\, \rho + \frac{2\,\sigma_2}{\delta}\, \frac{\sin\theta}{\rho} \right\}$$

$$+ \varepsilon^2 \left\{ e_{00,2} + e_{00,0}\, \frac{1}{4}\, \delta^2\, \rho^2 + (\, e_{00,2} + e_{00,0}\, \frac{1}{4}\, \delta^2\, \rho^2 \right.$$

$$\left. + (\, e_{02,0} \cos 2\theta + f_{02,0} \sin 2\theta\,)\, \frac{1}{8}\, \delta^2\, \rho^2 + \frac{2\,\kappa\,\sigma_2}{\delta}\, \theta + B_1\, e_{10,0} \right\}$$

$$+ \varepsilon^3 \ln\varepsilon \left\{ -\,\delta\, \sigma_2 \cosh 2\hat{\nu}\, \rho \sin\theta \right\}$$

$$+ \varepsilon^3 \left((\, e_{01,0} \cos\theta + f_{01,0} \sin\theta\,)\, \frac{1}{16}\, \delta^3\, \rho^3 + (\, e_{01,2} \cos\theta + f_{01,2} \sin\theta\,)\, \frac{1}{2}\, \delta\rho \right.$$

$$+ (e_{03,0} \cos 3\theta + f_{03,0} \sin 3\theta) \frac{1}{48} \delta^3 \rho^3 - \sigma_2 \delta \cosh 2\hat{v} \ln \delta\rho \, \rho \sin\theta + \sigma_2 b_1 \delta \rho \sin\theta$$

$$- \frac{2\kappa^2 \sigma_2}{\delta} \theta \rho \cos\theta + B_1 (e_{11,0} \cos\theta + f_{11,0} \sin\theta) \frac{1}{2} \delta \rho \Bigg). \quad (4.3.22)$$

Observe that there is no constant term at $O(\varepsilon)$ in $\Psi^{(2,3)}$ so that the constant P_1 appearing in (4.3.17) must be zero.

HIGHER-ORDER INNER SOLUTION

Further, (4.3.22) suggests that the inner solution must be continued as

$$\psi^{(3)} = e_{00,0} + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \varepsilon^3 \ln\varepsilon \, \psi_{31} + \varepsilon^3 \psi_3, \quad (4.3.23)$$

where ψ_2 satisfies

$$\nabla^2 \psi_2 = \delta^2 e_{00,0} \quad (4.3.24)$$

in the fluid region together with the free-surface condition

$$\frac{\partial\psi_2}{\partial\eta} = -\kappa \psi_1 = -\kappa P_2 (\xi^2 + 1)^{\frac{1}{2}} \text{ on } \eta = 0, \xi \neq 0, \quad (4.3.25)$$

ψ_{31} satisfies homogeneous equations and ψ_3 satisfies

$$\nabla^2 \psi_3 = \delta^2 \psi_1 = \delta^2 \left(-\kappa P_0 \eta + P_2 \operatorname{Re}_j(z^2 + 1)^{\frac{1}{2}} \right) \quad (4.3.26)$$

and

$$\frac{\partial \psi_3}{\partial \eta} = -\kappa\, \psi_2 + \kappa\, V\, P_0 \;\text{ on }\; \eta = 0\,,\, \xi \neq 0. \tag{4.3.27}$$

All of ψ_i, $i = 2, 31, 3$ satisfy the barrier condition (4.3.5).

TO COMPUTE ψ_2

Note that a particular solution of equation (4.3.24) is

$$\psi_{2,p} = \frac{\delta^2}{4}\, P_0\, \rho^2 \tag{4.3.28}$$

so write

$$\psi_2 = \psi_{2,p} + \psi_{2,1} \tag{4.3.29}$$

where $\psi_{2,1}$ is a harmonic function and from (4.3.5), (4.3.25) and (4.3.29) satisfies the boundary conditions

$$\frac{\partial \psi_{2,1}}{\partial \eta} = -\kappa\, P_2\, \mathrm{Re}_j\,(\, z^2 + 1\,)^{\frac{1}{2}} \;\text{ on }\; \eta = 0\,,\, \xi \neq 0 \tag{4.3.30}$$

and

$$\frac{\partial \psi_{2,1}}{\partial \xi} = 0 \;\text{ on }\; \xi = 0 \;\;,\;\; 0 < \eta < 1. \tag{4.3.31}$$

By integrating (4.3.30) and satisfying the boundary condition (4.3.31) we get

$$\psi_{2,1} = -\frac{1}{2}\kappa P_2 \, \mathrm{Im}_j \left[\ln(z + (z^2 + 1)^{\frac{1}{2}}) + z (z^2 + 1)^{\frac{1}{2}} \right]. \tag{4.3.32}$$

Here the notation Im_j means that the imaginary part with respect to j. Also homogeneous solutions are needed to match with (4.3.22), including these gives

$$\psi_2 = \frac{1}{4}\delta^2 P_0 \rho^2 - \frac{1}{2}\kappa P_2 \, \mathrm{Im}_j \left[\ln\left(z + (z^2 + 1)^{\frac{1}{2}} \right) + z (z^2 + 1)^{\frac{1}{2}} \right] + P_3 + P_4 \, \mathrm{Re}_j \, z^{2}. \tag{4.3.33}$$

From (4.3.17) and (4.3.33), the outer expansion of $\psi^{(2)}$ to $O(\varepsilon^2)$ as,

$$\psi^{(2,2)} = P_0 + \varepsilon \left\{ -\kappa P_0 \rho \cos\theta + P_2 \left(\rho \sin\theta + \frac{\sin\theta}{2\rho} \right) \right\}$$

$$+ \varepsilon^2 \left\{ \frac{1}{4}\delta^2 P_0 \rho^2 - \frac{1}{2}\kappa P_2 \left(-\theta + \frac{\pi}{2} + \rho^2 \sin 2\theta \right) + P_3 - P_4 \rho^2 \cos 2\theta \right\}. \tag{4.3.34}$$

Match (4.3.22) with (4.3.34) up to $O(\varepsilon^2)$ gives

$$\varepsilon \rho \cos\theta : \; P_0 = -e_{01,0} \frac{\delta}{2\kappa}, \qquad \varepsilon \rho \sin\theta : \; P_2 = f_{01,0} \frac{\delta}{2}, \tag{4.3.35}$$

$$\varepsilon \frac{\sin\theta}{\rho} : \; P_2 = \frac{4\sigma_2}{\delta}, \qquad \varepsilon^2 \rho^2 \sin 2\theta : \; P_2 = -f_{02,0} \frac{\delta^2}{4\kappa}, \tag{4.3.36}$$

and

$$\varepsilon^2 \rho^2 \cos 2\theta : \ P_4 = \frac{\delta^2}{8} e_{02,0}, \qquad \varepsilon^2 : -\frac{\pi}{4} \kappa P_2 + P_3 = e_{00,2} + B_1 e_{10,0} . \tag{4.3.37}$$

A careful examination of the definitions of the expansion coefficients e and f shows that the new expressions for P_0 and P_2 in equations(4.3.35) and (4.3.36) are consistent with equations (4.3.19). From (4.3.35) and (4.3.36)

$$\sigma_2 = \frac{1}{8} \delta^2 f_{01,0} \tag{4.3.38}$$

is a first order approximation to σ , defined in equation (4.1.2) is given by $\varepsilon^2 \sigma_2$.

As in section 4.2, the above approximation, to order ε^2 is not given for greater accuracy. In order to obtain greater accuracy, we need to consider the higher-order solution. By looking at the inner expansion of $\Psi^{(2)}$ to order $O(\varepsilon^3)$, that is from (4.3.21), the solution for ψ_{31} requires only suitable homogeneous solutions to match with the $\varepsilon^3 \ln \varepsilon$ in (4.3.22) and so

$$\psi_{31} = P_5 \, \mathrm{Re}_j \, (z^2 + 1)^{\frac{1}{2}}. \tag{4.3.39}$$

TO COMPUTE ψ_3

The solution for ψ_3 may be broken down into a number of stages. A particular solution of the field equation (4.3.26) may be found by writing $\psi_3 = \mathrm{Re}_j \, W$ where W is the solution of

$$4\,\frac{\partial^2 W}{\partial z \partial \bar{z}} = \delta^2 \left(\kappa\, P_0\, jz + P_2\,(\, z^2 + 1\,)^{\frac{1}{2}} \right) \tag{4.3.40}$$

and $\bar{z}$ denotes complex conjugate. Integrating twice gives the particular solution

$$\psi_{3,p} = \frac{1}{8}\,\delta^2\,\mathrm{Re}_j \left\{ \kappa P_0 j\, z^2 \bar{z} + P_2\, \bar{z} \left[\ln\left(z + (\, z^2 + 1\,)^{\frac{1}{2}} \right) + z\,(\, z^2 + 1\,)^{\frac{1}{2}} \right] \right\}. \tag{4.3.41}$$

Note that

$$\frac{\partial \psi_{3,p}}{\partial \eta} = -\,\frac{\kappa\,\delta^2\,P_0}{8}\,\xi^2 \quad \text{on} \quad \eta = 0\,,\ \xi \neq 0 \tag{4.3.42}$$

and

$$\frac{\partial \psi_{3,p}}{\partial \xi} = 0\,,\ \xi = 0\,,\ 0 < \eta < 1 \tag{4.3.43}$$

Now write

$$\psi_3 = \psi_{3,p} + \psi_{3,2} \tag{4.3.44}$$

where $\psi_{3,2}$ is a harmonic function and from (4.3.27), (4.3.42), and (4.3.44) must satisfies the boundary condition

$$\frac{\partial \psi_{3,2}}{\partial \eta} = -\,\kappa \left[(\,\frac{1}{8}\delta^2\,P_0\,)\,\xi^2 + P_3 - V\,P_0 \right] \text{on}\ \eta = 0\,,\ \ \xi \neq 0 \tag{4.3.45}$$

and also the barrier condition (4.3.5). It is easily verified that a suitable combination of harmonic functions satisfying (4.3.45) and (4.3.5) is

$$\psi_{3,2} = \frac{1}{3}\kappa\left(\frac{1}{8}\delta^2 P_0 + P_4\right)\left(\eta^3 - 3\xi^2\eta\right) - \kappa\left(P_3 - V P_0\right)\eta. \tag{4.3.46}$$

Including all the homogeneous solutions required to match with (4.3.22),

$$\psi_3 = \psi_{3,p} + \psi_{3,2} + P_6 \operatorname{Re}_j\left(z^2+1\right)^{\frac{1}{2}} + P_7 \operatorname{Re}_j\left(z^2+1\right)^{\frac{3}{2}} + P_8 \operatorname{Re}_j\left\{z\ln\left(z+\left(z^2+1\right)^{\frac{1}{2}}\right)\right\}. \tag{4.3.47}$$

It is not immediately apparent that the final term in (4.3.47) is required, it might be thought that the logarithmic term in (4.3.38) is sufficient to match with the $\ln\rho\,\rho\sin\theta$ term at $O(\varepsilon^3)$ in (4.3.22) but this is not the case. From (4.3.23) the outer expansion of $\psi^{(3)}$ to the $O(\varepsilon^4)$ can be calculated as

$$\psi^{(3,4)} = P_0 + \varepsilon\left\{-\kappa P_0\,\rho\cos\theta + P_2\left(\rho\sin\theta + \frac{1}{2}\frac{\sin\theta}{\rho} + \frac{1}{8}\frac{\sin 3\theta}{\rho^3}\right)\right\}$$

$$+\varepsilon^2\left\{\frac{\delta^2}{4}P_0\,\rho^2 - \frac{1}{2}\kappa P_2\left(\frac{\pi}{2} - \theta - \frac{1}{8}\frac{\sin 2\theta}{\rho^2} + \rho^2\sin 2\theta\right) + P_3 - P_4\,\rho^2\cos 2\theta\right\}$$

$$+\varepsilon^3\ln\varepsilon\, P_5\left(\rho\sin\theta + \frac{1}{2}\frac{\sin\theta}{\rho}\right)$$

$$+\varepsilon^3\left(\frac{\delta^2}{8}\left[-\kappa P_0\rho^3\cos\theta+P_2\left(\ln 2\rho\,\rho\sin\theta+\rho\cos\theta\,(\frac{\pi}{2}-\theta\right)+\rho^3\sin\theta\,)\right]\right.$$

$$+\frac{\kappa}{3}\,(P_4+\frac{\delta^2}{8}P_0)\,\rho^3\cos 3\theta-\kappa\,(P_3-VP_0)\,\rho\cos\theta+P_6\,(\rho\sin\theta+\frac{1}{2}\frac{\sin\theta}{\rho})$$

$$+P_7\,(\frac{3}{2}\rho\sin\theta+\frac{3}{8}\frac{\sin\theta}{\rho}-\rho^3\sin 3\theta\,)$$

$$\left.+P_8\left[\ln 2\rho\,\rho\sin\theta-\rho\cos\theta\,(\frac{\pi}{2}-\theta)+\frac{1}{4}\frac{\sin\theta}{\rho}-\rho^3\sin 3\theta\right]\right) \tag{4.3.48}$$

As in section 4.2, the outer expansion of $\psi^{(3)}$ can only match with the inner expansion of $\Psi^{(4)}$ if σ has the expansion

$$\sigma^{(4)}=\varepsilon^2\sigma_2+\varepsilon^3\ln\varepsilon\,\sigma_{31}+\varepsilon^3\sigma_3+\varepsilon^4\ln\varepsilon\,\sigma_{41}+\varepsilon^4\sigma_4\,. \tag{4.3.49}$$

By using the matching principle, $\Psi^{(2,3)}\equiv\psi^{(3,2)}$ we can easily prove that

$$\sigma_{31}=\sigma_3=0.$$

Therefore,

$$\sigma^{(4)}=\varepsilon^2\sigma_2+\varepsilon^4\ln\varepsilon\,\sigma_{41}+\varepsilon^4\sigma_4\,. \tag{4.3.50}$$

TO COMPUTE σ_{41} AND σ_4

To obtain the next two terms σ_{41} and σ_4 in the expansion of σ, we need to consider the matching $\Psi^{(4,3)} \equiv \psi^{(3,4)}$. The non-dipole singular terms in the outer expansion of $\psi^{(3)}$ imply that the outer solution must continue as

$$\Psi^{(4)} = h_0^{(4)} + \varepsilon^2 B_1 h_1^{(2)} + \varepsilon^4 \ln \varepsilon \, B_2 h_1^{(0)} + \varepsilon^4 \left\{ B_3 h_1^{(0)} + B_4 h_2^{(0)} \right\}. \tag{4.3.51}$$

Using the expansions of h_0 and h_1, the inner expansion of outer solution is

$$\Psi^{(4,3)} = e_{00,0} + \left\{ (e_{01,0} \cos\theta + f_{01,0} \sin\theta) \frac{1}{2} \delta \rho + \frac{2\sigma_2}{\delta} \frac{\sin\theta}{\rho} + \frac{8 B_1 \sigma_2}{\delta^3} \frac{\sin 3\theta}{\rho^3} \right\}$$

$$+ \varepsilon^2 \left\{ e_{00,0} + B_1 e_{10,0} + \frac{\delta^2 \rho^2}{4} e_{00,0} + \frac{2 \kappa \sigma_2}{\delta} \theta + \frac{4 B_1 \sigma_2 \kappa}{\delta^3} \frac{\sin 2\theta}{\rho^2} \right.$$

$$\left. + (e_{02,0} \cos 2\theta + f_{02,0} \sin 2\theta) \frac{\delta^2 \rho^2}{8} \right\} + \varepsilon^3 \ln \varepsilon \left\{ -\delta \sigma_2 \cosh 2\vartheta \, \rho \sin\theta + \frac{2\sigma_{41}}{\delta} \frac{\sin\theta}{\rho} \right\}$$

$$+ \varepsilon^3 \left((e_{01,2} \cos\theta + f_{01,2} \sin\theta) \frac{\delta \rho}{2} + (2\sigma_4 + B_1 \sigma_2) \frac{\sin\theta}{\delta \rho} \right.$$

$$+ (e_{01,0} \cos\theta + f_{01,0} \sin\theta) \frac{1}{16} \delta^3 \rho^3 + (e_{03,0} \cos 3\theta + f_{03,0} \sin 3\theta) \frac{1}{48} \delta^3 \rho^3$$

$$- \sigma_2 \delta \cosh 2\vartheta \ln \delta\rho \, \rho \sin\theta + \sigma_2 b_1 \delta \rho \sin\theta - \frac{2\kappa^2 \sigma_2}{\delta} \theta \rho \cos\theta$$

$$+ B_1 \frac{\delta}{2} (e_{11,0} \rho \cos\theta + f_{11,0} \rho \sin\theta) \Big\} \tag{4.3.52}$$

Match (4.3.48) with (4.3.52) gives,

$$\varepsilon \frac{\sin 3\theta}{\rho^3} : \ P_2 = \frac{64 B_1}{\delta^3} \sigma_2, \tag{4.3.53}$$

$$\varepsilon^3 \ln\varepsilon \, \rho \sin\theta : \ P_5 = - \sigma_2 \, \delta \, \cosh 2\hat{\nu}, \tag{4.3.54}$$

$$\varepsilon^3 \ln\varepsilon \frac{\sin\theta}{\rho} : \ P_5 = \frac{4 \sigma_{41}}{\delta}, \tag{4.3.55}$$

$$\varepsilon^3 \rho^3 \sin 3\theta : \ P_7 = - \frac{1}{48} \delta^3 f_{03,0}, \tag{4.3.56}$$

$$\varepsilon^3 \rho \cos\theta \, \theta : \ P_8 = - \sigma_2 \, \delta \cosh 2\hat{\nu} - \frac{\delta^2}{8} P_2, \tag{4.3.57}$$

$$\varepsilon^3 \rho \sin\theta : \ P_6 = \frac{1}{2} \delta f_{01,2} - \sigma_2 \, \delta \ln\delta \cosh 2\hat{\nu} + \sigma_2 \, \delta \, b_1,$$

$$+ \frac{1}{2} \delta f_{11,0} B_1 - \frac{1}{16} \delta^2 P_2 - \frac{3}{2} P_7 - \ln 2 \, (\frac{\delta^2}{8} P_2 + P_8), \tag{4.3.58}$$

and

$$\varepsilon^3 \frac{\sin\theta}{\rho} : \ \frac{P_6}{2} + \frac{3 P_7}{8} + \frac{P_8}{4} = \frac{1}{\delta} (2 \sigma_4 + B_1 \sigma_2) \tag{4.3.59}$$

from (4.3.36) and (4.3.53),

$$B_1 = \frac{\delta^2}{16}. \tag{4.3.60}$$

From equations (4.3.54) and (4.3.55) give,

$$\sigma_{41} = -\frac{\sigma_2 \delta^2}{4} \cosh 2\hat{v} = -\frac{1}{32} \delta^4 f_{01,0} \cosh 2\hat{v} \tag{4.3.61}$$

is a second approximation of O($\varepsilon^4 \ln \varepsilon$) for σ. Substitute for P_6, P_7, P_8, and B_1 in the equation (4.3.59) gives,

$$\sigma_4 = \frac{1}{2} \delta \left(\frac{1}{4} \delta f_{01,2} + \frac{1}{64} \delta^3 f_{11,0} + \frac{1}{128} \delta^3 f_{03,0} \right.$$

$$\left. + \frac{1}{2} \delta \sigma_2 \left[b_1 - (\frac{1}{2} + \ln \frac{\delta}{2}) \cosh 2\hat{v} - \frac{5}{8} \right] \right). \tag{4.3.62}$$

The coefficient f_{01} is the coefficient of $I_1 (p\, r) \sin\theta$ in the expansion of h_0 defined through (4.3.7-8). Using (4.2.38) it follows from (A4.1.32), (A4.2.3) and (A4.2.15) that

$$f_{01} = \frac{2\alpha}{p} \Delta_{00}^{(f)} (-1)^{M+1} (\cos 2\alpha x_0 - e^{2i\alpha p}) \cosh kh + \frac{\sin\sigma}{\delta^2} (S_1 + c_1) \tag{4.3.63}$$

where

$$S_1 = 2\pi \sum_{n=1}^{\infty} \frac{\alpha_m h \cos^2 k_m h}{N_m^2 \sinh 2\alpha_m h} \left(\cosh 2\alpha_m x_0 - e^{-2\alpha_m b} \right), \tag{4.3.64}$$

$$c_1 = -2h^2 \int_\Omega \left\{ \left(\frac{e^{\beta h}}{K \cosh \beta h - \beta \sinh \beta h} - \frac{2}{K - \beta} \right) + \frac{e^{-\beta h}}{K \cosh \beta h - \beta \sinh \beta h} \right\} t^2 \, dt$$

$$+ 2 \pi i \left\{ \frac{\alpha h \cosh^2 kh}{N_0^{\,2}} - \delta^2 \sinh 2\vartheta \right\} \tag{4.3.65}$$

and

$$\beta = (p^2 + t^2)^{\frac{1}{2}}.$$

Expanding f_{01} using (3.4.6-7) gives

$$f_{01,2}^{\;(2)} = f_{01,0} + \varepsilon^2 f_{01,2}\,, \tag{4.3.66}$$

where

$$f_{01,0} = \frac{2\,\alpha_M}{p} \Delta_{00,0}^{\;(f)} \left(1 - (-1)^M \cos 2\alpha_M x_0 \right) \cosh k_M h\,, \tag{4.3.67}$$

and

$$f_{01,2} = - f_{01,0}\, \sigma_2 \left\{ \frac{\alpha_M}{4k_M^{\,2}\, b\, N_{0,M}^{\;2}} \left(2K_M\, h + \frac{\sinh 2k_M h}{2\, k_M h} \right) \right.$$

$$\left. + \frac{x_0 (-1)^M \sin 2\alpha_M x_0 + ib}{b \left(1 - (-1)^M \cos 2\alpha_M x_0 \right)} \right\} + \frac{\sigma_2}{\delta^2} \left(S_{1,M} + c_{1,M} \right) \tag{4.3.68}$$

where $N_{0,M}$, $S_{1,M}$ and $c_{1,M}$ are now evaluated at $K = K_M$. Also

$$f_{11,0} = \frac{2\, k_M^{\;2}}{p^2 \cosh^2 k_M h}\, f_{01,0}\,,\quad f_{03,0} = \left(3 + \frac{4\alpha_M^{\;2}}{p^2} \right) f_{01,0}. \tag{4.3.69}$$

Gathering all of these results together in (4.3.50) gives the expansion of σ to O(ε^4) as

$$\sigma^{(4)} = \varepsilon^2\, \sigma_2 \Bigg[1 - \varepsilon^2 \ln \varepsilon\, \frac{\delta^2}{4} \cosh 2\hat{v}$$

$$+ \varepsilon^2 \Bigg\{ - \frac{\sigma_2\, \alpha_M}{4\, k_M^{\;2}\, b\, N_{0,M}^{\;2}} \left(2\, K_M h + \frac{\sinh 2k_M h}{2k_M h} \right)$$

$$- \frac{\pi x_0 \alpha_M h \cosh^2 k_M h}{4b\, N_{0,M}^{\;2}} (-1)^M \sin 2\alpha_M x_0$$

$$+ \frac{1}{8} \left(S_{1,M} + \mathrm{Re}_j\, c_{1,M} + \frac{k_M^{\;2}\, h^2}{\cosh^2 k_M h} + \frac{1}{2}\, \alpha_M^{\;2}\, h^2 \right)$$

$$- \frac{1}{4}\, p^2 h^2 \left(\left(\frac{-1}{2} + \gamma + \ln \frac{ph}{4} \right) \cosh 2\hat{v} + \hat{v} \sinh 2\hat{v} + \frac{1}{4} \right) \Bigg\} \Bigg] \tag{4.3.70}$$

where

$$\sigma_2 = \frac{\pi\, \alpha_M h}{4\, N_{0,M}^{\;2}} \cosh^2 k_M h \left(1 - (-1)^M \cos 2\alpha_M x_0 \right), \tag{4.3.71}$$

$\cosh \vartheta = \dfrac{K_M}{p}$ and γ is Euler's constant.

APPENDIX 4.1: Submerged multipole potentials

(a) Construction

The aim is to construct solutions of the modified Helmholtz equation that are singular at $(X,Y)=(x-x_0, y-y_0)$ and satisfy all of the conditions of the problem, equations (3.1.5-6) and (3.1.9), except for the condition on the body contour C. The construction is carried out in three stages. (i) Integral representations are obtained for the fundamental singular solutions of the modified Helmholtz equation, (ii) non-singular terms are added to satisfy the free-surface and bed conditions and (iii) further non-singular terms are added to satisfy the conditions on the vertical walls.

(i) Integral representations of fundamental singularities

From Twersky (1962, equation (31)), for $Y>0$

$$H_n(kr)i^n e^{in\theta} = \frac{2}{\pi i}\int_0^\infty \frac{e^{-(t^2-k^2)^{1/2}Y}\cos\left(tX + n\sin^{-1}(t/k)\right)}{(t^2-k^2)^{1/2}}\,dt \tag{A4.1.1}$$

where H_n denotes the Hankel function of the first kind and order n. The substitution $k=ip$ gives

$$K_n(pr)e^{in\theta} = \int_0^\infty \frac{e^{-\beta Y}\cos(tX - in\mu)}{\beta}\,dt \tag{A4.1.2}$$

where

$$\beta = (p^2+t^2)^{1/2} \tag{A4.1.3}$$

and μ is defined by

$$\sinh\mu = t/p \quad \text{and} \quad \cosh\mu = \beta/p. \tag{A4.1.4}$$

Now separate real and imaginary parts, and extend the definition to $Y<0$ by making use of the relevent symmetry or antisymmetry of each multipole, to obtain

$$K_n(pr)\cos n\theta = (\ \mathrm{sgn}\ Y)^n \int_0^\infty \frac{e^{-\beta|Y|}\cos tX\cosh n\mu}{\beta}\,dt, \quad n=0,1,2... \tag{A4.1.5}$$

and

$$K_n(pr)\sin n\theta = (\ \mathrm{sgn}\ Y)^{n+1} \int_0^\infty \frac{e^{-\beta|Y|}\sin tX\sinh n\mu}{\beta}\,dt, \quad n=1,2,3.... \tag{A4.1.6}$$

The singularities in equations (A4.1.5) and (A4.1.6) will be referred to as symmetric and antisymmetric (about $X = 0$) respectively.

(ii) *Free-surface and bed conditions*

To construct symmetric multipoles satisfying the free-surface and bed conditions write

$$\phi_n = K_n(pr)\cos n\theta + \int_0^\infty (A(t)\sinh\beta y + B(t)\cosh\beta y)\,\frac{\cos tX \cosh n\mu}{\beta}\,dt. \qquad (A4.1.7)$$

Substitution into the free-surface condition (3.1.6) and the zero flow condition on $y = h$, and making use of the integral representation (A4.1.5) for the singular part, gives simultaneous equations for A and B which when solved yield

$$\begin{aligned}\phi_n =& K_n(pr)\cos n\theta + \fint_0^\infty \Big\{ e^{-\beta(h-y_0)}(K\sinh\beta y - \beta\cosh\beta y) \\ &- (-1)^n(K+\beta)e^{-\beta y_0}\cosh\beta(h-y)\Big\}\frac{\cos tX\cosh n\mu}{(K\cosh\beta h - \beta\sinh\beta h)\beta}\,dt. (A4.1.8)\end{aligned}$$

There are poles of the integrand corresponding to the roots of

$$K = \beta\tanh\beta h. \qquad (A4.1.9)$$

Let k be the real positive root of $(A4.1.9)$ then the corresponding pole is at $t = (k^2-p^2)^{1/2}$ which lies on the path of integration for $k > p$. The path of integration is chosen to run beneath this pole in order to give outgoing waves at large distances. If $k < p$ there is no pole on the integration path and the multipoles are non-radiating. These non-radiating multipoles for infinite depth were used by Ursell (1951) to construct trapped wave solutions in the presence of a submerged horizontal cylinder.

Alternative forms for ϕ_n follow from replacing $K_n(pr)\cos n\theta$ by the integral representation (A4.1.5). For $y > y_0$ the result is

$$\phi_n = \fint_0^\infty \frac{(K-\beta)e^{\beta y_0} - (-1)^n(K+\beta)e^{-\beta y_0}}{(K\cosh\beta h - \beta\sinh\beta h)\beta}\cosh\beta(h-y)\cos tX\cosh n\mu\,dt \quad (A4.1.10)$$

and for $y < y_0$

$$\phi_n = \fint_0^\infty \frac{e^{-\beta(h-y_0)} + (-1)^n e^{\beta(h-y_0)}}{(K\cosh\beta h - \beta\sinh\beta h)\beta}(K\sinh\beta y - \beta\cosh\beta y)\cos tX\cosh n\mu\,dt. \qquad (A4.1.11)$$

For $n = 0$ equations (A4.1.10-11) are the results for the source solution of the modified Helmholtz equation given by MacCamy (1957).

Following a standard procedure (see, for example, Mei, 1983, pp. 380) the multipole expansions may be expressed as eigenfunction expansions. Thus (A4.1.10) is rewritten as

$$\phi_n = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{(K-\beta)e^{\beta y_0} - (-1)^n (K+\beta)e^{\beta y_0}}{(K\cosh\beta h - \beta\sinh\beta h)\beta} \cosh\beta(h-y)\, e^{it|X|} \cosh n\mu \, dt \quad (A4.1.12)$$

where now the path of integration runs below the pole at $t = (k^2 - p^2)^{1/2}$ and above that at $t = -(k^2 - p^2)^{1/2}$ when $k > p$. This integral may be evaluated using the residue theorem. There are further poles on the imaginary axis in the t-plane corresponding to the imaginary roots of (A4.1.9) denoted by $\beta = \pm ik_m$, $m = 1, 2, 3...$ giving poles at

$$t = \pm i(k_m^2 + p^2)^{1/2} = \pm i\alpha_m. \quad (A4.1.13)$$

There are also branch points at $t = \pm ip$ and suitable branch cuts must be inserted that do not cross the $\Re t$-axis, but these do not cause any difficulties. Evaluating the integral with the aid of the closing semi-circular contours described by Mei, modified to circumnavigate the relevant branch cuts, yields

$$\phi_n = \sum_{m=0}^{\infty} \Gamma_{nm} \cos k_m(h-y)\, e^{-\alpha_m|X|}, \quad (A4.1.14)$$

where

$$\Gamma_{nm} = \frac{\pi}{2\alpha_m h N_m^2}\left(e^{-ik_m(h-y_0)} + (-1)^n e^{ik_m(h-y_0)}\right)\cosh n\nu_m, \quad (A4.1.15)$$

$$N_m^2 = \tfrac{1}{2}\left(1 + \frac{\sin 2k_m h}{2k_m h}\right), \quad (A4.1.16)$$

$$k_0 = -ik, \quad \alpha_0 = -i\alpha = -i(p^2 - k^2)^{1/2} \quad (A4.1.17)$$

and ν_m is defined by

$$\sinh\nu_m = \frac{i\alpha_m}{p}, \quad \cosh\nu_m = \frac{ik_m}{p}. \quad (A4.1.18)$$

Equation (A4.1.14) is valid throughout the fluid, both (A4.1.10) and (A4.1.11) yield the same eigenfunction expansion (A4.1.14).

Similar calculations may be carried out for multipoles ψ_n that are antisymmetric in X. The form, equivalent to (A4.1.8), explicitly displaying the singularity is

$$\psi_n = K_n(pr)\sin n\theta + \int_0^\infty \Big\{ e^{-\beta(h-y_0)}(K\sinh\beta y - \beta\cosh\beta y) + (-1)^n(K+\beta)e^{-\beta y_0}\cosh\beta(h-y)\Big\} \frac{\sin tX \sinh n\mu}{(K\cosh\beta h - \beta\sinh\beta h)\beta}\, dt. \quad (A4.1.19)$$

As in (A4.1.10-11), the singular part may be incorporated into the integral using (A4.1.6). The eigenfunction expansion representation is

$$\psi_n = \operatorname{sgn} X \sum_{m=0}^{\infty} \Delta_{nm}\cos k_m(h-y)e^{-\alpha_m|X|}, \quad (A4.1.20)$$

where

$$\Delta_{nm} = \frac{\pi}{2i\alpha_m h N_m^2}\left(e^{-ik_m(h-y_0)} - (-1)^n e^{ik_m(h-y_0)}\right)\sinh n\nu_m, \quad (A4.1.21)$$

For $k > p$, the first terms in the series (A4.1.14) and (A4.1.20) give the propagating waves generated by the singularities at large distances.

(iii) *Side-wall conditions*

To obtain multipole potentials appropriate to a closed basin, that is having zero x-derivative on $x = \pm b$, a similar strategy to that used in (A4.1.7) is adopted. Write

$$\phi_n^{(b)} = \phi_n + \sum_{m=0}^{\infty}\Gamma_{nm}\cos k_m(h-y)\left\{A_m\cosh\alpha_m(x-x_0) + B_m\sinh\alpha_m(x-x_0)\right\}, \quad (A4.1.22)$$

using the eigenfunction representation (A4.1.14) for ϕ_n, and apply the boundary conditions on $x = \pm b$ to determine the unknown coefficients. The resulting multipole potentials are

$$\phi_n^{(b)} = \phi_n + \sum_{m=0}^{\infty}\frac{\Gamma_{nm}\cos k_m(h-y)}{\sinh 2\alpha_m b}\Big\{\left(\cosh 2\alpha_m x_0 + e^{-2\alpha_m b}\right)\cosh\alpha_m(x-x_0) + \sinh 2\alpha_m x_0 \sinh\alpha_m(x-x_0)\Big\} \quad (A4.1.23)$$

and the corresponding result for the antisymmetric multipoles is

$$\psi_n^{(b)} = \psi_n + \sum_{m=0}^{\infty}\frac{\Delta_{nm}\cos k_m(h-y)}{\sinh 2\alpha_m b}\Big\{\sinh 2\alpha_m x_0\cosh\alpha_m(x-x_0) + \left(\cosh 2\alpha_m x_0 - e^{-2\alpha_m b}\right)\sinh\alpha_m(x-x_0)\Big\}. \quad (A4.1.24)$$

(b) Expansion about singular point

The generating function for the modified Bessel functions I_q is

$$e^{\frac{1}{2}Z(T+T^{-1})} = \sum_{q=-\infty}^{\infty} T^q I_q(Z). \qquad (A4.1.25)$$

The substitutions $Z = pr$ and $T = \pm\exp(\mu + i\theta)$, where μ is defined in (A4.1.4), give

$$e^{\pm(\beta Y + itX)} = \sum_{q=0}^{\infty} \epsilon_q\,(\pm 1)^q \cosh q(\mu + i\theta)\, I_q(pr) \qquad (A4.1.26)$$

where $\epsilon_0 = 1$ and $\epsilon_q = 2,\ q \geq 1$. Equation (A4.1.26) may be used to expand the integral terms in (A4.1.8) and (A4.1.19) to obtain

$$\phi_n - K_n(pr)\cos n\theta = \tfrac{1}{2}\sum_{q=0}^{\infty} \epsilon_q\, I_q(pr)\cos q\theta \,⨍_0^{\infty} \Big\{ K\left(e^{2\beta y_0} - (-1)^q\right) - \beta\left(e^{2\beta y_0} + (-1)^q\right)$$

$$-(-1)^n (K+\beta)\left(1 + (-1)^q e^{2\beta(h-y_0)}\right)\Big\} \frac{e^{-\beta h}\cosh n\mu \cosh q\mu}{(K\cosh\beta h - \beta\sinh\beta h)\beta}\, dt \qquad (A4.1.27)$$

and

$$\psi_n - K_n(pr)\sin n\theta = \sum_{q=1}^{\infty} I_q(pr)\sin q\theta \,⨍_0^{\infty} \Big\{ K\left(e^{2\beta y_0} + (-1)^q\right) - \beta\left(e^{2\beta y_0} - (-1)^q\right)$$

$$+(-1)^n (K+\beta)\left(1 - (-1)^q e^{2\beta(h-y_0)}\right)\Big\} \frac{e^{-\beta h}\sinh n\mu \sinh q\mu}{(K\cosh\beta h - \beta\sinh\beta h)\beta}\, dt. \qquad (A4.1.28)$$

These expansions are valid for $0 < r < 2y_0$. As $h \to \infty$ in (A4.1.27) the result of Ursell (1951, equation (11)) is recovered.

To expand the summation terms in (A4.1.23) and (A4.1.24) a modification of the result (A4.1.26) is needed. Replace t by $i(k_m^2 + p^2)^{1/2} = i\alpha_m$ to give

$$e^{\pm(ik_m Y - \alpha_m X)} = \sum_{q=0}^{\infty} \epsilon_q\,(\pm 1)^q \cosh q(\nu_m + i\theta) I_q(pr) \qquad (A4.1.29)$$

where ν_m is defined by

$$\sinh\nu_m = i\alpha_m/p \quad \text{and} \quad \cosh\nu_m = ik_m/p. \qquad (A4.1.30)$$

Thus

$$\phi_n^{(b)} - \phi_n = \tfrac{1}{2}\sum_{q=0}^{\infty} \epsilon_q\, I_q(pr)$$

$$\times \left\{ \cos q\theta \sum_{m=0}^{\infty} \frac{\Gamma_{nm}\left(\cosh 2\alpha_m x_0 + e^{-2\alpha_m b}\right)}{\sinh 2\alpha_m b} \left((-1)^q e^{ik_m(h-y_0)} + e^{-ik_m(h-y_0)}\right) \cosh q\nu_m \right.$$

$$\left. + i \sin q\theta \sum_{m=0}^{\infty} \frac{\Gamma_{nm} \sinh 2\alpha_m x_0}{\sinh 2\alpha_m b} \left((-1)^q e^{ik_m(h-y_0)} - e^{-ik_m(h-y_0)}\right) \sinh q\nu_m \right\} \quad (A4.1.31)$$

and

$$\psi_n^{(b)} - \psi_n = \tfrac{1}{2}\sum_{q=0}^{\infty} \epsilon_q\, I_q(pr)$$

$$\times \left\{ \cos q\theta \sum_{m=0}^{\infty} \frac{\Delta_{nm} \sinh 2\alpha_m x_0}{\sinh 2\alpha_m b} \left((-1)^q e^{ik_m(h-y_0)} + e^{-ik_m(h-y_0)}\right) \cosh q\nu_m \right.$$

$$\left. + i \sin q\theta \sum_{m=0}^{\infty} \frac{\Delta_{nm}\left(\cosh 2\alpha_m x_0 - e^{-2\alpha_m b}\right)}{\sinh 2\alpha_m b} \left((-1)^q e^{ik_m(h-y_0)} - e^{-ik_m(h-y_0)}\right) \sinh q\nu_m \right\}. \quad (A4.1.32)$$

APPENDIX 4.2: Free-surface multipole potentials

(a) Construction

The main results for the source and horizontal dipole may be deduced from the results on submerged multipoles in appendix 4.1. From (A4.1.10) the free-surface source satisfying both the free-surface and bed conditions is

$$\phi_0^{(f)} = \phi_0\big|_{y_0=0} = -2\int_0^\infty \frac{\cosh\beta(h-y)\cos tX}{K\cosh\beta h - \beta\sinh\beta h}\,dt \tag{A4.2.1}$$

and the corresponding free-surface horizontal dipole is then defined by

$$\psi_0^{(f)} = \frac{1}{p}\frac{\partial\phi_0^{(f)}}{\partial x_0}. \tag{A4.2.2}$$

It is conventional to take the definition of the free-surface source to be half that given in (A4.2.1), this has not been done here to enable direct use to be made of the results for submerged singularities. The eigenfunction representations and the series' to be added in order to satisfy the side-wall conditions follow immediately from (A4.1.14), (A4.1.20) and (A4.1.23-24). The series coefficients for the free-surface source and dipole are defined by

$$\Gamma_{0m}^{(f)} = \Gamma_{0m}\big|_{\{y_0=0\}}, \quad \Delta_{0m}^{(f)} = \Delta_{1m}\big|_{\{y_0=0\}}, \quad m = 0,1,2\ldots \tag{A4.2.3}$$

where Γ_{0m} and Δ_{1m} are given by (A4.1.15) and (A4.1.21).

Following Ursell (1968) two sets of wave-free potentials are also defined. The symmetric potentials for deep water are

$$\begin{aligned}\hat\phi_n^{(f)} &= K_{2n}(pr)\cos 2n\theta + \frac{2K}{p}K_{2n-1}(pr)\cos(2n-1)\theta \\ &\qquad + K_{2n-2}(pr)\cos(2n-2)\theta, \\ &= \frac{2}{p}\int_0^\infty \frac{K+\beta}{\beta}e^{-\beta y}\cos tX\cosh(2n-1)\mu\,dt, \quad n = 1,2,3\ldots\end{aligned} \tag{A4.2.4}$$

where the integral representation (A4.1.5) has been used. The potentials appropriate to finite depth are found in a similar fashion to the derivation of (A4.1.8) and the result is

$$\begin{aligned}\phi_n^{(f)} &= \hat\phi_n^{(f)} + \frac{2}{p}\int_0^\infty \frac{(K+\beta)e^{-\beta h}(K\sinh\beta y - \beta\cosh\beta y)}{(K\cosh\beta h - \beta\sinh\beta h)\beta}\cos tX\cosh(2n-1)\mu\,dt \\ &= \frac{2}{p}\int_0^\infty \frac{(K^2-\beta^2)\cosh\beta(h-y)}{(K\cosh\beta h - \beta\sinh\beta h)\beta}\cos tX\cosh(2n-1)\mu\,dt\end{aligned} \tag{A4.2.5}$$

which has an eigenfunction representation

$$\phi_n^{(f)} = \sum_{m=0}^{\infty} \Gamma_{nm}^{(f)} \cos k_m(h-y)\, e^{-\alpha_m |X|} \tag{A4.2.6}$$

where

$$\Gamma_{nm}^{(f)} = \frac{\pi i k_m \cosh(2n-1)\nu_m}{ph\alpha_m N_m^2 \cos k_m h}, \quad n = 1,2,3... \tag{A4.2.7}$$

and N_m and ν_m are defined by equations (A4.1.16) and (A4.1.18) respectively. The antisymmetric wave-free potentials for deep water are

$$\begin{aligned}\hat{\psi}_n^{(f)} &= K_{2n+1}(pr)\sin(2n+1)\theta + \frac{2K}{p} K_{2n}(pr)\sin 2n\theta \\ &\quad + K_{2n-1}(pr)\sin(2n-1)\theta, \\ &= \frac{2}{p}\int_0^{\infty} \frac{K+\beta}{\beta} e^{-\beta y} \sin tX \sinh 2n\mu\, dt, \quad n = 1,2,3... \end{aligned} \tag{A4.2.8}$$

where the integral representation (A4.1.6) has been used. The finite depth potentials are

$$\begin{aligned}\psi_n^{(f)} &= \hat{\psi}_n^{(f)} + \frac{2}{p} \int_0^{\infty} \frac{(K+\beta)e^{-\beta h}(K\sinh\beta y - \beta\cosh\beta y)}{(K\cosh\beta h - \beta\sinh\beta h)\beta} \sin tX \sinh 2n\mu\, dt \\ &= \frac{2}{p} \int_0^{\infty} \frac{(K^2-\beta^2)\cosh\beta(h-y)}{(K\cosh\beta h - \beta\sinh\beta h)\beta} \sin tX \cosh 2n\mu\, dt \end{aligned} \tag{A4.2.9}$$

which has an eigenfunction representation

$$\psi_n^{(f)} = \operatorname{sgn} X \sum_{m=0}^{\infty} \Delta_{nm}^{(f)} \cos k_m(h-y)\, e^{-\alpha_m |X|} \tag{A4.2.10}$$

where

$$\Delta_{nm}^{(f)} = \frac{\pi k_m \sinh 2n\nu_m}{ph\alpha_m N_m^2 \cos k_m h}, \quad n = 1,2,3\dots \tag{4.2.11}$$

The eigenfunction representations (A4.2.6) and (A4.2.10) are identical in form to those for the submerged multipoles given by (A4.1.14) and (A4.1.20). Hence, the additional series' required to satisfy the side-wall conditions follow immediately from (A4.1.23) and (A4.1.24).

(b) Expansion about singular point

In deep water, the free-surface source is

$$\hat{\phi}_0^{(f)} = \lim_{h\to\infty} \phi_0^{(f)} = -2\int_0^{\infty} \frac{e^{-\beta y}\cos tX}{K-\beta}\, dt \tag{A4.2.12}$$

and so

$$\phi_0^{(f)} - \hat{\phi}_0^{(f)} = -\int_0^\infty \left\{ \left(\frac{e^{\beta h}}{K\cosh\beta h - \beta\sinh\beta h} - \frac{2}{K-\beta} \right) e^{-\beta y} + \frac{e^{-\beta h}e^{\beta y}}{K\cosh\beta h - \beta\sinh\beta h} \right\} \cos tX\, dt \qquad (A4.2.13)$$

which may be expanded using (A4.1.26) to give

$$\phi_0^{(f)} - \hat{\phi}_0^{(f)} = -\sum_{q=0}^{\infty} \epsilon_q I_q(pr)\cos q\theta \int_0^\infty \left\{ (-1)^q \left(\frac{e^{\beta h}}{K\cosh\beta h - \beta\sinh\beta h} - \frac{2}{K-\beta} \right) + \frac{e^{-\beta h}}{K\cosh\beta h - \beta\sinh\beta h} \right\} \cosh q\mu\, dt. \quad (A4.1.14)$$

The corresponding expression for the free-surface dipole follows from (A4.2.2) and is

$$\psi_0^{(f)} - \hat{\psi}_0^{(f)} = \tfrac{1}{2}\sum_{q=0}^{\infty} \epsilon_q \left(I_{q+1}(pr)\sin(q+1)\theta - I_{q-1}(pr)\sin(q-1)\theta \right) \times \int_0^\infty \left\{ (-1)^q \left(\frac{e^{\beta h}}{K\cosh\beta h - \beta\sinh\beta h} - \frac{2}{K-\beta} \right) + \frac{e^{-\beta h}}{K\cosh\beta h - \beta\sinh\beta h} \right\} \cosh q\mu\, dt. \qquad (A4.2.15)$$

The expansion about the singular point of the deep water source potential is given by Ursell (1962, pp.502); after a simple change of integration variable it may be seen that his Φ_0 is identical to $\hat{\phi}_0^{(f)}$. In particular

$$\hat{\phi}_0^{(f)} = -2\ln pr + 2a_1 - 2 + 2Kr\ln pr\cos\theta - 2a_1 Kr\cos\theta - 2Kr\,\theta\sin\theta - \tfrac{1}{2}(pr)^2\ln pr - (Kr)^2\ln pr\cos 2\theta + \tfrac{1}{2}a_1(pr)^2 + a_2(pr)^2\cos 2\theta + (Kr)^2\,\theta\sin 2\theta + O\left((pr)^3\ln pr\right) \qquad (A4.2.16)$$

where

$$a_1 = \ln 2 - \gamma + 1 + (\pi i - \hat{\nu})\coth\hat{\nu},$$

$$a_2 = (\ln 2 - \gamma + 3/2)\cosh^2\hat{\nu} + \tfrac{1}{2}(\pi i - \hat{\nu})\coth\hat{\nu}\cosh 2\hat{\nu}, \qquad (A4.2.17)$$

γ is Euler's constant and $\hat{\nu}$ is defined by

$$\cosh\hat{\nu} = K/p. \qquad (4.2.18)$$

Using the definition of the horizontal dipole, equation (A4.2.2), the expansion of $\hat{\psi}_0^{(f)}$ may be calculated from (A4.2.16) by differentiation as

$$\begin{aligned}\hat{\psi}_0^{(f)} =& 2\frac{\sin\theta}{pr} + 2\theta\cosh\hat{\nu} - pr\ln pr\sin\theta\cosh 2\hat{\nu} \\ &+ b_1 pr\sin\theta - 2pr\,\theta\cos\theta\cosh^2\hat{\nu} + O\left((pr)^2\ln pr\right) \qquad (A4.2.19)\end{aligned}$$

where

$$\begin{aligned}b_1 &= 1/2 - a_1 + 2a_2 - \cosh^2\hat{\nu} \\ &= 1/2 + (\ln 2 - \gamma + 1)\cosh 2\hat{\nu} + (\pi i - \hat{\nu})\sinh 2\hat{\nu}. \qquad (A4.2.20)\end{aligned}$$

CHAPTER 5

SLOSHING IN A RECTANGULAR TANK WITH INTERNAL BODIES: RESULTS

5.1 INTRODUCTION

In this chapter results are presented for the sloshing frequencies of fluid contained in a rectangular tank containing internal bodies. The boundary-value problem is defined by the modified Helmholtz equation (3.1.5), the linearized free-surface condition (3.1.6), and the condition of no flow through the solid boundaries (3.1.6-9). The eigenfrequencies are calculated by using various methods which have been described in chapters 3 and 4. My aim is to examine how these eigenfrequencies are influenced by the size, shape and position of the body. Here the bodies considered are vertical surface-piercing and bottom-mounted barriers, and circular and elliptic cylinders.

When a barrier is introduced only partly into the fluid it can be shown (Courant and Hilbert 1953) that the eigenfrequencies are decreased in general, except when the position of the barrier coincides with an antinode of an oscillation where the horizontal velocity is zero throughout the depth. The eigenfrequency then remains the same. As the barrier is introduced further and further into the fluid the nth eigenfrequency changes continuously from its corresponding value in the absence of the barrier to the closest eigenfrequency not greater than it corresponding to the two separate containers.

In section 3.2, the problem is solved for a vertical barrier by the eigenfunction expansion method and the sloshing frequencies were found to be the solutions of

$$\det (C - AB) = 0, \tag{5.1.1}$$

where

$$A = -\left(\tanh\left[\alpha\,(b - x_0)\right] + \tanh\left[\alpha\,(b + x_0)\right] \right),$$

$$C = cc^T \;,\; c^T = (\, c_{00}, c_{10}, c_{20}, \ldots, c_{m0}\,),\; c_{mn} = \langle \psi_m, \psi_n \rangle\,,$$

$$B = (\, B_{mn}\,)\,,\; B_{mp} = \sum_{n=1}^{\infty} c_{mn}\, c_{pn}\, s_n\,,$$

$$s_n = \alpha\, m_n^{-1} \left(\coth m_n(b - x_0) + \coth m_n(b + x_0) \right),$$

$$\psi_n(y) = N_n^{-1} \cos k_n(h - y),$$

k_n ($n = 1, 2 \ldots$) are the positive roots of the equation

$$K + k_n \tan k_n h = 0,$$

with $k_0 = ik$, and

$$2\, N_n^{\,2} = (\, h + \frac{\sin 2k_n h}{2k_n}\,)\,,$$

$$\alpha = (k^2 - p^2)^{\frac{1}{2}}, \quad p = \frac{N\pi}{l}\,,$$

$$m_n^{\,2} = p^2 + k_n^{\,2},$$

and the wave number k which is related to the frequency ω through

$$\frac{\omega^2}{g} = K = k \tanh kh \ .$$

In section 3.3, the body is taken to be arbitrary but of finite thickness and the motion is two-dimensional. By using the constant boundary element method, the original boundary-value problem becomes the solution of a matrix eigenvalue problem which is given by

$$(-\pi I + B + C)\,\phi_i = K\, D\, \phi_i,$$

where ϕ_i are the unknown nodal values of the potential,

$$B_{ij} = \int_{S_{B_j}} \frac{\partial G_{ij}}{\partial n}\, ds \ ,$$

$$C_{ij} = \int_{S_{F_j}} \frac{\partial G_{ij}}{\partial n}\, ds \ ,$$

$$D_{ij} = \int_{S_{F_j}} G_{ij}\, ds \ ,$$

and I is the identity matrix of order $N{\times}N$. Results are given for an elliptic cylinder whose equation is given by

$$\frac{x^2}{a^2} + \frac{y^2}{c^2} = 1, \tag{5.1.2}$$

where $2a$ and $2c$ are the lengths of the major and minor axes and the major axis is parallel to the tank bottom and the minor axis is fixed. This includes the special case of a circular cylinder.

In section 3.4, two approximate solutions are described, these are a "wide-spacing approximation" and a "small-body approximation". Here the body is either a vertical surface-piercing barrier or a submerged circular cylinder. By using the wide-spacing approximation, we obtained a general expression to determine the eigenfrequencies in terms of reflection and transmission coefficients for that body. For a circular cylinder, centred on (x_0 , y_0) with radius a and $y_0 >$ a,

$$\alpha = \frac{M\pi}{2b} - \frac{\pi k_M h}{2b\, N_{0,M}^{\;2}}\, \varepsilon^2 \left(\cosh\,[2k_M h(\frac{y_0}{h} + 1\,)] - (-1)^M \cos 2\alpha_M\, x_0 \right), \qquad (5.1.3)$$

where

$$\varepsilon = \frac{a}{h}\,, \quad \alpha_M = \frac{M\pi}{2b}\,, \qquad (5.1.4)$$

$$k_M = (\,\frac{M^2\pi^2}{4\,b^2} + p^2\,)^{\frac{1}{2}}\,, \qquad (5.1.5)$$

and

$$N_{0,M}^{\;2} = \frac{1}{4\,k_M h}\,(\,\sinh 2k_M h + 2k_M h\,)\,, \qquad (5.1.6)$$

and for a surface-piercing barrier

$$\alpha = \frac{M\pi}{2b} - \varepsilon^2\, \frac{M\pi^2 h}{16b^2\, N_{0,M}^{\;2}} \cosh^2 k_M h \left(1 - (-1)^M \cos 2\alpha_M x_0 \right). \qquad (5.1.7)$$

In the small-body approximation, by applying Green's theorem to the potentials with and without the body being present, explicit forms for the eigenfrequencies are obtained in terms of free-surface and body integrals. For a circular cylinder with two-dimensional wave motion, the eigenfrequency is given by

$$K = K_M - \frac{2\pi\, k_M^2 a^2}{b \cosh^2 k_M h} \Big(\sin^2 k_M(x_0 - b) \cosh^2 k_M(y_0 - h)$$

$$+ \cos^2 k_M(x_0 - b) \sinh^2 k_M(y_0 - h) \Big) \tag{5.1.8}$$

with $k_M = \dfrac{M\pi}{2b}$, and

$$K_M = k_M \tanh k_M h\,, \tag{5.1.9}$$

and for a thin vertical surface-piercing barrier, the eigenfrequency is given by

$$K = K_M - \frac{\pi\, \alpha_M a}{2bh}^2 \sin^2 k_M(x_0 - b)\,. \tag{5.1.10}$$

In section 4.2, the lowest-order solution was obtained in terms of the cross-sectional area and dipole strength of the body by using the method of matched asymptotic expansions for an arbitrary cross sectional body. Here a is small compare to wavelength and water depth. The lowest-order solution, σ_2 is given by

$$\sigma_2 = \frac{\pi h}{2\alpha_M {N_{0,M}}^2} \left(\frac{Sp^2}{\pi a^2} \left(1 + (-1)^M \cos 2\alpha_M x_0 \right) \cosh^2 k_M(h - y_0) \right.$$

$$+ 2\, \nu k_M^{\,2} \left(1 + (-1)^M \cos 2\alpha_M x_0 \right) \sinh^2 k_M (y - y_0)$$

$$- (\Upsilon - \lambda)\, \alpha_M k_M\, (-1)^M \sin 2\alpha_M x_0 \sinh 2k_M (h - y_0)$$

$$+ 2\Lambda\, \alpha_M^{\,2} \left(1 - (-1)^M \cos 2\alpha_M x_0 \right) \cosh^2 k_M (h - y_0) \Bigg) . \qquad (5.1.11)$$

Here S is a cross-sectional area of the body, ν, Υ, λ, and Λ are dipole strengths of the body and σ_2 is related to the frequency parameter K by the equation

$$K = K_M \left(1 - \varepsilon^2 V \right) , \qquad (5.1.12)$$

where

$$V = \frac{M \pi \sigma_2}{4 k_M^{\,2} b^2} \left(1 + \frac{k_M^{\,2} h}{K_M} - K_M h \right) . \qquad (5.1.13)$$

For a circular cylinder $S = \pi a^2$ and

$$\nu = \Lambda = 1 , \quad \lambda = \Upsilon = 0 .$$

When the body is elliptic cylinder whose equation is given by (5.1.2), the cross-sectional area is $S = \pi a c$. From Newman (1977, pp.144), the dipole strengths for an elliptic cylinder are given by

$$\nu = \frac{1}{2} \left(1 + \frac{c}{a} \right) , \quad \Lambda = \frac{c}{2a} \left(1 + \frac{c}{a} \right) , \qquad (5.1.14)$$

and

$$\Upsilon = \lambda = 0 \, . \tag{5.1.15}$$

The higher-order solution is very difficult for a body of arbitrary cross-section. Since first-order solution gives good agreement with the solution of boundary element method, the higher-order solution is not presented here.

In section 4.3, the body is a surface-piercing vertical barrier and the higher-order solution was obtained up to ε^4, that is

$$\sigma^{(4)} = \varepsilon^2 \sigma_2 \Bigg[1 + \varepsilon^2 \ln \varepsilon \, \sigma_{41}$$

$$+ \varepsilon^2 \Bigg\{ - \frac{\sigma_2 \alpha_M}{4 k_M^2 b N_{0,M}^2} \left(2K_M h + \frac{\sinh 2k_M h}{2k_M h} \right)$$

$$- \frac{\pi x_0 \alpha_M h \cosh^2 k_M h}{4 b N_{0,M}^2} (-1)^M \sin 2\alpha_M x_0$$

$$+ \frac{1}{8} \left(S_{1,M} + \mathrm{Re}_j \, c_{1,M} + \frac{k_M^2 h^2}{\cosh^2 k_M h} + \alpha_M^2 h^2 \right)$$

$$- \frac{p^2 h^2}{4} \left((\gamma + \ln \frac{ph}{4} - \frac{1}{2}) \cosh 2\vartheta + \vartheta \sinh 2\vartheta + \frac{1}{4} \right) \Bigg\} \Bigg] , \tag{5.1.16}$$

where

$$\sigma_2 = \frac{\pi \alpha_M h}{4 N_{0,M}^2} \cosh^2 k_M h \,(1 - (-1)^M \cos 2\alpha_M \, x_0) , \tag{5.1.17}$$

$$\sigma_{41} = -\frac{1}{4} \delta^2 \cosh 2\vartheta \, \sigma_2 , \tag{5.1.18}$$

$$S_{1,M} = 2\pi \sum_{n=1}^{\infty} \frac{\alpha_n h \cos^2 k_n h}{N_{n,M}^2 \sin 2\alpha_n b} \,(\cosh 2\alpha_n x_0 - e^{-2\alpha_n b}) , \tag{5.1.19}$$

$$c_{1,M} = -2h^2 \int_0^{\infty} \left\{ \left(\frac{e^{\beta h}}{K_M - \beta \sinh \beta h} - \frac{2}{K_M - \beta} \right) + \frac{e^{-\beta h}}{K_M \cosh \beta h - \beta \sinh \beta h} \right\} t^2 \, dt$$

$$+ 2\pi i \left\{ \frac{\alpha_M h \cosh^2 kh}{N_{0,M}^2} - \delta^2 \sinh 2\vartheta \right\} , \tag{5.1.20}$$

$$\beta = (p^2 + t^2)^{\frac{1}{2}} , \quad \delta = ph ,$$

$$\cosh \vartheta = \frac{K_M}{p} ,$$

and γ is Euler's constant, and $\mathrm{Re}_i \, c_{1,M}$ is a real part of $c_{1,M}$. This solution is based on a rational approximation for

$$\sigma = \varepsilon^2 \sigma_{22} \frac{1 + A\varepsilon^2 \ln \varepsilon + B\varepsilon^2}{1 + C\varepsilon^2 \ln \varepsilon + D\varepsilon^2} \approx \varepsilon^2 \sigma_2 \,(1 + (A - C)\varepsilon^2 \ln \varepsilon + (B - D)\varepsilon^2)$$

which gives the expansion for σ provided

$$A - C = \frac{\sigma_{41}}{\sigma_2}, \quad B - D = \frac{\sigma_4}{\sigma_2}.$$

The best results were obtained by choosing

$$A = B = 0 .$$

In section 5.2 results are calculated using the above methods are presented and, where appropriate, comparisons made between exact and approximate solutions.

5.2 Results

Here the following variables are used: a is a cross-sectional length of the body, h_0 is a submergence of cylinder, h is a water depth, and $p = N\pi/l$, where N is an integer and l is the width of the tank in the z – direction. The results are presented as curves of $kd(d = 2b)$ against a/d for different values of h/d, and pd using different methods; the eigenfunction expansion method, small-body and wide-spacing approximations, and the method of matched asymptotic expansions. The aim is to examine how the sloshing frequencies are influenced by the presence of the body. The bodies considered here are the vertical surface-piercing and bottom-mounted barrier, the circular cylinder and the elliptic cylinder. When bodies are either circular or elliptic cylinder the two-dimensional motion is considered.

5.2.1 THE VERTICAL BARRIER

Figures 5.1-5.2, show the values of kd corresponding to the lowest six modes for a surface-piercing barrier placed centrally in the tank for $pd =$ 1 and 3. The symmetric modes have zero horizontal velocity on the centre and so are unaffected by the barrier. The values of kd for the lowest mode is slowly reduced as the submergence of the barrier is increased, until the tank is almost fully divided when there is a very rapid drop to its lowest value which is given in the table 5.2.1. Also the higher-frequency modes are reduced to their lower limiting values given in the table 5.2.1. This gives a numerical confirmation of Courant and Hilbert's result. For these higher modes, the eigenfrequencies decrease much more rapidly than lower eigenfrequencies as the submergence increases. We would expect this given that these shorter waves do not penetrate deeply.

A comparison of the exact method with the lowest-order approximation of equation (5.1.7) is made on the same figures. For the lowest mode both methods agree well for values of $a/d <$ 0.15. For the higher modes, the results do not agree well as we assumed that the body is small compared to the wavelength in the approximate method.

In figures 5.1-2, a comparison of the exact method with the lowest and higher-order approximation of equation (5.1.16) is made when the barrier is centrally placed in the tank. Both methods agree well in the case of the lowest mode. But for higher modes there is good agreement only for the values of $a/d \leq 0.2$, because the assumption is made that the length scale of the motion is much larger than the body dimension. Also we can notice that the higher-order approximation gives better agreement than the first-order approximation.

When barrier is not centrally placed in the tank the results are displayed in figures 5.3-4. A comparison of the exact method with the lowest and higher-order approximations is made on the same figures. Again there will be some unaffected modes. As before the higher-order approximation gives better agreement than lowest mode. For higher modes the agreement is good only for values of $a/d \leq 0.15$. The reasons are given above.

In the case of a bottom-mounted barrier, the behaviour differs from the surface-piercing barrier, as can be seen from figures 5.5-5.8. Note that a is here the barrier length and that the horizontal axis is reversed so that the fully-divided tank again lies on the right of the figures. In this case the barrier must extend over a substantial part of the depth before there is a significant change in the eigenfrequencies. The higher modes are less affected by the barrier as we would expect for shorter wavelength. Good approximate solutions for the bottom-mounted barrier were found by Evans and McIver (1987).

So far all the calculations reported have been for a depth to width ratio of unity. Figures 5.9-10 give kd vs a/d for an off-centre surface-piercing barrier in a shallower tank with $h/d = 0.5$. The effects of the reduced depth do not became significant change until the gap is small. As before, the higher-order approximation gives good agreement for the lowest mode. In the case of bottom-mounted barrier which is not centrally in a shallower tank the graph is drawn for different values of kd vs $0.5 - a/d$ (see figures 5.11-12). Since $h/d = 0.5$, the range of values of $0.5 - a/d$ is chosen up to 0.5. Now that the bottom-mounted barrier is extended over a larger proportion of the depth it will produce bigger changes in kd. From above figures the changes of values of kd decreases with increasing values of pd.

5.2.2 A CIRCULAR CYLINDER

Figures 5.13-16 show the results for the eigenfrequency parameter Kb plotted against a/b for a circular cylinder with radius a placed centrally in the tank. Here the submerged depth of the cylinder, $h_0/b = 0.25$ and 0.5, and 0.5 and 1.0 for fixed depth of water, $h/b = 1.0$ and 2.0 respectively. In figures, 5.13 and 5.14 the maximum values of a/b are 0.2 and 0.4 respectively, otherwise the cylinder will overlap the walls or free surface. For the lowest mode, there can be a considerable change in eigenfrequencies with increasing a/b. But for higher modes there is usually only a small change in eigenfrequencies. A comparison is made with the approximation of equation (5.1.3) and good agreement is found for all modes. When the submergence increases for fixed depth, there is not good agreement in the case of higher modes for some range of values of a/b. The reason is that in this range the small-body assumption is not valid; the body is then not small compared with tank dimensions or the wavelength.

Figure 5.17-18 show the results for eigenfrequencies for a circular cylinder not centrally placed in the tank. The submergence depths, $h_0/b = 0.25$ and 0.5 and depths of water, $h/b = 1.0$ and 2.0 are considered. The range of values of a/b are 0.2 and 0.4, otherwise the cylinder will overlap the boundaries of the tank. As before, there is considerable change in eigenfrequencies when a/b increases. A comparison is made with the approximate solution and good agreement found, even when the cylinder is not in the middle and for all modes. The reason is given above. From these figures the comparison is good whether the cylinder is in the middle or not, so a higher-order approximation is not pursued.

5.2.3 A ELLIPTIC CYLINDER

Now the body is an elliptic cylinder and its equation is given by (5.1.2). In the results presented the major axis is parallel to the bottom of the tank and the minor axis is held fixed. Figures 5.19-20, shows the results for eigenfrequencies for an elliptic cylinder placed centrally in the tank. From the graphs some of modes are very little affected by the cylinder. A comparison is made with the approximate solution, equation (5.1.11) and there is reasonable agreement for some of the modes. For values of $a/b \geq 0.35$, there is not good agreement except the lowest mode. But in figure 5.20, they agree well for all values of a/b and all modes. The reason is given above.

Figure 5.21 shows that the result for eigen frequencies corresponding to various a/b for an elliptic cylinder which not centrally placed in the tank. Here submerged depth h_0/b is 0.25 and depth of water h/b is 1.0 for fixed $c/b = 0.05$. From the graph the comparison is good for all values of a/b and all modes. Finally, when the body is either a circular or elliptic cylinder the comparison is good for all values of a/b.

CONCLUSION:

When body introduces into three-dimensional rectangular tank, it will reduce the eigenfrequencies. Nearly, a half immersed surface-piercing barrier reduces the lowest eigen wave-number to less than half when $p \leq 1$. This similar result was observed by Evans and McIver for two-dimensional similar problem, i.e. $p = 0$. But for increasing values of $p > 1$, it is not true. In contrast, a bottom-mounted barrier of the same length has negligible effect on eigen wavenumbers. By using eigen-function expansion method, the eigenfrequencies can be computed for either a surface-piercing or bottom-mounted barrier of any length, position and values of p and for all tank dimensions.

We obtained the general formula, for determining the eigenfrequencies in terms of the reflection and transmission coefficients for the body in an infinite wavetrain, based on the wide-spacing approximation.

When the body is either a submerged circular or elliptic cylinder this considerably changes the lowest eigenfrequencies. By using the boundary element method the eigenfrequencies were determined for any geometry and depth and for all tank dimensions.

Using the method of matched asymptotic expansions, we obtained the general expression for eigenfrequencies in terms of dipole strengths and cross-sectional area of the body.

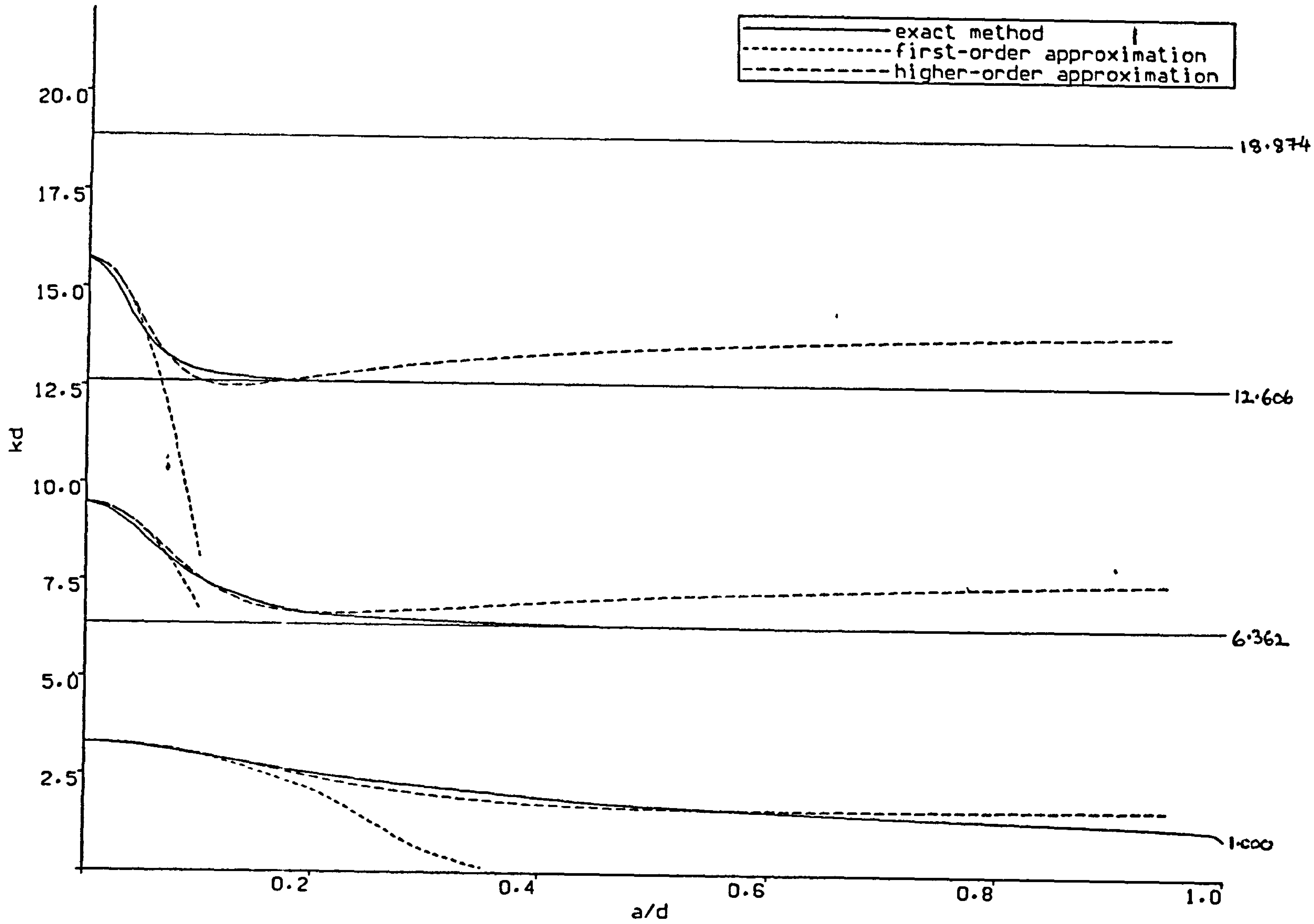


figure 5.1. surface piercing barrier in middle with pd =1   and h/d=1

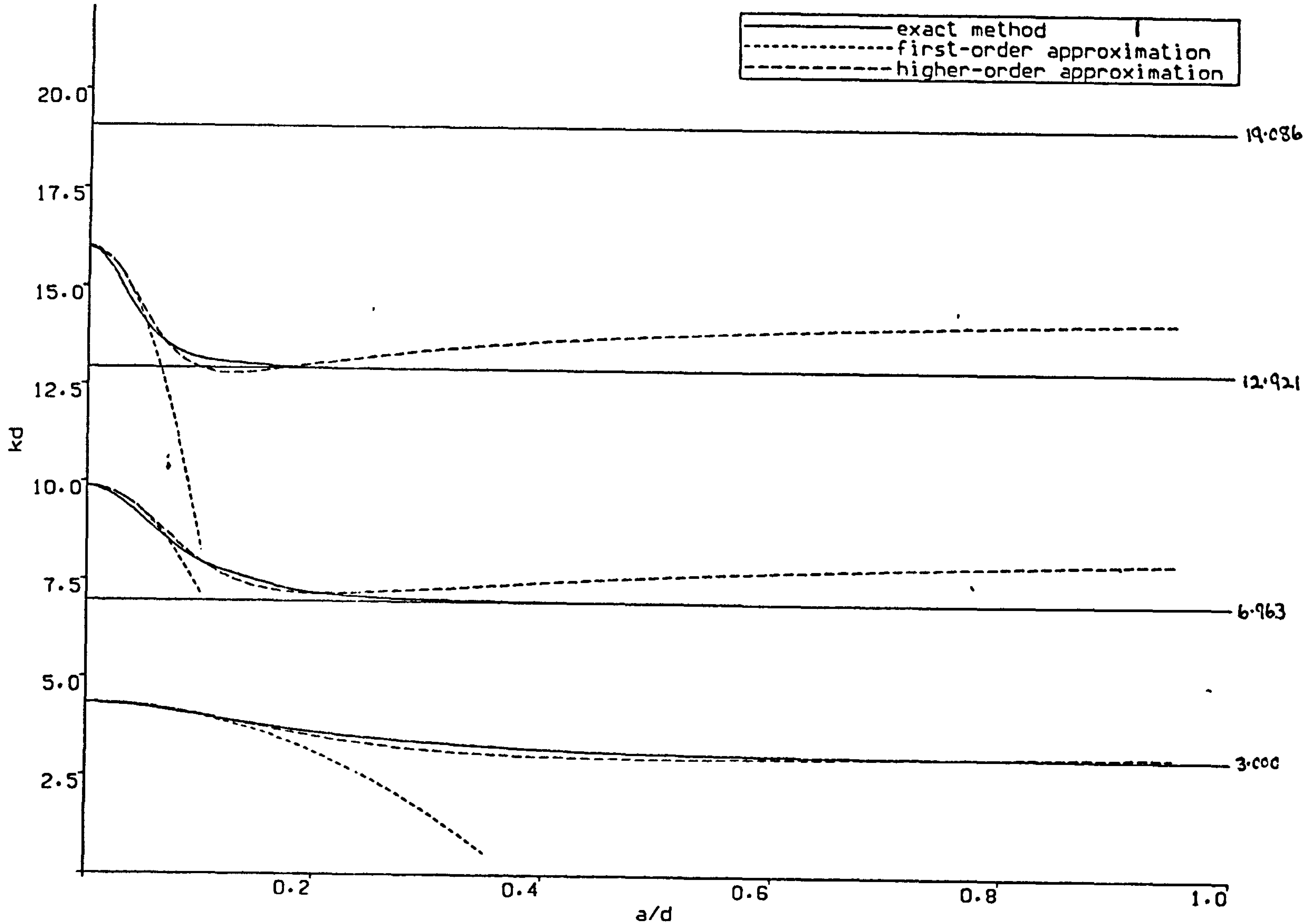


figure 5.2. surface-piercing barrier in middle with pd =3 and h/d=1

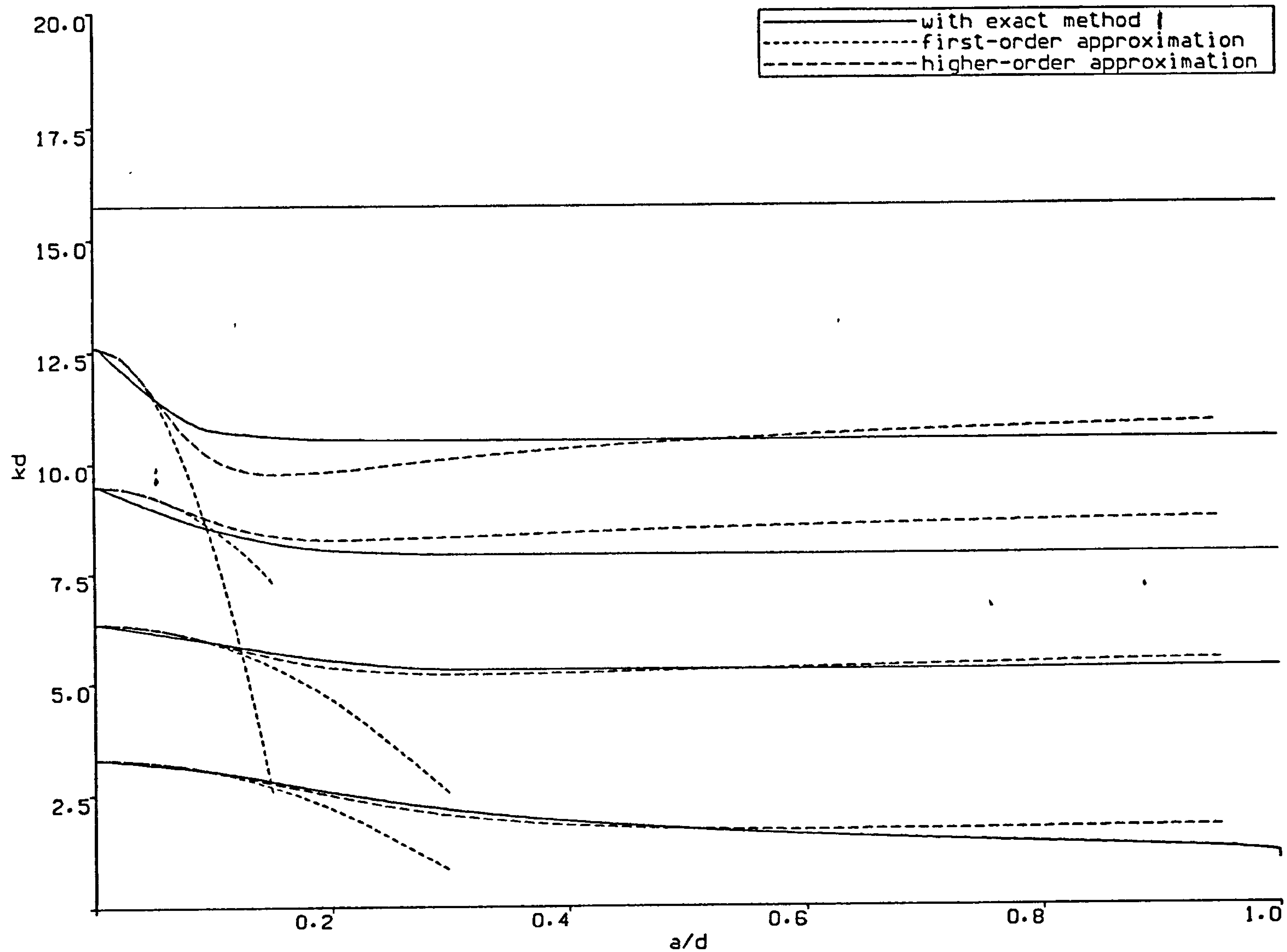


figure 5.3. surface-piering barrier not in middle with pd=1 and h/d=1

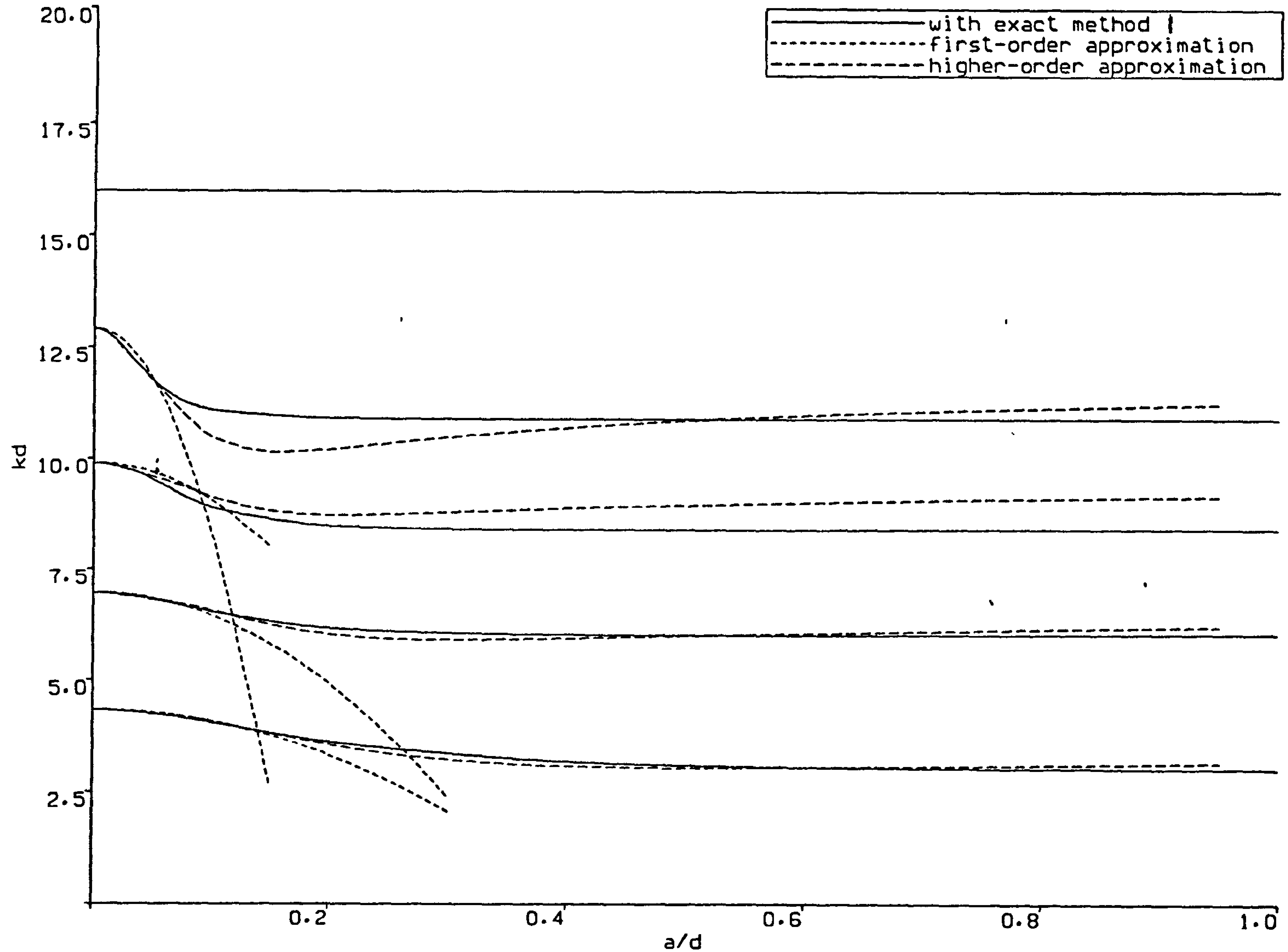


figure 5.4. surface-piering barrier not in middle with pd=3 and h/d=1

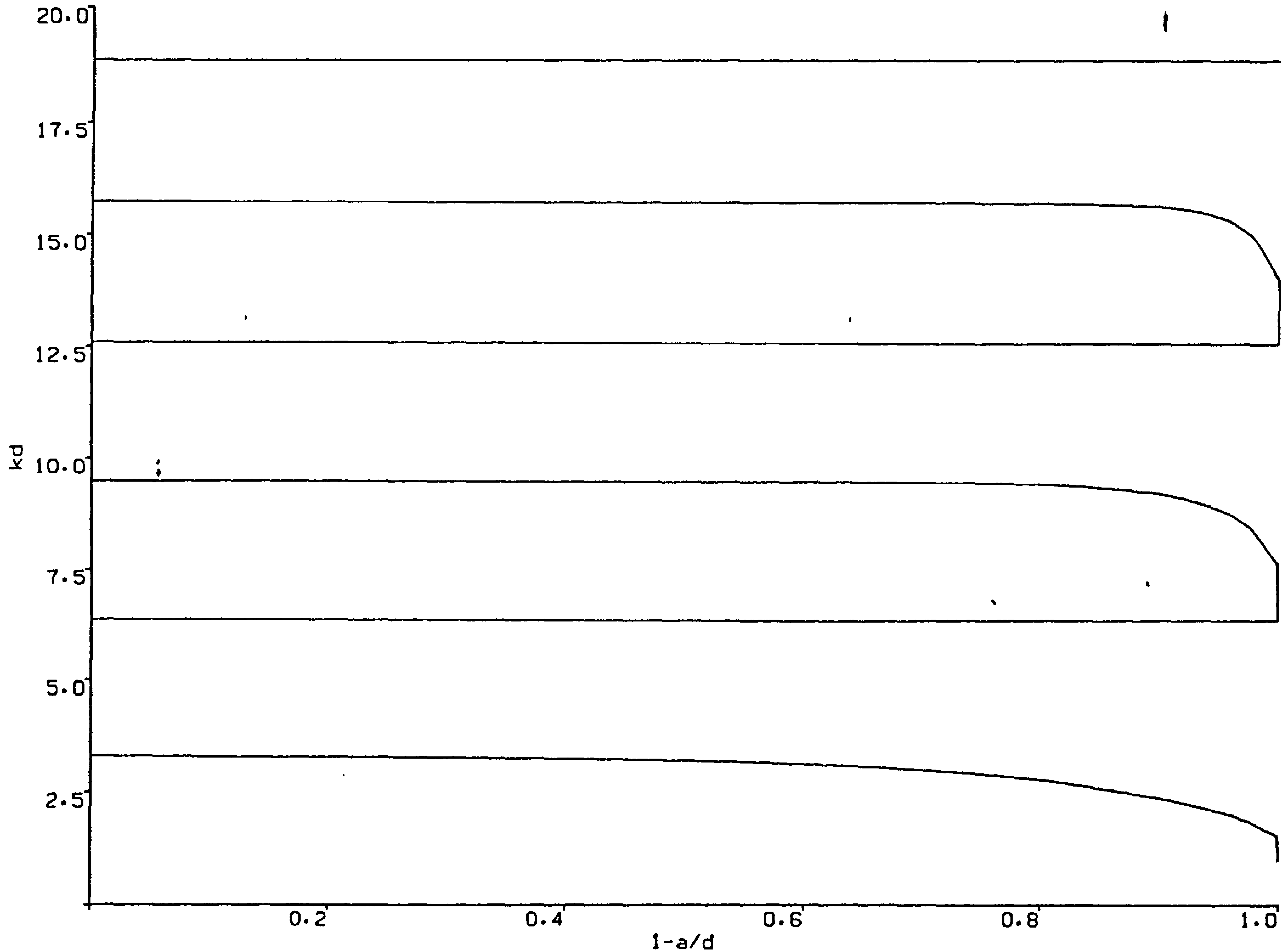


figure 5.5. bottom-mounted barrier in middle with pd=1 and h/d=1.0

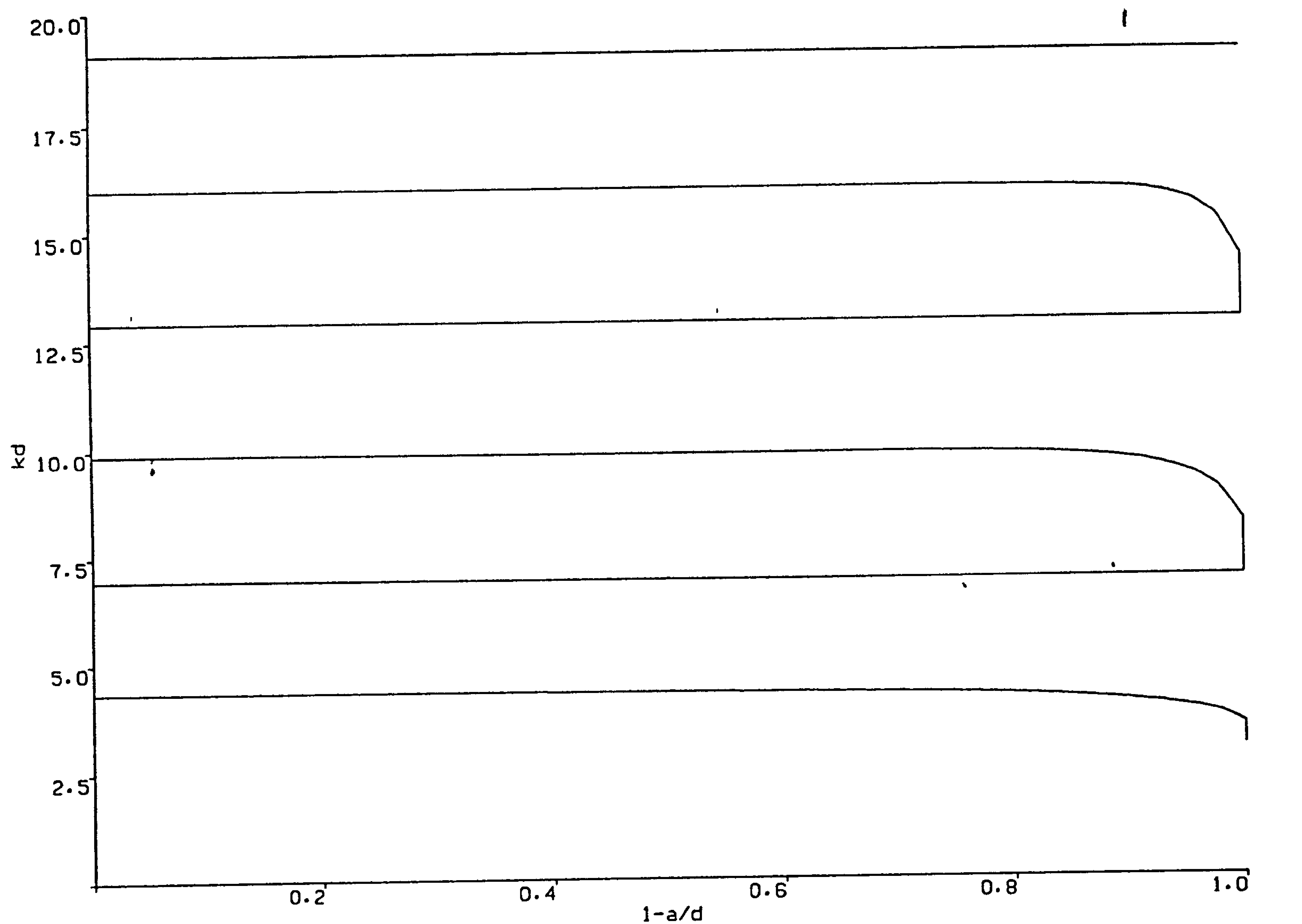


figure.5.6. bottom-mounted barrier in middle with pd=3 and h/d=1.0

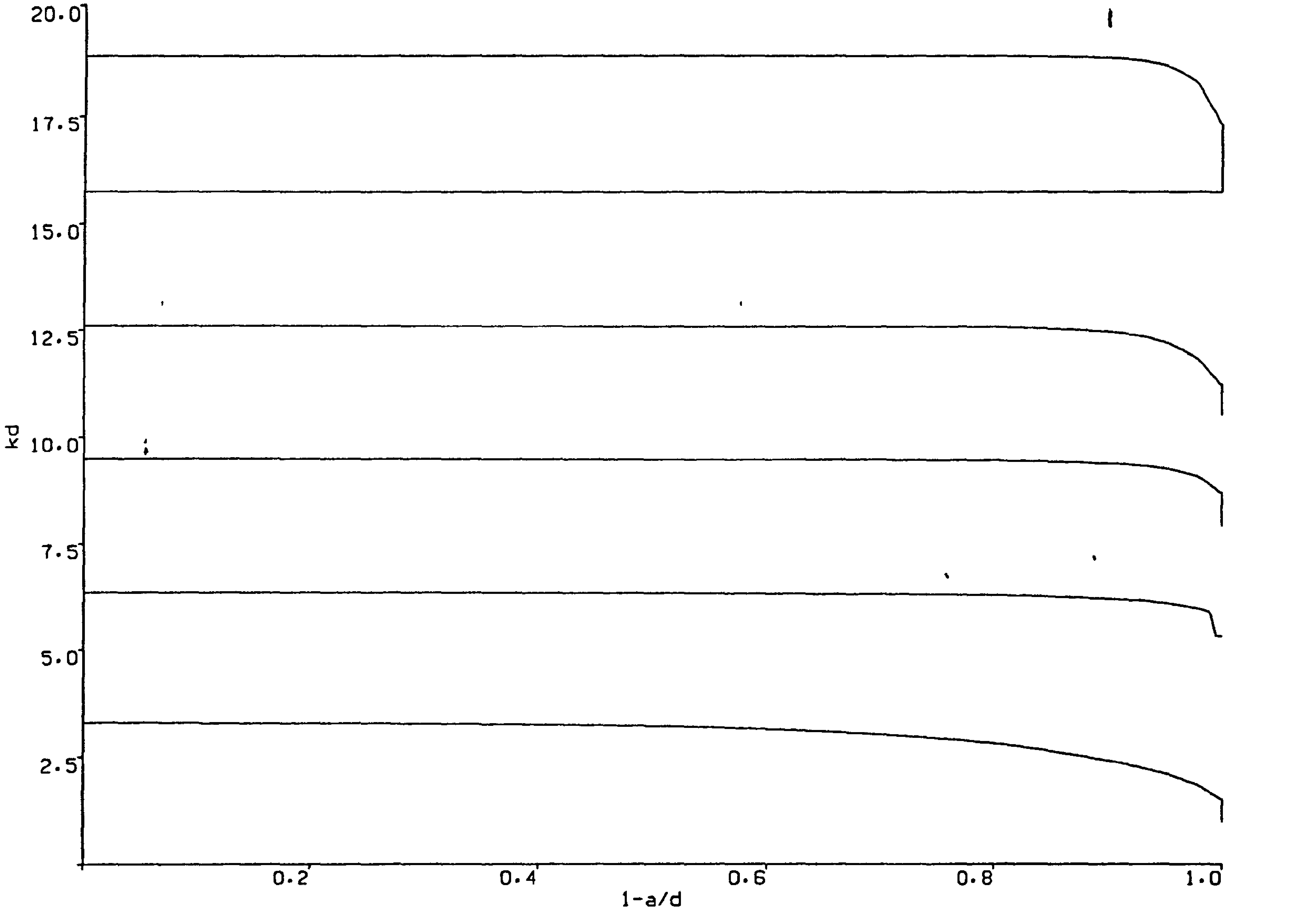


figure 5.7. bottom-mounted barrier not in middle with pd= 1 and h/d=1.0

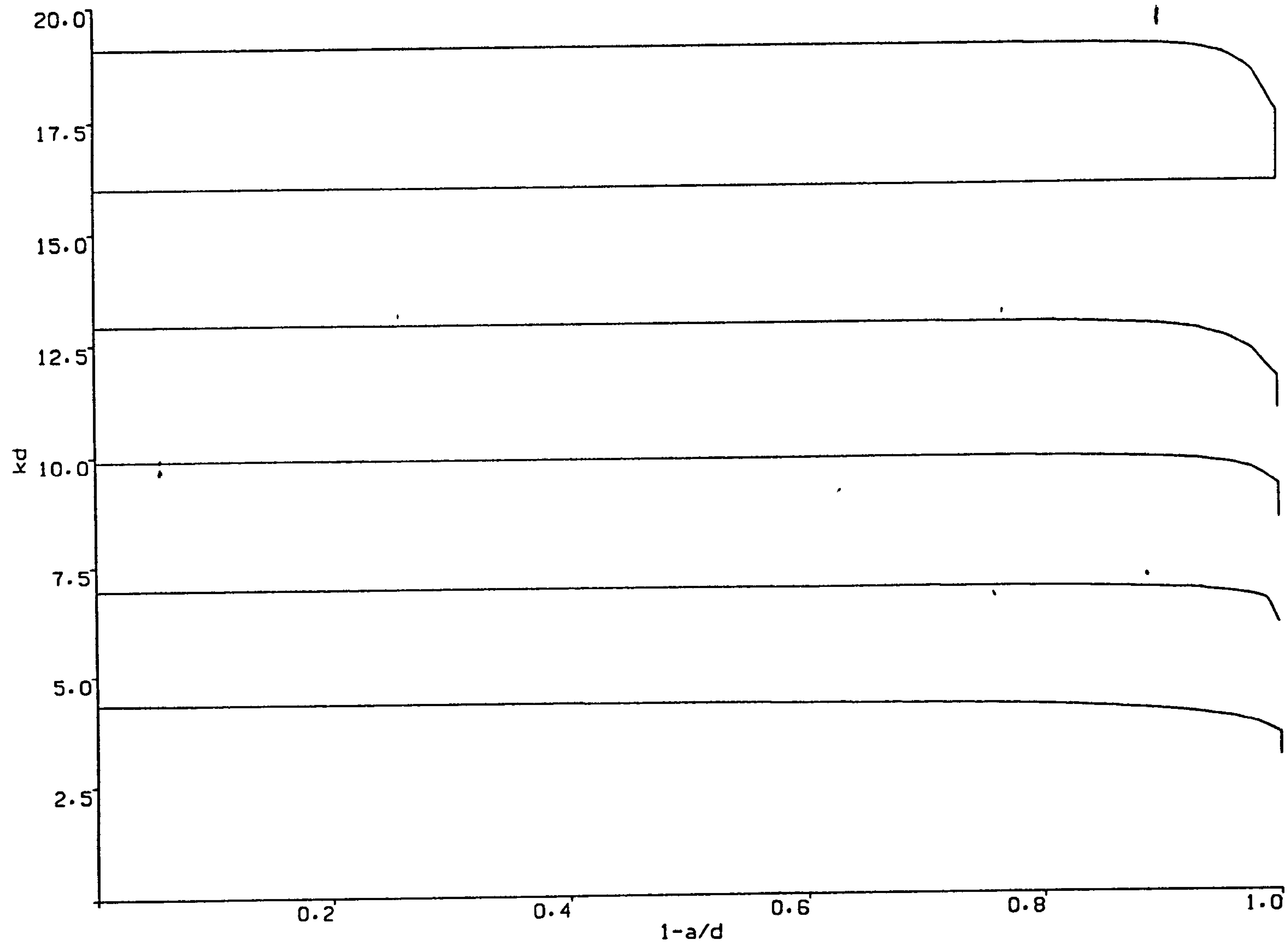


figure 5.8. bottom-mounted barrier not in middle with pd= 3 and h/d=1.0

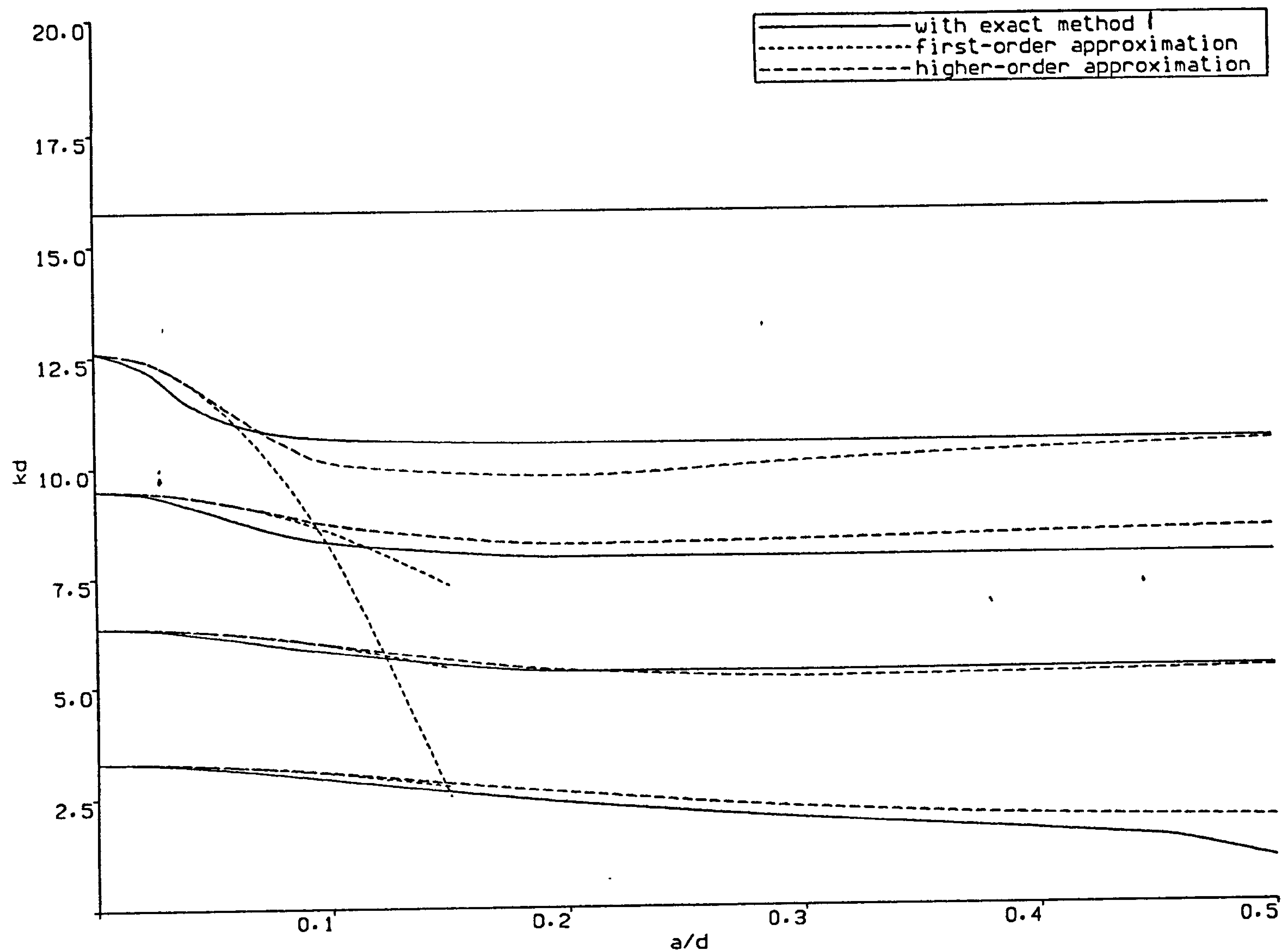


figure 5.9. surface-piering barrier not in middle with   pd=1 and h/d=0.5

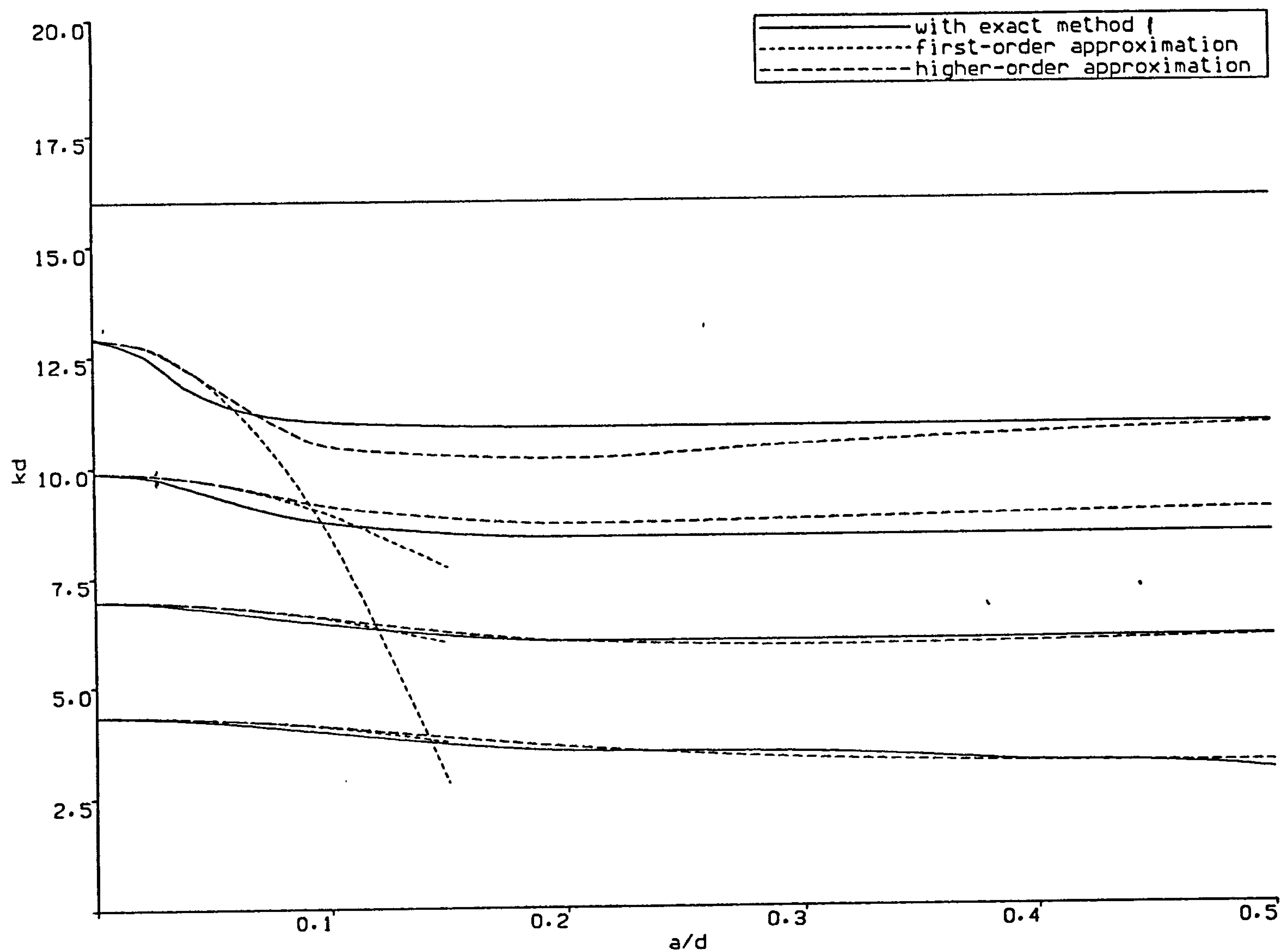


figure 5.10. surface-piering barrier not in middle with pd=3 and h/d=0.5

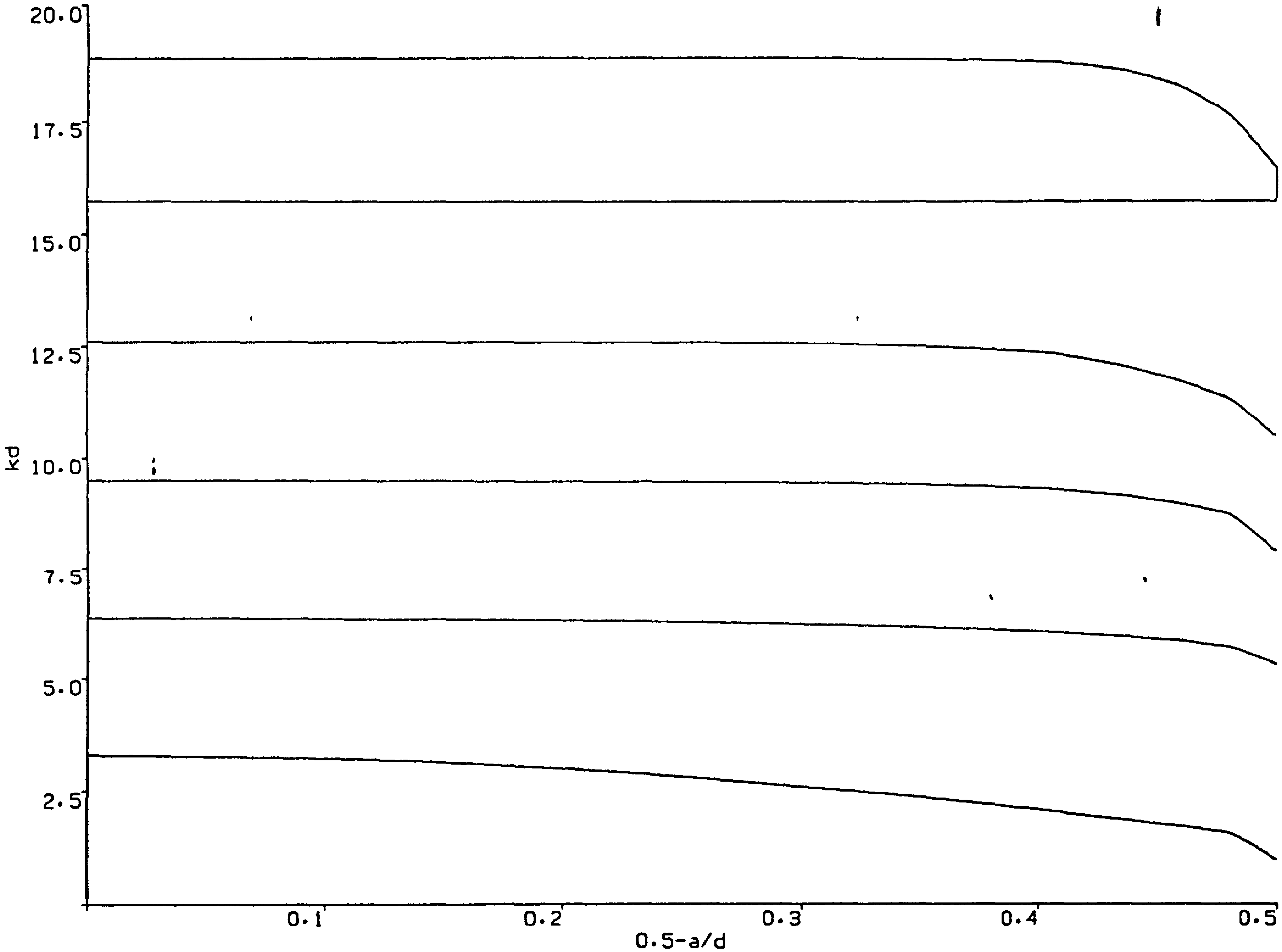


figure 5.11. bottom-mounted barrier not in middle with pd=1 and h/d=0.5

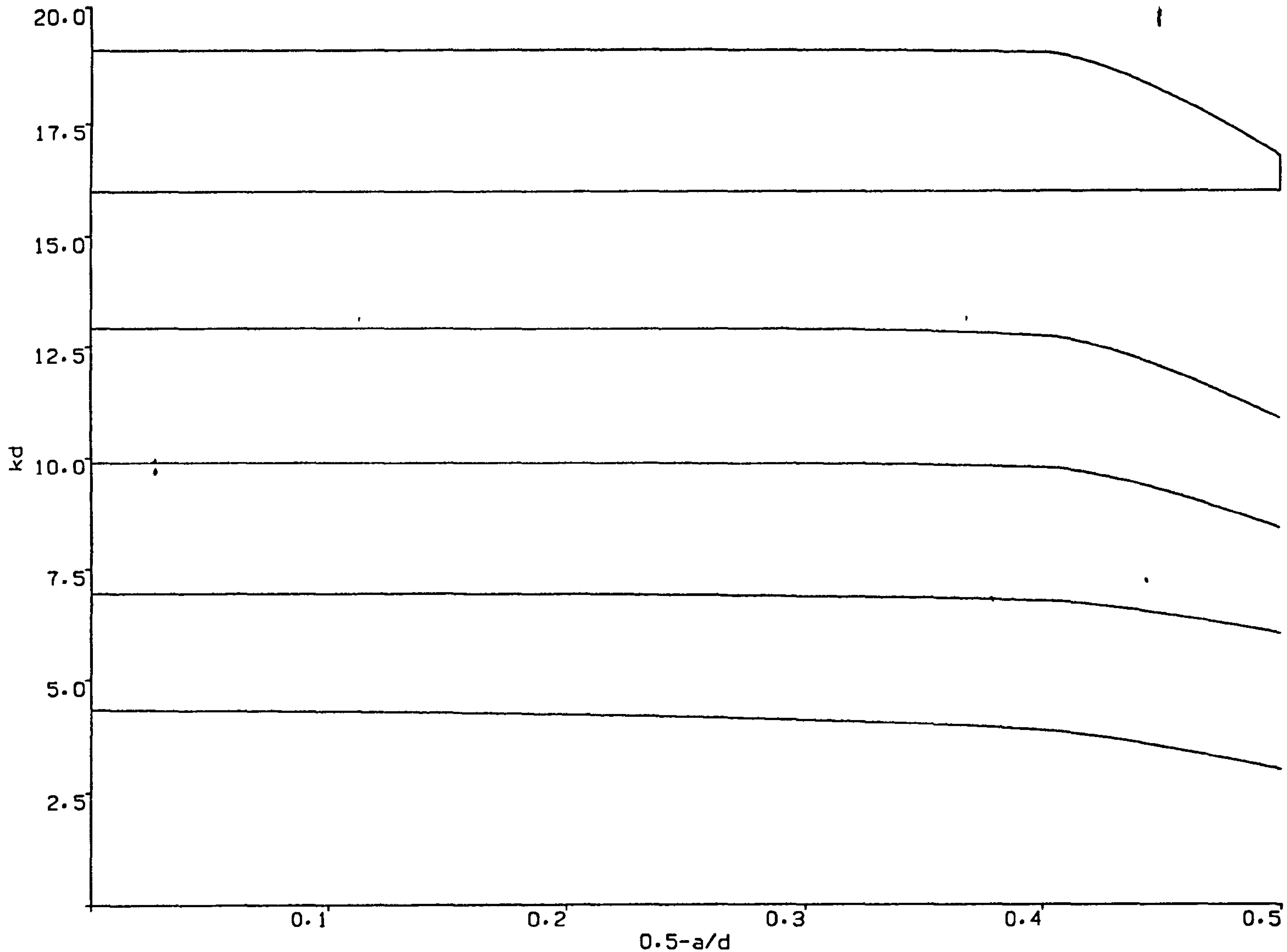


figure 5.12. bottom-mounted barrier not in middle with pd=3 and h/d=0.5

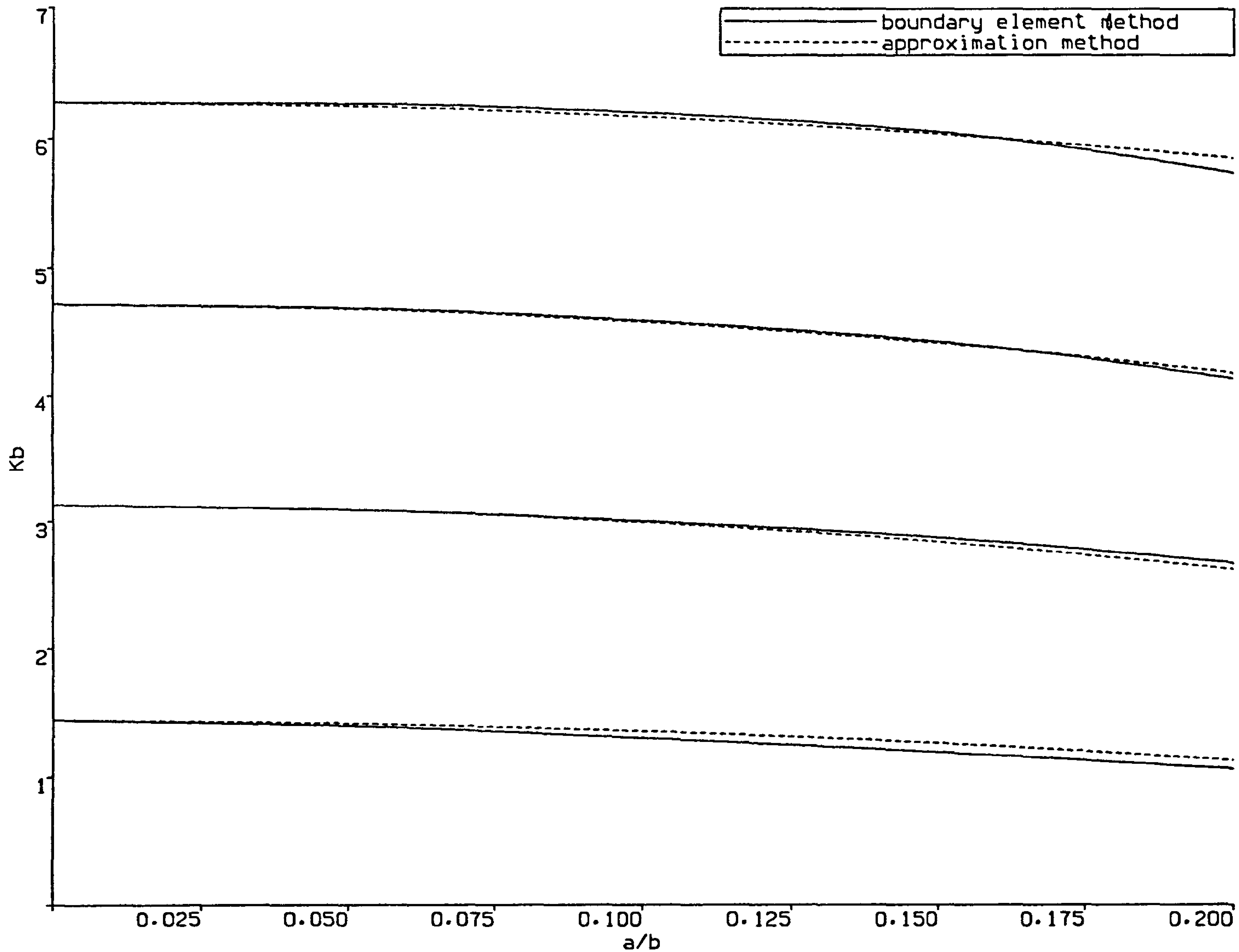


Figure 5.13. submerged circular cylinder with radius a and depth 0.25 in middle and depth of water 1.0

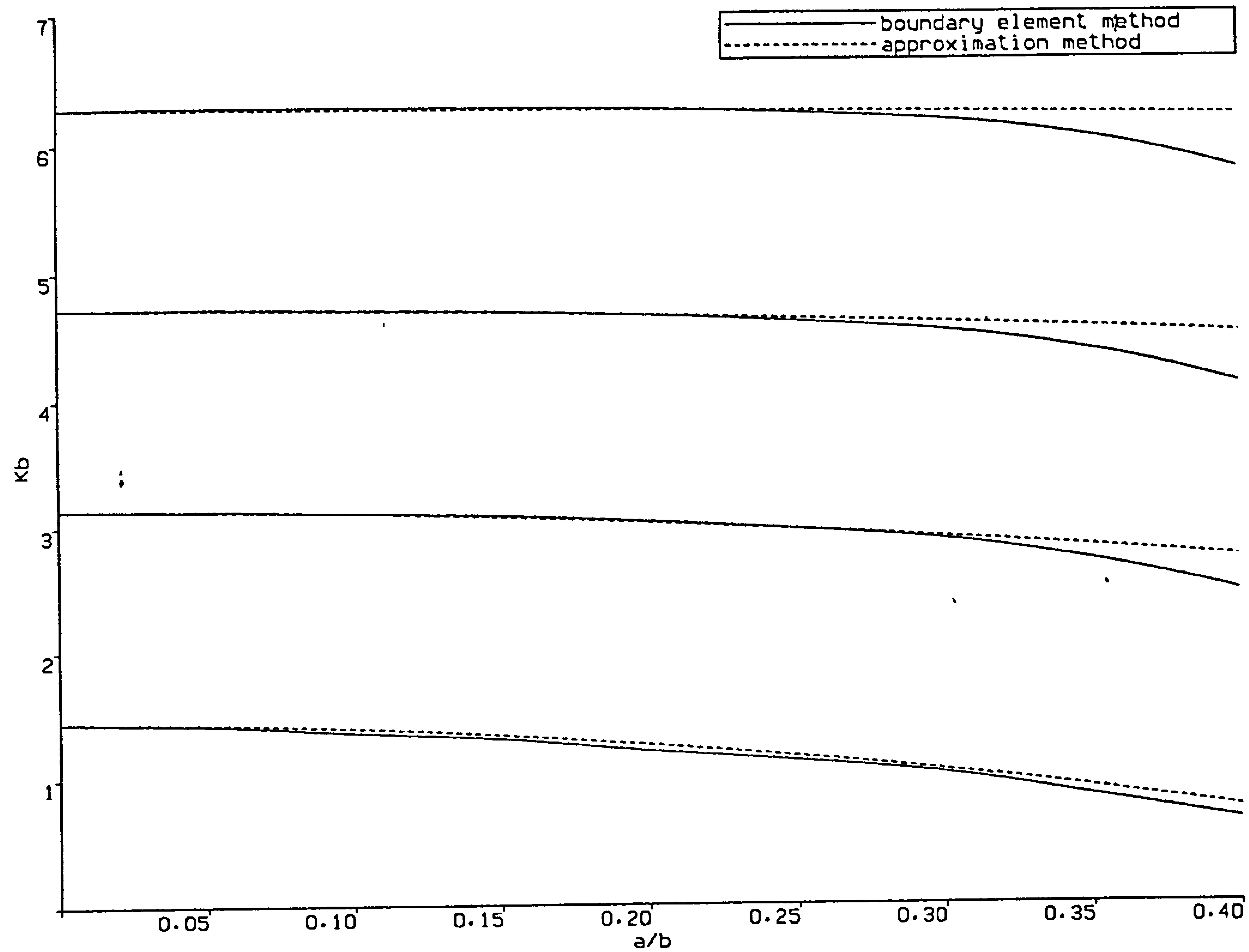


Figure 5.14. submerged circular cylinder with radius a and depth is 0.5 and is in middle and depth of water is 1.0

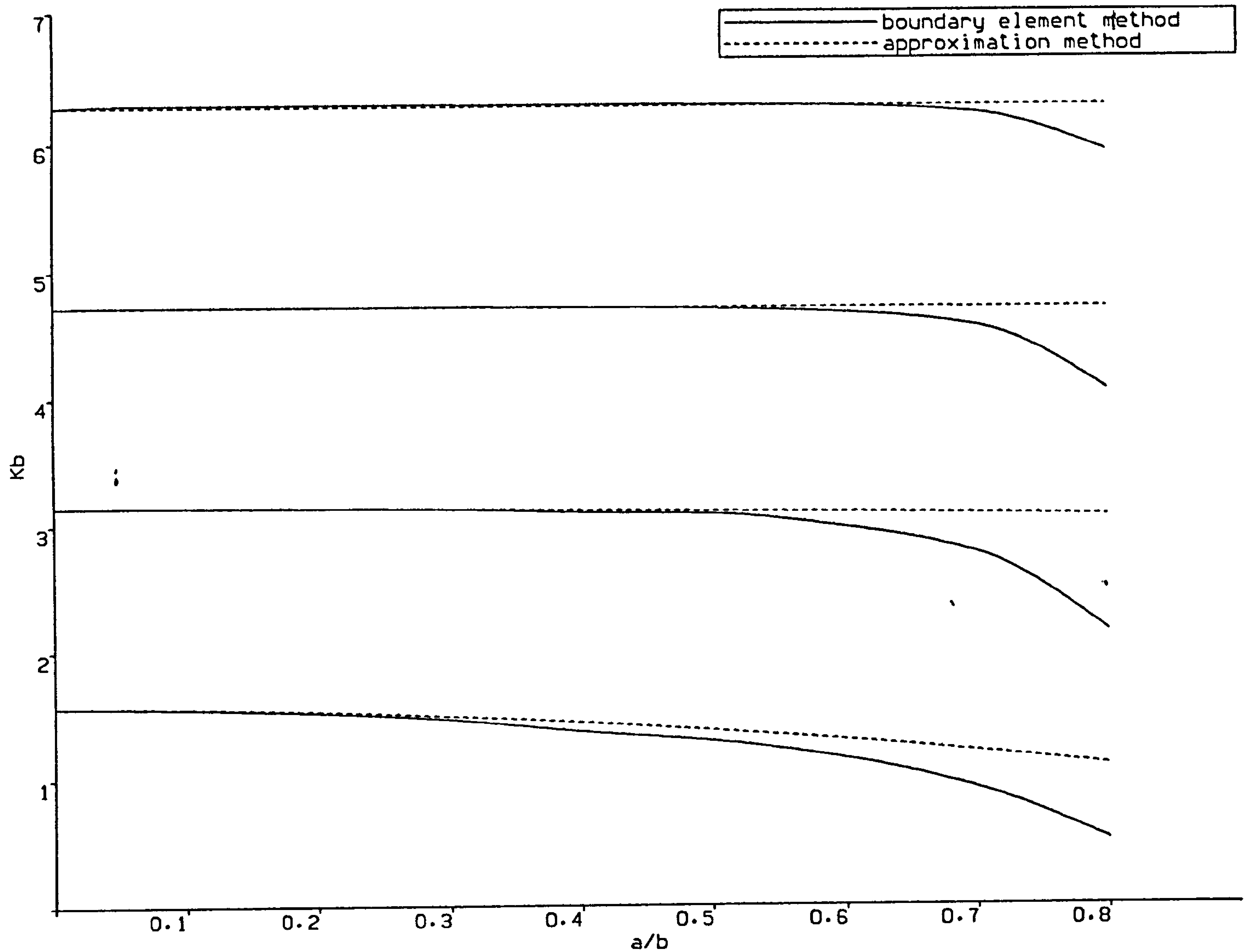


Figure 5.15. submerged circular cylinder with radius a and depth is 1.0 and is in middle and depth of water is 2.0

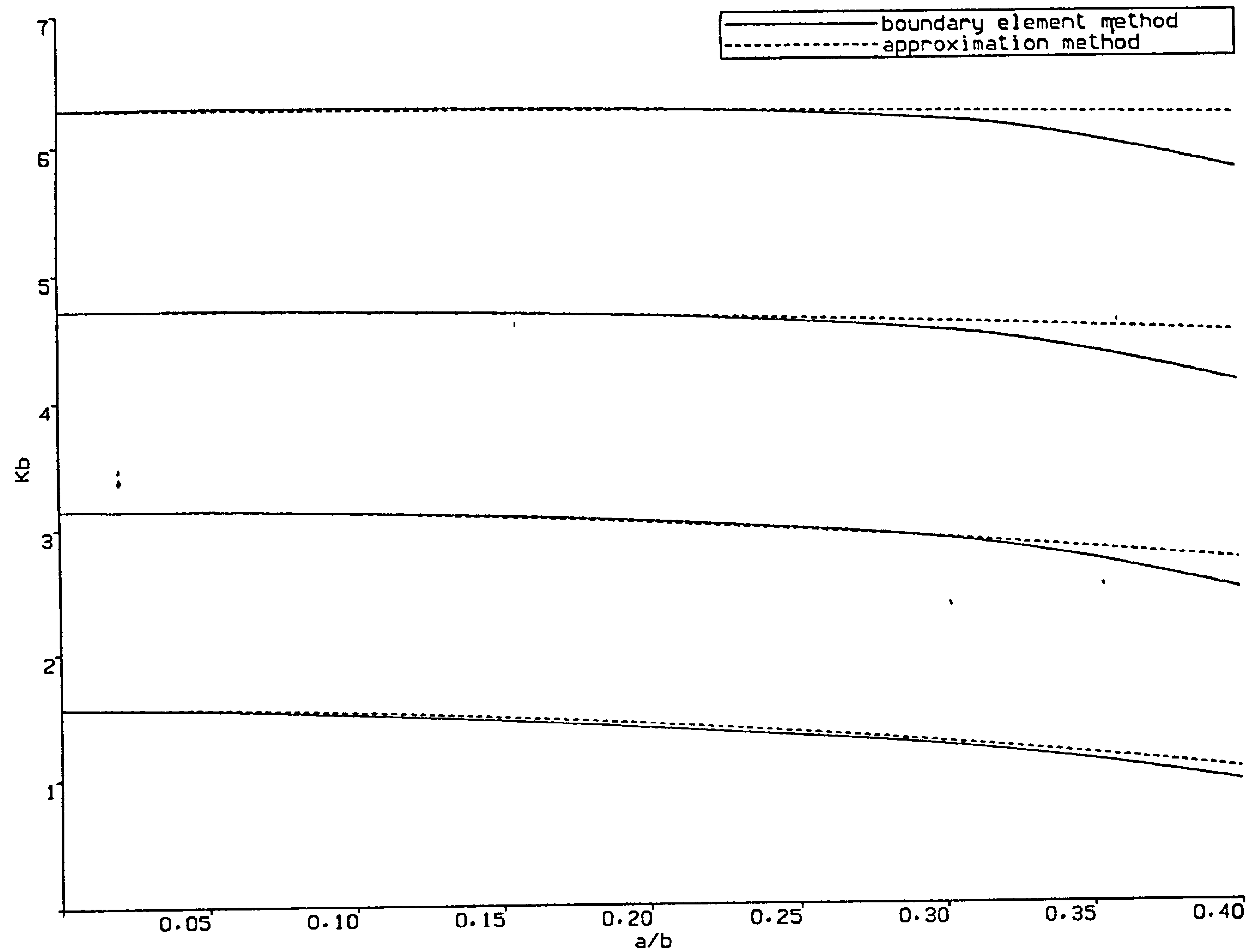


Figure 5.16. submerged circular cylinder with radius a and depth is 0.5 and is in middle and depth of water is 2.0

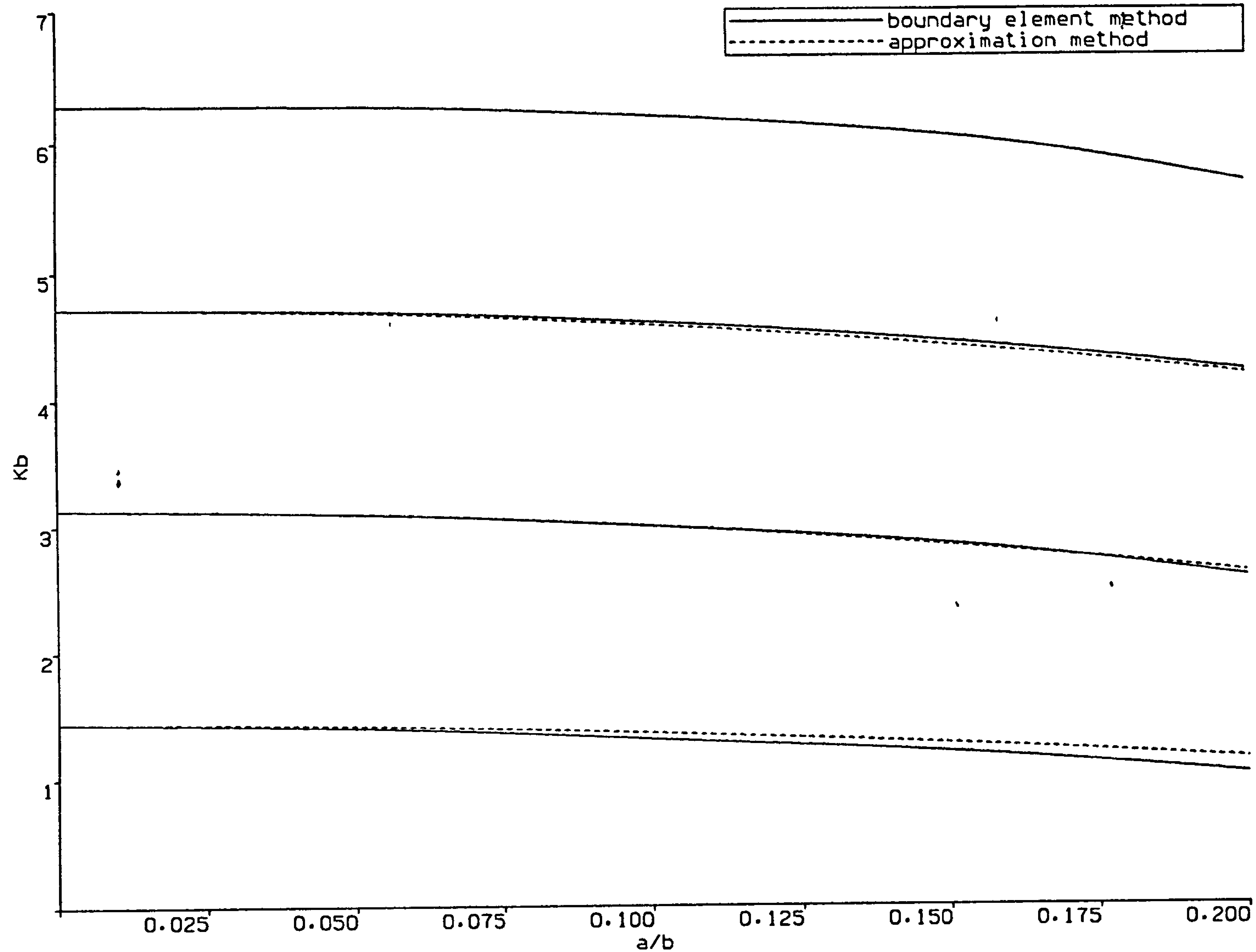


Figure 5.17. submerged circular cylinder with radius a and depth is 0.25 and is not in middle and depth of water is 1.0

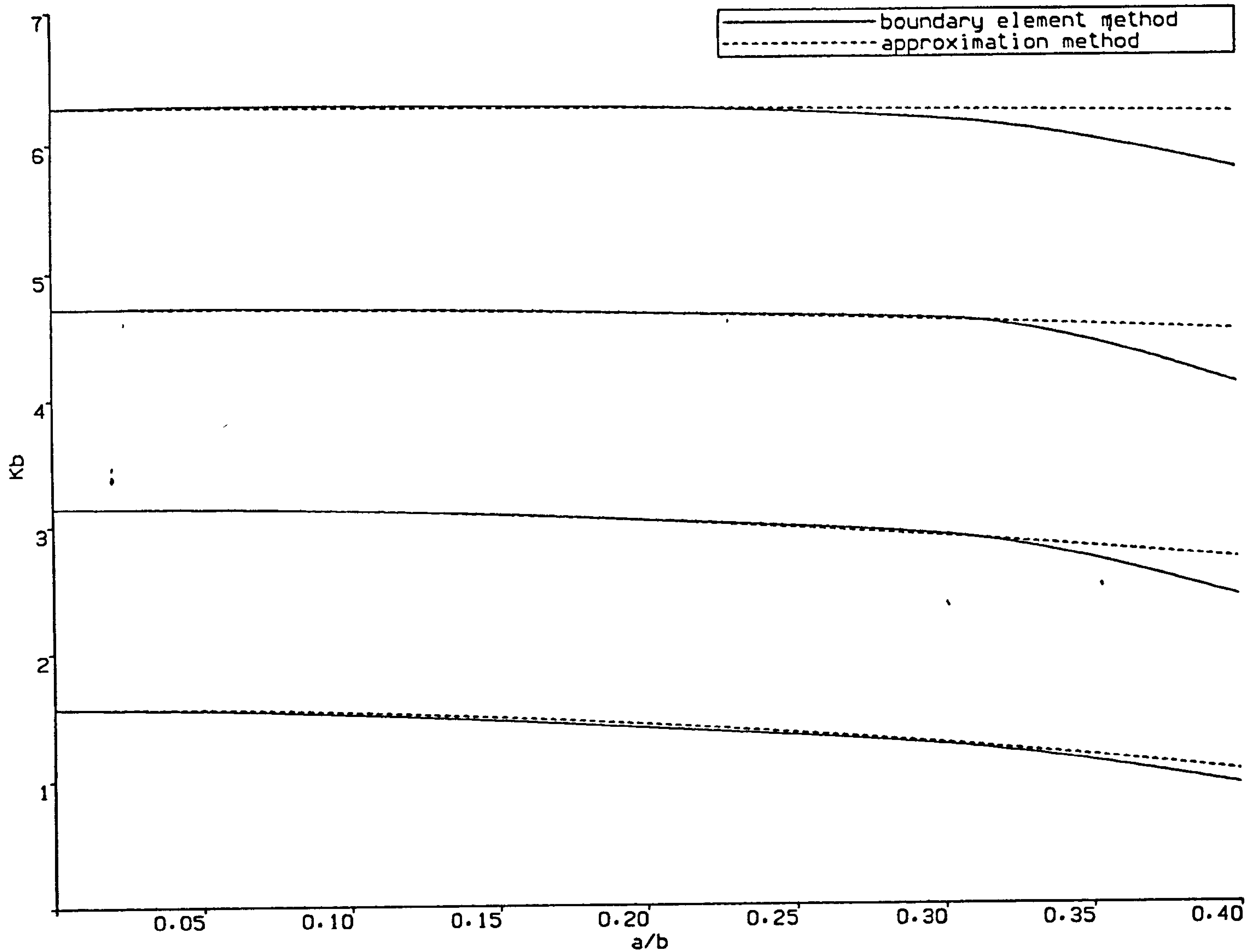


Figure 5.18. submerged circular cylinder with radius a and depth is 0.5 and is not in middle and depth of water is 2

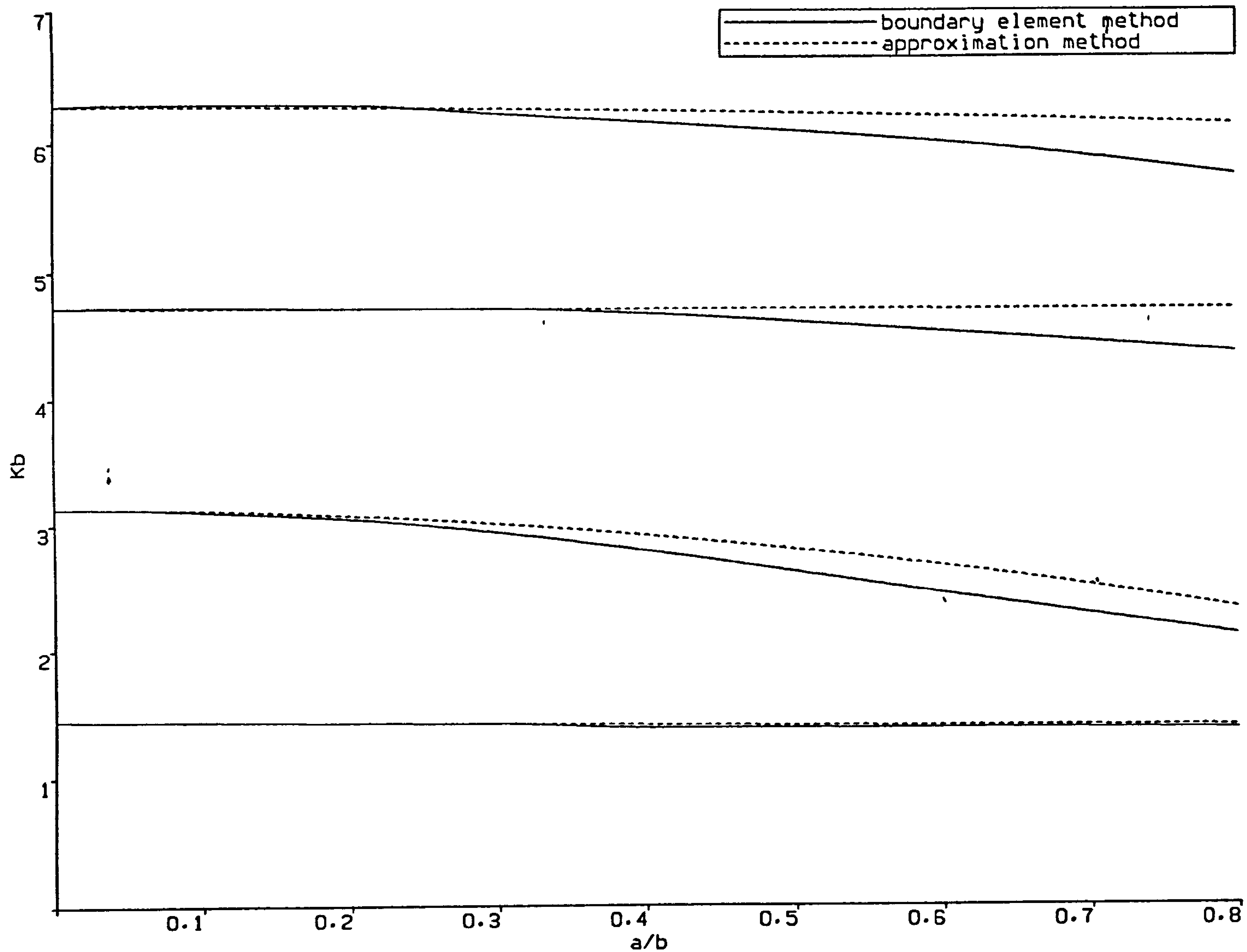


Figure 5.19. submerged elliptic cylinder and depth is 0.5 and is in middle and depth of water is 1.0

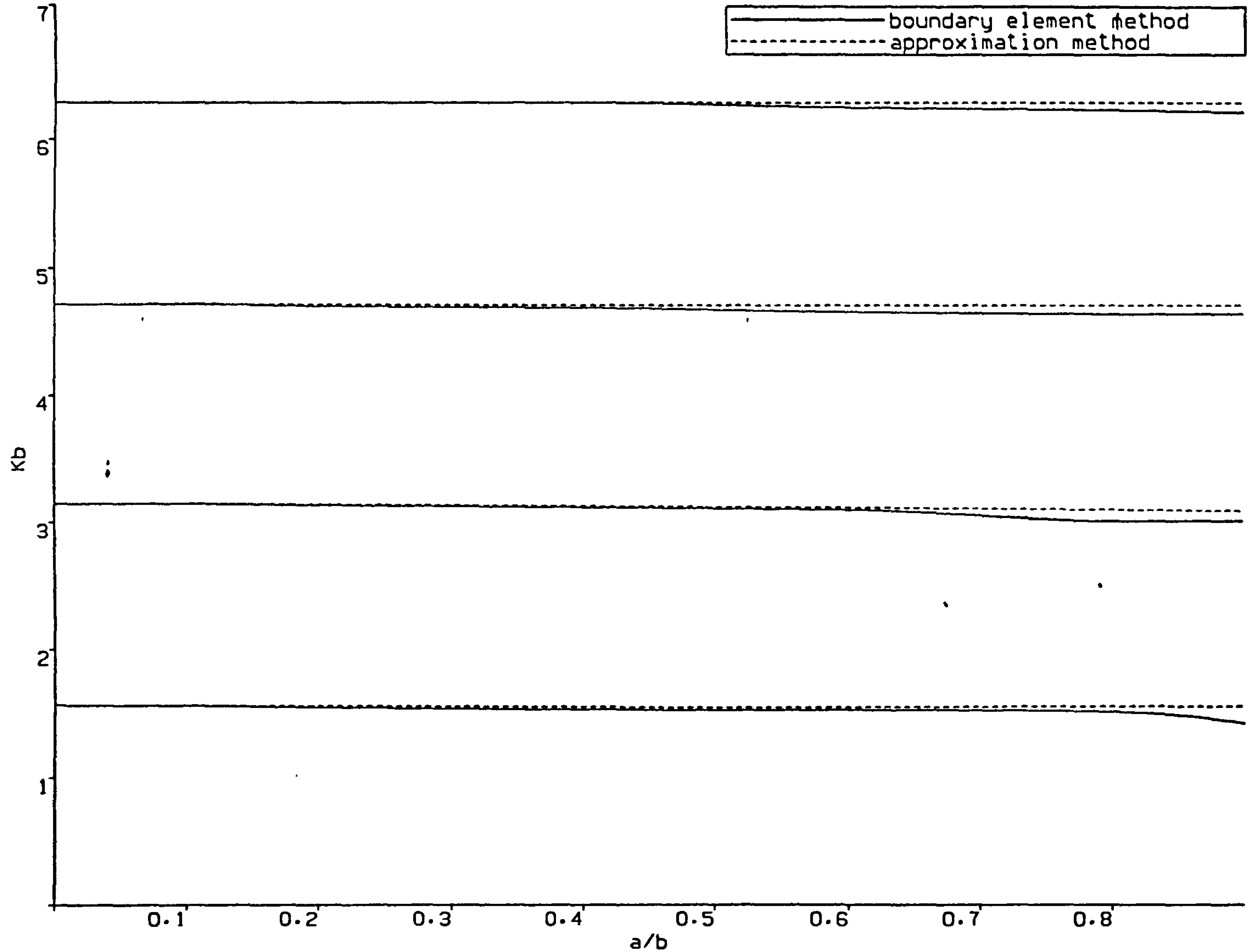


Figure 5.20. submerged elliptic cylinder and depth is 1.0 and is in middle and depth of water is 2.0

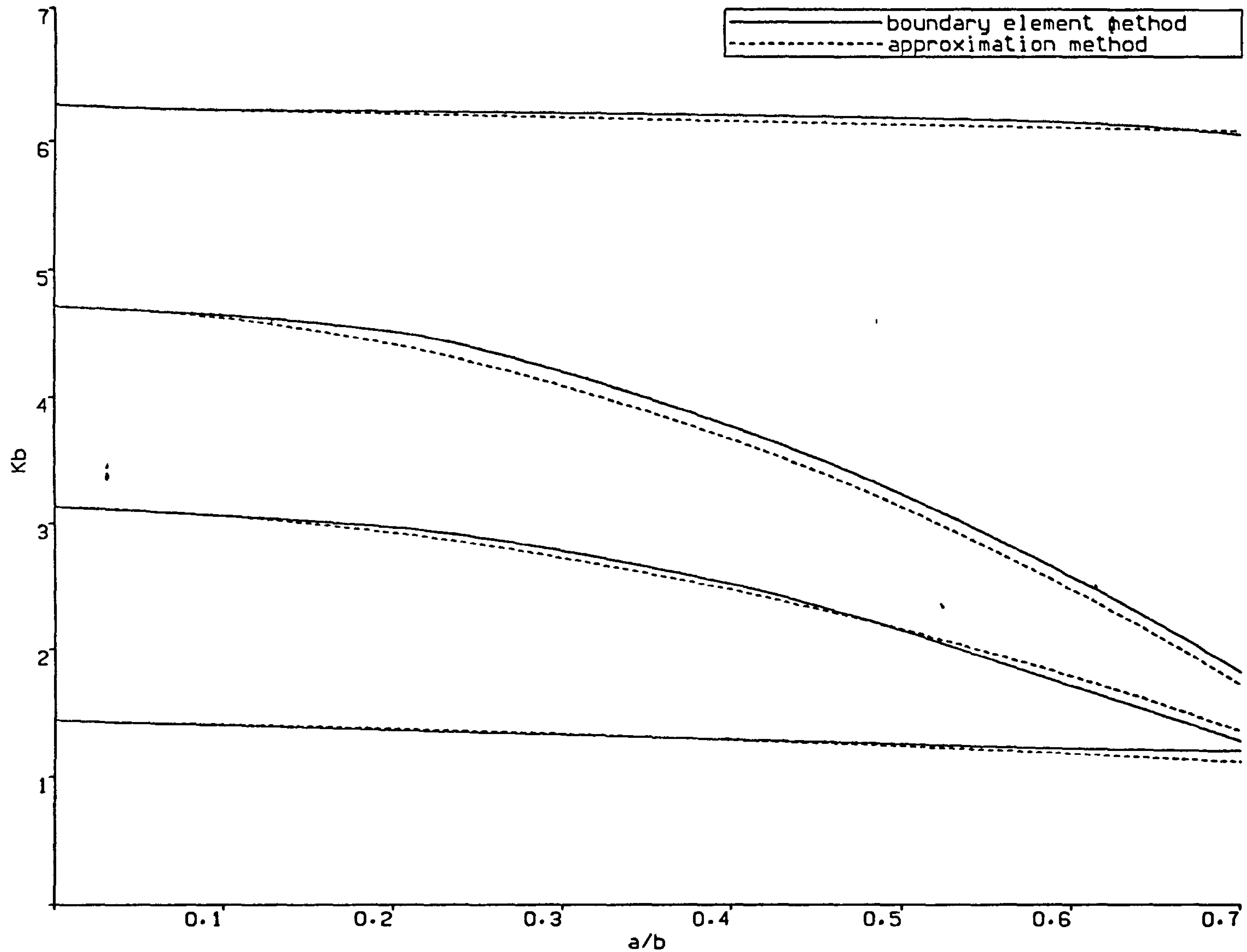


Figure 5.21. submerged elliptic cylinder and depth is 0.25 and is not in middle and depth of water is 1.0

CHAPTER 6

DRIFT FORCES ON MULTIPLE BODIES

6.1 Introduction

The scattering of an incident wave field by a group of bodies may give rise to wave forces on one of the bodies that differ significantly from the forces it would experience if in isolation. The modification of wave forces due to hydrodynamic interaction in arrays is the subject of the present chapter.

Let a perodic wave train with frequency ω whose first-order amplitude is A, approach a body from large negative x, and let $\varepsilon_0 = kA$ be the small wave slope. Then, using Stokes' expansion, the potential Φ can be expressed as

$$\Phi = \varepsilon_0 \Phi_1 + \varepsilon_0^2 \Phi_2 + \ldots\ldots \qquad (6.1.1)$$

where each $\Phi_1, \Phi_2, \ldots$ contain incident and scattered waves. The first two terms in (6.1.1) contribute to the first- and second- order exciting force on the body. The first-order force is just the linearized exciting force on the body. The second-order problem involves solving a boundary-value problem where the boundary conditions contain products of two terms from the first-order solution, each of which are harmonic oscillations with frequency ω. The products of the first-order terms will give one term that is time independent, and one term that is oscillating with frequency 2ω. The second-order potential will have the same time dependence as the boundary conditions for the problem. The pressure, and hence the force, associated with the second-order potential can be found from Bernoulli's equation. The time average of a periodic quantity is zero so that there is no contribution to the mean wave loads at second order from the second-order potential. However, the contribution from quadratic terms in the first-order potential to the mean wave forces is non-zero. Thus the second-order mean force, or drift force, may be calculated from knowledge of the first-order potential only.

The drift forces, can give rise to constant offsets from the equilibrium position, and are therefore of interest to the designers of rigs with flexible mooring systems. Furthermore for mixed seas with narrow-banded spectra, Newman (1974) has shown that the forces on a floating body give rise to difference-frequency oscillations which can be approximated in terms of the mean drift force on the body in regular waves.

The horizontal drift forces on multiple bodies in long waves have been considered by Eatock-Taylor and Hung (1985). They found numerically that, for certain geometries, the drift force on a group of N bodies is approximately N^2 times that on an isolated body when the incident waves are long compared with the group size. Eatock-Taylor and Hung (1986) also suggested that radiation damping coefficients will obey an N^2 enhancement law and the first-order exciting forces and vertical mean drift forces have an N-dependence for arrays of bodies in low-frequency waves.

Under the assumptions that the waves are long relative to body separation and the bodies are widely spaced relative to body size, McIver (1987) calculated the mean drift force on a group of N vertical circular cylinders by using the method of matched asymptotic expansions. He obtained an additional higher-order term, beside the N^2 term found previously and this additional term indicates how the geometry of an array affects wave forces.

In the present work, the work of McIver (1987) is extended in two ways. First of all the assumption that the cylinders are widely spaced relative to body size is relaxed, while retaining the assumption on the wave length. This is done in section 6.2, considering the case of two circular cylinders. Two flow regions are used with the inner region now containing both cylinders, and bipolar coordinates are used to obtain the inner solution. In section 6.3, the effects of body geometry are investigated by allowing the cylinders to be of arbitrary cross-section. Three flow regions are defined. These are: an outer region at large distances from the array where the lengthscale is k^{-1}(k is the wave number); an intermediate region within the array (but not 'close' to any body) where the length scale is the body spacing; and an inner region surrounding each body where the length scale is the body radius. In the outer region, the scattered wave appears to be the result of singularities at a single origin, whilst in the intermediate region the disturbance appears to be genearated by singularities at the origin of each body coordinate system. The basic procedure of this method is the same as that used by McIver (1987).

Consider a plane wavetrain of amplitude A and frequency ω, incident upon the fixed vertical cylinders, and standing in water of finite depth h. Cartesian coordinates (x , y , z) are defined so that the (x , y)- plane corresponds to the mean free surface and the z- axis is directed vertically downwards. The origin of the coordinate system j is at (x , y , z) $= (\xi_j, \eta_j, 0)$, while the position of body l relative to body j has polar coordinates $(r_j, \psi_j) = (R_{jl}, \alpha_{jl})$ as shown in figure 6.1.

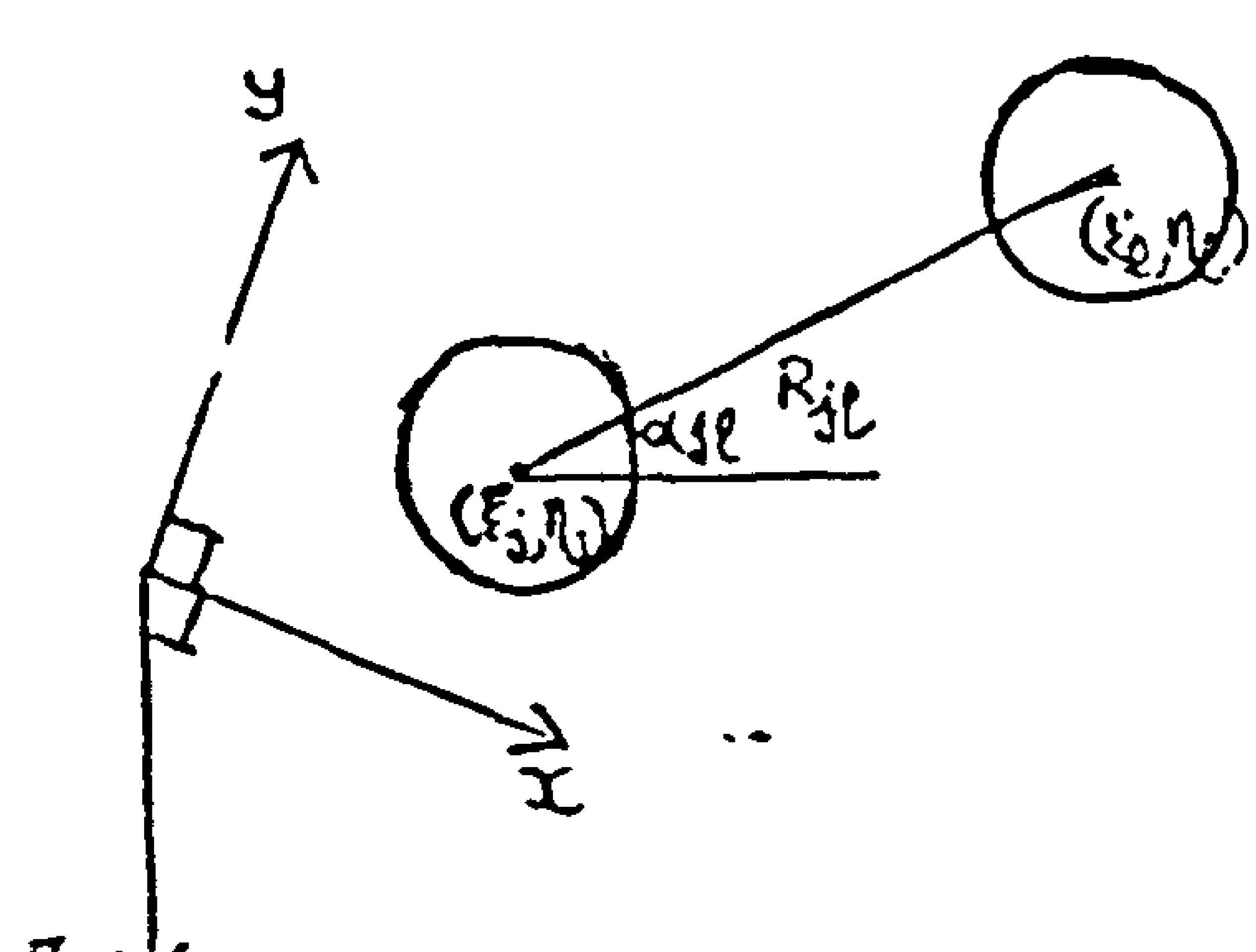


FIGURE 6.1. PLAN VIEW OF ARRAY SHOWING BODIES j AND l.

The usual assumptions of linearised water wave potential theory are used. Hence the fluid motion is described by a velocity potential

$$\Phi_T(x, y, z, t) = \mathrm{Re}\left\{ -\frac{ig\,A}{\omega}\,\phi_T(x, y, z)\,e^{-i\omega t} \right\}, \tag{6.1.2}$$

The complex-valued function $\phi_T(x, y, z)$ satisfies

$$\nabla^2 \phi_T = 0 \tag{6.1.3}$$

within the fluid, the linearized free-surface condition

$$\frac{\partial \phi_T}{\partial z} + \frac{\omega^2}{g}\,\phi_T = 0, \quad z = 0, \tag{6.1.4}$$

the zero-flux condition on the bed

$$\frac{\partial \phi_T}{\partial z} = 0, \quad z = h, \tag{6.1.5}$$

and the no-normal-flow condition on each body surface

$$\frac{\partial \phi_T}{\partial n} = 0, \tag{6.1.6}$$

where n is the normal measured into the fluid. We take the incident wave to travel in the direction of increasing x, so that ϕ_T may be written in the form

$$\phi_T(x, y, z) = \left(e^{ikx} + \phi(x, y) \right) \frac{\cosh k(z-h)}{\cosh kh}, \tag{6.1.7}$$

where the first term within the large bracket represents the incident wave and ϕ the scattered wave.

This separation of variables is possible since the cylinders extend throughout the fluid depth, and the form (6.1.7) clearly satisfies equations (6.1.4-5) with the usual dispersion relation,

$$\frac{\omega^2}{g} = k \tanh kh \,. \tag{6.1.8}$$

Substituting the expression for ϕ_T into (6.1.3) gives the Helmholtz equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k^2 \phi = 0. \tag{6.1.9}$$

From the body boundary condition (6.1.6), ϕ must satisfy

$$\frac{\partial \phi}{\partial n} = - \frac{\partial}{\partial n}\left(e^{ikx} \right), \tag{6.1.10}$$

on each body.

By using the ideas of Maruo (1960) and Newman (1967), Faltinsen (1990, pp 140), obtained the result for the mean horizonal drift force in the direction of wave advance as

$$f_{mx}^{(2)} = \frac{\rho\, g\, A^2}{2\pi k}\left(1 + \frac{2kh}{\sinh 2kh} \right) \int_0^{2\pi} (1 - \cos\theta)\, |f(\theta)|^2 \, d\theta. \tag{6.1.11}$$

Here ρ_0 is the fluid density, and the angle θ is defined by

$$x = r \cos\theta, \quad y = r \sin\theta \,.$$

6.2 Two circular cylinders with different radii

Each body is taken to be a vertical circular cylinder, extending throughout the depth of the fluid and with radii a and δa as shown in figure 6.2.

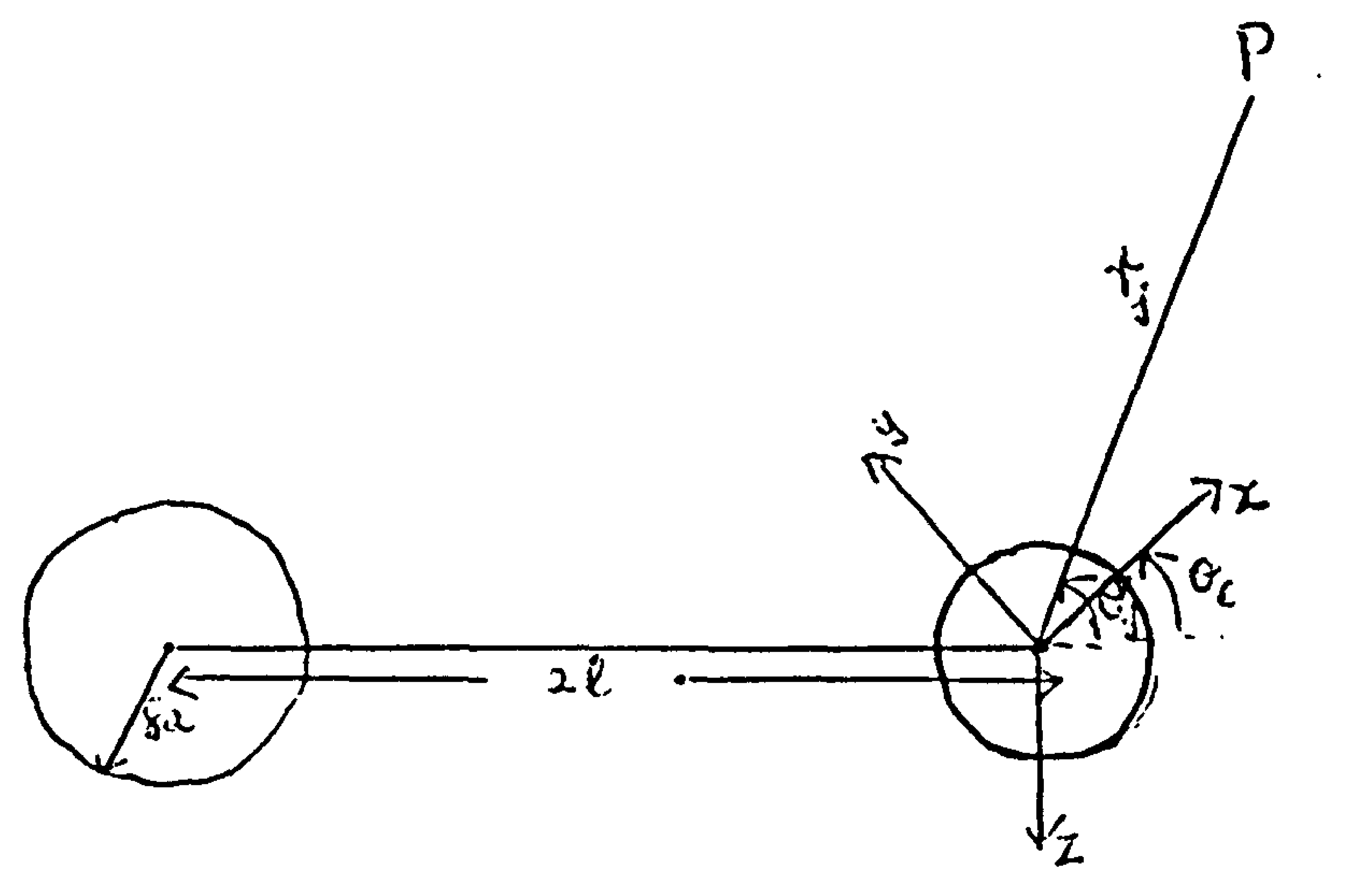


FIGURE 6.2 PLANE VIEW OF TWO CYLINDERS

Here the boundary-value problem to be solved is given by the Helmholtz equation (6.1.9), the free-surface condition (6.1.4), the bed condition (6.1.5), and the body condition (6.1.6). In terms of the scattered potential ϕ, the body boundary condition is

$$\frac{\partial \phi}{\partial n} = -\frac{\partial}{\partial n}\left(e^{ik\left(2l\,\delta_{j2}\cos\theta_0 + r_j\cos(\theta_j - \theta_0)\right)} \right), \quad r_1 = a,\ r_2 = \delta a,\ (j = 1,2) \qquad (6.2.1)$$

where

$$\delta_{j2} = 1\ , \text{ when } j = 2\ ,$$

$$= 0\ , \text{when } j \neq 2\ ,$$

$2l$ is a distance between centres of the two cylinders, θ_0 is the angle between the incident wave and a line joining the centres of the two cylinders, and polar coordinates (r_j , θ_j) are defined by

$$x_j = r_j \cos(\theta_j - \theta_0), \quad y_j = r_j \sin(\theta_j - \theta_0).$$

INNER REGION

Scaled coordinates for the inner region are defined by

$$\xi_j = \frac{x_j}{a}, \quad \eta_j = \frac{y_j}{a}.$$

From equation (6.1.9), the inner potential $\psi(\xi_j, \eta_j)$ satisfies

$$\frac{\partial^2 \psi}{\partial \xi_j^{\,2}} + \frac{\partial^2 \psi}{\partial \eta_j^{\,2}} + \varepsilon_1^2 \psi = 0. \tag{6.2.2}$$

Here $\varepsilon_1 = ka$.The body boundary condition (6.2.1) becomes

$$\frac{\partial \psi}{\partial n} = -\frac{\partial}{\partial n}\left(e^{i\varepsilon_1 \rho_j \cos(\theta_j - \theta_0)} \right) = -\frac{\partial}{\partial n}\left(i\varepsilon_1 \rho_j \cos(\theta_j - \theta_0) - \frac{\varepsilon_1^2}{2} \rho_j^2 \cos^2(\theta_j - \theta_0) + O(\varepsilon_1^3) \right) \tag{6.2.3}$$

on each cylinder. Here the local polar coordinates (ρ_j, θ_j) are defined by

$$\xi_j = \rho_j \cos\theta_j,$$

and

$$\eta_j = \rho_j \sin \theta_j.$$

The boundary condition (6.2.3) suggests that the leading-order inner potential $\psi^{(1)}$ is of order ε_1 and from (6.2.2-3) $\psi^{(1)}$ is a harmonic function and satisfies

$$\frac{\partial \psi^{(1)}}{\partial n} = - i \, \varepsilon_1 \frac{\partial}{\partial n} (\xi_j \cos \theta_0 + \eta_j \sin \theta_0) \tag{6.2.4}$$

on each cylinder. To find the solution for $\psi^{(1)}$, bi-polar coordinates are used. By using two-dimensional bipolar coordinates, Morse and Feshbach (1953, pp. 1210-1211) obtain the solution for Ξ_1 satisfying the boundary condition

$$\frac{\partial \Xi}{\partial n} = 0 \tag{6.2.5}$$

on each cylinder as

$$\Xi = \xi_j \cos \theta_0 + \eta_j \sin \theta_0 + f_1(\alpha , \beta) + f_2(\alpha , \beta) \tag{6.2.6}$$

where

$$f_1(\alpha , \beta) = \frac{2c_1}{a} \sum_{n=0}^{\infty} (-1)^n \, e^{-n \alpha_1} \frac{\cosh n(\alpha - \alpha_2)}{\sinh n(\alpha_1 - \alpha_2)} \cos (\theta_0 + n \beta), \tag{6.2.7}$$

$$f_2(\alpha , \beta) = - \frac{2c_2}{a} \sum_{n=0}^{\infty} (-1)^n \, e^{n \alpha_2} \frac{\cosh n(\alpha - \alpha_1)}{\sinh n(\alpha_1 - \alpha_2)} \cos (\theta_0 - n \beta), \tag{6.2.8}$$

and α and β are related to x and y by

$$x + jy = c \tanh \frac{(\alpha + j\beta)}{2}, \; (-\infty < \alpha < \infty, \; -\pi < \beta < \pi).$$

Also

$$\alpha_1 = \sinh^{-1}\left(\frac{c_1}{a}\right), \; c_1 = (l^2 - a^2)^{\frac{1}{2}}, \; \alpha_2 = \sinh^{-1}\left(\frac{c_2}{\delta a}\right), \; c_2 = (l^2 - \delta^2 a^2)^{\frac{1}{2}}.$$

Therefore from (6.2.4-6) the solution for $\psi^{(1)}$ is

$$\psi^{(1)} = i\,\varepsilon_1 (f_1 + f_2). \tag{6.2.9}$$

To find expansions of the series of $f_1(\alpha, \beta)$ and $f_2(\alpha, \beta)$, define

$$z = x + j\,y = c \coth \tau .$$

By using above equation, terms of series $f_1(\alpha, \beta)$ and $f_2(\alpha, \beta)$, can be written in the forms

$$(-1)^n \cosh n(\alpha - \alpha_2) \cos(\theta_0 + n\beta) = \frac{1}{2}\Big(\cosh 2\tau n \cosh t_n - \sinh 2n\tau \sinh t_n$$

$$+ \cosh 2n\tau^* \cosh t_n^* - \sinh 2n\tau^* \sinh t_n^* \Big),$$

and

$$(-1)^n \cosh n(\alpha - \alpha_1) \cos(\theta_0 - n\beta) = \frac{1}{2}\Big(\cosh 2n\tau \cosh v_n - \sinh 2n\tau \sinh v_n$$

$$+\cosh 2n\tau^* \cosh v_n^* - \sinh 2n\tau^* \sinh v_n^* \Big)$$

respectively. Here * denotes complex conjugate, and

$$t_n = n\,\alpha_2 - i\,\theta_0,\quad v_n = n\,\alpha_1 + i\,\theta_0 .$$

Expanding the above expressions for $f_1(\alpha, \beta)$ and $f_2(\alpha, \beta)$ as a series in inverse powers of ρ, and substitute into (6.2.9) gives, after some algebra,

$$\psi^{(1)} = i\,\varepsilon_1 \left(a_1 \cos\theta_0 + b_1 \cos\theta_0 \frac{\xi_j}{\rho_j^2} + b_2 \sin\theta_0 \frac{\eta_j}{\rho_j^2} + O\left(\frac{1}{\rho_j^4} \right) \right), \tag{6.2.10}$$

where

$$a_1 = \frac{2}{a} \sum_{n=0}^{\infty} \left(\frac{e^{-n\alpha_1} c_1 \cosh n\alpha_2}{\sinh(n(\alpha_1 - \alpha_2))} - \frac{e^{n\alpha_2} c_2 \cosh n\alpha_1}{\sinh(n(\alpha_1 - \alpha_2))} \right), \tag{6.2.11}$$

$$b_1 = -\frac{4c}{a^2} \sum_{n=0}^{\infty} \left(\frac{e^{-n\alpha_1} n\, c_1 \sinh n\alpha_2}{\sinh[n(\alpha_1 - \alpha_2)]} - \frac{e^{n\alpha_2} n\, c_2 \sinh n\alpha_1}{\sinh[n(\alpha_1 - \alpha_2)]} \right), \tag{6.2.12}$$

and

$$b_2 = \frac{4c}{a^2} \sum_{n=0}^{\infty} \left(\frac{e^{-n\alpha_1} n\, c_1 \cosh n\alpha_2}{\sinh[n(\alpha_1 - \alpha_2)]} - \frac{e^{n\alpha_2} n\, c_2 \cosh n\alpha_1}{\sinh[n(\alpha_1 - \alpha_2)]} \right). \tag{6.2.13}$$

Here c can be related by the equation,

$$2l = (a^2 + c^2)^{\frac{1}{2}} + (\delta^2 a^2 + c^2)^{\frac{1}{2}} .$$

Therefore from (6.2.10), the outer expansion of $\psi^{(1)}$ to order ε_1^2 is

$$\psi^{(1,2)} = i\,\varepsilon_1 \left(b_1 \frac{\cos\theta_j}{\rho_j} + b_2 \frac{\sin\theta_j}{\rho_j} \right). \tag{6.2.14}$$

OUTER REGION

Outer coordinates are defined by,

$$X = kx\,, \quad Y = ky \tag{6.2.15}$$

so that, from (6.1.9), Ψ (X,Y) satisfies

$$\frac{\partial^2\Psi}{\partial X^2} + \frac{\partial^2\Psi}{\partial Y^2} + \Psi = 0 \tag{6.2.16}$$

in the fluid region. Equation (6.2.14) suggests the leading-order outer potential

$$\Psi^{(2)} = \varepsilon_1^2\, \Psi_0 \tag{6.2.17}$$

where from (6.2.16), Ψ_0 satisfies

$$\frac{\partial^2\Psi_0}{\partial X^2} + \frac{\partial^2\Psi_0}{\partial Y^2} + \Psi_0 = 0 \tag{6.2.18}$$

in the fluid. The general solution for Ψ_0 satisfying the radiation condition and (6.2.14) is

$$\Psi_0 = A_0\, H_0(R) + (A_1 \cos\theta + B_1 \sin\theta)\, H_1(R), \tag{6.2.19}$$

where H_0 and H_1 denote the Hankel function of the first kind of order 0 and 1 respectively, and A_0, A_1, and B_1 are complex constants to be determined from the matching.

From (6.2.17) and (6.2.19) the inner expansion of $\Psi^{(2)}$ is

$$\Psi^{(2,1)} = -2\,\varepsilon_1\, A_1\, i\,\frac{\cos\theta_j}{\pi\,\rho_j} - 2\,\varepsilon_1\, B_1\, i\,\frac{\sin\theta_j}{\pi\,\rho_j}. \tag{6.2.20}$$

Matching (6.2.14) with (6.2.20) gives

$$A_1 = -\frac{\pi\, b_1}{2}, \tag{6.2.21}$$

and

$$B_1 = -\frac{\pi\, b_2}{2}. \tag{6.2.22}$$

Furthermore, the inner expansion of $\Psi^{(2)}$ to order ε_1^2 gives

$$\Psi^{(2,2)} = \varepsilon_1^2 A_0\,\Gamma - \frac{2\,\varepsilon_1\, i}{\pi}\left(A_1\,\frac{\cos\theta_j}{\rho_j} + B_1\,\frac{\sin\theta_j}{\rho_j}\right) + \frac{2\,A_0\, i}{\pi}\left(\varepsilon_1^2 \ln\varepsilon + \varepsilon_1^2 \ln\rho_j\right) \tag{6.2.23}$$

where

$$\Gamma = 1 + \frac{2i}{\pi} (\gamma - \ln 2), \tag{6.2.24}$$

and γ is Euler's constant. Equation (6.2.23) suggests that

$$\psi^{(2)} = \varepsilon_1 \, \psi_1 + \varepsilon_1^2 \ln \varepsilon_1 \, P_1 + \varepsilon_1^2 \, \psi_2 \tag{6.2.25}$$

where P_1 is an unknown constant and, from (6.2.2) and (6.2.3), ψ_2 is a harmonic function which satisfies

$$\frac{\partial \psi_2}{\partial n} = \frac{\partial}{\partial n} \left(\frac{\rho_j^2}{4} + \frac{\rho_j^2}{4} \cos [\, 2 \, \theta_j - \theta_0 \,] \right) \tag{6.2.26}$$

on each cylinder. The solution of ψ_2 is

$$\psi_2 = \left(a_2 \ln \rho_j + \kappa(\rho_j \, , \theta_j) \right) \tag{6.2.27}$$

where

$$a_2 = \frac{1 + \delta^2}{2} \tag{6.2.28}$$

and $\kappa(\rho_j \, , \theta_j)$ is a harmonic function satisfying

$$\frac{\partial \kappa}{\partial n} = \frac{\partial}{\partial n} (\frac{\rho_j^2}{4} \cos (2 \, \theta_j - \theta_0))$$

on each cylinder. The solution of κ is such that

$$\kappa = O\left(\frac{1}{\rho_j^2} \right)$$

and so is not necessary in the matching. i.e. $\psi^{(2,2)} \equiv \Psi^{(2,2)}$. Now writing the outer expansion of $\psi^{(2)}$ to order ε_1^2 and matching with (6.2.23) yields

$$A_0 = -\frac{\pi i (1+\delta^2)}{2}. \tag{6.2.29}$$

THE MEAN DRIFT FORCE:

Maruo (1960) and Newman (1967) show that the mean drift force on a body is related to the far field of the first-order scattering potential. Since

$$H_0(R) \approx \left(\frac{2}{\pi R} \right)^{\frac{1}{2}} e^{i(R-\frac{\pi}{4})}, \quad R -> \infty, \tag{6.2.30}$$

and

$$H_1(R) \approx -i \left(\frac{2}{\pi R} \right)^{\frac{1}{2}} e^{i(R-\frac{\pi}{4})}, \quad R -> \infty, \tag{6.2.31}$$

from (6.2.17), (6.2.19), (6.2.21-22), and (6.2.29-31) we have that

$$\Psi^{(2)} = \varepsilon_1^2 \left(\frac{2}{\pi R} \right)^{\frac{1}{2}} e^{i(R-\frac{\pi}{4})} f(\theta), \tag{6.2.32}$$

where

$$f(\theta) = -\frac{\pi i a_2 .}{2} - i\left((A_1 \cos\theta_0 + B_1 \sin\theta_0) \cos\theta + (B_1 \cos\theta_0 - A_1 \sin\theta_0) \sin\theta \right) \quad (6.2.33)$$

Evaluating the integral in (6.1.11) gives the mean horizontal drift force as

$$f_{mx}^{(2)} = \frac{\rho g A^2}{2k} \varepsilon_1^4 \pi^2 \left(1 + \frac{2kh}{\sinh 2kh} \right) \left(\frac{1}{8} (1 + \delta^2)^2 + A_1^2 + B_1^2 - A_1 \cos\theta_0 - B_1 \sin\theta_0 \right)$$

(6.2.34)

where

$$A_1 = \frac{2c}{a^2} \cos\theta_0 \sum_{n=0}^{\infty} \left(\frac{e^{-n\alpha_1} c_1 n \sinh n\alpha_2 - e^{n\alpha_2} n c_2 \sinh n\alpha_1}{\sinh [n(\alpha_1 - \alpha_2)]} \right), \quad (6.2.35)$$

and

$$B_1 = -\frac{2c}{a^2} \sin\theta_0 \sum_{n=0}^{\infty} \left(\frac{e^{-n\alpha_1} n c_1 \cosh n\alpha_2 + e^{n\alpha_2} n c_2 \cosh n\alpha_1}{\sinh [n(\alpha_1 - \alpha_2)]} \right). \quad (6.2.36)$$

McIver (1987) shows that the mean horizontal drift force on a single cylinder with radius a is given by

$$f_{1,mx}^{(2)} = \frac{5 \rho g A^2 \pi^2}{16 k} \left(1 + \frac{2kh}{\sinh 2kh} \right) \varepsilon_1^4 . \quad (6.2.37)$$

Therefore from (6.2.34) and (6.2.37), the ratio of the horizontal drift force on two cylinders to that on an isolated circular cylinder with radius a is given by

$$F_{mx}^{(2)} = \frac{8}{5}\left(\frac{1}{8}(1+\delta^2)^2 + A_1^2 + B_1^2 - \frac{(1+\delta^2)}{2} A_1 \cos\theta_0 + B_1 \sin\theta_0 \right). \qquad (6.2.38)$$

where θ_0 is the angle between incident wave and the line joining centres of two cylinders, whose radii are a and δa. A_1 and B_1 are given by equations (6.2.35) and (6.2.36) respectively.

6.3 N arbitrary cross-sectional cylinders

To calculate the ratio of the drift force on N cylinders to that on an isolated cylinder, we have to consider a single arbitrary cross-sectional cylinder first.

(a) Single cylinder:

The cylinder extends throughout the depth of the fluid and a typical radius of the cylinder cross-section C is denoted by a. The boundary value problem to be solved is given by the Helmholtz equation (6.1.9), the free-surface condition (6.1.5), bed condition (6.1.4), and the body condition (6.1.6). The body boundary condition is

$$\frac{\partial \phi}{\partial n} = -\frac{\partial}{\partial n}\left(e^{ikx} \right) \text{on } C, \qquad (6.3.1)$$

INNER REGION

Scaled coordinates for the inner region are defined as in section 6.2. Define a small parameter, $\varepsilon_1 = ka$. From equation (6.1.9) the inner potential $\psi(\xi, \eta)$ then satisfies

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} + \varepsilon_1 \psi = 0. \qquad (6.3.2)$$

and the body boundary condition (6.3.1) becomes

$$\frac{\partial \psi}{\partial n} = -\frac{\partial}{\partial n}\left(e^{i\varepsilon_1 \rho \cos\theta} \right) = -\frac{\partial}{\partial n}\left(i\varepsilon_1 \rho \cos\theta - \frac{\varepsilon_1^2}{2} \rho^2 \cos^2\theta + O(\varepsilon_1^3) \right) \text{on } C. \qquad (6.3.3)$$

Here the polar coordinates $(\rho\ ,\ \theta)$ are defined by

$$\xi = \rho \cos \theta,$$

and

$$\eta = \rho \sin \theta.$$

The boundary condition (6.3.3) suggests that the leading-order inner potential is

$$\psi^{(1)} = \varepsilon_1\, \psi_1 \tag{6.3.4}$$

where from (6.3.2-3), ψ_1 is a harmonic function and satisfies

$$\frac{\partial \psi_1}{\partial n} = -\, i\, \frac{\partial}{\partial n}\,(\, \rho \cos \theta\,) \text{ on } C. \tag{6.3.5}$$

The solution for ψ_1 can be written as

$$\psi_1 = i\, \tau_1 \tag{6.3.6}$$

where τ_1 is a harmonic function and satisfies

$$\frac{\partial \tau_1}{\partial n} = -\, \frac{\partial}{\partial n}\,(\, \rho \cos \theta\,) \text{ on } C. \tag{6.3.7}$$

Here the potential τ_1 is the disturbance to a uniform flow past C in the direction of wave incidence.

From Batchelor (1967, p.127), as $\rho \to \infty$

$$\tau_1 = \nu_0 \frac{\cos\theta}{\rho} + \lambda_0 \frac{\sin\theta}{\rho} + O(\frac{1}{\rho^2}), \tag{6.3.8}$$

where ν_0 and λ_0 are the dipole strengths of the cylinder in two orthogonal directions, and assumed to be known. Therefore from (6.3.4), (6.3.6), and (6.3.8), the outer expansion of $\psi^{(1)}$ to order ε^2 is

$$\psi^{(1,2)} = i\,\varepsilon_1 \left(\nu_0 \frac{\cos\theta}{\rho} + \lambda_0 \frac{\sin\theta}{\rho} \right) \tag{6.3.9}$$

OUTER REGION

Outer coordinates are defined as in section 6.2, equation (6.2.15), so that, from (6.1.9), Ψ (X,Y) satisfies

$$\frac{\partial^2\Psi}{\partial X^2} + \frac{\partial^2\Psi}{\partial Y^2} + \Psi = 0 \tag{6.3.10}$$

in the fluid region. Equation (6.3.9) suggests the leading-order outer potential

$$\Psi^{(2)} = \varepsilon_1^2\,\Psi_2 \tag{6.3.11}$$

where from (6.3.10), Ψ_2 satisfies

$$\frac{\partial^2\Psi_2}{\partial X^2} + \frac{\partial^2\Psi_2}{\partial Y^2} + \Psi_2 = 0 \tag{6.3.12}$$

in the fluid. As in section 6.2, the general solution for Ψ_2 satisfying the radiation condition and able to match with (6.3.9) is

$$\Psi_2 = A_0\, H_0(R) + (A_1 \cos\theta + B_1 \sin\theta)\, H_1(R)\,, \tag{6.3.13}$$

where H_0 and H_1 denote the Hankel function of the first kind and order 0 and 1 respectively, while A_0, A_1, and B_1 are complex constants to be determined from the matching. From (6.3.11) and (6.3.13), the inner expansion of $\Psi^{(2)}$ is

$$\Psi^{(2,1)} = -2\,\varepsilon_1\, A_1\, i\, \frac{\cos\theta}{\pi\,\rho} - 2\,\varepsilon_1\, B_1\, i\, \frac{\sin\theta}{\pi\,\rho}\,. \tag{6.3.14}$$

Matching (6.3.9) with (6.3.14) gives

$$A_1 = -\frac{\pi\, V_0}{2}\,, \tag{6.3.15}$$

and

$$B_1 = -\frac{\pi\,\lambda_0}{2}\,. \tag{6.3.16}$$

Furthermore, the inner expansion of $\Psi^{(2)}$, to order ε^2, gives

$$\Psi^{(2,2)} = \varepsilon_1^2 A_0\,\Gamma - \frac{2\,\varepsilon_1\, i}{\pi}\left(A_1\, \frac{\cos\theta}{\rho} + B_1\, \frac{\sin\theta}{\rho}\right) + \frac{2\,A_0\, i}{\pi}\left(\varepsilon_1^2 \ln\varepsilon_1 + \varepsilon_1^2 \ln\rho\right) \tag{6.3.17}$$

where Γ is defined in (6.2.24). Equation (6.3.17) suggests that

$$\psi^{(2)} = \varepsilon_1\,\psi_1 + \varepsilon_1^2 \ln\varepsilon_1\, P_1 + \varepsilon_1^2\,\psi_2 \tag{6.3.18}$$

where P_1 is an unknown constant and, from (6.3.2) and (6.3.3), ψ_2 is a harmonic function which satisfies

$$\frac{\partial \psi_2}{\partial n} = \frac{\partial}{\partial n}\left(\frac{\rho^2}{4} + \frac{\rho^2}{4} \cos 2\theta \right) \text{on } C. \tag{6.3.19}$$

The solution for ψ_2 is

$$\psi_2 = \tau_{21}(\rho , \theta) + \tau_{22}(\rho , \theta) \tag{6.3.20}$$

where τ_{21} and τ_{22} are harmonic functions and satisfying

$$\frac{\partial \tau_{21}}{\partial n} = \frac{1}{4} \frac{\partial}{\partial n} (\rho^2 \cos 2\theta) \text{ on } C, \tag{6.3.21}$$

and

$$\frac{\partial \tau_{22}}{\partial n} = \frac{1}{4} \frac{\partial \rho^2}{\partial n} \text{ on } C. \tag{6.3.22}$$

Since there is no term in θ because there is no circulation around the body and no logarithm ic term by the divergence theorem. Therefore from Batchelor (1967, pp.127), as $\rho -> \infty$

$$\tau_{21} = \nu_0 \frac{\cos\theta}{\rho} + \lambda_0 \frac{\sin\theta}{\rho} + \nu_1 \frac{\cos 2\theta}{\rho^2} + \lambda_1 \frac{\sin 2\theta}{\rho^2}, \tag{6.3.23}$$

and

$$\tau_{22} - \frac{S}{2\pi\, a^2} \ln \rho -> 0. \tag{6.3.24}$$

Here ν_1 and λ_1 are the quadrupole strengths which depend on C, and are assumed to be known, and S is the cross-sectional area of the body. Therefore from (6.3.18), (6.3.20), and (6.3.23-24), the outer expansion of $\psi^{(2)}$ to order ε_1^2 may be found. Matching with (6.3.17) yields

$$A_0 = - \frac{i\, S}{4\, a^2}. \tag{6.3.25}$$

Now the constants A_0, A_1, and B_1 are known, and the mean horizonal drift force on a single arbitrary cross-sectional cylinder may be calculated.

THE MEAN DRIFT FORCE

As before in section 6.2, by using Maruo's (1960) and Newman's (1967) theory, the mean drift force on a body can be calculated. From (6.3.13), (6.2.30-31), (6.3.15-16), and (6.3.25) we have that

$$\Psi^{(2)} = \varepsilon_1^2 \left(\frac{2}{\pi\, R} \right)^{\frac{1}{2}} e^{\,i\,(R - \frac{\pi}{4})} f(\theta), \tag{6.3.26}$$

where

$$f(\theta) = A_0 - i\,(\, A_1 \cos\theta + B_1 \sin\theta\,). \tag{6.3.27}$$

Therefore by integrating (6.1.11) with respect to θ, and after a little algebra, the mean horizonal drift force can be calculated as

$$f_{1,mx}^{(2)} = \frac{\rho g A^2}{2k}\,\varepsilon_1^4\,\pi^2\left(1+\frac{2kh}{\sinh 2kh}\right)\left(\frac{S^2}{8\,a^4}+\frac{\pi^2}{4}\,(\nu_0^{\,2}+\lambda_0^{\,2})+\frac{\pi\,\nu_0\,S}{4\,a^2}\right). \qquad (6.3.28)$$

(b) N cylinders:

The cylinders extend throughout the depth of the fluid and the typical cross-section of each cylinder is denoted by C_i($i = 1, 2, \ldots, N$). The boundary-value problem to be solved is given by the Helmholtz equation (6.1.9), the free-surface condition (6.1.4), bed condition (6.1.5), and the body condition (6.1.6). The body boundary condition is

$$\frac{\partial \phi}{\partial n_i} = -\frac{\partial}{\partial n_i}\left(e^{ikx} \right) \text{on } C_i. \tag{6.3.29}$$

OUTER REGION

Outer coordinates are defined by,

$$X = kx\,, \quad Y = ky\,. \tag{6.3.30}$$

Under the assumptions that, $\varepsilon = \dfrac{a}{l} \ll 1$, and $\mu = kl \ll 1$, from (6.1.9) the outer potential $\Psi(X, Y)$ satisfies equation (6.3.10). As before the leading-order outer potential is

$$\Psi^{(2)} = \mu^2\, \Psi_2\,. \tag{6.3.31}$$

The general solution for $\Psi^{(2)}$ satisfying (6.1.9) and the radiation condition, is given by

$$\Psi^{(2)} = A_0\, H_0(\, R\,) + (\, A_1 \cos\theta + B_1 \sin\theta\,)\, H_1(\, R\,), \tag{6.3.32}$$

where A_0, A_1, and B_1 are complex constants to be determined from the matching.

INTERMEDIATE REGION

Scaled coordinates for the intermediate region are defined by

$$\bar{x} = \frac{x}{l}\,,\ \bar{y} = \frac{y}{l}\,,\ \bar{r} = \frac{r}{l} \tag{6.3.33}$$

and from (6.1.9), $\bar{\psi}(\,\bar{x},\bar{y}\,)\ (\equiv \phi(\,x,y\,)\,)$ satisfies

$$\frac{\partial^2\bar{\psi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\psi}}{\partial\bar{y}^2} + \mu^2\,\bar{\psi} = 0 \tag{6.3.34}$$

within the fluid. Intermediate expansion of the outer solution suggests that the intermediate potential is expanded in μ as

$$\bar{\psi}^{(2)} = \mu\,\bar{\psi}_1 + \mu^2 \ln\mu\,\bar{\psi}_{21} + \mu^2\,\bar{\psi}_2\,. \tag{6.3.35}$$

Here only those terms required to determine the outer potential to leading order are given. Substitution of (6.3.35) into (6.3.34) shows that each term in the expansion of $\bar{\psi}$, to the order displayed, is a solution of the two-dimensional Laplace's equation. Therefore the solutions of each intermediate potential $\bar{\psi}_t$ ($t = 1, 21, 2$) are of the form

$$\bar{\psi}_t = \bar{A}_{t,0} + \sum_{j=1}^{N}\left(\bar{B}_{t,j0}\ln\bar{r}_j + (\,\bar{A}_{t,j1}\cos\theta_j + \bar{B}_{t,j1}\sin\theta_j\,)\,\frac{1}{\bar{r}_j}\right), \tag{6.3.36}$$

where $\bar{r}_j = \frac{r_j}{l}$ and $\bar{A}_{t,0}$, $\bar{A}_{t,j1}$, $\bar{B}_{t,j0}$, and $\bar{B}_{t,j1}$ are complex constants to be determined.

OUTER/INTERMEDIATE MATCHING:

The intermediate expansion of $\Psi^{(2)}$ to order μ^2 gives

$$\Psi^{(2,2)} = -\frac{2i\mu}{\pi}\left(A_1 \frac{\cos\theta}{\bar{r}} + B_1 \frac{\sin\theta}{\bar{r}} \right) + \mu^2 \ln\mu \frac{2A_0 i}{\pi} + \mu^2 \frac{2A_0 i}{\pi} \ln\bar{r} + \mu^2 A_0 \Gamma \quad (6.3.37)$$

where Γ is defined by equation (6.2.24). To obtain the outer expansion of the intermediate solution, define

$$z = r\,e^{i\theta}\,,\quad z_j = r_j\,e^{i\theta_j}\,,\quad \sigma_j = s_j\,e^{i\alpha_j}\,,$$

where (r, θ), and (r_j, θ_j) are polar coordinates of field point with respect to origin and body j respectively, and (s_j, α_j) are polar coordinates of body j with respect to origin. Therefore

$$z = \sigma_j + z_j\,.$$

In terms of outer variables, $Z = R\,e^{i\theta}$, the above equation can be rewritten as

$$\bar{z}_j = \frac{Z}{\mu}\left(1 - \frac{\mu\bar{\sigma}_j}{Z} \right),$$

where $\bar{z}_j = \frac{z_j}{l}$, and $\bar{\sigma}_j = \frac{\sigma_j}{l}$. Now using the above equation, the outer expansion of the intermediate potential up to order μ^2 is

$$\bar{\psi}^{(2,2)} = \mu\left(\bar{A}_{1,0} + \sum_{j=1}^{N}\left(\bar{B}_{1,j0}\left(\ln\bar{r} - \bar{r}_j \cos\alpha_j \frac{\cos\theta}{\bar{r}} - \bar{s}_j \sin\alpha_j \frac{\sin\theta}{\bar{r}} \right)\right.\right.$$

$$+ (\bar{A}_{1,j1} \cos\theta + \bar{B}_{1,j1} \sin\theta) \frac{1}{\bar{r}} \Bigg) \Bigg) + \mu^2 \ln\mu \left(\bar{A}_{21,0} + \sum_{j=1}^{N} \bar{B}_{21,j0} \ln\bar{r} \right)$$

$$+ \mu^2 \left(\bar{A}_{2,0} + \sum_{j=1}^{N} \bar{B}_{2,j0} \ln\bar{r} \right). \qquad (6.3.38)$$

Matching (6.3.37) with (6.3.38) gives

$$\bar{A}_{1,0} = \bar{B}_{1,j0} = \bar{B}_{21,j0} = 0 \,, \ A_1 = \frac{\pi i}{2} \sum_{j=1}^{N} \bar{A}_{1,j1} \,,$$

$$(6.3.39)$$

$$B_1 = \frac{\pi i}{2} \sum_{j=1}^{N} \bar{B}_{1,j1} \,, \ \frac{2iA_0}{\pi} = \bar{A}_{21,0} = \frac{2i}{\pi\,\Gamma} \bar{A}_{2,0} = \sum_{j=1}^{N} \bar{B}_{2,j0}.$$

INNER EXPANSION OF INTERMEDIATE SOLUTION:

Define

$$z_j = r_j e^{i\theta_j} \,, \ \sigma_{jp} = s_{jp} e^{i\alpha_{jp}} \,, \hat{z}_j = \rho_j e^{i\theta_j} \,.$$

where (s_{jp}, α_{jp}), and (ρ_j, θ_j) are polar coordinates of body p with respect to body j, and of the field point with respect to body j in terms of inner coordinates respectively, and (r_j, θ_j) are polar coordinates of body j with respect to origin. Then

$$z_j = \sigma_{jp} + z_p \,.$$

In terms of inner variables above equation can be written as

$$\overline{z}_p = -\,\overline{\sigma}_{jp}\,(\,1 - \frac{\varepsilon \hat{z}_j}{\overline{\sigma}_{jp}}\,)\,.$$

Here $\overline{z}_p = \frac{z_p}{l}$, and $\overline{\sigma}_{jp} = \frac{\sigma_{jp}}{l}$. Using the above equation, the inner expansion of the intermediate solution can be written as

$$\overline{\psi}^{(2)} = \mu \left(\frac{1}{\varepsilon \rho_j} \left(\overline{A}_{1,j1} \cos \theta_j + \overline{B}_{1,j1} \sin \theta_j \right) + \sum_{\substack{p=1 \\ p \neq j}}^{N} \left(\overline{A}_{1,p1} \left(-\frac{\cos \alpha_{jp}}{\overline{s}_{jp}} \right.\right.\right.$$

$$\left.\left.\left. - \frac{\varepsilon}{\overline{s}_{jp}^{\,2}}\, \rho_j \cos(\,\theta_j - 2\alpha_{jp}\,) \right) + \overline{B}_{1,p1} \left(-\frac{\sin \alpha_{jp}}{\overline{s}_{jp}} + \frac{\varepsilon}{\overline{s}_{jp}^{\,2}}\, \rho_j \sin(\,\theta_j - 2\alpha_{jp}\,) \right) \right) + O(\,\varepsilon^2\,) \right)$$

$$+ \mu^2 \ln \mu\, \overline{A}_{21,0} + \mu^2 \left(\overline{A}_{2,0} + \overline{B}_{2,j0} \ln \varepsilon \rho_j + \frac{1}{\varepsilon \rho_j}\,(\,\overline{A}_{2,j1} \cos \theta_j + \overline{B}_{2,j1} \sin \theta_j\,) \right.$$

$$+ \sum_{\substack{p=1 \\ p \neq j}}^{N} \left(\overline{B}_{2,j0} \left(\ln \overline{s}_{jp} - \frac{\varepsilon}{\overline{s}_{jp}}\, \rho_j \cos(\theta_j - \alpha_{jp}) + O(\varepsilon^2) \right) \right.$$

$$+ \overline{A}_{2,p1} \left(-\frac{\cos \alpha_{jp}}{\overline{s}_{jn}} - \frac{\varepsilon}{\overline{s}_{jp}^{\,2}}\, \rho_j \cos(\theta_j - 2\alpha_{jp}) \right)$$

$$+ \bar{B}_{2,p1}\left(-\frac{\sin\alpha_{jp}}{\bar{s}_{jp}} + \frac{\varepsilon}{\bar{s}_{jp}^{\,2}}\,\rho_j \sin(\theta_j - 2\alpha_{jp}) \right)\Bigg)\Bigg) + O(\varepsilon^2)\Bigg], \tag{6.3.40}$$

which will be used later. From equation (6.3.39), to calculate far field potential we need to compute $\bar{A}_{1,j1}$ and $\bar{B}_{1,j1}$, which are coefficients of dipole terms of order μ, and $\bar{B}_{2.,j0}$ which is coefficient of source term of order μ^2 These terms will be calculated by doing inner/intermediate matching.

INNER REGION

Scaled coordinates for the inner region of body j are defined by

$$\xi_j = \frac{x - x_j}{a},\ \eta_j = \frac{y - y_j}{a},\ \rho_j = (\xi_j^{\,2} + \eta_j^{\,2})^{\frac{1}{2}}. \tag{6.3.41}$$

From (6.1.9), the inner potential ψ_j satisfies

$$\frac{\partial^2 \psi_j}{\partial \xi_j^{\,2}} + \frac{\partial^2 \psi_j}{\partial \eta_j^{\,2}} + (\mu\varepsilon)^2\,\psi_j = 0. \tag{6.3.42}$$

From the boundary condition (6.3.29)

$$\frac{\partial \psi_j}{\partial n_i} = -\frac{\partial}{\partial n_i}\left(e^{i\mu(\varepsilon\xi_j + \bar{\zeta}_j)} \right) = -i\mu\varepsilon\frac{\partial \xi_j}{\partial n_i} + \mu^2\left(\varepsilon\,\bar{\zeta}_j\frac{\partial \xi_j}{\partial n_i} + \frac{\varepsilon^2}{2}\frac{\partial \xi_j^2}{\partial n_i} \right) + O(\mu^3\varepsilon) \text{ on } C_i,$$

(6.3.43)

where $\bar{\zeta}_j = \dfrac{x_j}{l}$, and n_i is a normal coordinate to the body.

The intermediate solution (6.3.35), and the body boundary condition (6.3.43), suggest that the inner solution can be written as an expansion in μ of the form

$$\psi_j^{(2)} = \mu\, \psi_{j,1} + \mu^2 \ln \mu\, \psi_{j,21} + \mu^2\, \psi_{j,2}\,. \tag{6.3.44}$$

By substituting (6.3.44) into (6.3.42), it is seen that each term in (6.3.44) is a solution of Laplace's equation and, from equations (6.3.43-44), the boundary conditions for $\psi_{j,1}$, $\psi_{j,21}$, and $\psi_{j,2}$ are

$$\frac{\partial \psi_{j,1}}{\partial n_i} = -i\varepsilon \frac{\partial}{\partial n_i}(\, \rho_j \cos \alpha_j\,)\,, \text{ on } C_i\,, \tag{6.3.45}$$

$$\frac{\partial \psi_{j,21}}{\partial n_i} = 0\,, \text{ on } C_i\,, \tag{6.3.46}$$

and

$$\frac{\partial \psi_{j,2}}{\partial n_i} = \varepsilon\, \overline{\zeta}_j \frac{\partial}{\partial n_i}(\rho_j \cos \alpha_j) + \frac{1}{2}\varepsilon^2 \frac{\partial}{\partial n_i}(\rho_j^2 \cos^2 \alpha_j)\,, \text{ on } C_i\,. \tag{6.3.47}$$

TO COMPUTE ψ_{j1}, $\psi_{j,21}$, AND $\psi_{j,2}$.

In order to match with (6.3.40), the solution of equation (6.3.46) is

$$\psi_{j,21} = \overline{A}_{21,0}\,. \tag{6.3.48}$$

Batchelor (1967, pp.127), shows that, as $\rho_j \to \infty$, the particular solution satisfying the boundary conditions (6.3.45) is

$$\psi_{j,1,p} = i\varepsilon \left(\nu_{j1} \frac{\cos\theta_j}{\rho_j} + \lambda_{j1} \frac{\sin\theta_j}{\rho_j} + O(\frac{1}{\rho_j^2}) \right), \tag{6.3.49}$$

where μ_{j1}, and λ_{j1} are dipole strengths which depend on the body contour C_i. The particular solution of (6.3.47) is

$$\psi_{j2,p} = \Omega_{j1}(\rho_j, \alpha_j) + \Omega_{j2}(\rho_j) \tag{6.3.50}$$

where Ω_{j1}, and Ω_{j2} are harmonic functions and satisfying

$$\frac{\partial\Omega_{j1}}{\partial n_i} = \varepsilon\, \overline{\zeta}_j \frac{\partial}{\partial n_i}(\rho_j \cos\theta_j) + \frac{\varepsilon^2}{4} \frac{\partial}{\partial n_i}(\rho_j^2 \cos 2\alpha_j), \tag{6.3.51}$$

and

$$\frac{\partial\Omega_{j2}}{\partial n_i} = \frac{1}{4} \frac{\partial}{\partial n_i}(\rho_j^2) \quad \text{on } C_i \tag{6.3.52}$$

However to calculate the far field potential we do not need to consider the solution Ω_{j1} to equation (6.3.51). From Batchelor (1967, pp.127), as $\rho_j -> \infty$, the solution for Ω_{j2} satisfies

$$\Omega_{j2} - \frac{S_j}{2\pi a^2} \ln \rho_j -> 0 \tag{6.3.53}$$

where S_j is the cross-sectional area of body i. The general form of $\psi_{j,1,c}$ ($t = 1,21,2$) which satisfies Laplace's equation, and has zero-normal derivative on body j, is

$$\psi_{j,\ell,c} = A_{j,\ell,0} + \sum_{m=1}^{\infty} \left(A_{j,\ell,m} \left(\rho_j^m \cos m\theta_j + \kappa_{j,\ell,m} \right) + B_{j,\ell,m} \left(\rho_j^m \sin m\theta_j + \chi_{j,\ell,m} \right) \right) \quad (6.3.54)$$

where $\kappa_{j,\ell,m}$ and $\chi_{j,\ell,m}$ are harmonic functions satisfying

$$\frac{\partial}{\partial n_j} \left(\rho_j^m \cos m\theta_j + \kappa_{j,\ell,m} \right) = 0 \text{ on } C_j \quad (6.3.55)$$

and

$$\frac{\partial}{\partial n_j} \left(\rho_j^m \sin m\theta_j + \chi_{j,\ell,m} \right) = 0 \text{ on } C_j. \quad (6.3.56)$$

To calculate the far-field potential we need to consider $\psi_{j,1}$. That is $m = 1$, and therefore, from Batchelor (1967, pp.127), as $\rho_j -> \infty$, we have

$$\kappa_{j,1,1} -> \nu_{j1} \frac{\cos\theta_j}{\rho_j} + \lambda_{j1} \frac{\sin\theta_j}{\rho_j} + O\left(\frac{1}{\rho_j^2}\right), \quad (6.3.57)$$

and

$$\chi_{j,1,1} -> \Upsilon_{j1} \frac{\cos\theta_j}{\rho_j} + \Lambda_{j1} \frac{\sin\theta_j}{\rho_j} + O\left(\frac{1}{\rho_j^2}\right). \quad (6.3.58)$$

INTERMEDIATE/INNER MATCHING:

In order to find constants A_0, A_1, and B_0, consider $\psi_{j,1}$ in the inner potential equation (6.3.44). From (6.3.49), (6.3.54), and (6.3.57-58), expanding each constant in (6.3.54) in powers of ε, the intermediate expansion of $\psi_{j,1}$ to order ε^3 is

$$\psi_{j,1} = \varepsilon\left(i\,\nu_{j1}\,\frac{\cos\theta_j}{\rho_j} + i\,\lambda_{j1}\,\frac{\sin\theta_j}{\rho_j} + A_{j1,1}\left(\rho_j\cos\theta_j + \nu_{j1}\,\frac{\cos\theta_j}{\rho_j} + \lambda_{j1}\,\frac{\sin\theta_j}{\rho_j}\right)\right.$$

$$\left. + B_{j1,1}\left(\rho_j\sin\theta_j + \Upsilon_{j1}\,\frac{\cos\theta_j}{\rho_j} + \Lambda_{j1}\,\frac{\sin\theta_j}{\rho_j}\right)\right)$$

$$+ \varepsilon^2\left(A_{j1,2}\left(\rho_j\cos\theta_j + \nu_{j1}\,\frac{\cos\theta_j}{\rho_j} + \lambda_{j1}\,\frac{\sin\theta_j}{\rho_j}\right)\right.$$

$$\left. + B_{j1,2}\left(\rho_j\sin\theta_j + \Upsilon_{j1}\,\frac{\cos\theta_j}{\rho_j} + \Lambda_{j1}\,\frac{\sin\theta_j}{\rho_j}\right)\right)$$

$$+ \varepsilon^3\left(A_{j1,3}\left(\rho_j\cos\theta_j + \nu_{j1}\,\frac{\cos\theta_j}{\rho_j} + \lambda_{j1}\,\frac{\sin\theta_j}{\rho_j}\right)\right.$$

$$\left. + B_{j1,3}\left(\rho_j\sin\theta_j + \Upsilon_{j1}\,\frac{\cos\theta_j}{\rho_j} + \Lambda_{j1}\,\frac{\sin\theta_j}{\rho_j}\right)\right) + O(\varepsilon^4). \qquad (6.3.59)$$

Similarly the expansion of the intermediate solution in powers of ε is equivalent to expanding the constant coefficients in (6.3.40), and this idea will be used here. To match with (6.3.59), consider the coefficient of μ in the inner expansion of intermediate solution, equation (6.3.40). The inner expansion of $\overline{\psi}_1$ to $O(\varepsilon^3)$ gives

$$\overline{\psi}_1 = \varepsilon\left(\overline{A}_{1,j1,1}\cos\theta_j + \overline{B}_{1,j1,1}\sin\theta_j\right)\frac{1}{\rho_j} + \varepsilon^2 \sum_{\substack{p=1\\ p\neq j}}^{N}\left(\frac{\overline{A}_{1,p1,2}}{\overline{s}_{jp}^{\,2}}\left(-\varepsilon\cos 2\alpha_{jp}\,\rho_j\cos\theta_j\right.\right.$$

$$\left. - \overline{s}_{jp}\cos\theta_{jp} - \varepsilon\sin 2\theta_{jp}\,\rho_j\sin\theta_j\right) + \frac{\overline{B}_{1,p1,2}}{\overline{s}_{jp}^{\,2}}\left(-\varepsilon\sin 2\theta_{jp}\,\rho_j\cos\theta_j\right.$$

$$\left.\left. + \varepsilon\cos 2\theta_{jp}\,\rho_j\sin\theta_j - \overline{s}_{jp}\sin\theta_{jp}\right)\right) + \varepsilon^3\left(\overline{A}_{1,j1,3}\cos\theta_j + \overline{B}_{1,j1,3}\sin\theta_j\right)\frac{1}{\rho_j}$$

$$+ O(\varepsilon^4), \tag{6.3.60}$$

Matching (6.3.59) with (6.3.60) yields

$$A_{j1,1} = A_{j1,2} = B_{j1,1} = B_{j1,2} = 0\,, \tag{6.3.61}$$

$$\overline{A}_{1,j1,1} = i\,\nu_{j1} + \nu_{j1}\,A_{j1,1} + \Upsilon_{j1}\,B_{j1,1},\;\; \overline{A}_{1,j1,3} = \nu_{j1}\,A_{j1,3} + B_{j1,3}\,\Upsilon_{j1}, \tag{6.3.62}$$

$$A_{j1,3} = -\sum_{\substack{p=1\\ p\neq j}}^{N}\left(\overline{A}_{1,p1,2}\cos 2\alpha_{jp} + \overline{B}_{1,p1,2}\sin 2\alpha_{jp}\right)\frac{1}{\overline{s}_{jp}^{\,2}}\,, \tag{6.3.63}$$

$$\overline{B}_{1,p1,1} = i\,\lambda_{j1},\;\; \overline{B}_{1,j1,3} = \lambda_{j1}\,A_{j1,3} + \Lambda_{j1}\,B_{j1,3}, \tag{6.3.64}$$

and

$$B_{j1,3} = - \sum_{\substack{p=1 \\ p \neq j}}^{N} \left(\bar{A}_{1,p1,2} \sin 2\alpha_{jp} - \bar{B}_{1,p1,2} \cos 2\alpha_{jp} \right) \frac{1}{\bar{s}_{jp}^{\,2}} . \qquad (6.3.65)$$

From equations (6.3.61-65)

$$\bar{A}_{1,p1,3} = - i\, v_{j1} \sum_{\substack{p=1 \\ p \neq j}}^{N} \left(v_{p1} \cos 2\alpha_{jp} + \lambda_{p1} \sin 2\alpha_{jp} \right) \frac{1}{\bar{s}_{jp}^{\,2}}$$

$$- i\, \Upsilon_{j1} \sum_{\substack{p=1 \\ p \neq j}}^{N} \left(v_{p1} \sin 2\alpha_{jp} - \lambda_{p1} \cos 2\alpha_{jp} \right) \frac{1}{\bar{s}_{jp}^{\,2}} , \qquad (6.3.66)$$

$$\bar{A}_{j1,1} = \varepsilon^{2} \left(i\, v_{j1} + \varepsilon^{2}\, \bar{A}_{1,j1,3} \right), \qquad (6.3.67)$$

$$\bar{B}_{1,j1,3} = - i\, \lambda_{j1} \sum_{\substack{p=1 \\ p \neq j}}^{N} \left(v_{p1} \cos 2\alpha_{jp} + \lambda_{p1} \sin 2\alpha_{jp} \right) \frac{1}{\bar{s}_{jp}^{\,2}}$$

$$- i\, \Lambda_{j1} \sum_{\substack{p=1 \\ p \neq j}}^{N} \left(v_{p1} \sin 2\alpha_{jp} - \lambda_{p1} \cos 2\alpha_{jp} \right) \frac{1}{\bar{s}_{jp}^{\,2}} , \qquad (6.3.68)$$

and

$$\bar{B}_{1,j1} = \varepsilon^{2} \left(i\, \lambda_{j1} + \varepsilon^{2}\, \bar{B}_{1,j1,3} \right). \qquad (6.3.69)$$

Similarly in the matching of the inner solution $\psi_{j,2}$ with the intermediate solution (6.3.35), the term in $\ln \rho_j$ gives

$$\bar{B}_{2,j0} = \varepsilon^2 \frac{S_j}{2\,\pi\,a^2} \tag{6.3.70}$$

Now from equations (6.3.39), and (6.3.66-70) A_0, A_1, and B_1 are known. That is

$$A_0 = -\frac{\varepsilon^2\, i}{4\, a^2} \sum_{j=1}^{N} S_j\,. \tag{6.3.71}$$

$$A_1 = -\varepsilon^2 \frac{\pi}{2} \left(\sum_{j=1}^{N} \nu_{j1} - \varepsilon^2 \sum_{j=1}^{N} (\,\nu_{j1}\, t_{j1} + \Upsilon_{j1}\, t_{j2}\,) \right), \tag{6.3.72}$$

and

$$B_1 = -\varepsilon^2 \frac{\pi}{2} \left(\sum_{j=1}^{N} \lambda_{j1} - \varepsilon^2 \sum_{j=1}^{N} (\,\lambda_{j1}\, t_{j1} + \Lambda_{j1}\, t_{j2}\,) \right). \tag{6.3.73}$$

where

$$t_{j1} = \sum_{\substack{p=1 \\ p \neq j}}^{N} \left(\nu_{p1} \cos 2\alpha_{jp} + \lambda_{p1} \sin 2\alpha_{jp} \right) \frac{1}{\bar{s}_{jp}^{\,2}}, \tag{6.3.74}$$

and

$$t_{j2} = \sum_{\substack{p=1 \\ p \neq j}}^{N} \left(\nu_{p1} \sin 2\alpha_{jp} - \lambda_{p1} \cos 2\alpha_{jp} \right) \frac{1}{\bar{s}_{jp}^{\,2}}. \tag{6.3.75}$$

From above equations (6.3.71-73) the mean horizonal drift force can be calculated.

THE MEAN DRIFT FORCE:

As in section 6.2, $f(\theta)$ can be defined as

$$f(\theta) = A_0 - i\,(A_1 \cos\theta + B_1 \sin\theta)\,. \tag{6.3.76}$$

Thus

$$\int_0^{2\pi} |f(\theta)|^2\,(1-\cos\theta)\,d\theta = \varepsilon^4 \left(\frac{\pi}{8\,a^4}\,(\sum_{j=1}^{N} S_j)^2 - \frac{\pi\,A_1}{2a^2\,\varepsilon^2}\sum_{j=1}^{N} S_j + \frac{\pi}{\varepsilon^4}\,(A_1{}^2 + B_1{}^2) \right), \tag{6.3.77}$$

where S_j is a cross-sectional area of the body j, and A_1 and B_1 are given in series form by equations (6.3.72) and (6.3.73) respectively. By substituting (6.3.77) into equation (6.1.10) and dividing by (6.3.27), the ratio of the mean horizonal drift force to that on an isolated cylinder is given by

$$F_{mx}{}^{(2)} = \frac{\left(\frac{1}{8\,a^4}\,(\sum_{j=1}^{N} S_j)^2 - \frac{A_1}{2a^2\,\varepsilon^2}\sum_{j=1}^{N} S_j + \frac{1}{\varepsilon^4}\,(A_1{}^2 + B_1{}^2) \right)}{\left(\frac{S^2}{8\,a^4} + \frac{\pi^2}{4}\,(\mu_0^2 + \lambda_0^2) + \frac{\pi\,\mu_0\,S}{4\,a^2} \right)}\,. \tag{6.3.78}$$

6.4 Results

Here results are presented for the long-wave limit of the drift force on two cylinders using the exact method in section 6.2 and the approximate theory derived under the assumption $a \ll l$ in section 6.3. Here a is a typical radius and $2l$ a typical spacing. The incident wave makes an angle θ_0 with the line joining the centres of the two cylinders. McIver (1987) considered only the long-wave limit of drift force on circular cylinders of equal radii under the assumption $a \ll l$. My aim is to investigate the effects of relaxing both these assumptions.

When $a/l \to 0$ the ratio of horizontal drift force is given by

$$F_{mx}^{(2)} = \frac{\left(\frac{1}{8a^4}(S_1+S_2)^2 + \frac{\pi}{4a^2}(S_1+S_2)(\nu_{11}+\nu_{21}) + \frac{\pi^2}{4}t^2 \right)}{\left(\frac{S^2}{8a^4} + \frac{\pi^2}{4}(\nu_0^2+\lambda_0^2) + \frac{\pi \nu_0 S}{4a^2} \right)}, \tag{6.4.1}$$

where

$$t^2 = (\nu_{11}+\nu_{21})^2 + (\lambda_{11}+\lambda_{21})^2 ,$$

S_1, S_2, and S are cross-sectional areas and ν_{11}, ν_{21}, λ_{11}, λ_{21}, λ_0, and ν_0 are dipole strengths of the body.

Firstly, two cylinders with the same radii were considered. Since $2l$ is the spacing of the two cylinders then the maximum value of a/l is one. From theory, when $a/l \to 0$ the ratio of horizontal drift force is four. Figure 6.1 is of drift force vs a/l for different angles of wave incidence θ_0 and comparison is made with McIver (1987). Figure 6.1 shows that the agreement is good when $a/l < 0.6$ and for all angles of incident wave, $0 \leq \theta_0 \leq \frac{\pi}{2}$. When $a/l > 0.6$, there is a difference in the graph since our solution gives the exact long-wave limit of the drift force, whereas McIver's solution is only an approximation valid for $a/l \ll 1$. As $\theta_0 \to \frac{\pi}{2}$ McIver's solution substantially underestimates the drift force.

Now cylinders with different radii are considered. The results in section 6.3 involve the cross-sectional area and dipole strengths for the cylinders. Newman (1977, pp.144) gives the dipole strength of a circular cylinder with radius δa as $(\delta a)^2$. Firstly, the ratio of horizontal drift force on two cylinders with radii a and $2a$ to that on an isolated circular cylinder with radius $2a$ was considered. From the result, as $a/l \to 0$ the drift force is 25. Figure 6.2 is of drift force vs a/l for different angles of wave incidence θ_0 and comparison is made between the two theories. In figure 6.2 the allowable range of values of a/l is $0 \leq a/l \leq 0.4$, since otherwise the two cylinders would overlap. Figure 6.2 shows that there is reasonable agreement up to about 0.15 for all angles of incident wave, $0 \leq \theta_0 \leq \frac{\pi}{2}$. Also from the figure, the difference between two solutions increases with increasing a/l and angles of incident wave, since the solution in section 6.2 gives the exact long-wave limit of the drift force whereas the solution in section 6.3 uses the approximation $a/l \ll 1$. Next, circular cylinders with radii a and $3a$ are considered. From result as $a/l \to 0$, the drift force is 100. The comparison is made with both results in sections 6.2 and 6.3 and results are displayed in figure 6.3. Here the range of values for a/l to avoid overlap is between 0.0 and 0.3. From figure 6.3 the agreement is good for values of a/l up to about 0.075. From figures 6.2 and 6.3 the results from each section for all angles of incident wave are same for values of $a/l \leq 0.1$, since when a/l decreases the distance between cylinders increases for fixed a. Therefore the drift force is nearly same for small values of a/l. That is bodies are not much affected when their distance increases. McIver (1987) shows that the drift force is proportional to $(\text{radius})^3$ for a single cylinder; therefore in figures 6.2 and 6.3, the drift force increases with size of the cylinder.

Now consider geometries other than circular cylinders. Here we consider one elliptic and one circular cylinder. The equation of the ellipse is given by

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \tag{6.4.2}$$

where $2b$ and $2a$ are lengths of major and minor axis of ellipse respectively. From Newman (1977, pp.144) the dipole coefficient for an elliptic cylinder is given by

$$v_{11} = \frac{1}{2}\left(1 + \frac{b}{a}\right). \tag{6.4.3}$$

Here the permitted range of values of a/l is

$$0 \le a/l \le \frac{2}{\left(1 + \frac{b}{a}\right)}, \tag{6.4.4}$$

otherwise the two cylinders would be overlapped. The range of values of a/l depends on b/a, here we consider $b/a = 0.5$, 1, and 2. From theory when $a/l \to 0$, the ratio of the drift force on two bodies to that on an isolated elliptic cylinder is given by 2.524 with $b/a = 0.5$. Similarly when $a/l \to 0$, the drift force is 4 and 7.300 with $b/a = 1$, and 2 respectively. My aim is to find out how the drift force changes with different body geometries. Figure 6.4 is drawn by using the result of section 6.3, and shows drift force vs a/l for different angles of incidence θ_0 with $b/a = 0.5$, 1, and 2. Here θ_0 is an angle between incident wave and the line joining centres of two cylinders. From the graph the drift force increases with body size and angles of incident wave. For small $a/l \le 0.1$, the drift force is same in each case for different angle of incident wave. However there is a big difference in the drift force with increasing a/l when $\theta_0 = \pi/2$. The results of figures 6.2 and 6.3 suggest that the curves in figure 6.4 underestimate the drift force.

CONCLUSION:

In section 6.2, the solution for the ratio of horizontal drift force was constructed for two different sizes of circular cylinder due to incident long waves. This solution gives the exact long wave limit but can be used only for two circular cylinders.

In section 6.3, a solution was constructed under assumption $\varepsilon = a/l \ll 1$ for N arbitrary cross-sectional cylinders and with results in terms of the cross-sectional area and dipole strengths of the bodies. This general solution could be used for finite number of any shaped bodies. The dipole strengths of various bodies, for example circular and elliptical cylinders were calculated by several authors, for example, Newman (1977, pp.144). The expression for ratio of the horizontal drift force confirms the N^2 behaviour found previously when the bodies are circular cylinders. Finally we note that close proximity of the cylinders can enhance the horizontal drift force by more than 50% compared to that of widely-spaced cylinders.

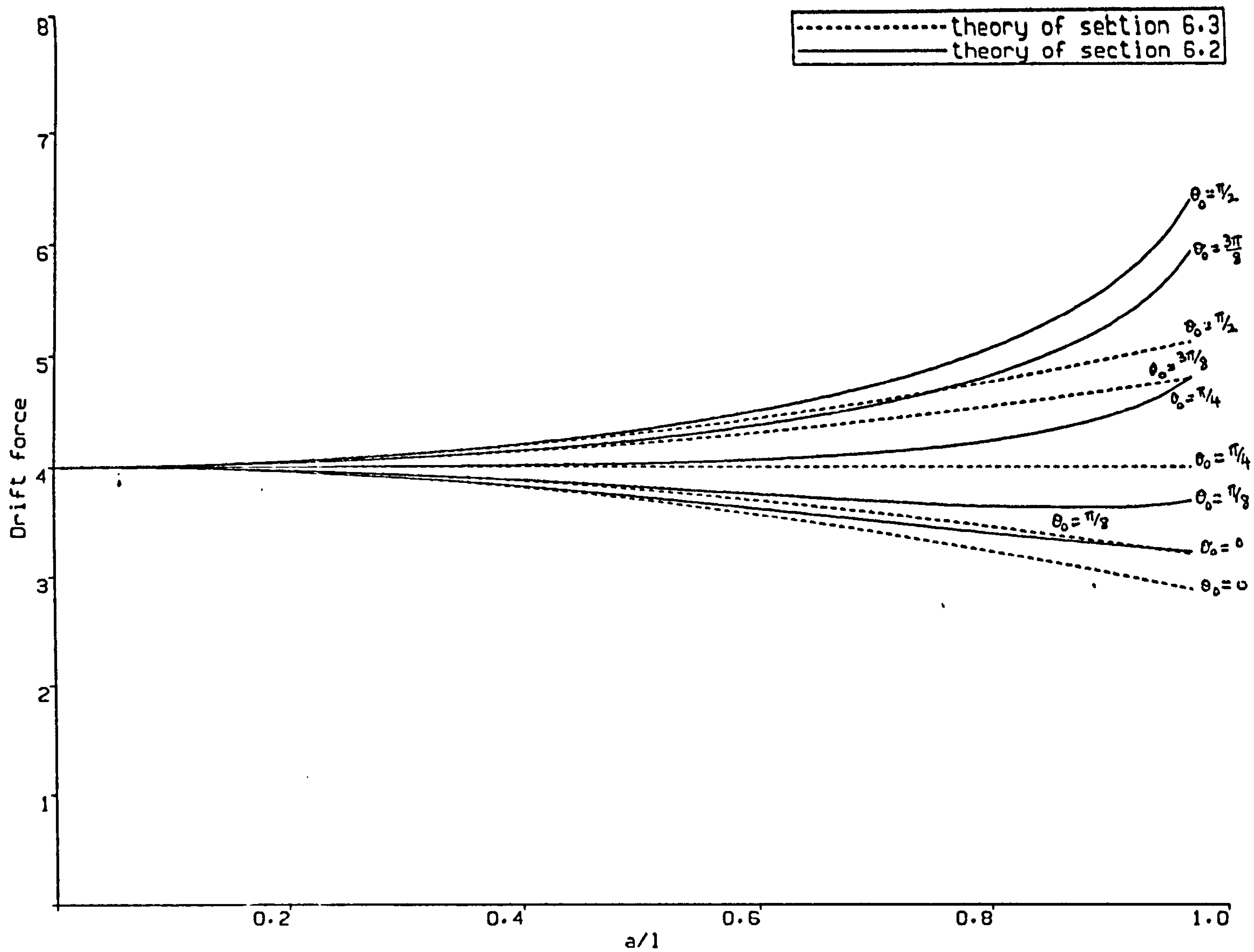


Figure 6.1. Two circular cylinders with same radius

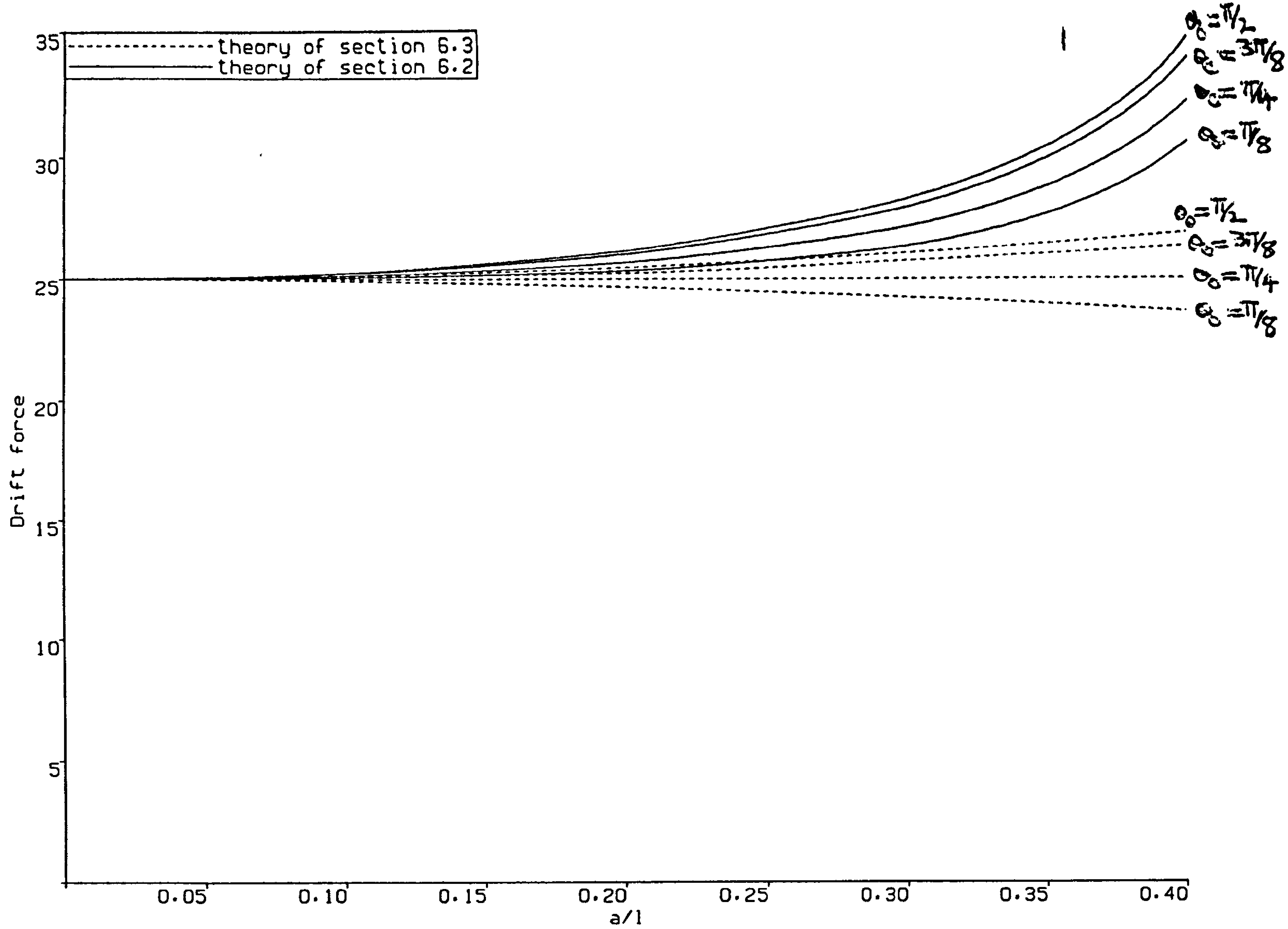


Figure 6.2. Two circular cylinders with different radii, a and 2a

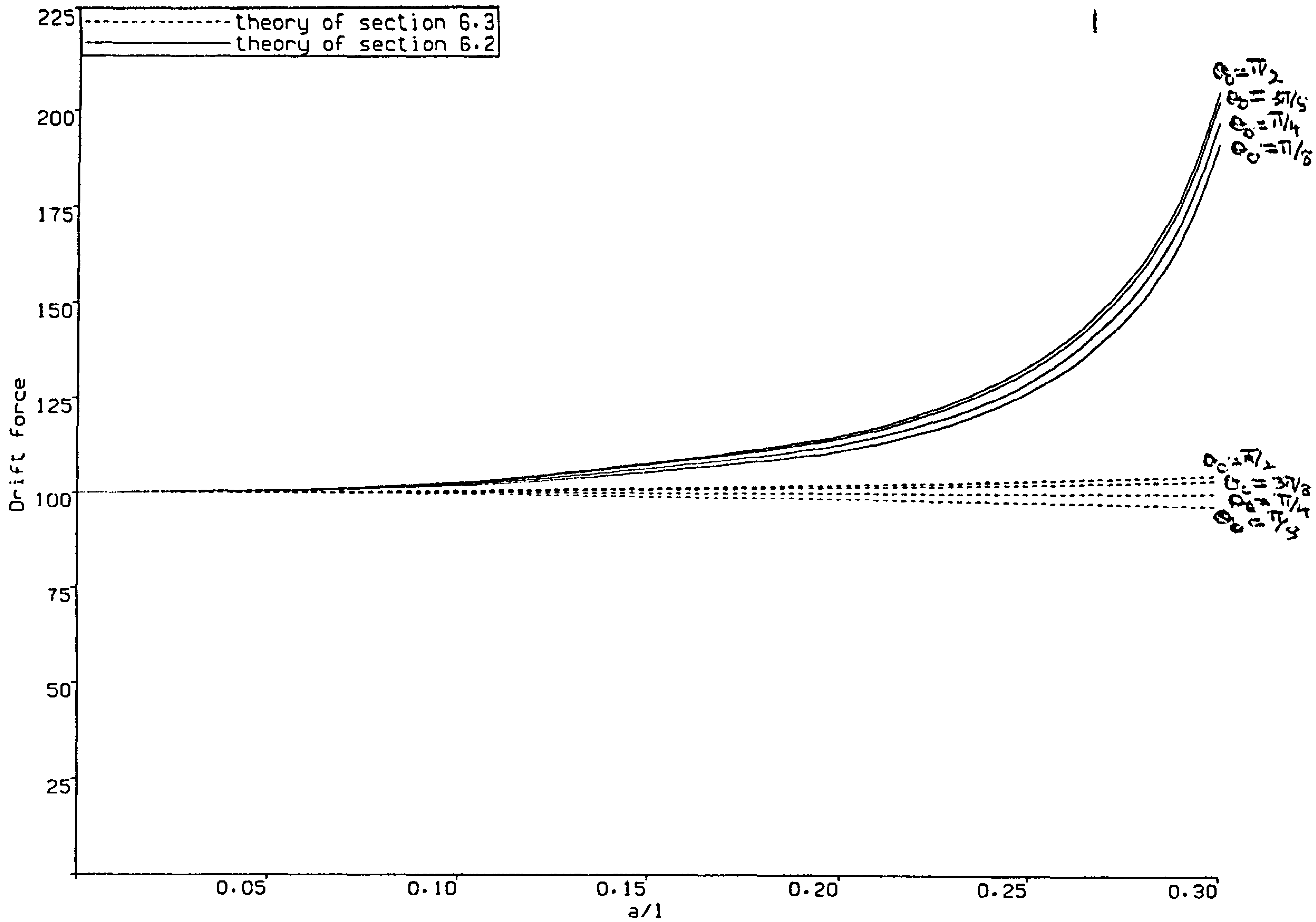


Figure 6.3. Two circular cylinders with different radii, a and 3a

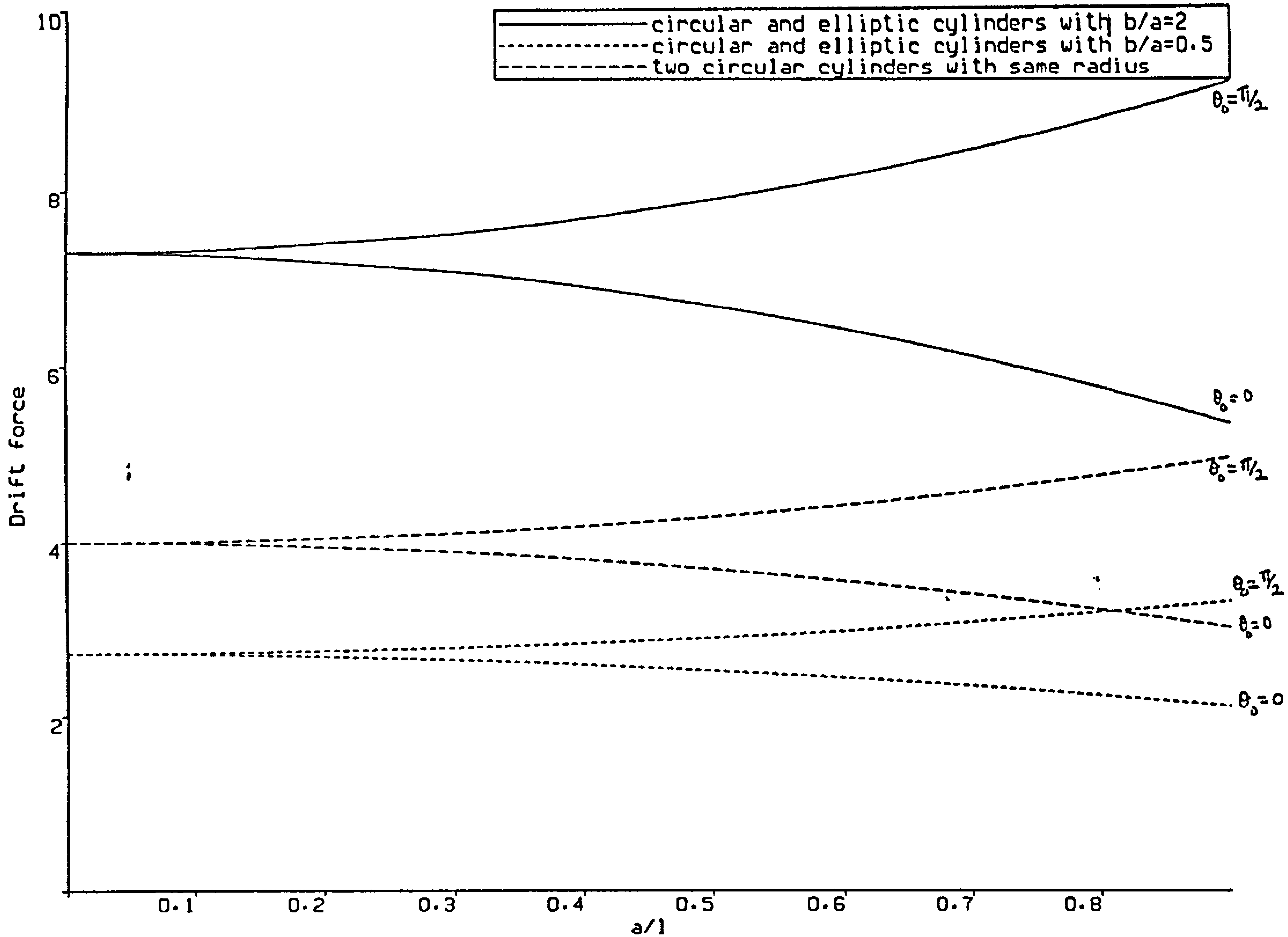

circular and elliptic cylinders with b/a=2
circular and elliptic cylinders with b/a=0.5
two circular cylinders with same radius
$\theta_0=\pi/2$
$\theta_0=0$
$\theta_0=\pi/2$
$\theta_0=\pi/2$
$\theta_0=0$
$\theta_0=0$
10
8
6
4
2
Drift force
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
a/l

Figure 6.4

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