Local minimizers in Micromagnetics

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Abstract

Let $\Omega \subset \mathbf{R}^3$ be a smooth bounded domain and consider the energy functional

$$J_\varepsilon(m) := \int_\Omega \left(\frac{1}{2\varepsilon}|Dm|^2 + \psi(m) + \frac{1}{2}|h-m|^2 \right) dx + \frac{1}{2}\int_{\mathbf{R}^3} |h_m|^2 \, dx.$$

Here ε is a small non negative parameter and the space of admissible functions for m is the Sobolev space of vector-valued functions $W^{1,2}(\Omega;\mathbf{R}^3)$ which satisfy the pointwise constraint $|m(x)|^2 - 1 = 0$ for a.e. $x \in \Omega$. The integrand $\psi : \mathbf{S}^2 \to \mathbf{R}$ is assumed to be a sufficiently smooth non negative density function with a multi-well structure. The function $h_m \in L^2(\mathbf{R}^3;\mathbf{R}^3)$ is related to m via Maxwells equations. Finally $h \in \mathbf{R}^3$ is a constant vector. The energy functional J_ε arises from the study of continuum models for ferromagnetic materials known as *micromagnetics* developed by W. Brown [7].

In this paper we aim to construct local energy minimizers for this functional. Our approach is based on studying the corresponding Euler-Lagrange equation and proving a *local existence* result for solutions close to a fixed constant solution. Our main device for doing this is a suitable version of the implicit function theorem. We then show that these solutions are local minimizers of J_ε in appropriate topologies by using certain sufficiency theorems for local minimizers.

Our analysis is applicable to a much broader class of functionals than the ones introduced above and on the way of proving our main results we reflect on some related problems.

1 Introduction

The micromagnetic theory of ferromagnetic materials as developed by Brown [7] consists of studying the minimizers of the energy functional

$$J_\varepsilon(m) = \int_\Omega \left(\frac{1}{2\varepsilon}|Dm|^2 + \psi(m) + \frac{1}{2}|h-m|^2 \right) dx + \frac{1}{2}\int_{\mathbf{R}^3} |h_m|^2 \, dx.$$

Here $\Omega \subset \mathbf{R}^3$ represents the region in space occupied by the body and the unknown function $m : \Omega \to \mathbf{S}^2$ denotes an arbitrary magnetization state for the body.

The parameter $\varepsilon > 0$ is related to the size of the body and is obtained by a simple rescaling argument (cf. e.g. DeSimone [8, 9]). The case $\varepsilon \to 0$ corresponds to the size of the body going to zero or what is known as the small particle limit. The various terms appearing in this energy functional are respectively

(i) The exchange energy: This term penalizes spatial changes in the magnetization m and hence describes the tendency of the body to maintain a spatially uniform magnetization state.
(ii) The anisotropy energy: This term describes the existence of preferred crystallographic directions or the so-called *easy axis* for the magnetic state of the material. To be more specific the anisotropy energy density $\psi : \mathbf{S}^2 \to \mathbf{R}$ is such that $\psi(m) \geq 0$ for all $m \in \mathbf{S}^2$ and $\psi(m) = 0$ if and only if $m \in K$ where K is a finite set of unit vectors representing the preferred crystallographic directions.
(iii) The external field energy: If the body lies in a region of space where an external magnetic field $h : \Omega \to \mathbf{R}^3$ is present, the magnetization m tends to align itself in the same direction as this field. The external field energy thus penalizes any deviation from this field inside the body. In this paper we assume that the applied field h is spatially uniform.
(iv) The field energy: The magnetization state m in the body generates a magnetic field $h_m : \mathbf{R}^3 \to \mathbf{R}^3$ that satisfies the Maxwell equations:

$$\left\{ \begin{array}{r} \mathbf{curl}\, h_m = 0, \\ \mathbf{div}\,(h_m + m\chi_\Omega) = 0. \end{array} \right.$$

The above equations show that the field h_m is nothing but the gradient part of the Helmholtz decomposition of $-m\chi_\Omega$. Thus in particular the field energy vanishes if and only if

$$\mathbf{div}\,(m\chi_\Omega) = 0.$$

We recall that recently James and Müller [18], following some earlier work by Lorentz [21] (cf. also Toupin [26]) obtained this field energy by studying the corresponding energy for a lattice of magnetic dipoles and passing to the continuum case by letting a typical lattice parameter go to zero.

In this paper we aim to construct local minimizers for the energy functional J_ε using a novel approach. We note that prior work on this problem due to DeSimone [9] employs ideas of De Giorgi or more precisely the notion of Gamma convergence which itself has been developed for the study of local minimizers by Kohn and Sternberg [19].

Our method is more direct. To be more specific we *construct* stationary points for the energy functional J_ε using an appropriate version of the implicit function theorem and then apply certain sufficiency theorems to establish the desired minimality property for these stationary points. It turns out that our results are stronger than the known ones in the sense that we show the stationary points constructed to be local minimizers of J_ε in weaker norms.

At this stage we should like to remark that the idea of applying versions of the implicit function theorem to achieve *local existence* for various equilibrium equations of continuum mechanics has been employed before in different contexts (cf. Ball *et. al.* [5], Valent [28], Zhang [30] for examples within elasticity theory). The novel idea in this paper however is to combine such local existence theorems together with certain sufficiency theorems to ensure the existence of a continuous branch of local energy minimizers.

Throughout this paper we assume that $\Omega \subset \mathbf{R}^n$ is a bounded domain (open connected set) with a smooth boundary $\partial\Omega$. We denote the unit outward normal to the boundary at a point x by $\nu(x)$. By $\mathcal{L}^n(\cdot)$ we denote the Lebesgue measure on $\mathbf{R}^n$. As regards the energy functional J_ε the dimension is $n = 3$. However we do not restrict our forthcoming analysis to this case only and therefore n can be any positive integer.

For the admissible class of functions at various stages of our work we use the Sobolev spaces of vector-valued functions $W^{m,p}(\Omega; \mathbf{R}^N)$ where m is a positive integer and the real exponent $p \geq 1$. Our terminology for these spaces throughout the article is in accordance with [1] and [31] and we refer the interested reader to these books for any reference on basic properties of these functions.

Assume now that $\mathcal{A} \subset W^{m,p}(\Omega; \mathbf{R}^N)$ is a given set of admissible functions and suppose that $J : \mathcal{A} \to \overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, \infty\}$ is a given functional. For later reference we state the following

Definition 1.1. *Let $1 \leq r \leq \infty$. The point $m_0 \in \mathcal{A}$ is an L^r local minimizer of J if and only if there exists $\delta > 0$ such that*

$$J(m_0) \leq J(m)$$

for all $m \in \mathcal{A}$ satisfying

$$||m - m_0||_{L^r(\Omega; \mathbf{R}^N)} < \delta.$$

To gain a clear understanding of the energy minimization problem described above we proceed by considering two related but slightly simplified problems each having *some* ingredients of the original micromagnetic energy functional.

In the first problem we consider the one parameter family of functionals

$$I_\varepsilon(u) := \int_\Omega \left(\frac{1}{2\varepsilon} |\nabla u|^2 + F(x, u) \right) \, dx,$$

where $F \in C^2(\overline{\Omega} \times \mathbf{R})$ and $\varepsilon > 0$ as before is a small parameter. The competing functions u are assumed to belong to the class

$$\mathcal{A}_1 := \{u \in W^{1,2}(\Omega) : I_\varepsilon \text{ is well defined}\}.$$

By well defined we mean that the function $F(\cdot, u(\cdot))$ has a well defined integral or more precisely at least one of the functions $F^+ := \max\{F(\cdot, u(\cdot)), 0\}$ or $F^- := \min\{F(\cdot, u(\cdot)), 0\}$ has a finite integral. It is therefore to be understood that $I : \mathcal{A}_1 \to \overline{\mathbf{R}}$. To specify the growth of F we assume that there are constants $C > 0$ and $p \geq 1$ such that

$$F(x, u) \geq -C(1 + |u|^p)$$

for all $x \in \Omega$ and all $u \in \mathbf{R}$.

As can be readily seen as this stage we have dropped the pointwise constraint $|u(x)| = 1$ from the admissible functions compared to the micromagnetic problem. In addition we have restricted attention to scalar valued functions, that is $N = 1$. We should however point out that this latter assumption is not a technical obstacle and almost all the statements and results in this case extend to $N > 1$ without any difficulty.

In our analysis special attention is made towards a limiting problem corresponding to the case $\varepsilon = 0$. We start by imposing conditions on the integrand F and a given point $\tilde{u} \in \mathcal{A}_1$ that turn out to be sufficient for $\tilde{u}$ to be a local minimizer of an appropriate functional corresponding to the $\varepsilon = 0$ problem. We then apply the implicit function theorem to prove the existence of a branch of stationary points u^ε for I_ε when $\varepsilon > 0$ is sufficiently small. Note that the Euler-Lagrange equation corresponding to I_ε takes the simple form

$$\begin{cases} \Delta u = \varepsilon F_u(x,u) & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu}(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Having established the existence of such stationary points we then proceed to studying the second variation of the functional I_ε at these points. Our starting assumptions on F and $\tilde{u}$ imply that the second variation at each u^ε is indeed positive and thus according to the sufficiency theorem in Section 2 (Theorem 2.2) these points are L^r local minimizers of the corresponding I_ε where the exponent r depends on the growth of F at infinity.

By imposing further assumptions on the integrand F we are able to show that for a sufficiently small range of the parameter ε the stationary points of I_ε obtained by the application of the implicit function theorem are the *only* stationary points of I_ε. This in particular means that if the limiting functional has only a finite number of *non degenerate* stationary points the same would hold true for I_ε when ε is small.

Having a clear understanding of the first problem we then proceed to the second family that consists of functionals in the form

$$E_\varepsilon(u) := \int_\Omega \left(\frac{1}{2\varepsilon} |Du|^2 + V(x,u) \right) dx,$$

where $V \in C^2(\overline{\Omega} \times \mathbf{R}^N)$. Here we aim to deal with the pointwise constraint $|u(x)| = 1$ and leave out the only remaining task i.e. handling the *non local* term in the original micromagnetics problem to the final stage. Thus we introduce the class of admissible functions

$$\mathcal{A}_2 := \{u \in W^{1,2}(\Omega; \mathbf{R}^N) : |u(x)| = 1 \quad \text{a.e.}\}.$$

It follows immediately from the constraint on u and the continuity assumption on V that E_ε is well defined and in fact finite over $\mathcal{A}_2$. In this setting it is also possible to assume without loss of generality that V has linear growth at infinity.

Similar to the first problem our analysis is linked to studying a limiting functional corresponding to the $\varepsilon = 0$ case. We impose conditions on the integrand V and a given $\tilde{u} \in \mathbf{S}^{N-1}$ that in turn would imply $\tilde{u}$ to be a *constrained* local minimizer of this latter functional.

It can be shown that here the Euler-Lagrange equation corresponding to E_ε takes the form

$$\begin{cases} \Delta u + |Du|^2 u - \varepsilon (I - u \otimes u) V_u(x,u) = 0, & \text{in } \Omega \\ Du\, \nu(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

We again apply the implicit function theorem to prove the existence of a continuous branch of solutions to the Euler-Lagrange equation corresponding to E_ε.

To deal with the pointwise constraint $|u(x)| = 1$ in applying Theorem 2.2, we extend the functional E_ε to $\tilde{E}_\varepsilon : W^{1,2}(\Omega; \mathbf{R}^N) \to \mathbf{R}$ in such a way that
(i) $\tilde{E}_\varepsilon(u) = E_\varepsilon(u)$ for every $u \in \mathcal{A}_2$,
(ii) If u^ε is a stationary point of E_ε it is also a stationary point of $\tilde{E}_\varepsilon$, and
(iii) $\delta^2 \tilde{E}_\varepsilon(u^\varepsilon) > 0$ for ε sufficiently small provided a similar condition hold for the solution to the $\varepsilon = 0$ problem.

It then follows from Theorem 2.2 that u^ε is an L^1 local minimizer of $\tilde{E}_\varepsilon$ and so (i) implies the same to be true for E_ε as $u \in \mathcal{A}_2$.

To end this introduction let us give a brief description of the plan of the present paper. In Section 2 we gather some known results and key tools that we will be frequently referring to throughout this article. This in particular includes the statements of both an appropriate version of the implicit function theorem and a sufficiency theorem for L^r local minimizers of a certain type of functionals. In Section 3 we study the first problem, namely the family of functionals I_ε. Section 4 continues with the first problem and includes a detailed analysis of the second variation of I_ε along the branch of stationary points constructed in Section 3. In addition we study the number of such solutions for fixed values of ε when this parameter is sufficiently small. In Section 5 we move on to the constrained problem, that is the study of the functionals E_ε. In the final section we return to the micromagnetics problem and apply the same ideas to construct L^1 local minimizers for the functional J_ε.

2 Preliminaries

In this section we gather some well-known results related to our analysis in the subsequent sections.

As pointed out in Section 1, our main tool for constructing solutions to the Euler-Lagrange equations is the implicit function theorem. We therefore present the following version, which is more suitable for the later applications and refer the interested reader to the books of Ambrosetti and Prodi [2] or Zeidler [29] for the proofs and further discussions.

Theorem 2.1. *Let X, Y, and Z be Banach spaces, U an open subset of $X \times Y$, and $T = T(\varepsilon, u)$ a C^1 map from U into Z. Let $(\varepsilon_0, u_0) \in U$ be such that $T(\varepsilon_0, u_0) = 0$ and $D_u T(\varepsilon_0, u_0)$ is a bijection of Y onto Z. Then there exist an open neighbourhood U_0 of (ε_0, u_0) in $X \times Y$, an open neighbourhood V_0 of ε_0 in X, and a C^1 function $\omega : V_0 \to Y$ such that*

$$\{(\varepsilon, u) \in U_0 : T(\varepsilon, u) = 0\} = \{(\varepsilon, u) : \varepsilon \in V_0, u = \omega(\varepsilon)\}.$$

Furthermore, U_0 can be chosen so that $D_u T(\varepsilon, u)$ is a bijection of Y onto Z for all $(\varepsilon, u) \in U_0$. In this case, if $\varepsilon \in V_0$ then

$$D\omega(\varepsilon) = -(D_u T(\varepsilon, \omega(\varepsilon)))^{-1} D_\varepsilon T(\varepsilon, \omega(\varepsilon)), \tag{2.1}$$

while if T is analytic at $(\varepsilon, \omega(\varepsilon))$ then ω is analytic at ε.

While the implicit function theorem can be applied to the Euler-Lagrange equation to establish the existence of a branch of solutions starting from a given function, we need certain sufficiency theorems to guarantee that such stationary points *can* be local minimizers for the corresponding functional.

We now state a sufficiency theorem for L^r local minimizers of functionals appearing in this article. For this let us assume that $F : \Omega \times \mathbf{R}^N \to \mathbf{R}$ is a given integrand and consider the functional

$$I(u) := \int_\Omega \left(\frac{1}{2} |Du|^2 + F(x, u) \right) dx,$$

over the class of admissible functions

$$\tilde{\mathcal{A}} := \{u \in W^{1,2}(\Omega; \mathbf{R}^N) : I \text{ is well defined}\}.$$

We can now state the following (cf. [24])

Theorem 2.2. *Let $F \in C^2(\overline{\Omega} \times \mathbf{R}^N)$ and assume that there are constants $C > 0$ and $p \geq 1$ such that*

$$F(x,u) \geq -C(1+|u|^p) \tag{2.2}$$

for all $x \in \Omega$ and all $u \in \mathbf{R}^N$. Furthermore let $u_0 \in \tilde{\mathcal{A}}$ be of class $L^\infty(\Omega;\mathbf{R}^N)$ and satisfy
(i) $\delta I(u_0,\varphi) := \frac{d}{dt} I(u_0 + t\varphi)|_{t=0} = 0$, and
(ii) $\delta^2 I(u_0,\varphi) := \frac{d^2}{dt^2} I(u_0 + t\varphi)|_{t=0} \geq \gamma ||\varphi||^2_{W^{1,2}(\Omega;\mathbf{R}^n)}$,
for all $\varphi \in C^\infty(\overline{\Omega};\mathbf{R}^N)$ and some $\gamma > 0$. Finally let $r = r(n,p) = \max(1, \frac{n}{2}(p-2))$. Then there exist $\sigma, \rho > 0$ such that

$$I(u) - I(u_0) \geq \sigma ||u - u_0||^2_{W^{1,2}(\Omega;\mathbf{R}^N)}$$

for all $u \in \tilde{\mathcal{A}}$ satisfying $||u - u_0||_{L^r(\Omega;\mathbf{R}^N)} < \rho$.

Note that the above theorem says more than u_0 being just an L^r local minimizer for I. The lower bound on $I(u) - I(u_0)$ shows that if $u \neq u_0$ the latter energy difference is strictly positive. In other words u_0 is a *strict* local minimizer of I. We refer the interested reader to [3] and [24] for more discussion on this and its connection to dynamic stability of u_0.

As pointed out earlier the magnetization m and the field h_m are related to one another by **Maxwells equations** which are as follows

$$\begin{cases} \mathbf{curl}\, h_m = 0, \\ \mathbf{div}\,(h_m + m\chi_\Omega) = 0. \end{cases} \tag{2.3}$$

In the next theorem we gather some of the important properties of the solution operator corresponding to this equation.

Theorem 2.3. *There exists a continuous linear operator $\mathcal{H} : L^2(\mathbf{R}^3;\mathbf{R}^3) \to L^2(\mathbf{R}^3;\mathbf{R}^3)$ such that*
(i) Given $m \in L^2(\Omega;\mathbf{R}^3)$, set $h_m := \mathcal{H}(m\chi_\Omega)$. Then equations (2.3) hold in the sense of distributions on $\mathbf{R}^3$.
(ii) For every m_1 and $m_2 \in L^2(\Omega;\mathbf{R}^3)$,

$$\int_{\mathbf{R}^3} h_{m_1} \cdot h_{m_2}\, dx = -\int_\Omega m_1 \cdot h_{m_2}\, dx = -\int_\Omega h_{m_1} \cdot m_2\, dx,$$

and so in particular

$$\int_{\mathbf{R}^3} |h_{m_1}|^2 = -\int_\Omega m_1 \cdot h_{m_1}.$$

(iii) There exists a positive definite, symmetric matrix D_e such that for every constant function m

$$\int_\Omega h_m\, dx = -D_e m.$$

For a proof of (i) we refer the reader to [16]. Part (ii) follows from (2.3) and a simple integration by parts argument. The proof of (iii) is a consequence of the linearity of $\mathcal{H}$ and (ii). See [9] for more details.

3 The unconstrained problem

We begin this section by formally deriving the Euler-Lagrange equation corresponding to the functional I_ε. For this we check the condition

$$\frac{d}{dt} I_\varepsilon(u + t\varphi)|_{t=0} = 0, \tag{3.1}$$

where the variation $\varphi \in C^\infty(\overline{\Omega})$. First, since equation (3.1) holds for all $\varphi \in C_0^\infty(\Omega)$ we deduce that

$$\Delta u = \varepsilon F_u(x, u). \tag{3.2}$$

Second, since equation (3.1) holds for all $\varphi \in C^\infty(\overline{\Omega})$ we get the natural boundary condition

$$\frac{\partial u}{\partial \nu} = 0. \tag{3.3}$$

Now we introduce the setting for the application of the implicit function theorem (cf. Theorem 2.1). A key point in the application of this theorem is the choice of the spaces X, Y and Z in order to ensure that the linearization of T at (ε_0, u_0) is a bijection. To exhibit this, as a first attempt in applying the implicit function theorem to (3.2) and (3.3) let us consider the map $T_1 : \mathbf{R} \times W^{2,s}(\Omega) \to L^s(\Omega) \times W^{1-1/s,s}(\partial\Omega)$ given by

$$T_1(\varepsilon, u) = \begin{pmatrix} \Delta u(x) - \varepsilon F_u(x, u(x)) \\ \frac{\partial u}{\partial \nu}(x) \end{pmatrix},$$

for some $s > \frac{n}{2}$. Clearly if $u^\varepsilon \in W^{2,s}(\Omega)$ is such that $T_1(\varepsilon, u^\varepsilon) = 0$, then u^ε would be the required branch of stationary points of I_ε, that is a continuous family of solutions to (3.2) and (3.3) in $W^{2,s}(\Omega)$. However it is a trivial matter to see that for the above choice of spaces the linearization of T_1 at any point $(0, u) \in \mathbf{R} \times W^{2,s}(\Omega)$ is not a bijection. To overcome this difficulty and also to motivate the proof of Theorem 3.1 let us formally seek a solution to (3.2) and (3.3) in the form

$$u(\varepsilon) = u + \varepsilon v + \varepsilon^2 w + ...$$

Substituting this into the equation it immediately follows that $u = \tilde{u}$ is constant. Moreover other powers of ε lead to further equations, namely

$$\begin{cases} \Delta v = F_u(x, \tilde{u}) \\ \frac{\partial v}{\partial \nu}(x) = 0, \end{cases} \tag{3.4}$$

for the coefficients of ε, and similarly

$$\begin{cases} \Delta w = F_{uu}(x, \tilde{u})\, v \\ \frac{\partial w}{\partial \nu}(x) = 0, \end{cases} \tag{3.5}$$

for the coefficients of ε^2. It follows that a necessary condition for solvability of (3.4) is that

$$\int_\Omega F_u(x, \tilde{u})\, dx = 0.$$

Moreover the solution obtained in this way is unique up to an additive constant. Substituting this solution v into (3.5) and using the necessary condition for solvability of (3.5), that is

$$\int_\Omega F_{uu}(x,\tilde{u})v(x)\,dx = 0,$$

it follows that this constant is uniquely determined provided

$$\int_\Omega F_{uu}(x,\tilde{u})\,dx \neq 0.$$

Following this informal discussion and to proceed with the detailed analysis let us introduce the map

$$T : \mathbf{R} \times W^{2,s}(\Omega) \to E^s(\Omega) \times \mathbf{R}$$

by

$$T(\varepsilon,u) = \begin{pmatrix} \Delta u(x) - \varepsilon\left(F_u(x,u(x)) - ⨍_\Omega F_u(x,u(x))\,dx\right) \\ \frac{\partial u}{\partial \nu}(x) \\ ⨍_\Omega F_u(x,u(x))dx \end{pmatrix},$$

where

$$E^s(\Omega) = \left\{(f,g) \in L^s(\Omega) \times W^{1-1/s,s}(\partial\Omega) : \int_\Omega f\,dx = \int_{\partial\Omega} g\,d\mathcal{H}^{n-1}\right\}, \tag{3.6}$$

and we set

$$s > \frac{n}{2}. \tag{3.7}$$

It is clear that for $\varepsilon \neq 0$ a function $u \in W^{2,s}(\Omega)$ is a solution of the Euler-Lagrange equation (3.2) with boundary condition (3.3) if and only if it satisfies

$$T(\varepsilon,u) = 0.$$

We now claim that for the choice of s given by (3.7), $T \in C^1(\mathbf{R} \times W^{2,s}(\Omega); E^s(\Omega) \times \mathbf{R})$. To show this we look at the Gateaux derivative D_uT at an arbitrary point $(\varepsilon,u) \in \mathbf{R} \times W^{2,s}(\Omega)$. Indeed we have

$$D_uT(\varepsilon,u)(w) = \begin{pmatrix} \Delta w - \varepsilon\left(F_{uu}(x,u)w - ⨍_\Omega F_{uu}(x,u)w\,dx\right) \\ \frac{\partial w}{\partial \nu}(x) \\ ⨍_\Omega F_{uu}(x,u)w\,dx \end{pmatrix}$$

for each $w \in W^{2,s}(\Omega)$. So according to [27] D_uT is continuous if and only if for every sequence $u^{(k)} \to u$ in $W^{2,s}(\Omega)$,

$$\sup\left\{||(D_uT(\varepsilon,u^{(k)}) - D_uT(\varepsilon,u))(w)||_{(E^s(\Omega)\times\mathbf{R})} : ||w||_{W^{2,s}(\Omega)} \leq 1\right\} \to 0.$$

But this follows immediately by recalling that $u^{(k)} \to u$ in $L^\infty(\Omega)$ as a result of (3.7). A similar argument can be applied to $D_\varepsilon T$ and so the claim is justified.

To check that the remaining assumptions of Theorem 2.1 are true we begin by solving the equation $T(0,u) = 0$, i.e.

$$\begin{cases} \Delta u = 0, \\ \frac{\partial u}{\partial \nu} = 0, \\ ⨍_\Omega F_u(x,u(x))dx = 0. \end{cases}$$

It follows from the first two equations that u is a constant. Call this constant $\tilde{u}$. We are therefore left with the third equation,

$$\fint_\Omega F_u(x,\tilde{u})\,dx = 0. \tag{3.8}$$

Assume there exists $\tilde{u}$ such that (3.8) holds. To check the second assumption of Theorem 2.1 we need to show that the linear operator $D_uT(0,\tilde{u}) : W^{2,s}(\Omega) \to E^s(\Omega)\times\mathbf{R}$ is bijective. This amounts to proving that the system

$$\begin{cases} \Delta w = f, \\ \frac{\partial w}{\partial \nu} = g, \\ \fint_\Omega F_{uu}(x,\tilde{u})w(x)dx = t \end{cases}$$

has a unique solution $w \in W^{2,s}(\Omega)$ for all $(f,g,t) \in E^s(\Omega)\times R$. It is well-known (see e.g. [28]) that given $(f,g) \in E^s(\Omega)$, the system

$$\begin{cases} \Delta w = f, \\ \frac{\partial w}{\partial \nu} = g \end{cases}$$

has a solution $w \in W^{2,s}(\Omega)$, which is unique up to an additive constant. If

$$\fint_\Omega F_{uu}(x,\tilde{u})\,dx \neq 0$$

this constant can be determined in a unique way by solving the third equation,

$$\fint_\Omega F_{uu}(x,\tilde{u})w(x)dx = t.$$

Thus we have proved

Theorem 3.1. *Suppose there exists a real constant $\tilde{u}$ such that*

$$\int_\Omega F_u(x,\tilde{u})dx = 0 \tag{3.9}$$

and that

$$\int_\Omega F_{uu}(x,\tilde{u})dx \neq 0. \tag{3.10}$$

Then for ε small enough the Euler-Lagrange equation (3.2) *subject to the boundary condition* (3.3) *has a solution u^ε which is contained in the Sobolev space $W^{2,s}(\Omega)$ and is close to $\tilde{u}$ in the corresponding norm. Furthermore if the neighbourhood of $\tilde{u}$ in $W^{2,s}(\Omega)$ is taken small enough, u^ε is the only solution to* (3.2), (3.3) *that lying in this neighbourhood.*

Having proved the existence of a continuous branch of stationary points for I_ε, we proceed by addressing the question under what conditions on the integrand F and the point $\tilde{u}$ would u^ε provide a local minimizer for I_ε. We pursue this in the following section.

4 Local minimizers and the positivity of the second variation

We now proceed by considering the question of positivity for the quadratic functional

$$\tilde{J}(\varphi) = \int_\Omega \left(|\nabla\varphi|^2 + a(x)\varphi^2\right)\, dx, \tag{4.1}$$

over $W^{1,2}(\Omega)$ for given $a \in L^\infty(\Omega)$. Setting φ to be constant it follows immediately that the condition

$$\int_\Omega a\, dx > 0 \tag{4.2}$$

is necessary. We can however prove

Proposition 4.1. *Let $\tilde{J}$ be as in* (4.1) *and a satisfy* (4.2)*. Then there exists $\gamma > 0$ such that*

$$\tilde{J}(\varphi) \geq \gamma ||\varphi||^2_{W^{1,2}(\Omega)} \tag{4.3}$$

provided $||a||_{L^\infty(\Omega)}$ is sufficiently small.

Proof. Given $\varphi \in W^{1,2}(\Omega)$ we can write $\varphi = \tilde{\varphi} + \fint_\Omega \varphi\, dx$, where $\fint_\Omega \tilde{\varphi}\, dx = 0$. Thus setting $c = \fint_\Omega \varphi\, dx$ we can write

$$\begin{aligned} \tilde{J}(\varphi) &= \int_\Omega \left(|\nabla\tilde{\varphi}|^2 + a\,(\tilde{\varphi} + c)^2\right)\, dx \\ &= \int_\Omega \left(|\nabla\tilde{\varphi}|^2 + a\tilde{\varphi}^2 + 2ac\tilde{\varphi} + ac^2\right)\, dx \\ &\geq \int_\Omega \left(|\nabla\tilde{\varphi}|^2 + a\tilde{\varphi}^2\right)\, dx - \tau\int_\Omega \tilde{\varphi}^2\, dx \\ &\quad + c^2\int_\Omega \left(a(1 - \frac{a}{\tau})\right)\, dx, \end{aligned}$$

where we have used

$$-2\int_\Omega ac\tilde{\varphi} \leq \int_\Omega \left(\tau\tilde{\varphi}^2 + \frac{1}{\tau}a^2c^2\right)\, dx,$$

that holds for every $\tau > 0$. If now $||a||_{L^\infty(\Omega)} < \lambda_2$ where $\lambda_2 > 0$ denotes the second eigenvalue of the Laplacian subject to Neumann boundary conditions on $\partial\Omega$ and τ is sufficiently small the sum of the first two terms would be positive. Thus there exists $\alpha > 0$ such that

$$\tilde{J}(\varphi) \geq \alpha||\varphi||^2_{L^2(\Omega)}.$$

Now we use an argument similar to Proposition 4.4 in [25] to show that this implies (4.3). For this let

$$\tilde{J}_1(\varphi) := \tilde{J}(\varphi) - \frac{\alpha}{2}||\varphi||^2_{L^2(\Omega)} \geq \frac{\alpha}{2}||\varphi||^2_{L^2(\Omega)}.$$

Then (4.3) follows if we show

$$\tilde{J}_1(\varphi) \geq \beta||\nabla\varphi||^2_{L^2(\Omega;\mathbf{R}^n)}$$

for some $\beta > 0$. Indeed if this where not true there would be a sequence of nonzero functions $\{\varphi^{(j)}\}$ such that

$$\frac{1}{j}||\nabla\varphi^{(j)}||^2_{L^2(\Omega;\mathbf{R}^n)} > \tilde{J}_1(\varphi^{(j)}) \geq \frac{\alpha}{2}||\varphi^{(j)}||^2_{L^2(\Omega)} > 0.$$

Note that from this it follows that $||\nabla\varphi^{(j)}||_{L^2(\Omega;\mathbf{R}^n)} \neq 0$ and so letting

$$\psi^{(j)} = \varphi^{(j)}/||\nabla\varphi^{(j)}||_{L^2(\Omega;\mathbf{R}^n)}$$

and appealing to the quadratic nature of $\tilde{J}_1$ we get

$$\frac{1}{j} > \int_\Omega \left(1 + (a(x) - \frac{\alpha}{2})(\psi^{(j)})^2\right) dx \geq \frac{\alpha}{2}||\psi^{(j)}||^2_{L^2(\Omega)}. \tag{4.4}$$

The boundedness of the sequence $\{\psi^{(j)}\}$ in $W^{1,2}(\Omega)$ implies that by passing to a subsequence if necessary

$$\psi^{(j)} \rightharpoonup \psi \quad \text{in} \quad W^{1,2}(\Omega), \qquad \psi^{(j)} \to \psi \quad \text{in} \quad L^2(\Omega),$$

and so by (4.4) $\psi = 0$. The contradiction now follows since $\|\nabla\psi^{(j)}\|_{L^2(\Omega;\mathbf{R}^n)}$. This completes the proof. □

As a consequence of the above proposition and Theorem 2.2 we can state the following

Theorem 4.1. *Assume that the hypotheses of Theorem* 3.1 *hold and that*

$$\int_\Omega F_{uu}(x, \tilde{u})\, dx > 0. \tag{4.5}$$

Then the solution u^ε given by Theorem (3.1) *is an L^∞ local minimizer of I_ε. Furthermore if the growth of F from below is restricted by*

$$F(x, u) \geq -C(1 + |u|^p)$$

for some $C > 0$ and $p \geq 1$, then u^ε is an L^r local minimizer with $r(n, p) = \max(1, \frac{n}{2}(p-2))$. In particular if F is bounded from below then u^ε is an L^1 local minimizer of I_ε.

Proof. We start by calculating the second variation of I_ε at the stationary point u^ε. Indeed for $\varphi \in C^\infty(\overline{\Omega})$

$$\begin{aligned} \delta^2 I_\varepsilon(u^\varepsilon, \varphi) &= \frac{d^2}{dt^2} I_\varepsilon(u^\varepsilon + t\varphi)\,|_{t=0} \\ &= \frac{1}{\varepsilon}\int_\Omega \left(|\nabla\varphi|^2 + \varepsilon F_{uu}(x, u^\varepsilon)\varphi^2\right) dx. \end{aligned} \tag{4.6}$$

Note that

$$\int_\Omega F_{uu}(x, u^\varepsilon)\, dx \geq \int_\Omega F_{uu}(x, \tilde{u})\, dx - \int_\Omega |F_{uu}(x, u^\varepsilon) - F_{uu}(x, \tilde{u})|\, dx > 0$$

provided ε is sufficiently small. Thus it follows from Proposition 4.1 that for ε small enough

$$\delta^2 I_\varepsilon(u^\varepsilon, \varphi) \geq \gamma ||\varphi||^2_{W^{1,2}(\Omega)}$$

for some $\gamma = \gamma(\varepsilon) > 0$ and all $\varphi \in W^{1,2}(\Omega)$. The result is now a consequence of Theorem 2.2. $\square$

Remark 4.1. Consider the function $I_0 : \mathbf{R} \to \mathbf{R}$ given by

$$I_0(u) := \int_\Omega F(x,u)\,dx.$$

It is clear that conditions (3.9) and (4.5) are sufficient for $\tilde{u}$ to be a local minimum of I_0. In Theorem 4.1 we have shown that under these conditions one can construct a continuous branch of local minimizers for I_ε that starts off from a local minimum of I_0.

We now wish to make a simple observation regarding the global minimizers of I_ε and their possible connection to those of I_0.

Proposition 4.2. *Let $F(x,u) \geq C_1 + C_2|u|$ for some $C_2 > 0$ and let $\{u^\varepsilon\}$ be a sequence such that $I_\varepsilon(u^\varepsilon) < M$ for some constant M. Then by passing to a subsequence if necessary $u^\varepsilon \to \tilde{u}$ in $W^{1,2}(\Omega)$ where $\tilde{u}$ is a constant.*

Proof. It follows from the coercivity condition above that

$$\begin{cases} \{u^\varepsilon\} & \text{is bounded in } L^1(\Omega), \\ \{\nabla u^\varepsilon\} & \text{is bounded in } L^2(\Omega; \mathbf{R}^n). \end{cases}$$

Hence $\{u^\varepsilon\}$ is bounded in $W^{1,2}(\Omega)$ and therefore by passing to a subsequence

$$u^\varepsilon \rightharpoonup \tilde{u} \quad \text{in } W^{1,2}(\Omega), \qquad u^\varepsilon \to \tilde{u} \quad \text{a.e.} \tag{4.7}$$

for some $\tilde{u} \in W^{1,2}(\Omega)$. Also it follows that

$$\frac{1}{2\varepsilon} \int_\Omega |\nabla u^\varepsilon|^2\,dx \leq M - C_1$$

and so $\nabla u^\varepsilon \to 0$ in $L^2(\Omega; \mathbf{R}^n)$. Hence $\nabla \tilde{u} = 0$ which means $\tilde{u}$ is constant and consequently the weak convergence in (4.7) is strong. $\square$

Remark 4.2. It can be easily checked that under the assumptions of Proposition 4.2 from every sequence of global minimizers of I_ε we can extract a subsequence that converges strongly in $W^{1,2}(\Omega)$ to a global minimizer of I_0. Indeed let $\{u^\varepsilon\}$ be such a sequence, then

$$I_\varepsilon(u^\varepsilon) \leq I_\varepsilon(u) = I_0(u) \tag{4.8}$$

where u is an arbitrary constant. It now follows from the above proposition that by passing to a subsequence $u^\varepsilon \to \tilde{u}$ in $W^{1,2}(\Omega)$ for some constant $\tilde{u}$. According to Fatou's Lemma

$$\int_\Omega F(x,\tilde{u})\,dx \leq \liminf_{\varepsilon \to 0} \int_\Omega F(x,u^\varepsilon)\,dx$$

and therefore $I_0(\tilde{u}) \leq \liminf_{\varepsilon \to 0} I_\varepsilon(u^\varepsilon)$, which together with (4.8) gives the result.

Proposition 4.3. *Let the partial derivative of F with respect to u satisfy*

$$(\mathbf{G}) \qquad \begin{cases} F_u(x,u) \to +\infty & \text{as } u \to +\infty \text{ uniformly in } x, \\ F_u(x,u) \to -\infty & \text{as } u \to -\infty \text{ uniformly in } x. \end{cases}$$

Furthermore let for some $1 \le q \le 2^*$

$$|F_u(x,u)| \le C(1+|u|^q), \tag{4.9}$$

for all $x \in \overline{\Omega}$ and all $u \in \mathbf{R}$ where $C > 0$. Then if $\{u^\varepsilon\}$ is a sequence of stationary points of I_ε in $W^{1,2}(\Omega)$, by passing to a subsequence if necessary $u^\varepsilon \to \tilde{u}$ in $W^{1,2}(\Omega)$ where $\tilde{u}$ is a constant.

Proof. It follows from $(\mathbf{G})$ that there exist a constant $C_0 > 0$ such that $F_u(x,u)\,u \ge -C_0$ for all $x \in \overline{\Omega}$ and all $u \in \mathbf{R}$. As u^ε is a stationary point of I_ε, it satisfies (3.2) and (3.3). This in particular implies that

$$\int_\Omega |\nabla u^\varepsilon|^2\,dx = -\varepsilon \int_\Omega F_u(x,u^\varepsilon)\,u^\varepsilon\,dx \le \varepsilon C_0\,\mathcal{L}^n(\Omega). \tag{4.10}$$

Hence $u^\varepsilon = v^\varepsilon + c^\varepsilon$ with $c^\varepsilon = ⨍_\Omega u^\varepsilon\,dx$ and $v^\varepsilon \to 0$ in $W^{1,2}(\Omega)$. We now claim that $\{c^\varepsilon\}$ is bounded and therefore by passing to a subsequence if necessary $c^\varepsilon \to \tilde{u}$. Indeed if $\{c^\varepsilon\}$ is unbounded without loss of generality we can extract a subsequence such that $c^\varepsilon \to +\infty$. Now let $K > 0$ be such that $F_u(x,u) \ge 1$ when $u \ge K$. We can write

$$\int_\Omega F_u(x,u^\varepsilon)\,dx = \int_{\{u^\varepsilon \ge K\}} F_u(x,u^\varepsilon)\,dx + \int_{\{u^\varepsilon < K\}} F_u(x,u^\varepsilon)\,dx,$$

where the first integral

$$\int_{\{u^\varepsilon \ge K\}} F_u(x,u^\varepsilon)\,dx \ge \mathcal{L}^n(\{u^\varepsilon \ge K\}) \to \mathcal{L}^n(\Omega),$$

and using the fact that $u^\varepsilon(x) < K \implies |u^\varepsilon(x)| \le \max(K, |v^\varepsilon(x)|)$, the second integral converges to zero as

$$\int_{\{u^\varepsilon < K\}} |F_u(x,u^\varepsilon)|\,dx \le C \int_{\{u^\varepsilon < K\}} (1 + \max(K, |v^\varepsilon|^q))\,dx \to 0.$$

The contradiction now follows by recalling that

$$\int_\Omega F_u(x,u^\varepsilon)\,dx = 0 \tag{4.11}$$

for all $\varepsilon > 0$ as u^ε is a stationary point of I_ε. □

We now look at the set of stationary points of I_ε when $\varepsilon > 0$. Let us assume that I_0 has at most finitely many critical points all of which satisfy (3.10). In other words there is a finite set $\mathbf{P^0} \subset \mathbf{R}$ containing all the points $\tilde{u}$ satisfying (3.9) and such that (3.10) holds for every $\tilde{u} \in \mathbf{P^0}$. According to Theorem 3.1, for any such $\tilde{u}$ there is a continuous branch of solutions starting from $\tilde{u}$ (cf. Fig. 1). Moreover as $\mathbf{P^0}$ is finite there exists an

Figure 1: The branches of stationary points corresponding to I_ε.

$\varepsilon_0 > 0$ such that for any $\tilde{u} \in \mathbf{P^0}$ the solution u^ε obtained by the application of the implicit function theorem exists for all $\varepsilon \leq \varepsilon_0$. We denote the set of all such solutions for each fixed $0 \leq \varepsilon \leq \varepsilon_0$ by $\mathbf{P}^\varepsilon$.

The following result shows that under a certain growth condition on F_u, the above class contains all possible solutions when ε is sufficiently small.

Proposition 4.4. *Let F satisfy condition* $(\mathbf{G})$ *in Proposition* 4.3 *and let*

$$|F_u(x,u)| \leq C(1+|u|^q) \tag{4.12}$$

for some $1 \leq q < \frac{n+2}{n-2}$. Then for $\varepsilon_0 > 0$ as above the complete set of stationary points of I_ε for $0 \leq \varepsilon \leq \varepsilon_0$ is given by $\mathbf{P}^\varepsilon$.

Proof. We argue by contradiction. Assume the conclusion of the proposition does not hold. Then there exists a sequence of stationary points of I_{ε_k} denoted by $\{u^{\varepsilon_k}\}$ such that u^{ε_k} does not lie in $\mathbf{P}^{\varepsilon_k}$. According to Theorem 3.1 we can further assume that this sequence is bounded away from $\mathbf{P}^{\varepsilon_k}$. In other words there exists $\rho > 0$ independent of k such that

$$||u^{\varepsilon_k} - v||_{W^{2,s}(\Omega)} \geq \rho \tag{4.13}$$

for all $v \in \mathbf{P}^{\varepsilon_k}$ where s is the same as in (3.7). It follows from Proposition 4.3 that $u^{\varepsilon_k} \to \tilde{u}$ in $W^{1,2}(\Omega)$ for some constant $\tilde{u}$, and so the contradiction is immediate if we show that $\tilde{u}$ is a stationary point of I_0 which is not in $\mathbf{P^0}$.

It is clear that u^{ε_k} satisfies

$$\begin{cases} \Delta u^{\varepsilon_k} = f^{\varepsilon_k} & \text{in } \Omega, \\ \frac{\partial u^{\varepsilon_k}}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}, \tag{4.14}$$

for each k with $f^{\varepsilon_k} = \varepsilon_k F_u(x, u^{\varepsilon_k})$. As $\{u^{\varepsilon_k}\}$ is bounded in $W^{1,2}(\Omega)$, using (4.12) we can bootstrap this to $\{u^{\varepsilon_k}\}$ being bounded in $W^{2,p}(\Omega)$ for every $p < \infty$ and thus by passing to a subsequence if necessary $u^{\varepsilon_k} \to \tilde{u}$ in $W^{1,\infty}(\Omega)$. A further application of the bootstrap argument now shows that $u^{\varepsilon_k} \to \tilde{u}$ in $W^{2,s}(\Omega)$ and this together with (4.13) implies that $\tilde{u}$ does not lie in $\mathbf{P}^0$.

Finally by integrating (i) in (4.14) and using the boundary condition (ii), we deduce that

$$\int_\Omega F_u(x, u^{\varepsilon_k})\, dx = 0$$

for all k. Thus by passing to the limit the same holds for $\tilde{u}$, giving the desired contradiction. * □

5 The constrained problem

This section is devoted to the study of the second problem introduced earlier in Section 1. Here the energy functional is defined over the space of vector-valued functions $u : \Omega \to \mathbf{R}^N$ whose values are restricted to lie on the unit sphere $\mathbf{S}^{N-1}$. Let us recall that the energy functional in this case is given by

$$E_\varepsilon(u) = \int_\Omega \left(\frac{1}{2\varepsilon} |Du|^2 + V(x, u) \right) dx, \tag{5.1}$$

where the class of admissible functions u is

$$\mathcal{A}_2 = \{u \in W^{1,2}(\Omega; \mathbf{R}^N) : |u(x)| = 1 \quad \text{a.e.}\}.$$

The integrand V is initially assumed to belong to the class $C^2(\overline{\Omega} \times \mathbf{S}^{N-1})$. However it is always possible to extend V to any neighbourhood of $\mathbf{S}^{N-1}$ in particular to $\mathbf{R}^N$. In addition one can arrange this in such a way that the extended V grows linearly at infinity e.g. by setting $V(x, ru) := r\, V(x, u)$ for every $x \in \Omega$, $u \in \mathbf{S}^{N-1}$ and $r \geq 0$. We can furthermore assume that the extended $V \in C^2(\overline{\Omega} \times \mathbf{R}^N)$ and that there is a $C > 0$ such that

$$|V(x, u)| \leq C(1 + |u|)$$

for all $x \in \Omega$ and all $u \in \mathbf{R}^N$. In what follows we always speak of V in this extended sense.

We start this section by formally deriving the Euler-Lagrange equation associated to E_ε. or this we consider variations $\varphi \in C^\infty(\overline{\Omega}; \mathbf{R}^N)$ such that $u(x) \cdot \varphi(x) = 0$ for a.e. $x \in \Omega$. This orthogonality is to ensure that $|u(x) + t\varphi(x)|^2 - 1$ is of order t^2 and hence as $t \to 0$ the point $u(x) + t\varphi(x)$ approaches $u(x)$ along the tangent plane to the sphere $\mathbf{S}^{N-1}$ at $u(x)$. We now check the condition

$$\frac{d}{dt} E_\varepsilon(u + t\varphi)|_{t=0} = 0. \tag{5.2}$$

First, since equation (5.2) holds for all $\varphi \in C_0^\infty(\Omega; \mathbf{R}^N)$ satisfying $u \cdot \varphi = 0$ a.e. in Ω, we deduce that

$$(I - u \otimes u)(\Delta u - \varepsilon V_u(x, u)) = 0 \tag{5.3}$$

where I denotes the identity matrix. Using the fact that

$$\nabla(|u|^2) = 0 \tag{5.4}$$

we can rewrite (5.3) as

$$\Delta u + |Du|^2 u - \varepsilon(I - u \otimes u)V_u(x, u) = 0. \tag{5.5}$$

Second, since equation (5.2) holds for all $\varphi \in C^\infty(\overline{\Omega}; \mathbf{R}^N)$ satisfying $u \cdot \varphi = 0$ a.e. in Ω, we get the natural boundary condition

$$(I - u \otimes u)(Du)\,\nu = 0.$$

It follows from (5.4) that on the boundary

$$(u \otimes u)(Du)\,\nu = 0$$

and thus we are led to the Neumann boundary condition

$$(Du)\,\nu = 0. \tag{5.6}$$

When $\varepsilon = 0$ the Euler-Lagrange equation reduces to

$$\begin{cases} \Delta u + |Du|^2 u = 0 & \text{in } \Omega, \\ (Du)\,\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.7}$$

which is the well-known equation of harmonic maps onto the unit sphere.

It is clear that in this case any function $u = \tilde{u}$ with $\tilde{u} \in \mathbf{S}^{N-1}$ is a solution to this system in $\mathcal{A}_2$. However such functions are far from being the only solutions to this system. For example when $\Omega = B_1$ is the unit ball in $\mathbf{R}^n$ with $n \geq 3$ the function $u(x) = x/|x|$ is a solution to (5.7) that lies in $\mathcal{A}_2$. In fact this function is the global minimizer of the Dirichlet integral over $\mathcal{A}_2$ subject to the linear boundary condition $u = x$ on $\partial\Omega$ (cf. Lin [20]).

Similar to Section 3 we proceed by formally seeking a solution to (5.5) and (5.6) in the form

$$u(\varepsilon) = u + \varepsilon v + \varepsilon^2 w + \ldots, \tag{5.8}$$

where $u = \tilde{u}$ for some $\tilde{u} \in \mathbf{S}^{N-1}$. Notice that unlike for the problem studied in Section 3, the fact that $u = \tilde{u}$ is constant does not follow by substituting the above ansatz in the equation and solving it for u. Indeed as explained in the previous paragraph the system (5.7) in general has non constant solutions.

Substituting (5.8) into the equation (5.5) and the boundary condition (5.6) we get

$$\begin{cases} \Delta v = (I - \tilde{u} \otimes \tilde{u})V_u(x, \tilde{u}) & \text{in } \Omega, \\ v \cdot \tilde{u} = 0 & \text{in } \Omega, \\ (Dv)\,\nu = 0, & \text{on } \partial\Omega \end{cases} \tag{5.9}$$

for the coefficients of ε. It follows that a necessary condition for the solvability of the system (5.9) is that

$$\int_\Omega (I - \tilde{u} \otimes \tilde{u}) V_u(x, \tilde{u})\, dx = 0.$$

Moreover in this case the solution is unique up to an additive constant. Note that the second equation in (5.9) implies that this constant is normal to $\tilde{u}$. The coefficient of ε^2 gives

$$\begin{cases} \Delta w + |Dv|^2 \tilde{u} = (I - \tilde{u} \otimes \tilde{u}) V_{uu}(x, \tilde{u}) v \\ \qquad\qquad -(\tilde{u} \otimes v + v \otimes \tilde{u}) V_u(x, \tilde{u}) \\ |v|^2 + 2w \cdot \tilde{u} = 0 \\ (Dw)\, \nu = 0. \end{cases} \tag{5.10}$$

Again a necessary condition for the solvability of (5.10) is that

$$\int_\Omega \left((I - \tilde{u} \otimes \tilde{u}) V_{uu}(x, \tilde{u}) v - (\tilde{u} \otimes v + v \otimes \tilde{u}) V_u(x, \tilde{u}) - |Dv|^2\, \tilde{u} \right)\, dx = 0. \tag{5.11}$$

Multiplying the first equation in (5.9) by v and integrating over Ω we get

$$\int_\Omega \left(|Dv|^2 + V_u(x, \tilde{u}) \cdot v \right)\, dx = 0,$$

and therefore (5.11) can be written as

$$\int_\Omega \left((I - \tilde{u} \otimes \tilde{u}) V_{uu}(x, \tilde{u}) - \tilde{u} \cdot V_u(x, \tilde{u})\, I \right)\, v\, dx = 0.$$

Thus the constant mentioned above (normal to $\tilde{u}$) can be uniquely determined provided the matrix

$$V_0 := \fint_\Omega \left((I - \tilde{u} \otimes \tilde{u}) V_{uu}(x, \tilde{u}) - \tilde{u} \cdot V_u(x, \tilde{u})\, I \right)\, dx$$

satisfies the following invertibility condition
Inv: Given $u \in \mathbf{R}^N$ satisfying $u \cdot \tilde{u} = 0$, there exist a unique $v \in \mathbf{R}^N$ with $v \cdot \tilde{u} = 0$ such that $V_0\, v = u$.

Following this informal discussion and to establish rigorously the existence of a continuous branch of solutions to (5.5) and (5.6) we proceed as follows. Assume that the coordinate system is such that

$$\tilde{u} = e_N = (0, 0, ..., 1).$$

Let $u(x) = (u_1(x), u_2(x), ..., u_N(x))$ where

$$u_N(x) = \sqrt{1 - \Sigma_{i=1}^{N-1} u_i^2(x)}$$

and $||u_i||_{L^\infty(\Omega)}$ are small for all $1 \leq i \leq N - 1$. Then clearly

$$u(x) \in \mathbf{S}^{N-1}$$

for a.e. $x \in \Omega$. Furthermore we claim that if $(u_1, ..., u_N)$ satisfy the first $N-1$ equations then the last one is automatically true. Indeed proceeding formally, it follows from the constraint $\Sigma_{i=1}^N u_i^2 = 1$ that $\Sigma_{i=1}^N u_i \nabla u_i = 0$ and so

$$\sum_{i=1}^{N} (|\nabla u_i|^2 + u_i \Delta u_i) = 0.$$

As $|Du|^2 = \Sigma_{i=1}^N |\nabla u_i|^2$, we have that $u_N \Delta u_N = -\Sigma_{i=1}^{N-1} u_i \Delta u_i - |Du|^2$. The result now follows by multiplying the i-th equation by u_i, summing over $i=1$ to $N-1$ and recalling that $((I - u \otimes u)\, V_u(x,u)) \cdot u = 0$.

Let us now set $w = (u - e_N)/\varepsilon$ and solve the first $N-1$ equations of the system

$$\left\{ \begin{array}{c} \Delta w + \varepsilon^2 |Dw|^2 w - (I - (e_N + \varepsilon w) \otimes (e_N + \varepsilon w))\, V_u(x, e_N + \varepsilon w) = 0, \\ (Dw)\, \nu = 0. \end{array} \right. \tag{5.12}$$

For this we introduce the map

$$T : \mathbf{R} \times (W^{2,s}(\Omega))^{N-1} \to (E^s(\Omega))^{N-1} \times \mathbf{R}^{N-1}$$

by

$$T(\varepsilon, w') = \left(\begin{array}{c} \Delta w_1 - \varepsilon (h_1(\varepsilon, x, w') - ⨍_\Omega h_1(\varepsilon, x, w')\, dx) \\ \frac{\partial w_1}{\partial \nu}(x) \\ ... \\ \Delta w_{N-1} - \varepsilon (h_{N-1}(\varepsilon, x, w') - ⨍_\Omega h_{N-1}(\varepsilon, x, w')\, dx) \\ \frac{\partial w_{N-1}}{\partial \nu}(x) \\ ⨍_\Omega h_1(\varepsilon, x, w')\, dx \\ ... \\ ⨍_\Omega h_{N-1}(\varepsilon, x, w')\, dx \end{array} \right) \tag{5.13}$$

first for $\varepsilon \neq 0$, where $w' = (w_1, ..., w_{N-1})$ and

$$h(\varepsilon, x, w') := -\varepsilon |Dw|^2 w + \frac{1}{\varepsilon} \left((I - (e_N + \varepsilon w) \otimes (e_N + \varepsilon w))\ V_u(x, e_N + \varepsilon w) \right).$$

As a simple Taylor expansion shows

$$V_u(x, e_N + \varepsilon w) = V_u(x, e_N) + \varepsilon V_{uu}(x, e_N) w + o(\varepsilon^2).$$

We can therefore write

$$⨍_\Omega h(\varepsilon, x, w')\, dx =$$

$$⨍_\Omega - \varepsilon |Dw|^2 w + (I - e_N \otimes e_N) V_{uu}(x, e_N) w - (e_N \otimes w + w \otimes e_N) V_u(x, e_N) + o(\varepsilon) \tag{5.14}$$

where we have assumed the following to hold

$$\int_\Omega (I - (e_N \otimes e_N) V_u(x, e_N))\ dx = 0.$$

This suggests that we can extend the map T to $\varepsilon = 0$ by substituting (5.14) into the last $N-1$ columns in (5.13). In particular $T \in C^1(\mathbf{R} \times (W^{2,s}(\Omega))^{N-1}; (E^s(\Omega))^{N-1} \times \mathbf{R}^{N-1})$ and

$$T(0, w') = \begin{pmatrix} \Delta w_1 \\ \frac{\partial w_1}{\partial \nu}(x) \\ \cdots \\ \Delta w_{N-1} \\ \frac{\partial w_{N-1}}{\partial \nu}(x) \\ \fint_\Omega \big(V_{u_1 u_j}(x, e_N) w_j(x) - V_{u_N}(x, e_N) w_1(x)\big)\, dx \\ \cdots \\ \fint_\Omega \big(V_{u_{N-1}\, u_j}(x, e_N) w_j(x) - V_{u_N}(x, e_N) w_{N-1}(x)\big)\, dx \end{pmatrix}.$$

At this point we add the further assumption that the matrix

$$\fint_\Omega \big(V_{u_i u_j}(x, e_N) - V_{u_N}(x, e_N)\delta_{ij}\big)\, dx \tag{5.15}$$

with $1 \le i, j \le N-1$ is nonsingular.

Now consider the equation $T(0, w') = 0$. It follows immediately that $w' = 0$ is the only solution. Indeed the first $2(N-1)$ equations imply that w' is a constant. Substituting this constant into the last $N-1$ equations and making use of (5.15) we have that $w' = 0$. It can now be easily verified that

$$D_w T(0, w')(v') = \begin{pmatrix} \Delta v_1 \\ \dfrac{\partial v_1}{\partial \nu} \\ \cdots \\ \Delta v_{N-1} \\ \dfrac{\partial v_{N-1}}{\partial \nu} \\ \displaystyle\fint_\Omega \big(V_{u_1 u_j}(x, e_N) v_j(x) - V_{u_N}(x, e_N) v_1(x)\big)\, dx \\ \cdots \\ \displaystyle\fint_\Omega \big(V_{u_{N-1}\, u_j}(x, e_N) v_j(x) - V_{u_N}(x, e_N) v_{N-1}(x)\big)\, dx \end{pmatrix}$$

and so to apply the implicit function theorem we need to show that the linear map

$$D_w T(0, 0) : (W^{2,s}(\Omega))^{N-1} \to (E^s(\Omega))^{N-1} \times \mathbf{R}^{N-1}$$

is a bijection. In other words we need to show that the system

$$
\begin{cases}
\Delta v_1 = f_1, \\
\dfrac{\partial v_1}{\partial \nu} = g_1, \\
\dots \\
\Delta v_{N-1} = f_{N-1}, \\
\dfrac{\partial v_{N-1}}{\partial \nu} = g_{N-1}, \\
\fint_\Omega \left(V_{u_1 u_j}(x, e_N) v_j(x) - V_{u_N}(x, e_N) v_1 \right) \, dx = t_1, \\
\dots \\
\fint_\Omega \left(V_{u_{N-1} u_j}(x, e_N) v_j(x) - V_{u_N}(x, e_N) v_{N-1} \right) \, dx = t_{N-1}
\end{cases}
$$

has a unique solution $v \in (W^{2,s}(\Omega))^{N-1}$ for all $(f, g, t) \in (E^s(\Omega))^{N-1} \times \mathbf{R}^{N-1}$. It is well-known that for all $(f, g) \in (E^s(\Omega))^{N-1}$ the system

$$
\begin{cases}
\Delta v = f, \\
\frac{\partial v}{\partial \nu} = g
\end{cases}
$$

has a solution $v \in (W^{2,s}(\Omega))^{N-1}$, which is unique up to an additive constant. But again according to (5.15), this constant can be determined in a unique way by solving the last $N-1$ equations. Therefore we have proved the following

Theorem 5.1. *Let $\tilde{u} \in \mathbf{S}^{N-1}$ satisfy*

$$
\int_\Omega (I - \tilde{u} \otimes \tilde{u}) \, V_u(x, \tilde{u}) \, dx = 0 \tag{5.16}
$$

and let the linear map $V_0 : \mathbf{R}^N \to \mathbf{R}^N$ corresponding to the matrix

$$
V_0 := \fint_\Omega \left((I - \tilde{u} \otimes \tilde{u}) V_{uu}(x, \tilde{u}) - \tilde{u} \cdot V_u(x, \tilde{u}) I \right) \, dx \tag{5.17}
$$

satisfy the condition **Inv**. *Then for ε small enough the Euler-Lagrange equation* (5.5) *subject to the boundary condition* (5.6) *has a solution u^ε which is contained in the space $W^{2,s}(\Omega, \mathbf{R}^N)$ and is close to $\tilde{u}$ in the corresponding norm.*

Remark 5.1. Note that it follows from the proof of this theorem that $\frac{1}{\varepsilon} Du^\varepsilon \to 0$ in $L^\infty(\Omega; \mathbf{R}^{N \times n})$.

Remark 5.2. Contrary to Theorem 3.1 it does not follow from the proof presented here that the solution u^ε corresponding to $\tilde{u}$ is unique even if we restrict to a small neighbourhood of $\tilde{u}$.

Remark 5.3. The invertibility condition **Inv** on the matrix V_0 given earlier states that for any $u \in \mathbf{R}^N$ satisfying $u \cdot \tilde{u} = 0$, there exist a unique $v \in \mathbf{R}^N$ with $v \cdot \tilde{u} = 0$ such that $V_0 \, v = u$.

Now let $\{w^1, ..., w^N\}$ be an orthonormal basis for $\mathbf{R}^N$ with $w_N = \tilde{u}$ and

$$V := \fint_\Omega (V_{uu}(x,\tilde{u}) - \tilde{u} \cdot V_u(x,\tilde{u})\, I)\; dx.$$

Then clearly for u and v as above $v = \alpha_i w^i$ and $u = \beta_i w^i$ for some $\alpha = (\alpha_1, ..., \alpha_{N-1})$ and $\beta = (\beta_1, ..., \beta_{N-1})$. Thus setting $A_{ij} = (V\, w^j) \cdot w^i$ for $1 \le i,j \le N-1$, the equation $V_0\, v = u$ can be written as $A\, \alpha = \beta$, and this is clearly solvable for $\alpha \in \mathbf{R}^{N-1}$ when the $(N-1)\times(N-1)$ matrix A is invertible. Moreover extending A to an $N \times N$ matrix via $A_{ij} = (V\, w^j) \cdot w^i$ (now for $1 \le i,j \le N$) it follows that the required invertibility condition is satisfied if A or equivalently V is positive definite. In other words there exists $\gamma > 0$ such that

$$V_{ij}\lambda_i\lambda_j \ge \gamma|\lambda|^2 \tag{5.18}$$

for all $\lambda \in \mathbf{R}^N$.

Having proved the existence of a continuous branch of stationary points for the functional E_ε, we now study conditions under which u^ε would provide a local minimizer for E_ε.

Theorem 5.2. *Assume that the hypotheses of Theorem* 5.1 *hold and that the matrix*

$$V = \fint_\Omega (V_{uu}(x,\tilde{u}) - \tilde{u} \cdot V_u(x,\tilde{u})\, I)\; dx \tag{5.19}$$

is positive definite. Then the solution branch given by Theorem 5.1 *is an* L^1 *local minimizer of* E_ε*.*

Proof. Consider the functional

$$\tilde{E}_\varepsilon(u) = \int_\Omega \left(\frac{1}{2\varepsilon}|Du|^2 + V(x,u) + \frac{1}{2}(|u|^2 - 1)(-\frac{1}{\varepsilon}|Du^\varepsilon|^2 - u^\varepsilon \cdot V_u(x,u^\varepsilon)) \right) dx.$$

As the integrand V has a linear growth at infinity, $\tilde{E}_\varepsilon$ is well defined and finite over the class of admissible functions $W^{1,2}(\Omega, \mathbf{R}^N)$. Moreover it is clear that $\tilde{E}_\varepsilon(u) = E_\varepsilon(u)$ for every $u \in \mathcal{A}_2$. The Euler-Lagrange equation associated with this functional can be easily checked to be

$$\begin{cases} \Delta u + |Du^\varepsilon|^2 u - \varepsilon(V_u(x,u) - u^\varepsilon \cdot V_u(x,u^\varepsilon)u) = 0, \\ (Du)\,\nu = 0. \end{cases}$$

Thus u^ε is a stationary point of $\tilde{E}_\varepsilon$ as a consequence of being a solution to the system (5.5) and (5.6).

Let us now consider the second variation of $\tilde{E}_\varepsilon$ at u^ε. Indeed for $\varphi \in C^\infty(\overline{\Omega}; \mathbf{R}^N)$ we can write

$$\delta^2 \tilde{E}_\varepsilon(u^\varepsilon, \varphi) = \frac{d^2}{dt^2}\tilde{E}_\varepsilon(u^\varepsilon + t\varphi)|_{t=0}$$

$$\begin{aligned} &= \int_\Omega \left(\frac{1}{\varepsilon}|D\varphi|^2 + V_{u_iu_j}(x,u^\varepsilon)\varphi_i\varphi_j + |\varphi|^2(-\frac{1}{\varepsilon}|Du^\varepsilon|^2 - u^\varepsilon \cdot V_u(x,u^\varepsilon) \right) dx \\ &= \frac{1}{\varepsilon}\int_\Omega \left(|D\varphi|^2 + \varepsilon\left((V_{u_iu_j}(x,u^\varepsilon) - u^\varepsilon \cdot V_u(x,u^\varepsilon)\delta_{ij}) - \frac{1}{\varepsilon}|Du^\varepsilon|^2\delta_{ij} \right) \varphi_i\varphi_j \right) dx, \end{aligned}$$

Proceeding in a similar way as in Proposition 4.1 we can show this to be uniformly positive if and only if the matrix

$$V_{ij}^{\varepsilon} = \int_{\Omega} \left((V_{u_i u_j}(x, u^{\varepsilon}) - u^{\varepsilon} \cdot V_u(x, u^{\varepsilon}) \delta_{ij}) - \frac{1}{\varepsilon} |Du^{\varepsilon}|^2 \delta_{ij} \right) dx$$

is positive definite. But this follows immediately from (5.19) and Remark 5.1. $\square$

Remark 5.4. Consider the function $E_0 : \mathbf{R}^N \to \mathbf{R}$ given by

$$E_0(u) := \int_{\Omega} V(x, u) \, dx.$$

It can be verified that $\tilde{u}$ would provide a local minimizer for E_0 subject to the constraint $u \in \mathbf{S}^{N-1}$ whenever (5.16) and (5.18) hold with λ in (5.18) being such that $\lambda \cdot \tilde{u} = 0$. Our assumption in Theorem 5.2 however is stronger than this. In particular if

$$\int_{\Omega} (\tilde{u} \otimes \tilde{u}) \, V_u(x, \tilde{u}) \, dx = 0$$

then $\tilde{u}$ would be a free local minimum of E_0 i.e. regardless of the constraint $|u| = 1$.

6 The energy functional of micromagnetics

In this section we focus on the energy functional of micromagnetics in the case of a spatially uniform applied field

$$J_{\varepsilon}(m) = \int_{\Omega} \left(\frac{1}{2\varepsilon} |Dm|^2 + W(m) \right) dx + \frac{1}{2} \int_{\mathbf{R}^3} |h_m|^2 \, dx. \tag{6.1}$$

Here we have set $W(m) := \psi(m) + \frac{1}{2}|h - m|^2$ for the functional to agree with the original form introduced in Section 1. It is initially assumed that the anisotropy energy density $\psi \in C^2(\mathbf{S}^2)$. However as outlined in Section 5 we can extend ψ to any neighbourhood of $\mathbf{S}^2$ in particular to $\mathbf{R}^3$. Moreover ψ can be extended to an element of $C^2(\mathbf{R}^3)$ that satisfies a linear growth condition at infinity. So for the rest of this section we assume that the corresponding integrand $W \in C^2(\mathbf{R}^3)$ and has at most quadratic growth at infinity.

Recall that here $\Omega \subset \mathbf{R}^3$ is a bounded domain with smooth boundary. Regarding the admissible functions we set

$$\mathcal{A}_3 := \{ m \in W^{1,2}(\Omega; \mathbf{R}^3) : |m(x)| = 1 \quad \text{a.e.} \}.$$

It is clear that J_{ε} is well defined and finite over this class. We now proceed by formally deriving the Euler-Lagrange equation associated to J_{ε}. For this we consider variations $\varphi \in C^{\infty}(\overline{\Omega}; \mathbf{R}^3)$ such that $m \cdot \varphi = 0$ in Ω. We now check the condition

$$\frac{d}{dt} J_{\varepsilon}(m + t\varphi)|_{t=0} = 0. \tag{6.2}$$

First, since equation (6.2) holds for all $\varphi \in C_0^{\infty}(\Omega; \mathbf{R}^3)$ with $m \cdot \varphi = 0$ in Ω, we deduce that

$$(I - m \otimes m)(\Delta m - \varepsilon(W_m(m) - h_m)) = 0. \tag{6.3}$$

Recalling the normalization constraint on the magnetization m it follows that

$$\nabla(|m|^2) = 0. \tag{6.4}$$

We can therefore rewrite (6.3) as

$$\Delta m + |Dm|^2 m - \varepsilon(I - m \otimes m)(W_m(m) - h_m) = 0. \tag{6.5}$$

Second, since equation (6.2) holds for all $\varphi \in C^\infty(\overline{\Omega}; \mathbf{R}^N)$ satisfying $m \cdot \varphi = 0$ in Ω, we get the natural boundary condition

$$(I - m \otimes m)(Dm)\,\nu = 0.$$

By (6.4) we further have

$$(m \otimes m)(Dm)\,\nu = 0.$$

This implies the Neumann boundary condition

$$(Dm)\,\nu = 0. \tag{6.6}$$

Similar to Section 5 we see that when $\varepsilon = 0$ the Euler-Lagrange equation reduces to

$$\begin{cases} \Delta m + |Dm|^2 m = 0 & \text{in } \Omega, \\ (Dm)\,\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.7}$$

which is the equation of harmonic maps to the unit sphere in $\mathbf{R}^3$. It is clear that in this case any function $m = \tilde{m}$ where $\tilde{m} \in \mathbf{S}^2$ is a solution of (6.7) in $\mathcal{A}_3$.

Similar to Section 3 we proceed by formally seeking a solution to (6.5) and (6.6) in the form

$$m(\varepsilon) = \tilde{m} + \varepsilon v + \varepsilon^2 w + ..., \tag{6.8}$$

where $\tilde{m} \in \mathbf{S}^2$. Substituting this into equations (6.5) and (6.7) we get

$$\begin{cases} \Delta v = (I - \tilde{m} \otimes \tilde{m})(W_m(\tilde{m}) - h_{\tilde{m}}) \\ v \cdot \tilde{m} = 0 \\ (Dv)\,\nu = 0, \end{cases} \tag{6.9}$$

for the coefficients of ε. It follows that a necessary condition for the solvability of (6.9) is that

$$\int_\Omega (I - \tilde{m} \otimes \tilde{m})(W_m(\tilde{m}) - h_{\tilde{m}})\,dx = 0.$$

Moreover in this case the solution is unique up to an additive constant. Note that the second equation in (6.9) implies that this constant is normal to $\tilde{m}$. The coefficient of ε^2 gives

$$\begin{cases} \Delta w + |Dv|^2 \tilde{m} = (I - \tilde{m} \otimes \tilde{m})(W_{mm}(\tilde{m})v - h_v) \\ \qquad -(\tilde{m} \otimes v + v \otimes \tilde{m})(W_m(\tilde{m}) - h_{\tilde{m}}) \\ |v|^2 + 2w \cdot \tilde{m} = 0 \\ (Dw)\,\nu = 0. \end{cases} \tag{6.10}$$

Again a necessary condition for the solvability of (6.10) is that

$$\int_\Omega \big((I-\tilde m\otimes\tilde m)(W_{mm}(\tilde m)v-h_v)-(\tilde m\otimes v+v\otimes\tilde m)(W_m(\tilde m)-h_{\tilde m})-|Dv|^2\tilde m\big)\ dx=0. \tag{6.11}$$

Multiplying the first equation in (6.9) by v and integrating over Ω we get

$$\int_\Omega \big(|Dv|^2+(W_m(\tilde m)-h_{\tilde m})\cdot v\big)\ dx=0,$$

and thus (6.11) can be written as

$$\int_\Omega ((I-\tilde m\otimes\tilde m)(W_{mm}(\tilde m)v-h_v)-\tilde m\cdot(W_m(\tilde m)-h_{\tilde m})v)\ dx=0.$$

Thus the constant mentioned above (normal to $\tilde m$) can be uniquely determined provided the matrix

$$W_0:=\fint_\Omega ((I-\tilde m\otimes\tilde m)(W_{mm}(\tilde m)+D_e)-\tilde m\cdot(W_m(\tilde m)-h_{\tilde m})\,I)\ dx$$

satisfies the invertibility condition **Inv** in Section 5 and D_e is the tensor which was introduce in Theorem 2.3.

Being motivated by this informal discussion we now proceed on to showing rigorously the existence of a continuous branch of solutions to (6.5) and (6.6). For this assume that the coordinate system is such that

$$\tilde m=e_3=(0,0,1).$$

Let $m(x)=(m_1(x),m_2(x),m_3(x))$ where

$$m_3(x)=\sqrt{1-\Sigma_{i=1}^2 m_i^2(x)}$$

and $||m_i||_{L^\infty(\Omega)}$ are small for $i=1,2$. Then clearly $m(x)\in\mathbf{S}^2$ for a.e. $x\in\Omega$. Furthermore we claim that if (m_1,m_2,m_3) satisfy the first two equations in the system then the last one is automatically satisfied. Indeed proceeding formally, it follows from the constraint $\Sigma_{i=1}^3 m_i^2=1$ that $\Sigma_{i=1}^3 m_i\nabla m_i=0$ and so

$$\sum_{i=1}^3(|\nabla m_i|^2+m_i\Delta m_i)=0.$$

As $|Dm|^2=\Sigma_{i=1}^3|\nabla m_i|^2$, we have that $m_3\Delta m_3=-\Sigma_{i=1}^2 m_i\Delta m_i-|Dm|^2$. The result now follows by multiplying the i-th equation by m_i, summing over $i=1$ to 2 and recalling that $(I-m\otimes m)\,(W_m(m)-h_m)\cdot m=0$.

We now set $w=(m-e_3)/\varepsilon$ and solve the first two equations of the system

$$\begin{cases}\Delta w+\varepsilon^2|Dw|^2w-(I-(e_3+\varepsilon w)\otimes(e_3+\varepsilon w))\,(W_m(e_3+\varepsilon w)-h_{(e_3+\varepsilon w)})=0\\ (Dw)\,\nu=0.\end{cases} \tag{6.12}$$

For this we introduce the map

$$T : \mathbf{R} \times (W^{2,s}(\Omega))^2 \to (E^s(\Omega))^2 \times \mathbf{R}^2$$

by

$$T(\varepsilon, w') = \begin{pmatrix} \Delta w_1 - \varepsilon(h_1(\varepsilon, x, w') - \fint_\Omega h_1(\varepsilon, x, w')\,dx) \\ \frac{\partial w_1}{\partial \nu}(x) \\ \Delta w_2 - \varepsilon(h_2(\varepsilon, x, w') - \fint_\Omega h_2(\varepsilon, x, w')\,dx) \\ \frac{\partial w_2}{\partial \nu}(x) \\ \fint_\Omega h_1(\varepsilon, x, w')\,dx \\ \fint_\Omega h_2(\varepsilon, x, w')\,dx \end{pmatrix} \tag{6.13}$$

first for $\varepsilon \neq 0$, where $w' = (w_1, w_2)$ and

$$h(\varepsilon, x, w') := -\varepsilon |Dw|^2 w + \frac{1}{\varepsilon}\big((I - (e_3 + \varepsilon w) \otimes (e_3 + \varepsilon w))\,(W_m(e_3 + \varepsilon w) - h_{(e_3+\varepsilon w)})\big)\,.$$

As a simple Taylor expansion shows

$$W_m(e_3 + \varepsilon w) = W_m(e_3) + \varepsilon W_{mm}(e_3) w + o(\varepsilon^2).$$

In addition by linearity $h_{(e_3+\varepsilon w)} = h_{e_3} + \varepsilon h_w$ and so we can write

$$\fint_\Omega h(\varepsilon, x, w')\,dx = o(\varepsilon) +$$

$$\fint_\Omega \big(-\varepsilon|Dw|^2 w + (I - e_3 \otimes e_3)(W_{mm}(e_3)w - h_w) - (e_3 \otimes w + w \otimes e_3)(W_m(e_3) - h_{e_3})\big)\,dx, \tag{6.14}$$

where we have assumed the following to hold

$$\int_\Omega (I - (e_3 \otimes e_3)(W_m(e_3) - h_{e_3}))\,dx = 0.$$

This suggests that we can extend the map T to $\varepsilon = 0$ by substituting (6.14) into the last 2 columns in (6.13). In particular $T \in C^1(\mathbf{R} \times (W^{2,s}(\Omega))^2; (E^s(\Omega))^2 \times \mathbf{R}^2)$ and

$$T(0, w') = \begin{pmatrix} \Delta w_1 \\ \frac{\partial w_1}{\partial \nu}(x) \\ \Delta w_2 \\ \frac{\partial w_2}{\partial \nu}(x) \\ \fint_\Omega \big(W_{m_1 m_j}(e_3) w_j - e_1 \cdot h_w - e_3 \cdot (W_m(e_3) - h_{e_3}) w_1\big)\,dx \\ \fint_\Omega \big(W_{m_2 m_j}(e_3) w_j - e_2 \cdot h_w - e_3 \cdot (W_m(e_3) - h_{e_3}) w_2\big)\,dx \end{pmatrix}.$$

At this point we add the further assumption that the matrix

$$\fint_\Omega ((I - e_3 \otimes e_3)(W_{mm}(e_3) + D_e) - e_3 \cdot (W_m(e_3) - h_{e_3}) I)\,dx \tag{6.15}$$

with $1 \le i, j \le 2$ is nonsingular.

Now consider the equation $T(0, w') = 0$. It follows immediately that $w' = 0$ is the only solution. Indeed the first 4 equations imply that w' is a constant. Substituting this constant into the last 2 equations and making use of (6.15) we have that $w' = 0$. It can now be easily verified that

$$D_w T(0, w')(v') = \begin{pmatrix} \Delta v_1 \\ \frac{\partial v_1}{\partial \nu}(x) \\ \Delta v_2 \\ \frac{\partial v_2}{\partial \nu}(x) \\ \fint_\Omega \left(W_{m_1 m_j}(e_3) v_j - e_1 \cdot h_v - e_3 \cdot (W_m(e_3) - h_{e_3}) v_1 \right) \, dx \\ \fint_\Omega \left(W_{m_2 m_j}(e_3) v_j - e_2 \cdot h_v - e_3 \cdot (W_m(e_3) - h_{e_3}) v_2 \right) \, dx \end{pmatrix}.$$

and so to apply the implicit function theorem we need to show that the linear map

$$D_w T(0,0) : (W^{2,s}(\Omega))^2 \to (E^s(\Omega))^2 \times \mathbf{R}^2$$

is a bijection. In other words we need to show that the system

$$\begin{cases} \Delta v_1 = f_1, \\ \dfrac{\partial v_1}{\partial \nu} = g_1, \\ \Delta v_2 = f_2, \\ \dfrac{\partial v_2}{\partial \nu} = g_2, \\ \fint_\Omega \left(W_{m_1 m_j}(e_3) v_j - e_1 \cdot h_v - e_3 \cdot (W_m(e_3) - h_{e_3}) v_1 \right) \, dx = t_1, \\ \fint_\Omega \left(W_{m_2 m_j}(e_3) v_j - e_2 \cdot h_v - e_3 \cdot (W_m(e_3) - h_{e_3}) v_2 \right) \, dx = t_2 \end{cases}$$

has a unique solution $v \in (W^{2,s}(\Omega))^2$ for all $(f, g, t) \in (E^s(\Omega))^2 \times \mathbf{R}^2$. Again we use the well-known fact that for all $(f, g) \in (E^s(\Omega))^2$ the system

$$\begin{cases} \Delta v = f, \\ \frac{\partial v}{\partial \nu} = g \end{cases}$$

has a solution $v \in (W^{2,s}(\Omega))^2$, which is unique up to an additive constant. But again according to (6.15) this constant can be determined in a unique way by solving the last 2 equations. Therefore we have proved the following

Theorem 6.1. *Let $\tilde{m} \in \mathbf{S}^2$ satisfy*

$$\int_\Omega (I - \tilde{m} \otimes \tilde{m}) \left(W_m(\tilde{m}) - h_{\tilde{m}} \right) dx = 0 \tag{6.16}$$

and assume that the linear map $W_0 : \mathbf{R}^3 \to \mathbf{R}^3$ corresponding to the matrix

$$W_0 := \fint_\Omega \left((I - \tilde{m} \otimes \tilde{m})(W_{mm}(\tilde{m}) + D_e) \right) - \tilde{m} \cdot (W_m(\tilde{m}) - h_{\tilde{m}} I) \, dx \tag{6.17}$$

satisfy the condition **Inv**. *Then for ε small enough the Euler-Lagrange equation* (6.5) *subject to the boundary condition* (6.6) *has a solution m^ε which is contained in the space $W^{2,s}(\Omega;\mathbf{R}^3)$ and is close to $\tilde m$ in the corresponding norm.*

Remark 6.1. Note that it follows from the proof of this theorem that $\frac{1}{\varepsilon}Dm^\varepsilon \to 0$ in $L^\infty(\Omega;\mathbf{R}^{3\times 3})$. Moreover equations (2.3) together with elliptic regularity theory and the fact that $m^\varepsilon \to \tilde m$ in $W^{1,\infty}(\Omega;\mathbf{R}^3)$ imply that $m^\varepsilon \cdot h_{m_\varepsilon} \to \tilde m \cdot h_{\tilde m}$ in $L^\infty(\Omega)$.

Remark 6.2. Contrary to Theorem 3.1 it does not follow from the proof presented here that the solution u^ε corresponding to $\tilde u$ is unique even if we restrict to a small neighbourhood of $\tilde u$.

Remark 6.3. The invertibility condition on the matrix W_0 given by **Inv** states that for any $m \in \mathbf{R}^3$ satisfying $m \cdot \tilde m = 0$, there exist a unique $v \in \mathbf{R}^3$ with $v \cdot \tilde m = 0$ such that $W_0\, v = m$.

Now let $\{w^1, w^2, w^3\}$ be an orthonormal basis for $\mathbf{R}^3$ with $w_3 = \tilde m$ and

$$W := \fint_\Omega \left(W_{mm}(\tilde m) - \tilde m \cdot (W_m(\tilde m) - h_{\tilde m})\, I\right)\, dx.$$

Then clearly for u and v as above $v = \alpha_i w^i$ and $m = \beta_i w^i$ for some $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$. Thus setting $A_{ij} = (W\, w^j) \cdot w^i$ for $1 \le i, j \le 2$, the equation $W_0\, v = u$ can be written as $A\,\alpha = \beta$, and this is clearly solvable for $\alpha \in \mathbf{R}^2$ when the 2×2 matrix A is invertible. Moreover extending A to a 3×3 matrix via $A_{ij} = (V\, w^j) \cdot w^i$ (now for $1 \le i, j \le 3$) it follows that the required invertibility condition is satisfied if A or equivalently W is positive definite. In other words there exists $\gamma > 0$ such that

$$W_{ij}\lambda_i\lambda_j \ge \gamma|\lambda|^2 \tag{6.18}$$

for all $\lambda \in \mathbf{R}^3$.

Having proved the existence of a continuous branch of stationary points for the functional J_ε we now study conditions under which m^ε would provide a branch of local minimizers for J_ε.

Theorem 6.2. *Assume that the hypotheses of Theorem* (6.1) *hold and that the matrix*

$$W := \int_\Omega \left(W_{m_i m_j}(\tilde m) - \tilde m \cdot (W_m(\tilde m) - h_{\tilde m}) I\right)\, dx \tag{6.19}$$

is positive definite. Then the solution branch given by Theorem 6.1 *is an L^1 local minimizer of J_ε.*

Proof. Consider the function

$$\tilde J_\varepsilon(m) := \frac{1}{2}\int_{\mathbf{R}^3} |h_m|^2\, dx$$
$$+ \int_\Omega \left(\frac{1}{2\varepsilon}|Dm|^2 + W(m) - \frac{1}{2}(|m|^2 - 1)(\frac{1}{\varepsilon}|Dm^\varepsilon|^2 + m^\varepsilon \cdot (W_m(m^\varepsilon) - h_{m^\varepsilon}))\right)\, dx.$$

As the integrand W has a quadratic growth at infinity, $\tilde{J}_\varepsilon$ is well defined and finite over the class of admissible functions $W^{1,2}(\Omega;\mathbf{R}^3)$. Moreover it is clear that $\tilde{J}_\varepsilon(m) = J_\varepsilon(m)$ for every $m \in \mathcal{A}_3$. Furthermore the Euler-Lagrange equation associated with this functional is

$$\begin{cases} \Delta m + |Dm^\varepsilon|^2 m - \varepsilon\left(W_m(m) - h_m - (m^\varepsilon \cdot (W_m(m^\varepsilon) - h_{m^\varepsilon}))\, m\right) = 0 \\ (Dm)\,\nu = 0. \end{cases}$$

Thus m^ε is a stationary point of $\tilde{J}_\varepsilon$ as a consequence of being a solution to the system (6.5) and (6.6).

We now look at the second variation of $\tilde{J}_\varepsilon$ at m^ε. For this let $\varphi \in C^\infty(\overline{\Omega};\mathbf{R}^3)$ and consider

$$\delta^2 \tilde{J}_\varepsilon(m^\varepsilon,\varphi) = \frac{d^2}{dt^2}\tilde{J}_\varepsilon(m^\varepsilon + t\varphi)|_{t=0} =$$
$$\int_{\mathbf{R}^3} |h_\varphi|^2\,dx$$
$$+\int_\Omega \left(\frac{1}{\varepsilon}|D\varphi|^2 + W_{m_i m_j}(m^\varepsilon)\varphi_i\varphi_j - (\frac{1}{\varepsilon}|Dm^\varepsilon|^2 + m^\varepsilon \cdot (W_m(m^\varepsilon) - h_{m^\varepsilon}))|\varphi|^2\right)dx$$
$$\geq \frac{1}{\varepsilon}\int_\Omega \left(|D\varphi|^2 + \varepsilon\left((W_{m_i m_j}(m^\varepsilon) - m^\varepsilon \cdot (W_m(m^\varepsilon) - h_{m^\varepsilon})\delta_{ij}) - \frac{1}{\varepsilon}|Dm^\varepsilon|^2\delta_{ij}\right)\varphi_i\varphi_j\right)dx.$$

Proceeding in a similar way as in Proposition 4.1 we can show this to be uniformly positive if and only if the matrix

$$W^\varepsilon_{ij} = \int_\Omega \left((W_{m_i m_j}(m^\varepsilon) - m^\varepsilon \cdot (W_m(m^\varepsilon) - h_{m^\varepsilon})\delta_{ij}) - \frac{1}{\varepsilon}|Dm^\varepsilon|^2\delta_{ij}\right)dx$$

is positive definite. But this follows immediately from (6.19) and Remark 6.1. $\square$

Remark 6.4. Consider the function $J_0 : \mathbf{R}^3 \to \mathbf{R}$ given by

$$J_0(m) := \mathcal{L}^3(\Omega)\,W(m) + \frac{1}{2}\int_{\mathbf{R}^3} |h_m|^2\,dx,$$

where as before $h_m = \mathcal{H}(m\chi_\Omega)$ (cf. Theorem 2.3). It can be verified that $\tilde{m}$ would provide a local minimizer for J_0 subject to the constraint $m \in \mathbf{S}^2$ whenever (6.16) and (6.18) hold with λ in (6.18) being such that $\lambda \cdot \tilde{m} = 0$. Our assumption in Theorem 6.2 however is stronger. In particular if

$$\int_\Omega (\tilde{m}\otimes\tilde{m})(W_m(\tilde{m}) - h_{\tilde{m}})\,dx = 0$$

then $\tilde{m}$ would be a free local minimum of J_0 i.e. regardless of the constraint $m \in \mathbf{S}^2$.

7 Acknowledgements

The research of AT was funded by a university scholarship from Heriot-Watt and an NSF grant from Courant Institute both gratefully acknowledged. An earlier version of this work appeared in the second author's Ph.D thesis at Heriot-Watt University. This research was initiated in Spring 1996 when MW held a post doctoral fellowship at Heriot-Watt University under the EU Programme "Human Capital and Mobility" (contract no. ERBCHBICT930744).

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