ON THE STATIONARY CAHN-HILLIARD EQUATION: BUBBLE SOLUTIONS *

JUNCHENG WEI † AND MATTHIAS WINTER ‡

Abstract. We study stationary solutions of the Cahn–Hilliard equation in a bounded smooth domain which have an interior spherical interface (bubbles). We show that a large class of interior points (the "nondegenerate peak" points) have the following property: there exists such a solution whose bubble center lies close to a given nondegenerate peak point. Our construction uses among others the Liapunov-Schmidt reduction method and exponential asymptotics.

Key words. Bubbles, Exponential Asymptotics, Phase transition

AMS subject classifications. Primary 35B40, 35B45; Secondary 35J40

1. Introduction. In this paper, we continue our investigation of stationary solutions of the Cahn-Hilliard equation.

The Cahn-Hilliard equation is the simplest model for the separaton of a binary mixture in the presence of a mass constraint (see [7]). It can be derived from a Helmholtz free energy

$$E(u) = \int_\Omega \left[F\big(u(x)\big) + \frac{1}{2}\epsilon^2 |\nabla u(x)|^2\right] dx \tag{1.1}$$

subject to the constraint $\frac{1}{|\Omega|}\int_\Omega u dx = m$. Here Ω is a bounded smooth domain corresponding to the region occupied by the body, $u(x)$ is a conserved order parameter representing for example the concentration, ϵ is the range of intermolecular forces, the gradient term is a contribution to the free energy coming from spatial fluctuations of the order parameter and $F(u)$ is the free energy density which has a double–well structure at low temperatures. The simplest one is $F(u) = \frac{1}{4}(1-u^2)^2$. Hence $f(u) := F^{'}(u) = u^3 - u$. For the rest of the paper we often write $u^3 - u$ instead of $f(u)$. However, since we are looking for solutions of (1.2) with $\|u\|_{L^\infty(\Omega)} \le C$, we can modify the nonlinearity $f(u) = u^3 - u$ for u large so that the mapping $u \mapsto u^3$, $H^2(\Omega) \to L^2(\Omega)$ is compact regardless of the dimension N. See [32] and [34] for more general nonlinearities.

A stationary solution of $E(u)$ satisfies the following Euler-Lagrange equation

$$\begin{cases} \epsilon^2 \triangle u - f(u) = \sigma_\epsilon & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \frac{1}{\Omega}\int_\Omega u dx = m \end{cases} \tag{1.2}$$

where $f(u) = F'(u)$, σ_ϵ is a constant and $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$.

*This work was supported by an Earmarked Grant from RGC of Hong Kong.

†Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong (wei@math.cuhk.edu.hk).

‡Mathematisches Institut, Universität Stuttgart, D-70511 Stuttgart, Germany (winter@mathematik.uni-stuttgart.de)

Equation (1.2) has been studied extensively by many authors. It was first observed by Modica in [19] that global minimizers u_ϵ of $E(u)$ under $m = \frac{1}{|\Omega|}\int_\Omega u dx$ have a transition layer. Namely, there exists an open set $\Gamma \subset \Omega$ such that if a sequence u_ϵ converges then $u_\epsilon \longrightarrow 1$ on $\Omega\backslash\bar{\Gamma}, u_\epsilon \longrightarrow -1$ on Γ as $\epsilon \longrightarrow 0$ and $\partial\Gamma \cap \bar{\Omega}$ is a minimal surface having constant mean curvature. Kohn and Sternberg in [16] studied local minimizers of the functional without mass conservation by using Γ-convergence. Chen and Kowalczyk [9] proved the existence of local minimizers using a geometric approach. The dynamics of the transition layer solution has been studied by many authors, e.g. Chen [8], Alikakos, Bates and Fusco [3], Alikakos, Bates and Chen [2], Alikakos, Fusco and Kowalczyk [4], Pego [25], etc.

The study of the solution set of (1.2) is the key in understanding the global dynamics as this has been illustrated by Bates and Fife [6], Alikakos, Fusco and Kowalczyk [4], Grinfeld and Novick–Cohen [13], [14].

In the one dimensional case, Grinfeld and Novick-Cohen [13] and [14] completely determined all stationary solutions and proved some properties of their connecting orbits. In the higher dimensional case ($N \geqslant 2$), little is known about stationary solutions except for the transition layer solution. In [32], we first established the existence of boundary spike layer solutions, namely solutions which are "almost" constant and have a spike on the boundary. More precisely, suppose that $\sqrt{\frac{1}{3}} < m < 1$ and $P_0 \in \partial\Omega$ such that $\nabla_{\tau_{P_0}} H(P_0) = 0, (\nabla^2_{\tau_{P_0}} H(P_0)) := G_B(P_0)$ is nondegenerate, where $H(P_0)$ is the mean curvature function at P_0 and $\nabla_{\tau_{P_0}}$ is the tangential derivative at P_0. Then for ϵ sufficiently small there exists a solution u_ϵ of (1.2) such that $u_\epsilon(x) \to m$ for $x \in \bar{\Omega}\backslash\{P_0\}$. Moreover, u_ϵ has only one local minimum P_ϵ where $P_\epsilon \in \partial\Omega, P_\epsilon \longrightarrow P_0$ and $u_\epsilon(P_\epsilon) \longrightarrow \beta < m$. Multiple boundary spikes are also constructed in [33].

In [34], we established the existence of interior spike layer solutions under some geometric conditions on the domain.

We first introduced the following set: For each $P \in \Omega$, we define

$$\Lambda_P := \left\{ d\mu_P(z) \in M(\partial\Omega) \left| \begin{array}{l} \exists \epsilon_k \longrightarrow 0 \text{ such that} \\ d\mu_P(z) = \lim_{\epsilon_k \to 0} \frac{e^{-\frac{|z-P|}{\epsilon_k}} dz}{\int_{\partial\Omega} e^{-\frac{|z-P|}{\epsilon_k}} dz} \end{array} \right. \right\} \tag{1.3}$$

where $M(\partial\Omega)$ are the bounded Borel measures on $\partial\Omega$ and the convergence is weak convergence of measures.

A point $P_0 \in \Omega$ is called a *nondegenerate peak* point if it satisfies the following conditions:

(1) $\Lambda_{P_0} = \{d\mu_{P_0}(z)\}$.

(2) There exists $a \in R^N$ such that $\int_{\partial\Omega} e^{<z-P_0,a>}(z-P_0)d\mu_{P_0}(z) = 0$ and $\int_{\partial\Omega} \left\{ \frac{e^{-\frac{|z-P_0|}{\epsilon}} e^{<z-P_0,a>}}{\int_{\partial\Omega} e^{-\frac{|z-P_0|}{\epsilon}} dz} \right\} (z-P_0)dz = O(\epsilon^{\alpha_0})$ for some $\alpha_0 > 0$. Here and throughout the paper $< A, B >$ means the inner product of $A \in R^N$ and $B \in R^N$.

(3) The matrix $G(P) := \left(\int_{\partial\Omega} e^{<z-P_0,a>}(z-P_0)_i(z-P_0)_j d\mu_{P_0}(z)\right)$ is nondegenerate, where a is given in (2).

Remark: The vector $a \in R^N$ in (2) and (3) is unique. A more geometric characterization of a nondegenerate peak point is the following fact: P_0 is a nondegenerate peak point if and only if $P_0 \in \text{int }(\text{conv}(\text{supp }(d\mu_{P_0})))$ where $\text{int }(\text{conv}(\text{supp }(d\mu_{P_0})))$ is the interior of the convex hull of the support of $d\mu_{P_0}$. Moreover, when Ω is strictly convex, the maximum point of the distance function-$d(x, \partial\Omega)$- is a nondegenerate peak point. See [29]. This is much in line with the formal analysis done in [27] (but here we don't need $N = 2$).

Under conditions (1) – (3), we proved in [34] that if $\sqrt{\frac{1}{3}} < m < 1$ then for ϵ sufficiently small, there exist solutions u_ϵ of (1.2) with the property that u_ϵ has only one local minimum P_ϵ and $u_\epsilon \to m$ for $x \in \overline{\Omega} \setminus \{P_0\}$, $u_\epsilon(P_\epsilon) \to \beta < m$, $P_\epsilon \to P_0$.

In this paper, we shall construct another kind of solutions: bubbles. A bubble solution is a transition layer solution with a spherical interface. More precisely, u_ϵ is a bubble solution if there exists an open ball (with center x_0 and radius r_b) $B_{r_b}(x_0) \subset \Omega$ such that $u_\epsilon \to +1$ in $B_{r_b}(x_0)$ and $u_\epsilon \to -1$ in $\overline{\Omega}\backslash\overline{B_{r_b}(x_0)}$.

Bubble like solutions have been studied recently by some authors. N. Alikakos and G. Fusco [5] and M.J. Ward [27] studied the dynamics of bubbles. It was proved that bubble solutions are metastable and the bubble drifts across the domain with exponentially small velocity without changing shape while maintaining a constant radius to conserve mass. In [27], M. J. Ward used matched asymptotics expansions to give a careful but formal (*non-rigorous*) analysis on stationary bubbles for equation (1.2) in a strictly convex domain in R^2 and some special domains in R^3. More precisely, it was shown in [27] that for a strictly convex domain Ω in R^2 the center of a bubble is at an $O(\epsilon)$ distance from the center of the largest inscribed circle in Ω. Some special results for R^3 were also contained in [27]. As far as we know, a rigorous proof of the existence of stationary bubbles in general domains has not been given.

The goal of this paper is to give an *explicit* and *rigorous* construction of bubble-like solutions in general domains. Our analysis is based on the Liapunov–Schmidt reduction method which was used in a similar context by Floer and Weinstein ([11]) and extended by Oh ([23], [24]) in the study of semi-classical states of the following nonlinear Schrödinger equation

$$\epsilon^2 \Delta u - V(x)u + u^p = 0, x \in R^N.$$

There they studied the role of the potential $V(x)$ for the existence of concentrated solutions and the order of the error is algebraic (i.e., $O(\epsilon)$). Here we have to overcome two additional difficulties. First, the error term is exponentially small, and we use the method of viscosity solutions as introduced in [18] and used in [22] to estimate exponentially small terms. Second, the linearized operator, modulo its approximate kernel, is not uniformly invertible with respect to ϵ (it is uniformly invertible in [11], [23], [24] and [34]). We have to estimate the order of small eigenvalues of the linearized operator (modulo its kernel).

The following is the main result of this paper.

THEOREM 1.1. *Let $P_0 \in \Omega$ and $m \in (-1, \frac{2|B_{d(P_0,\partial\Omega)}(P_0)|}{|\Omega|} - 1)$. Suppose P_0 is a "nondegenerate peak" point. Then for ϵ sufficiently small there exists a solution u_ϵ of*

(1.2) such that $u_\epsilon \to 1$ in $B_{r_b}(P_0)$ and $u_\epsilon \to -1$ in $\overline{\Omega} \setminus \overline{B_{r_b}(P_0)}$ where r_b is such that

$$|B_{r_b}(P_0)| = \frac{m+1}{2}|\Omega|. \tag{1.4}$$

Examples. (1) A bubble in a dumbell domain (see Fig. 1.1).

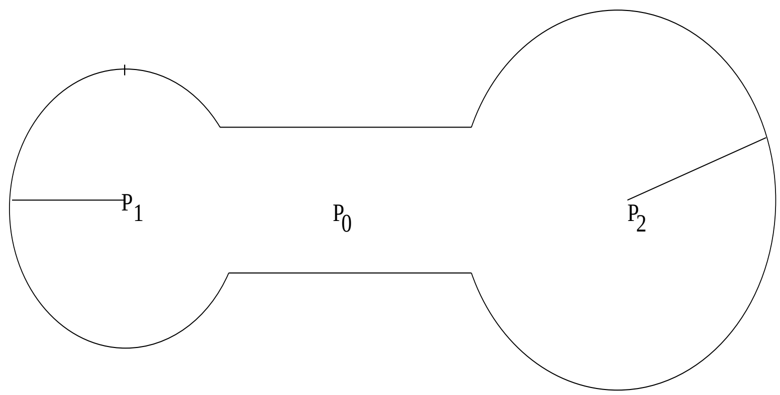

FIG. 1.1. *Dumbell Domain*

By explicit computation, we know that P_1 and P_2 are nondegenerate peak points. There are two bubble solutions for (1.2).

(2) Let $\Omega \subset R^2$. If the support of $d\mu_{P_0}(z)$ contains more than two points then P_0 is a nondegenerate peak point (see Fig. 1.2).

To lay down the proof of Theorem 1.1, we first transform equation (1.2). It is easy to see that equation (1.2) is equivalent to the following

$$\begin{cases} \epsilon^2\triangle u + u - u^3 = m - \frac{1}{|\Omega|}\int_\Omega u(x)^3 dx & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \frac{1}{|\Omega|}\int_\Omega u dx = m. \end{cases} \tag{1.5}$$

We prove Theorem 1.1 in the following steps.

We first study a problem in R^N, namely the following

$$\begin{cases} \triangle v + v - v^3 = \sigma & \text{in } R^N, \\ v(0) = \max_{y\in R^N} v(y), v \geqslant \tau_\sigma, v(y) \longrightarrow \tau_\sigma & \text{as } |y| \to +\infty \end{cases} \tag{1.6}$$

where τ_σ is such that

$$v - v^3 - \sigma = (v - \tau_\sigma)(v - a_\sigma)(b_\sigma - v), \tau_\sigma < a_\sigma < b_\sigma.$$

Note that as $\sigma \to 0$, $\tau_\sigma \to -1$, $a_\sigma \to 0$, $b_\sigma \to 1$. Moreover, if $\sigma > 0$, we have

$$\int_{\tau_\sigma}^{b_\sigma} [v - v^3 - \sigma] dv > 0.$$

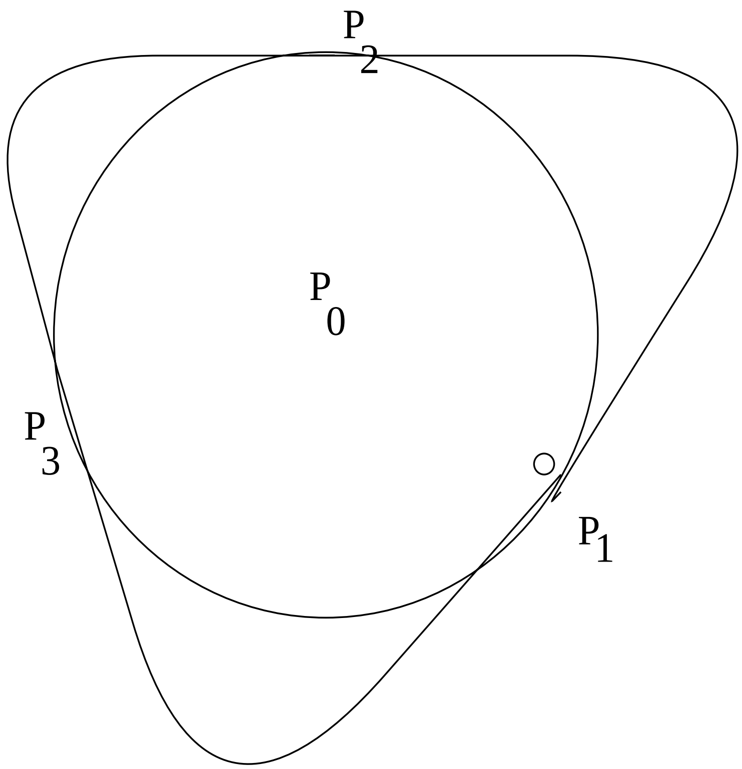

FIG. 1.2. *Support of $d\mu_{P_0}$ contains exactly 3 points*

It is well–known (see [10] and [26]) that the following equation

$$\begin{cases} \triangle w + w(w-a)(b-w) = 0 & in R^N, \\ w(0) = \max_{z\in R^N} w(z),\, w(z) > 0,\, w(z) \to 0 & \text{as } |z| \to \infty \end{cases} \tag{1.7}$$

has a unique solution which is radial if

$$0 < a < b$$

and

$$\int_0^b w(w-a)(b-w)dw > 0.$$

Hence $\sigma > 0$ fixed and small (1.6) has a unique solution v_σ which is radial.

In Section 2, we study the asymptotic behavior of v_σ as $\sigma \to 0$. By a special choice of σ (namely $\sigma = O(\epsilon)$), we have

$$v_\sigma\left(\frac{|x-P_0|}{\epsilon}\right) \to +1 \text{ in } B_{r_b}(P_0), v_\sigma\left(\frac{|x-P_0|}{\epsilon}\right) \to -1 \text{ in } \overline{\Omega} \setminus \overline{B_{r_b}(P_0)}.$$

for some $r_b > 0$. Hence v_σ is a bubble solution to (1.6). However, v_σ does not satisfy the boundary condition (which is why we need to introduce the geometric conditions (1)-(3)).

Set

$$\Omega_\epsilon = \{y|\epsilon y \in \Omega\}, \Omega_{\epsilon,P} = \{y|\epsilon y + P \in \Omega\}.$$

In Section 3, we study a function $P_{\Omega_{\epsilon,P}} v_\sigma$ which is a modification of v_σ. It satisfies the Neumann boundary condition on $\partial\Omega_{\epsilon,P}$.

In Section 4, we choose σ such that

$$\sigma = m - \frac{1}{|\Omega|} \int_\Omega (P_{\Omega_{\epsilon,P_0}} v_\sigma)^3 dx. \tag{1.8}$$

We set $P_{\Omega_{\epsilon,P}} v_\sigma = w_{\epsilon,P}$. We use $w_{\epsilon,P}$ as our approximate solution.

In Section 5, we set

$$u_\epsilon = w_{\epsilon,P_0+z} + \Phi_{\epsilon,z} \tag{1.9}$$

where

$$z = \epsilon(\frac{1}{2\sqrt{2}} d(P_0, \partial\Omega) a + \tilde{z})$$

and substitute into equation (1.2). We linearize equation (1.2) around w_{ϵ,P_0+z}. The linearized operator is

$$L_\epsilon \Phi = \triangle \Phi + (1 - 3w^2_{\epsilon,P_0+z})\Phi + 3\frac{1}{|\Omega|} \int_\Omega w^2_{\epsilon,P_0+z} \Phi dx.$$

The error term $\Phi_{\epsilon,z}$ is exponentially small. We need to obtain the precise exponential asymptotics. This is done in Section 5.

In Sections 6, we use the classical Liapunov-Schmidt reduction procedure. We first define the approximate kernel

$$K_{\epsilon,z} = \text{span}\,\{\frac{\partial w_{\epsilon,P_0+z}}{\partial z_i} | i = 1, ..., N\} \subset H^2(\Omega_\epsilon)$$

and approximate cokernel

$$C_{\epsilon,z} = \text{span}\,\{\frac{\partial w_{\epsilon,P_0+z}}{\partial z_i} | i = 1, ..., N\} \subset L^2(\Omega_\epsilon).$$

We solve $\Phi_{\epsilon,z}$ in the approximate kernel. To this end, we need to analyze the small eigenvalues of L_ϵ (modulo $K_{\epsilon,z}$). We will show that these small eigenvalues are of order $O(\epsilon^2)$. Thus $\Phi_{\epsilon,z}$ can be solved. Equation (1.2) is reduced to finite dimensions.

In Section 7 we apply a degree-theoretic argument to solve the reduced finite dimensional problem (in which the nondegeneracy of the peak point P_0 is essential) and complete the proof of Theorem 7.1.

We note that M.J. Ward in [27] obtained identities similar to condition (2) about bubbles. In [28], he also derived a similar identity for the location of peaks of localized solutions for a semilinear elliptic equations with Robin boundary conditions. Such kind of identities have also appeared in the analysis of interior spike solutions for the stationary reaction–diffusion equation

$$\begin{cases} \epsilon^2 \triangle u + f(u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ or } u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

See [22], [29], [30], [31], [34], etc.

Throughout this paper, we use C, C_0, C_N, c, etc. to denote various generic constants. The symbols $O(A), o(A)$ mean that $|O(A)| \le C|A|, o(A)/|A| \to 0$ respectively. $A \sim B$ means $A/B \to C$ in some limit. The numbers μ, δ are small positive numbers.

2. Equation in R^N. In this section, we study a parametrized semilinear elliptic equation in R^N.

Let v_σ be the unique solution of the problem

$$\begin{cases} \triangle v + v - v^3 = \sigma & \text{in } R^N, \\ v(0) = \max_{y \in R^N} v(y), v \geqslant \tau_\sigma, v(y) \longrightarrow \tau_\sigma & \text{as } |y| \to +\infty. \end{cases} \tag{2.1}$$

For σ small, let $v - v^3 - \sigma = (v - \tau_\sigma)(v - a_\sigma)(b_\sigma - v)$ where $\tau_\sigma < a_\sigma < b_\sigma$. Then

$$\tau_\sigma = -1 + c_0\sigma + O(\sigma^2), a_\sigma = 0 + c_1\sigma + O(\sigma^2), b_\sigma = 1 + c_2\sigma + O(\sigma^2). \tag{2.2}$$

where c_0, c_1, c_2 are constants.

Let R_σ be the radius such that

$$v_\sigma(R_\sigma) = 0. \tag{2.3}$$

We have

Lemma 2.1.

$$\sigma R_\sigma = c_b + O(\sigma) \tag{2.4}$$

as $\sigma \to 0$ where $c_b > 0$ is a positive constant.

Proof: We divide the proof into the following steps.

Step 1: $R_\sigma \to \infty$ as $\sigma \to 0$.

We have $v_\sigma \to v_0$ uniformly in any compact set where v_0 satisfies

$$\begin{cases} \triangle v_0 + v_0 - v_0^3 = 0, \\ v_0(0) = 1, v_0'(0) = 0. \end{cases} \tag{2.5}$$

This implies $v_0 \equiv 1$ (since v_0 is radial). Therefore, $R_\sigma \to \infty$ as $\sigma \to 0$ and Step 1 is proved.

Step 2: $v_\sigma(R_\sigma + s) \to U_0(s)$ in $C^2_{loc}(R)$ as $\sigma \to 0$ where $U_0(s)$ is the unique solution of the ODE

$$\begin{cases} u'' + u - u^3 = 0, -\infty < r < +\infty, \\ u(0) = 0, \lim_{r \to -\infty} u(r) = -1, \lim_{r \to +\infty} u(r) = +1. \end{cases} \tag{2.6}$$

Set $\hat{v}_\sigma(|x|) := v_\sigma(x)$ and $\tilde{v}_\sigma(s) := \hat{v}_\sigma(R_\sigma + s)$. Note that $\tilde{v}_\sigma$ satisfies

$$\tilde{v}_\sigma'' + \frac{N-1}{R_\sigma + s}\tilde{v}_\sigma' + \tilde{v}_\sigma - \tilde{v}_\sigma^3 = \sigma. \tag{2.7}$$

Now

$$\frac{1}{R_\sigma + s} \to 0 \tag{2.8}$$

uniformly with respect to s in any compact subset of the real line R since $R_\sigma \to \infty$.

This implies that $\tilde{v}_\sigma \to U_0$ in $C^2_{loc}(R)$ where U_0 satisfies (2.6). Step 2 is thus proved.

Step 3: $\sigma R_\sigma = c_b + O(\sigma)$ *as* $\sigma \to 0$.

Set $\Phi_\sigma(s) = \tilde{v}_\sigma(s) - U_0(s)$. Then Φ_σ satisfies

$$\Phi_\sigma'' + (1 - 3U_0^2)\Phi_\sigma + O(|\Phi_\sigma|)\Phi_\sigma = \sigma - \frac{N-1}{R_\sigma + s}\tilde{v}_\sigma' \tag{2.9}$$

uniformly in any compact subset of R. This implies

$$\|\Phi_\sigma\|_{C^2_{loc}[-R_\sigma,\infty)} \le C\mathrm{Max}(\sigma, R_\sigma^{-1}). \tag{2.10}$$

Furthermore, U_0' satisfies

$$(U_0')'' + (1 - 3U_0^2)U_0' = 0. \tag{2.11}$$

Multiplying equation (2.9) by U_0' and (2.11) by Φ_σ, integrating and taking the difference, we get

$$\Phi_\sigma' U_0' - \Phi_\sigma U_0''|_{-R_\sigma}^{\infty} + \int_{-R_\sigma}^{\infty} O(|\Phi_\sigma|^2) U_0'\, ds =$$

$$\sigma \int_{-R_\sigma}^{\infty} U_0'\, ds - \int_{-R_\sigma}^{\infty} \frac{N-1}{R_\sigma + s}\tilde{v}_\sigma' U_0'\, ds. \tag{2.12}$$

This implies

$$\sigma R_\sigma = \frac{N-1}{2}\int_{-\infty}^{\infty} (U_0')^2\, ds + O(R_\sigma \mathrm{Max}(\sigma^2, R_\sigma^{-2})) \tag{2.13}$$

as $\sigma \to 0$. Therefore Step 3 is proved and Lemma 2.1 follows. □

Let $U_0(r)$ be the solution of (2.6). We then have

LEMMA 2.2.

$$v_\sigma(r) = U_0(r - R_\sigma) + O(\sigma). \tag{2.14}$$

Proof. Lemma 2.2 follows by Lemma 2.1 and equation (2.10). □

Next we shall study the eigenvalues associated with the linearized operator

$$L_\sigma \Phi := \triangle\Phi + (1 - 3v_\sigma^2)\Phi,$$

$$L_\sigma : H^2_N(\Omega_{\epsilon,P}) \to L^2(\Omega_{\epsilon,P})$$

where

$$\Omega_{\epsilon,P} = \{y|\epsilon y + P \in \Omega\}$$

and

$$H^2_N(\Omega_{\epsilon,P}) = \left\{ u \in H^2(\Omega_{\epsilon,P})\Big| \ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega_{\epsilon,P} \right\}.$$

We first consider the operator on R^N:

$$L\Phi := \triangle\Phi + (1 - 3v_\sigma^2)\Phi,$$

$$L : H^2(R^N) \to L^2(R^N).$$

LEMMA 2.3. *For $\sigma > 0$ sufficiently small*

$$\mathit{Kernel}(L) := X = \mathit{span}\left\{ \frac{\partial v_\sigma}{\partial y_j} \middle| \, j = 1, 2, \ldots, N \right\} \subset H^2(R^N).$$

Proof. By [26], L_σ is invertible in the space $H^2_r(R^N) = \{u = u(|y|) \in H^2(R^N)\}$. Similar to the proof of Lemma B.2 in [21], we have Lemma 2.3. □

We now use a perturbation analysis to extend Lemma 2.3 to the operator defined on $\Omega_{\epsilon,P}$. Similar to [32], we introduce a notion of "distance" between two closed subspaces E, F of a Hilbert space $H := L^2(\Omega_\epsilon)$. Following [15], we set

$$\vec{d}\,(E, F) = \sup\{d(x, F)|x \in E, \|x\|_H = 1\}$$

It is easy to see that $\vec{d}$ is non-symmetric, $\vec{d}\,(E, F) \leqslant 1$ and that

$$\vec{d}(E, F) = 1 \ \text{ if and only if } E \perp F. \tag{2.15}$$

Moreover, it is not hard to show that

$$\vec{d}\,(E, F) = \vec{d}\,(F^\perp, E^\perp).$$

Then the following two lemmata are proved in [15].

LEMMA 2.4. *Let A be a selfadjoint operator on a Hilbert space H, I a compact interval in R, $\{\Psi_1, \ldots, \Psi_N\}$ linearly independent normalized elements in $D(A)$. Assume that the following conditions are true*
(i)

$$\begin{cases} A\Psi_j = \mu_j\Psi_j + r_j, \ \|r_j\| < \epsilon' \\ \quad \mu_j \in I, \ j = 1, \ldots, N. \end{cases}$$

(ii) There is a number $a > 0$ such that I is a-isolated in the spectrum of A:

$$(\sigma(A) \setminus I) \cap (I + (-a, a)) = \emptyset.$$

Then

$$\vec{d}\,(E,F) = \sup\{d(x,F)|x \in E, \|x\|_H = 1\} \leq \frac{N^{1/2}\epsilon'}{a(\lambda_{min})^{1/2}}$$

where

$$E = span\{\Psi_1, \ldots, \Psi_N\},$$

$$F = closed\ subspace\ associated\ to\ \sigma(A) \cap I,$$

$$\lambda_{min} = the\ smallest\ eigenvalue\ of\ the\ matrix\ (<\Psi_i,\ \Psi_j>).$$

LEMMA 2.5. *Let $K > 0$ and consider that part of the spectra of two linear operators L and M which lie in $I(\epsilon) = (-\infty, K\epsilon^2)$. Let E and F be the corresponding spectral subspaces. Assume moreover that $I(\epsilon)$ is ϵ^2-isolated in $\sigma(L)$ for $\epsilon < \epsilon_0$:*

$$\sigma(L) \cap [K\epsilon^2,\ (K + \overline{a})\epsilon^2) = \emptyset$$

for some $\overline{a} > 0$. Then there is a bijection

$$b :\ \sigma(L) \cap I(\epsilon) \to \sigma(M) \cap I(\epsilon)$$

(counting multiplicities) such that for $\epsilon < \epsilon_0$ the following estimates hold:

$$b(\lambda) - \lambda = O(e^{-C/\epsilon}), \tag{2.16}$$

$$\vec{d}\,(E,F) = O(e^{-C/\epsilon}), \tag{2.17}$$

$$\vec{d}\,(F,E) = O(e^{-C/\epsilon}) \tag{2.18}$$

for some $C > 0$.

The following result gives an approximation of the kernel of the linear operator L_σ defined on $\Omega_{\epsilon,P}$.

LEMMA 2.6. *Suppose that $\sigma = c\epsilon + O(\epsilon^2)$ where $c > 0$ is constant. For $\epsilon > 0$ sufficiently small there exists $C > 0$ such that*

$$\vec{d}(Kernel(L), X_\sigma) = O(e^{-C/\epsilon})$$

and

$$\vec{d}(X_\sigma, Kernel(L)) = O(e^{-C/\epsilon})$$

where

$$X_\sigma = span\left\{ \frac{\partial v_\sigma}{\partial y_j} \in L^2(\Omega_{\epsilon,P}) \middle| j = 1, 2, \ldots, N \right\}$$

is the kernel of L_σ defined on $\Omega_{\epsilon,P}$.

Proof. The lemma is an immediate consequence of Lemma 2.5. $\square$

Now we estimate the eigenvalues of the operator defined on $\Omega_{\epsilon,P}$.

LEMMA 2.7. *Let (τ,Φ_τ) with $\Phi_\tau \in H^2(\Omega_{\epsilon,P})$ be a solution of the following eigenvalue problem*

$$\begin{cases} \Delta\Phi + (1-3v_\sigma^2)\Phi = \tau\Phi & in\ \Omega_{\epsilon,P}, \\ \frac{\partial\Phi}{\partial\nu} = 0 & on\ \partial\Omega_{\epsilon,P}. \end{cases} \tag{2.19}$$

Suppose that $\sigma = c\epsilon + O(\epsilon^2)$ and $\Phi_\tau \perp X_\sigma$ where $c>0$ and

$$X_\sigma := span\left\{\frac{\partial v_\sigma}{\partial y_j} \in L^2(\Omega_{\epsilon,P}) \middle| j=1,2,\ldots,N\right\}.$$

Then $|\tau| \geq C\sigma^2$ where C is independent of $\sigma << 1$.

Proof: Suppose Lemma 2.7 is not true. Then there exist sequences τ_k and σ_k, $k=1,2,\ldots$ such that $\frac{\tau_k}{\sigma_k^2} \to 0$ as $k\to\infty$. Here τ_k is an eigenvalue of L_{σ_k} and $\tau_k \neq 0$, i.e.

$$L_{\sigma_k}\Phi_k = \tau_k\Phi_k,\ \Phi_k \perp X_{\sigma_k}$$

where

$$X_{\sigma_k} = \left\{\frac{\partial v_{\sigma_k}}{\partial y_j},\ j=1,2,\ldots,N\right\} \subset L^2(\Omega_{\epsilon,P}).$$

Φ_k satisfies

$$\Phi_k'' + \frac{N-1}{r}\Phi_k' + \frac{1}{r^2}\triangle_{S^{N-1}}\Phi_k + (1-3v_{\sigma_k}^2)\Phi_k = \tau_k\Phi_k. \tag{2.20}$$

Assume that

$$\|\Phi_k\|_{H^2(\Omega_{\epsilon,P})} = 1. \tag{2.21}$$

Extend Φ_k from $\Omega_{\epsilon,P}$ to a function in R^N such that $\Phi_k = O(e^{-C|y|})$ for $y \in R^N\setminus\Omega_{\epsilon,P}$ and such that the same result holds for the first and second derivatives of Φ_k.

We make the following decomposition

$$\Phi_k(r) = \sum_{m=1}^{\infty}\Phi_{k,m}(r-R_{\sigma_k})e_m(\theta) \tag{2.22}$$

where $r=|y|$. Here $e_m(\theta)$ are the eigenfunctions of $\Delta_{S^{N-1}}$, i.e.,

$$\triangle_{S^{N-1}}e_m + \mu_m e_m = 0.$$

Note that $\Phi_k(r) = O(e^{-\delta R_\sigma})$ for $|r-R_\sigma| \geq \beta\delta_0 > 0$. Hence there exists $\delta>0$ such that

$$\Phi_{k,m}(r) = \int_{|\theta|=1}\Phi_k(r)e_m(\theta)d\theta = O(e^{-\delta R_\sigma})\ \text{ for } |r-R_\sigma| \geq \beta\delta_0 > 0.$$

It is well–known that

$$\mu_0 = 0,\ \mu_1 = \ldots = \mu_N = N-1, \mu_{N+1} > N-1, \mu_m \sim m^2 \text{ as } m \to \infty.$$

Furthermore, $\Phi_{k,m}$ satisfies

$$\Phi''_{k,m} + \frac{N-1}{R_{\sigma_k}+s}\Phi'_{k,m} - \frac{\mu_m}{(R_{\sigma_k}+s)^2}\Phi_{k,m} + (1-3\tilde{v}^2_{\sigma_k})\Phi_{k,m} = \tau_k \Phi_{k,m} \tag{2.23}$$

in $[-R_\sigma, \infty)$. Note that $\tilde{v}'_{\sigma_k}$ satisfies

$$(\tilde{v}'_{\sigma_k})'' + \frac{N-1}{R_{\sigma_k}+s}(\tilde{v}'_{\sigma_k})' + (1-3\tilde{v}^2_{\sigma_k})\tilde{v}'_{\sigma_k} = \frac{N-1}{(R_{\sigma_k}+s)^2}\tilde{v}'_{\sigma_k} \text{ in } [-R_\sigma, \infty). \tag{2.24}$$

We next decompose $\Phi_{k,m}$ into

$$\Phi_{k,m} = C_{k,m}\tilde{v}'_{\sigma_k} + \Phi^2_{k,m}$$

where

$$\Phi^2_{k,m} \perp \tilde{v}'_{\sigma_k}.$$

Multiplying (2.23) by $\tilde{v}'_{\sigma_k}$, multiplying (2.24) by $\Phi_{k,m}$, taking the difference and integrating we obtain

$$\int_{-R_\sigma}^{\infty} \left(\tau_k + \frac{\mu_m - (N-1)}{(R_{\sigma_k}+s)^2}\right) \Phi_{k,m}\tilde{v}'_{\sigma_k}\, ds = O(e^{-\delta R_\sigma}). \tag{2.25}$$

Since $\tau_k = o(1)\sigma_k^2$, we have

$$C_{k,m} = O(\frac{R_\sigma^2 e^{-\delta R_\sigma}}{\mu_m}). \tag{2.26}$$

Note that $\Phi^2_{k,m}$ satisfies

$$(\Phi^2_{k,m})'' + \frac{N-1}{R_{\sigma_k}+s}(\Phi^2_{k,m})' + (1-3\tilde{v}^2_{\sigma_k})(\Phi^2_{k,m}) = \frac{\mu_m}{(R_{\sigma_k}+s)^2}\Phi^2_{k,m}$$

$$+\tau_k\Phi^2_{k,m} + \frac{\mu_m-(N-1)}{(R_{\sigma_k}+s)^2}C_{k,m}\tilde{v}'_{\sigma_k} + \tau_k C_{k,m}\tilde{v}'_{\sigma_k} \text{ in } [-R_\sigma, \infty). \tag{2.27}$$

Multiplying (2.27) by $\Phi^2_{k,m}$ and integrating by parts, we have

$$\int_{-R_\sigma}^{\infty} [((\Phi^2_{k,m})')^2 - (1-3\tilde{v}^2_{\sigma_k})(\Phi^2_{k,m})^2 + (\frac{\mu_m}{(R_{\sigma_k}+s)^2} + \tau_k)(\Phi^2_{k,m})^2$$

$$-\frac{N-1}{R_{\sigma_k}+s}(\Phi^2_{k,m})'\Phi^2_{k,m}]\, ds = O(e^{-\delta R_\sigma}). \tag{2.28}$$

Since $\Phi^2_{k,m} \perp \tilde{v}'_{\sigma_k}$, we have that

$$\int_{-R_\sigma}^{\infty} [((\Phi^2_{k,m})')^2 - (1-3\tilde{v}^2_{\sigma_k})(\Phi^2_{k,m})^2 + (\frac{\mu_m}{(R_{\sigma_k}+s)^2} + \tau_k)(\Phi^2_{k,m})^2$$

$$-\frac{N-1}{R_{\sigma_k}+s}(\Phi_{k,m}^2)'\Phi_{k,m}^2]\,ds \geq \int_{-R_\sigma}^{\infty} \sigma_0[((\Phi_{k,m}^2)')^2 + (\Phi_{k,m}^2)^2]\,ds. \tag{2.29}$$

(Suppose not. Then there exists a subsequence, again denoted by $\Phi_{k,m}^2$, such that $\Phi_{k,m}^2 \to \Phi_0$ in $H^1(-\infty,\infty)$ where $\int_{-\infty}^{\infty}((\Phi_0)')^2 + (\Phi_0)^2 = 1$ and $\Phi_0 \perp U_0'$. Furthermore, Φ_0 satisfies

$$\int_{-\infty}^{\infty}[((\Phi_0)')^2 - (1-3(U_0)^2)(\Phi_0)^2]\,ds = 0.$$

This is a contradiction since the operator $-\Delta + (1-3U_0^2)$ is positive and has the kernel span(U_0').)

Hence, combining (2.28) and (2.29),

$$\int_{-R_\sigma}^{\infty}[((\Phi_{k,m}^2)')^2 + (\Phi_{k,m}^2)^2]\,ds = O(\frac{e^{-\delta R_\sigma}}{R_\sigma^2+\mu_m}) = O(\frac{e^{-\delta R_\sigma}}{\mu_m}),$$

or, in other words,

$$\|\Phi_{k,m}^2\|_{H^1([-R_\sigma,\infty))}^2 = O(e^{-\delta R_\sigma}/\mu_m).$$

By elliptic regularity theory we also know that

$$\|\Phi_{k,m}^2\|_{H^2([-R_\sigma,\infty))} = O(e^{-\delta R_\sigma}/\mu_m).$$

Hence

$$\|\Phi_{k,m}^2\|_{H^2(R^N)}^2 = O(R_\sigma^{N-1}e^{-\delta R_\sigma}/\mu_m). \tag{2.30}$$

By (2.26) and (2.30),

$$\|\Phi_k\|_{H^2(\Omega_{\epsilon,P})}^2 \leq \sum_{m=N+1}^{\infty}\|\Phi_{k,m}\|_{H^2(R^N)}^2 = O(R_\sigma^{N+1}e^{-\delta R_\sigma})\sum_{m=N+1}^{\infty}\frac{1}{\mu_m} = o(1).$$

This is a contradiction! The proof is finished. □

Corollary 2.1. *For all $\Phi \in H_N^2(\Omega_{\epsilon,P})$ where Φ is orthogonal to the kernel of L_σ, we have*

$$\|L_\sigma\Phi\|_{L^2(\Omega_{\epsilon,P})} \geq C\sigma^2\|\Phi\|_{H^2(\Omega_{\epsilon,P})} \tag{2.31}$$

where $C>0$ is independent of $\sigma << 1$.

Proof: Let $L_\sigma\Phi = \sigma^2 f$, then by Lemma 2.4, we have

$$\|\sigma^2 f\|_{L^2(\Omega_{\epsilon,P})} \geq C\sigma^2\|\Phi\|_{L^2(\Omega_{\epsilon,P})}.$$

On the other hand, Φ satisfies

$$\Delta\Phi - 2\Phi = (3v_\sigma^2 - 3)\Phi + \sigma^2 f.$$

Hence by elliptic regularity estimates, we have

$$\|\Phi\|_{H^2(\Omega_{\epsilon,P})} \leq C(\|\Phi\|_{L^2(\Omega_{\epsilon,P})} + \sigma^2\|f\|_{L^2(\Omega_{\epsilon,P})})$$

$$\leq C\|f\|_{L^2(\Omega_{\epsilon,P})} \leq C\sigma^{-2}\|L_\sigma\Phi\|_{L^2(\Omega_{\epsilon,P})}.$$

The Corollary is thus proved. □

Finally, we study the asymptotic behavior of v_σ.

LEMMA 2.8. *For σ sufficiently small, we have*

$$v_\sigma - \tau_\sigma = C(\frac{r}{R_\sigma})^{-\frac{N-1}{2}} e^{\bar{\nu}_\sigma(R_\sigma - r)}(1+O(\sigma)) \text{ for } r \geq R_\sigma \tag{2.32}$$

where τ_σ is defined in Section 2 (note that $\tau_\sigma \to -1$ as $\sigma \to 0$), $C \neq 0$ is a generic constant and

$$\bar{\nu}_\sigma = \sqrt{3\tau_\sigma^2 - 1}.$$

Proof: We use matched asymptotics as in [27], although the proof can be made rigorous by ODE arguments and the maximum principle.

Let $\hat{v}_\sigma = v_\sigma - \tau_\sigma$. Linearizing equation (2.1) around τ_σ, we have that $\hat{v}_\sigma$ satisfies

$$\hat{v}_\sigma^{''} + \frac{N-1}{r}\hat{v}_\sigma^{'} - \bar{\nu}_\sigma^2\hat{v}_\sigma + O(\hat{v}_\sigma^2) = 0.$$

Note that $\bar{\nu}_\sigma = \sqrt{2} + O(\sigma)$ and the exact solution of the following problem

$$u^{''} + \frac{N-1}{r}u^{'} - \bar{\nu}_\sigma^2 u = 0, u(R_\sigma) = -\tau_\sigma, r \geq R_\sigma, u(r) \to 0 \text{ as } r \to \infty$$

is $(-\tau_\sigma)(\frac{r}{R_\sigma})^{1-N/2}K_m(\bar{\nu}_\sigma r)(K_m(\bar{\nu}_\sigma R_\sigma))^{-1}$ where $m = (N-2)/2$ and $K_m(z)$ is the modified Bessel function of the second kind of order m.

Since

$$K_m(z) = (1 + O(\frac{1}{z}))(\pi/(2z))^{1/2}e^{-z}$$

as $z \to \infty$, we have

$$\hat{v}_\sigma = C_\sigma(\frac{r}{R_\sigma})^{1-\frac{N}{2}}(\frac{\pi}{2r})^{\frac{1}{2}}e^{-\bar{\nu}_\sigma r}(1+O(\sigma)) \qquad \text{as } r \to \infty \tag{2.33}$$

where C_σ may depend on σ. On the other hand, let $r = R_\sigma + s$, then

$$\hat{v}_\sigma = C_0 e^{-\bar{\nu}_\sigma s}(1+O(\sigma)) \tag{2.34}$$

for s large, where $C_0 \neq 0$ is a generic constant. Combining (2.33) and (2.34), we have

$$C_\sigma = C_0\pi^{-1/2}(2\bar{\nu}_\sigma R_\sigma)^{1/2}e^{\bar{\nu}_\sigma R_\sigma}.$$

Hence Lemma 2.8 is proved. □

In the following, it will be more convenient to rewrite equation (2.32) as follows

$$v_\sigma - \tau_\sigma = C\sigma^l r^{-\frac{N-1}{2}}e^{\bar{\nu}_\sigma(R_\sigma - r)}(1+O(\sigma)) \qquad \text{for } r \geq R_\sigma. \tag{2.35}$$

where $l = -(N-1)/2$.

3. The projection of v_σ. In this section, we construct a modified function $P_{\Omega_{\epsilon,P}} v_\sigma$. It is close to v_σ and satisfies the Neumann boundary condition. Furthermore, we provide an error estimate for $\Psi_{\epsilon,P} = v_\sigma - P_{\Omega_{\epsilon,P}} v_\sigma$.

Let $\Psi_{\epsilon,P}$ be the unique solution of

$$\left\{ \begin{array}{ll} \epsilon^2 \triangle u - \bar{\nu}_\sigma^2 u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v_\sigma(\frac{x-P}{\epsilon})}{\partial \nu} & \text{on } \partial\Omega. \end{array} \right. \tag{3.1}$$

Define $P_{\Omega_{\epsilon,P}} v_\sigma := v_\sigma - \Psi_{\epsilon,P}$. Later, in Section 4, we will show that for every small $\epsilon > 0$ there exists exactly one $\sigma = \sigma(\epsilon)$ satisfying a certain nonlinear equation, and, furthermore, we have $\sigma(\epsilon) = \gamma_0 \epsilon + O(\epsilon^2)$ as $\epsilon \to 0$ where γ_0 is some positive constant. In this section we will write σ and ϵ with the understanding that this relation holds. We set

$$\nu_\epsilon = \bar{\nu}_{\sigma(\epsilon)}.$$

Note that by (2.35) on the boundary of $\partial\Omega$,

$$v_\sigma(\frac{x-P}{\epsilon}) = \tau_\sigma + C\sigma^l \left(\frac{|x-P|}{\epsilon} \right)^{-\frac{N-1}{2}} e^{-\nu_\epsilon(|x-P|/\epsilon - R_\sigma)} (1 + O(\sigma)).$$

In particular, we have the following asymptotic expansion of $\Psi_{\epsilon,P}$. A proof can be found in [34].

LEMMA 3.1. *For ϵ sufficiently small, we have*

$$\begin{aligned} &\Psi_{\epsilon,P}(x) = (C_N + O(\epsilon)) \epsilon^{l_1} e^{\nu_\epsilon R_\sigma} \\ &\times \int_{\partial\Omega} \left\{ e^{-\nu_\epsilon \frac{|t-P|+|t-x|}{\epsilon}} |t-P|^{-\frac{N-1}{2}} |t-x|^{-\frac{N-1}{2}} \frac{\langle t-x, \nu \rangle}{|t-x|} \right\} dt \end{aligned} \tag{3.2}$$

where l_1 is a rational number.

Let us introduce the following notation

$$\tilde{\varphi}_{\epsilon,P}(P) := \left[\int_0^\infty (\tau_\sigma^2 - v_\sigma^2(r)) v_\sigma^{'}(r) u_\sigma'(r) r^{N-1} dr \right] \Psi_{\epsilon,P}(P) \tag{3.3}$$

where u_σ is the unique solution of

$$\Delta u - \nu_\epsilon^2 u = 0, u(0) = 1, u > 0, u = u(r) \qquad \text{for } r \in [0, \infty). \tag{3.4}$$

We have the following key computations.

LEMMA 3.2. *Let P_0 be a nondegenerate peak point of Ω and $\alpha_0 > 0$ is given by condition (2) in Section 1. Suppose $P_\epsilon = P_0 + \epsilon(\frac{a}{2\sqrt{2}} d(P_0, \partial\Omega) + \tilde{z})$ with $|\tilde{z}| = O(\epsilon^\alpha), 0 < \alpha < \alpha_0$. Then*

$$\begin{aligned} L_j(\epsilon, \tilde{z}) &:= \textstyle\int_{\Omega_{\epsilon,P_\epsilon}} (\tau_\sigma^2 - v_\sigma^2) \Psi_{\epsilon,P_\epsilon} \frac{\partial v_\sigma}{\partial y_j} \\ &= L_j(\tilde{z}) \tilde{\varphi}_{\epsilon,P_\epsilon}(P_\epsilon) + O\big(\tilde{\varphi}_{\epsilon,P_\epsilon}(P_\epsilon) \epsilon^{\min(1,2\alpha,\alpha_0)}\big) \end{aligned} \tag{3.5}$$

where $L(\tilde{z}) := (L_1(\tilde{z}), ..., L_N(\tilde{z}))$ is a matrix which satisfies

$$L_j(\tilde{z}) = \gamma \frac{\int_{\partial\Omega} e^{<t-P_0,a>} \langle t - P_0, \tilde{z} \rangle \big(t_j - P_{0,j}\big) d\mu_{P_0}(t)}{\int_{\partial\Omega} e^{<t-P_0,a>} d\mu_{P_0}(t)}$$

where $\gamma \neq 0$ is a constant depending on N and $d(P_0, \partial\Omega)$ only.

Proof. Since the proof is quite similar to the proof of Lemma 3.4 in [34], we will merely sketch it. Note that

$$L_j(\epsilon, \tilde{z}) = \int_{\Omega_{\epsilon,P_\epsilon}} (\tau_\sigma^2 - v_\sigma^2)\Psi_{\epsilon,P_\epsilon} \frac{\partial v_\sigma}{\partial y_j}$$

$$= \int_0^\infty (\tau_\sigma^2 - v_\sigma^2) v_\sigma' r^{N-1} dr \int_{|\theta|=1} \theta_j \Psi_{\epsilon,P_\epsilon}(\epsilon y + P_\epsilon) d\theta + O(\tilde{\varphi}_{\epsilon,P_\epsilon}^{1+\mu}(P_\epsilon)).$$

But (let $x = \epsilon y + P_\epsilon$)

$$\Psi_{\epsilon,P_\epsilon}(\epsilon y + P_\epsilon) = \Psi_{\epsilon,P_\epsilon}(P_\epsilon) \frac{\int_{\partial\Omega} \left\{ e^{-\nu_\epsilon \frac{|t-P_\epsilon|+|t-x|}{\epsilon}} |t-P_\epsilon|^{-\frac{N-1}{2}} |t-x|^{-\frac{N-1}{2}} \frac{\langle t-x,\nu\rangle}{|t-x|} \right\} dt}{\int_{\partial\Omega} \left\{ e^{-\nu_\epsilon \frac{2|t-P_\epsilon|}{\epsilon}} |t-P_\epsilon|^{-\frac{N-1}{2}} |t-x|^{-\frac{N-1}{2}} \frac{\langle t-x,\nu\rangle}{|t-x|} \right\} dt}$$

$$= \Psi_{\epsilon,P_\epsilon}(P_\epsilon) \frac{\int_{\partial\Omega} \left\{ e^{-\nu_\epsilon \frac{2|t-P_\epsilon|}{\epsilon}} e^{\nu_\epsilon < \frac{t-P_\epsilon}{|t-P_\epsilon|}, y>} |t-P_\epsilon|^{-\frac{N-1}{2}} |t-x|^{-\frac{N-1}{2}} \frac{\langle t-x,\nu\rangle}{|t-x|} \right\} dt}{\int_{\partial\Omega} \left\{ e^{-\nu_\epsilon \frac{2|t-P_\epsilon|}{\epsilon}} |t-P_\epsilon|^{-\frac{N-1}{2}} |t-x|^{-\frac{N-1}{2}} \frac{\langle t-x,\nu\rangle}{|t-x|} \right\} dt}$$

$$= \Psi_{\epsilon,P_\epsilon}(P_\epsilon) \int_{\partial\Omega} e^{<t-P_0,a>} e^{\nu_\epsilon < \frac{t-P_0}{|t-P_0|}, y>} d\mu_{P_0}^a(t)(1 + O(\epsilon^{\alpha_0}))$$

by condition (2) on page 2, where

$$d\mu_P^a(t) = \lim_{\epsilon \to 0} \frac{e^{-2\nu_\epsilon |t-P_\epsilon|/\epsilon} dt}{\int_{\partial\Omega} e^{-2\nu_\epsilon |t-P_\epsilon|/\epsilon} dt}.$$

Hence

$$L_j(\epsilon, \tilde{z}) = \left[\int_0^\infty (\tau_\sigma^2 - v_\sigma^2(r)) v_\sigma'(r) u_\sigma'(r) r^{N-1} dr \right] \Psi_{\epsilon,P_\epsilon}(P_\epsilon) L_j(\tilde{z})$$

$$+ O(\tilde{\varphi}_{\epsilon,P_\epsilon}(P_\epsilon)\epsilon^{\min(1,2\alpha,\alpha_0)}) = L_j(\tilde{z})\tilde{\varphi}_{\epsilon,P_\epsilon}(P_\epsilon) + O\big(\tilde{\varphi}_{\epsilon,P_\epsilon}(P_\epsilon)\epsilon^{\min(1,2\alpha,\alpha_0)}\big).$$

□

4. Choosing σ. In this section we choose σ and give an asymptotic expansion including error estimate for its behavior as $\epsilon \to 0$.

Let $P_{\Omega_{\epsilon,P}} v_\sigma$ be defined as in Section 3. Set

$$\sigma = m - \frac{1}{|\Omega|} \int_\Omega P_{\Omega_{\epsilon,P}} v_\sigma dx. \tag{4.1}$$

We show that this equation has a unique solution σ if ϵ is small enough.

Note that

$$\int_\Omega (P_{\Omega_{\epsilon,P}} v_\sigma)^3 \, dx = \int_\Omega v_\sigma^3 \, dx + \int_\Omega [(P_{\Omega_{\epsilon,P}} v_\sigma)^3 - v_\sigma^3] \, dx.$$

Now choose R_σ such that for $r_b = \epsilon R_\sigma$

$$\frac{|B_{r_b}| - |\Omega \setminus B_{r_b}|}{|\Omega|} = m + O(\sigma) + O(\epsilon) \tag{4.2}$$

as $\sigma, \epsilon \to 0$. This implies

$$\frac{1}{|\Omega|} \int_\Omega v_\sigma^3 \, dx = m + c\sigma + O(\sigma^2)$$

for some constant $c > 0$. Furthermore, there exists $C > 0$ such that

$$\int_\Omega [(P_{\Omega_{\epsilon,P}} v_\sigma)^3 - v_\sigma^3] \, dx \le C \int_\Omega |\Psi_{\epsilon,P}| = O(e^{-C/\epsilon}).$$

Therefore by the implicit function theorem, if ϵ is small enough, there exists exactly one solution σ of (4.1). Furthermore, this σ satisfies

$$\sigma = \gamma_0 \epsilon + O(\epsilon^2) \tag{4.3}$$

as $\epsilon \to 0$, where $\gamma_0 = \frac{c_b}{r_b}$.

5. Technical Framework. In this section, we set up the technical framework to solve equation (1.2). As we mentioned in Section 1, this framework was originated by Floer and Weinstein [11] and later used by Oh [23], [24]. We modified their approach to the Cahn-Hilliard equation in [32], [33] and [34]. We shall follow [34].

Without loss of generality, we assume that $P_0 = 0 \in \Omega$ is a nondegenerate peak point, i.e.

(1) $\Lambda_0 = \{d\mu_0(t)\}$,

(2) $\exists a \in R^N$ such that

$$\int_{\partial\Omega} e^{<t,a>} t d\mu_0(t) = 0$$

and

$$\int_{\partial\Omega} \left\{ \frac{e^{-\frac{|t|}{\epsilon}} e^{<t,a>}}{\int_{\partial\Omega} e^{-\frac{|t|}{\epsilon}} dt} \right\} t dt = O(\epsilon^{\alpha_0})$$

for some $\alpha_0 > 0$,

(3) the matrix $G(0) := \left(\int_{\partial\Omega} e^{<t,a>} (t_i t_j) d\mu_0(t) \right)$ is nondegenerate.

Let $z = \epsilon(\frac{a}{2\sqrt{2}} d(0, \partial\Omega) + \tilde{z})$ where $|\tilde{z}| < \epsilon^\alpha$ with $0 < \alpha < 1$ to be chosen later.

We assume that $\sigma = \sigma(\epsilon)$ where $\sigma(\epsilon)$ is defined in Section 4.

Define $H_\epsilon : H_N^2(\Omega_\epsilon) \to L^2(\Omega_\epsilon)$ by

$$H_\epsilon(u) := \triangle u + u - u^3 - m + \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} u^3 \, dy \tag{5.1}$$

where

$$H_N^2(\Omega_\epsilon) := \left\{ u \in H^2(\Omega_\epsilon) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\epsilon \right\}.$$

We are looking for a nontrivial zero of (5.1). We make the ansatz

$$u = P_{\Omega_{\epsilon,z}} v_\sigma + \Phi_\epsilon$$

where Φ_ϵ is now the unknown. Recall that we set $w_{\epsilon,z} = P_{\Omega_{\epsilon,z}} v_\sigma$. We assume that $\epsilon > 0$ is small and Φ_ϵ is small in $C^2_{loc}(\Omega_\epsilon)$. We shall see that solutions of this particular form correspond to bubble solutions of (1.2) where the center of the bubble is located near 0. Inserting this into the equation gives

$$\triangle\Phi_\epsilon + \Phi_\epsilon + \triangle(P_{\Omega_{\epsilon,z}} v_\sigma) + P_{\Omega_{\epsilon,z}} v_\sigma - (P_{\Omega_{\epsilon,z}} v_\sigma + \Phi_\epsilon)^3 =$$

$$m - \frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon} (P_{\Omega_{\epsilon,z}} v_\sigma + \Phi_\epsilon)^3 \, dy.$$

Recall that

$$\triangle(P_{\Omega_{\epsilon,z}} v_\sigma) + P_{\Omega_{\epsilon,z}} v_\sigma = \Delta v_\sigma - \triangle\Psi_{\epsilon,z} + v_\sigma - \Psi_{\epsilon,z}$$

$$= v_\sigma^3 + \sigma - 3\tau_\sigma^2 \Psi_{\epsilon,z}.$$

This implies

$$\triangle\Phi_\epsilon + \Phi_\epsilon + v_\sigma^3 + \sigma - 3\tau_\sigma^2\Psi_{\epsilon,z} - (P_{\Omega_{\epsilon,z}} v_\sigma + \Phi_\epsilon)^3$$

$$= m - \frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon} (P_{\Omega_{\epsilon,z}} v_\sigma + \Phi_\epsilon)^3 \, dy.$$

By the choice of σ,

$$L_\epsilon \Phi_\epsilon + v_\sigma^3 - 3\tau_\sigma^2\Psi_{\epsilon,z} - (v_\sigma - \Psi_{\epsilon,z})^3 + N_{\epsilon,z}(\Phi_\epsilon) = 0$$

where

$$L_\epsilon\Phi_\epsilon := \triangle\Phi_\epsilon + \Phi_\epsilon - 3(P_{\Omega_{\epsilon,z}} v_\sigma)^2\Phi_\epsilon + 3\frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon} (P_{\Omega_{\epsilon,z}} v_\sigma)^2 \Phi_\epsilon \, dy$$

and

$$N_{\epsilon,z}(\Phi_\epsilon) = -3P_{\Omega_{\epsilon,z}} v_\sigma \Phi_\epsilon^2 - \Phi_\epsilon^3 + \frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon} [3P_{\Omega_{\epsilon,z}} v_\sigma \Phi_\epsilon^2 + \Phi_\epsilon^3] \, dy.$$

Recalling that $\Phi_\epsilon \to 0$ as $\epsilon \to 0$ in $C^2_{loc}(\Omega_\epsilon)$ we finally arrive at

$$L_\epsilon\Phi_\epsilon + 3(v_\sigma^2 - \tau_\sigma^2)\Psi_{\epsilon,z} + N_{\epsilon,z}(\Phi_\epsilon) + M_{\epsilon,z}(\Psi_{\epsilon,z}) = 0$$

where

$$M_{\epsilon,z}(\Psi_{\epsilon,z}) = -3v_\sigma \Psi_{\epsilon,z}^2 + \Psi_{\epsilon,z}^3.$$

It is easy to see that

LEMMA 5.1. *For ϵ sufficiently small*

$$\|N_{\epsilon,z}(\Phi_\epsilon)\|_{L^2(\Omega_{\epsilon,z})} \leqslant c\|\Phi_\epsilon\|^2_{H^2(\Omega_{\epsilon,z})},$$
$$\|M_{\epsilon,z}(\Psi_{\epsilon,z})\|_{L^2(\Omega_{\epsilon,z})} \leqslant c\|\Psi_{\epsilon,z}\|^2_{L^2(\Omega_{\epsilon,z})} \leq c|\tilde{\varphi}_{\epsilon,z}(z)|.$$

Furthermore,

$$\|N_{\epsilon,z}(\Phi^{(1)}_\epsilon) - N_{\epsilon,z}(\Phi^{(2)}_\epsilon)\|_{L^2(\Omega_{\epsilon,z})} \leq c\|\Phi^{(1)}_\epsilon - \Phi^{(2)}_\epsilon\|^2_{H^2(\Omega_{\epsilon,z})}.$$

It remains then to estimate the term $3(v_\sigma^2 - \tau_\sigma^2)\Psi_{\epsilon,z}$. We have

LEMMA 5.2. *For ϵ sufficiently small, we have*

$$\|(v_\sigma^2 - \tau_\sigma^2)\Psi_{\epsilon,z}\|^2_{L^2(\Omega_{\epsilon,z})} \leq C|\tilde{\varphi}_{\epsilon,z}(z)|^{1.5}. \tag{5.2}$$

Proof: In fact,

$$(v_\sigma^2 - \tau_\sigma^2)\Psi_{\epsilon,z} = e^{\nu_\epsilon R_\sigma} u_\sigma (v_\sigma^2 - \tau_\sigma^2) u_\sigma^{-1} e^{-\nu_\epsilon R_\sigma} \Psi_{\epsilon,z}$$

where u_σ is the unique radial solution of $\Delta u - \nu_\epsilon^2 u = 0, u(0) = 1, u > 0$.

Now

$$|u_\sigma(v_\sigma^2 - \tau_\sigma)| \leq e^{(\nu_\epsilon + \delta)R_\sigma} \tag{5.3}$$

where $\delta > 0$ is small. Furthermore, by Lemma 3.1, (note that $\epsilon y + z = x$),

$$e^{-\nu_\epsilon R_\sigma}\Psi_{\epsilon,z}$$

$$= (C_N + O(\epsilon)) \int_{\partial\Omega} \left\{ e^{-\nu_\epsilon \frac{|t-z|+|t-x|}{\epsilon}} |t-z|^{-\frac{N-1}{2}} |t-x|^{-\frac{N-1}{2}} \frac{\langle t-x, \nu\rangle}{|t-x|} \right\} dt.$$

$$\leq e^{\nu_\epsilon R_\sigma} e^{-2\nu_\epsilon d(z,\partial\Omega)/\epsilon} e^{(\nu_\epsilon+\delta)|y|}.$$

Therefore,

$$|u_\sigma^{-1} e^{-\nu_\epsilon R_\sigma} \Psi_{\epsilon,z}| \leq C e^{-2\nu_\epsilon d(z,\partial\Omega)/\epsilon} e^{(\nu_\epsilon+\delta)R_\sigma}. \tag{5.4}$$

Combining (5.3) and (5.4), we obtain

$$|(v_\sigma^2 - \tau_\sigma^2)\Psi_{\epsilon,z}| \leq C e^{-2\nu_\epsilon(d(z,\partial\Omega) - \epsilon R_\sigma) + 2(\delta+\nu_\epsilon)R_\sigma}$$

$$\leq C(\tilde{\varphi}_{\epsilon,z}(z))^{0.8}.$$

This implies

$$\|(v_\sigma^2 - \tau_\sigma^2)\Psi_{\epsilon,z}\|^2_{L^2(\Omega_{\epsilon,z})} \leq \tilde{\varphi}_{\epsilon,z}(z)^{1.5}.$$

The Lemma is thus proved. □

6. Reduction to Finite Dimensions: Fredholm Inverses. In this section, we show that $H'_\epsilon(w_{\epsilon,z})$, modulo its approximate kernel, is an invertible linear operator if ϵ is small enough. Moreover we show that the operator norm of the inverse operator is bounded by $C\epsilon^{-2}$. (Note that in [11], [23], [24] and [34], the operator norm of the inverse operator is uniformly bounded).

Set

$$K_{\epsilon,z} = \text{ span } \Big\{ \frac{\partial w_{\epsilon,z}}{\partial z_i} \Big| i = 1, \cdots, N \Big\} \subset H^2_N(\Omega_\epsilon) \tag{6.1}$$

and

$$C_{\epsilon,z} = \text{ span } \Big\{ \frac{\partial w_{\epsilon,z}}{\partial z_i} \Big| i = 1, \cdots, N \Big\} \subset L^2(\Omega_\epsilon). \tag{6.2}$$

$K_{\epsilon,z}$ is called the approximate kernel, while $C_{\epsilon,z}$ is called the approximate co-kernel. Note that a function $\Phi \in$ co-kernel of $H'_\epsilon(w_{\epsilon,z})$ iff for all $\psi \in H^2_N(\Omega_\epsilon)$ we have

$$\int_{\Omega_\epsilon} \Phi H'_\epsilon(w_{\epsilon,z})\psi \, dy = 0.$$

Integrating by parts, we have

$$\int_{\partial\Omega_\epsilon} \psi \frac{\partial \Phi}{\partial \nu} \, do + \int_{\Omega_\epsilon} [(\Delta\Phi + (1 - 3w^2_{\epsilon,z})\Phi)\psi] \, dy$$

$$+3\frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} \Phi \, dy \int_{\Omega_\epsilon} w^2_{\epsilon,z}\psi \, dy = 0, \ \forall \psi \in H^2_N(\Omega_\epsilon).$$

Hence $\Phi \in$ co-kernel of $H'_\epsilon(w_{\epsilon,z})$ if and only if

$$\begin{cases} \Delta\Phi + (1 - 3w^2_{\epsilon,z})\Phi + 3w^2_{\epsilon,z} \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} \Phi \, dy = 0 & \text{in } \Omega_\epsilon, \\ \frac{\partial\Phi}{\partial\nu} = 0 & \text{in } \partial\Omega_\epsilon. \end{cases}$$

Observe also that span$\{ \frac{\partial v_\sigma}{\partial y_i} | i = 1, \cdots, N\}$ is the kernel of L, where L is the linear operator defined as

$$L\Phi := \Delta\Phi + \Phi - 3v^2_\sigma\Phi, \qquad \Phi \in H^2(R^N).$$

Our main result in this section can be stated as follows.

Proposition 6.1. *There exist positive constants ϵ_1, λ such that for all $\epsilon \in (0, \epsilon_1)$*

$$\|L_{\epsilon,z}\Phi\|_{L^2(\Omega_\epsilon)} \geqslant \lambda\sigma^2 \|\Phi\|_{H^2(\Omega_\epsilon)} \tag{6.3}$$

for all $|z| \le C\epsilon$ and for all $\Phi \in K^\perp_{\epsilon,z}$ where

$$L_{\epsilon,z} = \pi_{\epsilon,z} \circ H'_\epsilon(w_{\epsilon,z}) \tag{6.4}$$

and $\pi_{\epsilon,z}$ is the L^2-orthogonal projection from $L^2(\Omega_\epsilon)$ to $C^\perp_{\epsilon,z}$.

The next proposition gives the surjectivity of $L_{\epsilon,z}$.

PROPOSITION 6.2. *There exists a positive constant ϵ_2 such that for all $\epsilon \in (0, \epsilon_2)$ and $|z| \leq C\epsilon$, the map*

$$L_{\epsilon,z} = \pi_{\epsilon,z} \circ H'_\epsilon(w_{\epsilon,z}) : K^\perp_{\epsilon,z} \longrightarrow C^\perp_{\epsilon,z}$$

is surjective.

Combining Propositions 6.1 and 6.2 gives us the invertibility of $L_{\epsilon,z}$.

PROPOSITION 6.3.

$$L_{\epsilon,z} : K^\perp_{\epsilon,z} \longrightarrow C^\perp_{\epsilon,z}$$

is invertible, namely,

$$L^{-1}_{\epsilon,z} : C^\perp_{\epsilon,z} \longrightarrow K^\perp_{\epsilon,z}$$

exists. Furthermore, $L^{-1}_{\epsilon,z}$ is bounded in the operator norm by $C\epsilon^{-2}$.

We now begin to prove Proposition 6.1.

Proof of Proposition 6.1: We use a different strategy than in [32].

Suppose (6.3) is false. Then there exist sequences $\{\epsilon_k\}, \{z_k\}$ and $\{\Phi_k\}$, with $|z_k| \leqslant C\epsilon_k$ and $\epsilon_k \to 0$ as $k \to \infty$ such that

$$\Phi_k \in K^\perp_{\epsilon_k,z_k} \quad \text{and}$$

$$\|L_{\epsilon_k,z_k}(\Phi_k)\|_{L^2(\Omega_{\epsilon_k})} = o(1)\epsilon_k^2, \quad \|\Phi_k\|_{H^2(\Omega_{\epsilon_k})} = 1. \tag{6.5}$$

We denote, for $i = 1, \cdots, N$

$$e_{k,i} = \frac{\frac{\partial w_{\epsilon_k,z_k}}{\partial z_i}}{\|\frac{\partial w_{\epsilon_k,\epsilon_k}}{\partial z_i}\|_{L^2(\Omega_{\epsilon_k})}}, e^*_{k,i} = \frac{\frac{\partial v_{\sigma_k}}{\partial y_i}}{\|\frac{\partial v_{\sigma_k}}{\partial y_i}\|_{L^2(\Omega_{\epsilon_k})}}. \tag{6.6}$$

Note that the difference between $e_{k,i}$ and $e^*_{k,i}$ is exponentially small. Hence, after applying the Gram-Schmidt process to $\{e_{k,i}|i = 1, \cdots, N\}$ we obtain a family of orthonormal functions $\{\tilde{e}_{k,i}|i = 1, \cdots, N\}$ with

$$\tilde{e}_{k,i} = e_{k,i} + \delta_{k,i}, \ i = 1, \cdots, N$$

where $\delta_{k,i} = O(e^{-\delta/\epsilon})$ in $L^2(\Omega_{\epsilon_k})$ as $k \to \infty$ for each $i = 1, \cdots, N$.

Hence,

$$L_{\epsilon_k,z_k}\Phi_k = H'_{\epsilon_k}(w_{\epsilon_k,z_k})\Phi_k - \sum_{i=1}^{N-1} \Big(\int_{\Omega_{\epsilon_k}} [H'_{\epsilon_k}(w_{\epsilon_k,z_k})\Phi_k] e_{k,i}\, dy\Big) e_{k,i} + E_k \tag{6.7}$$

where E_k is defined by (6.7) and it is easy to see that $\|E_k\|_{L^2(\Omega_{\epsilon_k})} = O(e^{-\delta/\epsilon_k})$ as $k \to \infty$.

Note that

$$\begin{aligned} \|L_{\epsilon_k,z_k}\Phi_k\|^2_{L^2(\Omega_{\epsilon_k})} &= \|H'_{\epsilon_k}(w_{\epsilon_k,z_k})\Phi_k\|^2_{L^2(\Omega_{\epsilon_k})} \\ &- \sum_{i=1}^{n} \Big(\int_{\Omega_{\epsilon_k}} [H'_{\epsilon_k}(w_{\epsilon_k,z_k})\Phi_k] e_{k,i}\, dy\Big)^2 + O(e^{-\delta/\epsilon_k}) \end{aligned} \tag{6.8}$$

as $k \to \infty$.

Let us denote

$$\Delta\Phi_k + (1 - 3w^2_{\epsilon_k,z_k})\Phi_k + 3\frac{1}{|\Omega_{\epsilon_k}|}\int_{\Omega_{\epsilon_k}} w^2_{\epsilon_k,z_k}\Phi_k\,dy = \sigma_k^2 f_k.$$

By Corollary 2.1, we have

$$\|f_k - 3\frac{1}{|\Omega_{\epsilon_k}|\sigma_k^2}\int_{\Omega_{\epsilon_k}} w^2_{\epsilon_k,z_k}\Phi_k\,dy\|_{L^2(\Omega_{\epsilon_k})} \geq C\|\Phi_k\|_{H^2(\Omega_{\epsilon_k})}. \tag{6.9}$$

Note that since Φ_k satisfies the Neumann boundary condition, we have

$$|\int_{\Omega_{\epsilon_k}} \Phi_k| = |\sigma_k^2\int_{\Omega_{\epsilon_k}} f_k\,dy| \leq C\epsilon_k^{2-\frac{N}{2}}\|f_k\|_{L^2(\Omega_{\epsilon_k})}.$$

Hence

$$3\frac{1}{|\Omega_{\epsilon_k}|\sigma_k^2}\int_{\Omega_{\epsilon_k}} w^2_{\epsilon_k,z_k}\Phi_k\,dy \leq C\epsilon_k^{\frac{N}{2}}\|f_k\|_{L^2(\Omega_{\epsilon_k})}.$$

Thus

$$\|3\frac{1}{|\Omega_{\epsilon_k}|\sigma_k^2}\int_{\Omega_{\epsilon_k}} w^2_{\epsilon_k,z_k}\Phi_k\,dy\|_{L^2(\Omega_{\epsilon_k})} \leq C\|f_k\|_{L^2(\Omega_{\epsilon_k})}.$$

The last inequality and (6.9) imply that

$$\|f_k\|_{L^2(\Omega_{\epsilon_k})} \geq C\|\Phi_k\|_{H^2(\Omega_{\epsilon_k})} \geq C.$$

Therefore

$$\|H'_{\epsilon_k}(w_{\epsilon_k,z_k})\Phi_k\|^2_{L^2(\Omega_{\epsilon_k})} \geq C\sigma_k^2. \tag{6.10}$$

Now we estimate

$$\int_{\Omega_{\epsilon_k}} [H'_{\epsilon_k}(w_{\epsilon_k,z_k})\Phi_k]e_{k,i}\,dy$$

$$= \int_{\Omega_{\epsilon_k}} [H'_{\epsilon_k}(w_{\epsilon_k,z_k})\Phi_k]\frac{\partial w_{\epsilon_k,z_k}}{\partial z_i}\,dy + O(e^{-\delta/\epsilon_k})$$

$$= \int_{\Omega_{\epsilon_k}} [\Delta\Phi_k + (1 - 3v^2_{\sigma_k})\Phi_k + 3\frac{1}{|\Omega_{\epsilon_k}|}\int_{\Omega_{\epsilon_k}} v^2_{\sigma_k}\Phi_k\,dy]\frac{\partial v_{\sigma_k}}{\partial y_i}\,dy + O(e^{-\delta/\epsilon_k})$$

$$= \int_{\partial\Omega_{\epsilon_k}} \left[\frac{\partial v_{\sigma_k}}{\partial y_i}\frac{\partial \Phi_k}{\partial \nu} - \Phi_k\frac{\partial}{\partial \nu}\left(\frac{\partial v_{\sigma_k}}{\partial y_i}\right)\right] do + 3\frac{1}{|\Omega_{\epsilon_k}|}\int_{\Omega_{\epsilon_k}} v^2_{\sigma_k}\Phi_k\,dy\int_{\Omega_{\epsilon_k}}\frac{\partial v_{\sigma_k}}{\partial y_i}\,dy$$

$$+ O(e^{-\delta/\epsilon_k}) = O(e^{-\delta/\epsilon_k}).$$

Therefore (6.8) implies that

$$o(1)\epsilon_k^2 \geq C\sigma_k^2 - o(e^{-\delta/\epsilon_k}). \tag{6.11}$$

This is a contradiction! Proposition 6.1 is thus proved. □

The following lemma, which can be found in [15], will be needed in the proof of Proposition 6.2.

LEMMA 6.1. *([15]; Lemma 1.3) If* $\overrightarrow{d}\,(E,F) := sup\{d(x,F)|x \in E, \|x\|_H = 1\} < 1$*, then* $\pi_{F|E} : E \to F$ *is injective and* $\pi_{E|F} : F \to E$ *has a bounded right inverse, where* $\pi_E(\pi_F$*, resp.) is the orthogonal projection from* H *to* $E(F$*, resp.). In particular,* $\pi_{E|F} : F \to E$ *is surjective.*

We are now ready to prove Proposition 6.2.

Proof of Proposition 6.2:

Let $CK_{\epsilon,z}$ = co-kernel of $H'_\epsilon(w_{\epsilon,z})$. We first claim that

$$\vec{d}(CK_{\epsilon,z}, C_{\epsilon,z}) < 1 \tag{6.12}$$

for all $\epsilon > 0$ sufficiently small.

In fact, suppose (6.12) is not true. Then there exist $\epsilon_k \to 0$ and $\Phi_k \in CK_{\epsilon_k,z_k}$ such that

$$\Delta\Phi_k + (1 - 3w^2_{\epsilon_k,z_k})\Phi_k + 3w^2_{\epsilon_k,z_k}\frac{1}{|\Omega_{\epsilon_k}|}\int_{\Omega_{\epsilon_k}}\Phi_k\,dy = 0 \text{ in } \Omega_{\epsilon_k}, \tag{6.13}$$

$$\frac{\partial\Phi_k}{\partial\nu} = 0 \text{ on } \partial\Omega_{\epsilon_k}, \tag{6.14}$$

$$\|\Phi_k\|_{L^2(\Omega_{\epsilon_k})} = 1, \tag{6.15}$$

$$\int_{\Omega_{\epsilon_k}}\Phi_k\frac{\partial(w_{\epsilon_k,z_k})}{\partial z_i}\,dy = 0, \quad i = 1, \cdots, N. \tag{6.16}$$

By (6.13), (6.14), we have

$$\int_{\Omega_{\epsilon_k}}(1 - 3w^2_{\epsilon_k,z_k})\Phi_k\,dy + 3\int_{\Omega_{\epsilon_k}}w^2_{\epsilon_k,z_k}\,dy\frac{1}{|\Omega_{\epsilon_k}|}\int_{\Omega_{\epsilon_k}}\Phi_k\,dy = 0.$$

Note that

$$\int_{\Omega_{\epsilon_k}}w^2_{\epsilon_k,z_k}\,dy = |\Omega_{\epsilon_k}|(1 + O(\epsilon_k)).$$

Hence, we have

$$\int_{\Omega_{\epsilon_k}}\Phi_k\,dy = \int_{\Omega_{\epsilon_k}}(1/3 - w^2_{\epsilon_k,z_k})\Phi_k\,dy(1 + O(\epsilon_k)) \leq O(\epsilon_k^{\frac{N+1}{2}})\|\Phi_k\|_{L^2(\Omega_{\epsilon_k})}.$$

Similar to the proof of Proposition 6.1, we conclude that

$$\|\Phi_k\|_{H^2(\Omega_{\epsilon_k})} = o(1). \tag{6.17}$$

This is a contradiction! Hence (6.12) is true.

Now by the fact that $\vec{d}(E,F) = \vec{d}(F^\perp, E^\perp)$, we have

$$\vec{d}(\overline{C}^\perp_{\epsilon,z}, \overline{CK}^\perp_{\epsilon,z}) < 1$$

where $\overline{C}^\perp_{\epsilon,z}(\overline{CK}^\perp_{\epsilon,z}$, resp.) is the orthogonal complement of $C_{\epsilon,z}(CK_{\epsilon,z}$, resp.) in $L^2(\Omega_\epsilon)$. Thus the map

$$\pi_{\overline{C}^\perp_{\epsilon,z}}\Big|_{\overline{CK}^\perp_{\epsilon,z}} : \overline{CK}^\perp_{\epsilon,z} \to \overline{C}^\perp_{\epsilon,z} \tag{6.18}$$

is surjective, by Lemma 6.1.

Since $\overline{CK}^\perp_{\epsilon,z}$ is the range of L_ϵ, it suffices to show that the map in (6.18) when restricted to $CK^\perp_{\epsilon,z}$, which is just $\pi_{\epsilon,z}$ is onto $C^\perp_{\epsilon,z}$. However, this follows easily from the expression

$$\pi_{\overline{C}^\perp_{\epsilon,z}}(\Phi) = \Phi - \pi_{C_{\epsilon,z}}\Phi. \qquad \square$$

Finally in this section, we solve the following equation for $\Phi_\epsilon \in K^\perp_{\epsilon,z}$.

$$\pi_{\epsilon,z} \circ H_\epsilon(w_{\epsilon,z})(w_{\epsilon,z} + \Phi_\epsilon) = 0. \tag{6.19}$$

Since $L_{\epsilon,z}\big|_{K^\perp_{\epsilon,z}}$ is invertible (and we shall denote its inverse just by $L^{-1}_{\epsilon,z}$) by Proposition 6.3, this is equivalent to solving

$$\Phi_\epsilon = L^{-1}_{\epsilon,z} \circ \pi_{\epsilon,z}(L_\epsilon(\Phi_\epsilon)) = -L^{-1}_{\epsilon,z} \circ \pi_{\epsilon,z}(3(v_\sigma^2 - \tau_\sigma^2)\Psi_{\epsilon,z} + N_{\epsilon,z}(\Phi_\epsilon) + M_{\epsilon,z}(\Psi_{\epsilon,z}))$$

$$:\equiv Q_{\epsilon,z}(\Phi_\epsilon)$$

where $Q_{\epsilon,z}$ is defined in the last equality for every $\Phi_\epsilon \in H^2_N(\Omega_\epsilon)$.

By Proposition 6.3, we have

$$\|L^{-1}_{\epsilon,z}\| \le C\epsilon^{-2}.$$

Hence,

$$\begin{aligned}\|Q_{\epsilon,z}(\Phi_\epsilon)\|_{H^2(\Omega_\epsilon)} &\leqslant C\epsilon^{-2}(\|(v_\sigma^2 - \tau_\sigma^2)\Psi_{\epsilon,z}\|_{L^2(\Omega_\epsilon)} + \|N_{z,\epsilon}(\Phi_\epsilon)\|_{L^2(\Omega_\epsilon)} \\ &\quad + \|M_{z,\epsilon}(\Psi_{\epsilon,z})\|_{L^2(\Omega_\epsilon)}) \\ &\leqslant c\epsilon^{-2}(\tilde{\varphi}^{\frac{1}{2}+\tilde{\eta}}_{\epsilon,z} + \ \delta\|\Phi_\epsilon\|_{H^2(\Omega_\epsilon)})\end{aligned}$$

for some $\tilde{\eta} > 0$ (in fact, we can take $\tilde{\eta} = 1/4$ by Lemma 5.1).

Take $\delta = |\tilde{\varphi}_{\epsilon,z}(z)|^{\frac{1+\eta}{2}}$ for $0 < \eta < 2\tilde{\eta}$. Then we have (since $\delta\epsilon^{-2} = o(1)$)

$$\|Q_{\epsilon,z}(\Phi_\epsilon)\|_{H^2(\Omega_\epsilon)} \leqslant C(\tilde{\varphi}^{\frac{1+\eta}{2}}_{\epsilon,z}(z)). \tag{6.20}$$

Equation (6.20) says that $Q_{\epsilon,z}(\Phi)$ is a continuous map:

$$B_\delta(0) \cap H^2_N(\Omega_\epsilon) \longrightarrow B_\delta(0) \cap H^2_N(\Omega_\epsilon).$$

Furthermore, $Q_{\epsilon,z}(\Phi)$ is a contracting map if ϵ is small by Lemma 5.1. Hence by the Contraction Mapping Principle we have the following proposition.

PROPOSITION 6.4. *There exists $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0, |z| \le C\epsilon$ there is a unique $\Phi_{\epsilon,z} \in K_{\epsilon,z}^{\perp}$ such that*

$$H_\epsilon(w_{\epsilon,z} + \Phi_{\epsilon,z}) \in C_{\epsilon,z}. \tag{6.21}$$

Furthermore,

$$\|\Phi_{\epsilon,z}\|_{H^2(\Omega_\epsilon)} \le C\tilde{\varphi}_{\epsilon,z}^{\frac{1+\mu}{2}}(z). \tag{6.22}$$

7. The Reduced Problem. In this section, we shall prove our main result Theorem 1.1.

By Proposition 6.4, for $\epsilon \leqslant \epsilon_0$ and $|z| \le C\epsilon$, there exists a unique $\Phi_{\epsilon,z}$ such that

$$H_\epsilon(w_{\epsilon,z} + \Phi_{\epsilon,z}) \in C_{\epsilon,z}. \tag{7.1}$$

Therefore it is enough to show that for some $|z| \le C\epsilon$, we have

$$H_\epsilon(w_{\epsilon,z} + \Phi_{\epsilon,z}) \perp C_{\epsilon,z}.$$

To this end, we now define a vector field

$$V_{\epsilon,j}(\tilde{z}) := \frac{1}{\epsilon^{\alpha-1}\tilde{\varphi}_{\epsilon,z}(z)}\Big[\int_{\Omega_\epsilon} H_\epsilon(w_{\epsilon,z} + \Phi_{\epsilon,z})\frac{\partial w_{\epsilon,z}}{\partial z_j}\,dy\Big] \tag{7.2}$$

where $z = \epsilon\frac{a}{2\sqrt{2}}d(0,\partial\Omega) + \epsilon^{\alpha+1}\tilde{z}, |\tilde{z}| \leqslant 1$, and $\vec{a}$ is given by conditions (2) and (3) in Section 1.

The main estimate of this section is

LEMMA 7.1. *For every $0 < \alpha < \alpha_0$, the vector field V_ϵ converges uniformly to V_0 in $B_1(0)$ as $\epsilon \to 0$, where*

$$\begin{aligned} V_0 &= (V_{0,1}, \cdots, V_{0,N}),\\ V_{0,j} &= \tfrac{\gamma}{\int_{\partial\Omega} e^{<t-P_0,a>}d\mu_{P_0}(t)}\textstyle\sum_{i=1}^N\big(\int_{\partial\Omega} e^{<x-P_0,a>}x_ix_jd\mu_{P_0}(x)\tilde{z}_i\big), j = 1, ..., N, \end{aligned}$$

and γ is given by Lemma 3.2.

Once Lemma 7.1 is proved, then Theorem 1.1 follows easily. In fact, since 0 is a nondegenerate peak point, V_0 has a nondegenerate zero at 0 (with degree different from 0). Then Lemma 7.1 and a simple degree theoretic argument imply that V_ϵ has a zero $\tilde{z}(\epsilon) \in B_{\frac{1}{2}}(0)$ for every ϵ sufficiently small. This solves the equation $H_\epsilon(w_{\epsilon,z}+\Phi_{\epsilon,z}) = 0$ for every ϵ sufficiently small. Setting $z(\epsilon) = \epsilon\frac{a}{2\sqrt{2}}d(0,\partial\Omega) + \epsilon^{\alpha+1}\tilde{z}(\epsilon)$ and

$$v_\epsilon = w_{\epsilon,z(\epsilon)} + \Phi_{\epsilon,z(\epsilon)}$$

for $x \in \Omega$ and ϵ sufficiently small, it follows then

$$v_\epsilon \not\equiv 0 \text{ since } \Phi_{\epsilon,z(\epsilon)} \to 0 \text{ in } H^2(\Omega_\epsilon) \text{ as } \epsilon \to 0$$

while $w_{\epsilon,z(\epsilon)}$ remains bounded away from 0 in $H^2(\Omega_\epsilon)$ as $\epsilon \to 0$.

That is, v_ϵ is a non-trivial solution of (1.2). By the structure of v_ϵ, v_ϵ has all the properties of Theorem 1.1.

It remains to prove Lemma 7.1. To this end, we have

$$\begin{aligned}
&\int_{\Omega_{\epsilon,z}} H_\epsilon(w_{\epsilon,z}+\Phi_{\epsilon,z})\tfrac{\partial w_{\epsilon,z}}{\partial z_j}\\
&= \int_{\Omega_{\epsilon,z}} [H_\epsilon^{'}(w_{\epsilon,z})\Phi_{\epsilon,z}]\tfrac{\partial w_{\epsilon,z}}{\partial z_j}\\
&+ \int_{\Omega_{\epsilon,z}} [N_{\epsilon,z}(\Phi_{\epsilon,z})]\tfrac{\partial w_{\epsilon,z}}{\partial z_j}\\
&+ \int_{\Omega_{\epsilon,z}} M_{\epsilon,z}(\Psi_{\epsilon,z})\tfrac{\partial w_{z,\epsilon}}{\partial z_j}\\
&+ \int_{\Omega_{\epsilon,z}} 3[v_\sigma^2-\tau_\sigma^2]\Psi_{\epsilon,z}\tfrac{\partial w_{\epsilon,z}}{\partial z_j}\\
&= I_1+I_2+I_3+I_4
\end{aligned}$$

where $I_i, i=1,2,3,4$ are defined by the last equality.

Note that

$$I_1 = 3\int_{\Omega_{\epsilon,z}} \big[(P_{\Omega_{\epsilon,z}}v_\sigma)^2 - v_\sigma^2\big]\Phi_{\epsilon,z}\frac{\partial w_{\epsilon,z}}{\partial z_j}\,dy$$

$$+3\int_{\Omega_{\epsilon,z}}\frac{\partial w_{\epsilon,z}}{\partial z_j}\,dy\int_{\Omega_{\epsilon,z}}(P_{\Omega_{\epsilon,z}}v_\sigma)^2\Phi_{\epsilon,z}\,dy$$

$$\leq C\left\|(P_{\Omega_{\epsilon,z}}v_\sigma - v_\sigma)\frac{\partial w_{\epsilon,z}}{\partial z_j}\right\|_{L^2(\Omega_{\epsilon,z})}\|\Phi_{\epsilon,z}\|_{L^2(\Omega_{\epsilon,z})}$$

$$+3\int_{\Omega_{\epsilon,z}}\frac{\partial w_{\epsilon,z}}{\partial z_j}\,dy\epsilon^{-N/2}\|\Phi_{\epsilon,z}\|_{L^2(\Omega_{\epsilon,z})}$$

$$\leq C\tilde{\varphi}_{\epsilon,z}(z)^{\frac{1+\mu}{2}}\tilde{\varphi}_{\epsilon,z}(z)^{\frac{1+\mu}{2}}$$

$$= O(\tilde{\varphi}_{\epsilon,z}^{1+\mu}(z))$$

where $\mu > 0$ is some small number. By Lemma 5.1 and Proposition 6.4 we have

$$|I_2| \leq C|\tilde{\varphi}_{\epsilon,z}(z)|^{1+\mu}$$

and

$$|I_3| \leq C|\tilde{\varphi}_{\epsilon,z}(z)|^{1+\mu}$$

since $N_{\epsilon,z}(\cdot)$ and $M_{\epsilon,z}(\cdot)$ depend on their arguments only in the second or higher powers. So we just need to compute I_4. In fact,

$$\begin{aligned}
I_4 &= -\int_{\Omega_{\epsilon,z}} 3\big[\tau_\sigma^2 - v_\sigma^2\big]\Psi_{\epsilon,z}\tfrac{\partial P_{\Omega_{\epsilon,z}}v_\sigma}{\partial z_j}\\
&= -\epsilon\int_{\Omega_{\epsilon,z}} 3[\tau_\sigma^2-v_\sigma^2]\Psi_{\epsilon,z}\tfrac{\partial v_\sigma}{\partial y_j}\\
&\quad + O\big(e^{-\sqrt{\nu_\epsilon}\frac{(2+\mu)d(z,\partial\Omega)}{\epsilon}}\big).
\end{aligned}$$

By Lemma 3.2, we conclude the proof of Lemma 7.1. □

Acknowledgments. The first author wishes to thank Professor Wei-Ming Ni for his constant encouragement. This work was done while the second author visited the Department of Mathematics at The Chinese University of Hong Kong. The work of the first author is supported by an Earmarked Grant from RGC of Hong Kong.

REFERENCES

[1] S. Agmon, *Lectures on Elliptic Boundary Value Problems,* Van Nostrand, Princeton, 1965.

[2] N. Alikakos, P.W. Bates and X. Chen, *Convergence of the Cahn-Hilliard equation to the Hele-Shaw model*, Arch. Rat. Mech. Anal., 128 (1994), pp. 165-205.

[3] N. Alikakos, P.W. Bates and G. Fusco, *Slow motion for Cahn-Hilliard equation in one space dimension*, J. Diff. Eqns., 90 (1991), pp. 81-134.

[4] N. Alikakos, G. Fusco and M. Kowalczyk, *Finite dimensional dynamics and interfaces intersecting the boundary I*, preprint.

[5] N. Alikakos and G. Fusco, *Slow dynamics for the Cahn-Hilliard equation in higher space dimension, Part I: Spectral estimates*, Comm. PDE, 19 (1994), pp. 1397-1447.

[6] P. Bates and P. Fife, *The dynamics of nucleation for the Cahn-Hilliard equation*, SIAM J. Appl. Math., 53 (1993), pp. 990-1008.

[7] J.W. Cahn and J.E. Hilliard, *Free energy of a nonuniform system, I. Interfacial free energy*, J. Chem. Phys., 28 (1958), pp. 258-267.

[8] X. Chen, *Generation and propagation of interfaces in reaction diffusion systems*, J. Diff. Equ., 96 (1992), pp. 116-141.

[9] X. Chen and M. Kowalczyk, *Existence of equilibria for the Cahn-Hilliard equation via local minimizers of the perimeter*, Comm. PDE, 21 (1996), pp. 1207-1233.

[10] E.N. Dancer, *A note on asymptotic uniqueness for some nonlinearities which change sign*, preprint.

[11] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal., 69 (1986), pp. 397-408.

[12] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in R^n*, Mathematical Analysis and Applications, Part A, Adv. Math. Suppl. Studies Vol. 7A, Academic Press, New York, 1981, pp. 369-402.

[13] M. Grinfeld and A. Novick-Cohen, *Counting stationary solutions of the Cahn-Hilliard equation by transversality arguments*, Proc. Roy. Soc. Edinburgh Sect. A, 125 (1995), pp. 351-370.

[14] M. Grinfeld and A. Novick-Cohen, *The viscous Cahn-Hilliard equation: Morse decomposition and structure of the global attractor*, submitted.

[15] B. Helffer and J. Sjöstrand, *Multiple wells in the semi-classical limit I*, Comm. PDE, 9 (1984), pp. 337-408.

[16] R.V. Kohn and P. Sternberg, *Local minimizers and singular perturbations*, Proc. Roy. Soc. Edinburgh Sect. A, 111 (1989), pp. 69-84.

[17] J.L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Vol. 1, Springer-Verlag, New York/Heidelberg/Berlin, 1972.

[18] P.L. Lions, *Generalized Solutions of Hamilton-Jacobi Equations*, Pitman Research Notes Series, Pitman, London, 1982.

[19] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal., 107 (1989), pp. 71-83.

[20] W.-M. Ni and I. Takagi, *On the shape of least energy solution to a semilinear Neumann problem*, Comm. Pure Appl. Math., 41 (1991), pp. 819-851.

[21] W.-M. Ni and I. Takagi, *Locating the peaks of least energy solutions to a semilinear Neumann problem*, Duke Math. J., 70 (1993), pp. 247-281.

[22] W.-M. Ni and J. Wei, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems*, Comm. Pure Appl. Math., 48 (1995), pp. 731-768.

[23] Y.G. Oh, *Existence of semi-classical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_a$*, Comm. PDE, 13 (1988), pp. 1499-1519.

[24] Y.G. Oh, *On positive multi-bump bound states of nonlinear Schrödinger equations under multiple-well potentials*, Comm. Math. Phys., 131, (1990), pp. 223-253.

[25] R.L. Pego, *Front migration in the nonlinear Cahn-Hilliard equation*, Proc. Roy. Soc. London A, 422 (1989), pp. 261-278.

[26] L.A. Peletier and J. Serrin, *Uniqueness of positive solutions of semilinear equations in* R^n, Arch. Rational Mech. Anal., 81, (1983), pp. 181-197.

[27] M.J. Ward, *Metastable bubble solutions for the Allen-Cahn equation with mass conservation*, SIAM J. Appl. Math., 1996, to appear.

[28] M. J. Ward, *An asymptotic analysis of localized solutions for some reaction-diffusion models in multi-dimensional domains*, Studies in Appl. Math. 97 (1996), 103-126.

[29] J. Wei, *On the interior spike layer solutions to a singularly perturbed semilinear Neumann problem*, Tohoku Math. J., to appear.

[30] J. Wei, *On the effect of domain geometry in some singular perturbation problems*, Diff. Int. Eqns, to appear.

[31] J. Wei, *On the interior spike solutions for some singular perturbation problems*, Proc. Roy. Soc. Edinburgh Sect. A., to appear.

[32] J. Wei and M. Winter, *Stationary solutions of the Cahn-Hilliard equation*, Ann. Inst. H. Poincaré, Anal. Non Lin., to appear.

[33] J. Wei and M. Winter, *Multiple spike layer solutions for a wide class of singular perturbation problems*, J. London Math. Soc., to appear.

[34] J. Wei and M. Winter, *On the stationary Cahn-Hilliard equation: Interior spike solutions*, submitted.