ON THE STATIONARY CAHN-HILLIARD EQUATION: INTERIOR SPIKE SOLUTIONS

JUNCHENG WEI AND MATTHIAS WINTER

ABSTRACT. We study solutions of the stationary Cahn-Hilliard equation in a bounded smooth domain which have a spike in the interior. We show that a large class of interior points (the "nondegenerate peak" points) have the following property: there exist such solutions whose spike lies close to a given nondegenerate peak point. Our construction uses among others the methods of viscosity solution, weak convergence of measures and Liapunov-Schmidt reduction.

1. INTRODUCTION

In this paper, we continue our investigation on stationary solutions of the Cahn-Hilliard Equation.

The Cahn-Hilliard equation is a well-known macroscopic field theoretical model of processes such as phase separation in a binary alloy (see [7]). It is derived from a Helmholtz free energy

$$E(u) = \int_\Omega \left[F\big(u(x)\big) + \frac{1}{2}\epsilon^2 |\nabla u(x)|^2\right] dx$$

where Ω is a bounded smooth domain corresponding to the region occupied by the body, $u(x)$ is a conserved order parameter representing for example the concentration; ϵ is the range of intermolecular forces, the gradient term is a contribution to the free energy coming from spatial fluctuations of the order parameter and $F(u)$ is the free energy

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density which has a double well structure at low temperatures (for example, $F(u) = (1-u^2)^2$).

We assume that the mass $m = \frac{1}{|\Omega|}\int_\Omega u dx$ is conserved. Thus, a stationary solution of $E(u)$ under $m = \frac{1}{|\Omega|}\int_\Omega u dx$ satisfies the following Euler-Lagrange equation

$$
\begin{cases} \epsilon^2 \triangle u - f(u) = \lambda_\epsilon & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \int_\Omega u = m|\Omega| \end{cases} \tag{1.1}
$$

where $f(u) = F'(u)$, λ_ϵ is a constant and $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$.

Equation (1.1) has been studied extensively by many authors. It was first observed by Modica in [22] that global minimizers u_ϵ of $E(u)$ under $m = \frac{1}{|\Omega|}\int_\Omega u dx$ have a transition layer. Namely, there exists an open set $\Gamma \subset \Omega$ such that if a sequence u_ϵ converges then $u_\epsilon \longrightarrow 1$ on $\Omega\backslash\bar{\Gamma}$, $u_\epsilon \longrightarrow -1$ on Γ as $\epsilon \longrightarrow 0$ and $\partial\Gamma \cap \Omega$ is a minimal surface having constant mean curvature. Kohn and Sternberg in [18] studied local minimizers of the functional without mass conservation by using Γ-convergence. Caffarelli and Córdoba proved that in this situation the level sets of global minimizers converge uniformly to the limit surface [6]. Chen and Kowalczyk [9] proved the existence of local minimizers using a geometric approach. The dynamics of the transition layer solution has been studied by many authors, e.g. Chen [8], Alikakos, Bates and Fusco [3], Alikakos, Bates and Chen [2], Alikakos, Fusco and Kowalczyk [4], Pego [28], etc.

To study the global dynamics associated with (1.1), it is very important to study stationary solutions of (1.1), as this has been illustrated by Bates and Fife [5], Alikakos, Fusco and Kowalczyk [4].

In particular, Bates and Fife [5] studied nucleation phenomena and proved the existence of three monotone nondecreasing stationary solutions when m is in the metastable region ($\sqrt{\frac{1}{3}} < m < 1$)

(a) the constant solution $u \equiv m$,

(b) a boundary layer (spike) solution where the layer is located at the left-hand endpoint,

(c) a transition layer solution with a layer in the interior of the material.

In the one dimensional case, Grinfeld and Novick-Cohen in [14] and [15] completely determined all stationary solutions and proved some of their qualitative properties. In the higher dimensional case ($N \geqslant 2$), little is known about stationary solutions except for the transition layer solution. In [35], we first established the existence of a boundary spike layer solution under some condition for the boundary. More precisely, suppose P_0 is a boundary point such that $\nabla_{\tau_{P_0}} H(P_0) = 0, (\nabla^2_{\tau_{P_0}} H(P_0)) := G_B(P_0)$ is nondegenerate, where $H(P_0)$ is the mean curvature function at P_0 and $\nabla_{\tau_{P_0}}$ is the tangential derivative at P_0, then for ϵ sufficiently small there exists a solution u_ϵ of (1.1) such that $u_\epsilon(x) \to m$ for $\bar{\Omega} \backslash \{P_0\}$. Moreover, u_ϵ has only one local minimum P_ϵ where $P_\epsilon \in \partial\Omega, P_\epsilon \longrightarrow P_0$ and $u_\epsilon(P_\epsilon) \longrightarrow \beta < m$. Later in [36] we constructed multiple boundary spike solutions at multiple nondegenerate critical points of $H(P)$.

In this paper, we are concerned with the existence of *interior* spike layer solutions. Intuitively interior spike layer solutions are more related to the geometry of Ω while boundary spike layer solutions are more related to the geometry of $\partial\Omega$. We shall establish the existence of interior spike layer solutions under some geometric assumptions.

From now on, we always assume that $m > 0$ and that m is in the metastable region, i.e., $f'(m) > 0$. For $F(u) = (1 - u^2)^2$ this means that $\sqrt{\frac{1}{3}} < m < 1$. For $m < 0$ results analogous to ours are true, but with the signs of the values reversed.

To state our results, we first transform equation (1.1) as follows. For σ small enough let τ_σ be the unique solution of

$$f(m - \tau_\sigma) - f(m) - \sigma = 0$$

which lies near zero. Obviously

$$\tau_\sigma = -\frac{\sigma}{f'(m)} + O(\sigma^2).$$

With this notation we further define

$$\begin{aligned} g_\sigma(v) &= f(m - \tau_\sigma - v) - f(m) - \sigma \\ &= -p_\sigma v + h_\sigma(v) \end{aligned}$$

where

$$v = m - \tau_\sigma - u,$$

$$p_\sigma = f'(m - \tau_\sigma),$$

$$h_\sigma(v) = f(m - \tau_\sigma - v) - f(m) - \sigma + f'(m - \tau_\sigma)v.$$

By the choice of h_σ

$$h_\sigma(v) = O(v^2)$$

as $v \to 0$. Note that in particular

$$\begin{aligned} g_0(v) &= f(m - v) - f(m) \\ &= -p_0 v + h_0(v) \end{aligned}$$

where

$$v = m - u,$$

$$p_0 = f'(m),$$

$$h_0(v) = f(m - v) - f(m) + f'(m)v.$$

Then equation (1.1) becomes

$$\begin{cases} \epsilon^2 \triangle v - p_0 v + h_0(v) - \frac{1}{|\Omega|} \int_\Omega h_0(v) = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We next introduce some notations. Assume that Ω is such that for each $P \in \Omega$ the set $B_{d(P,\partial\Omega)}(P) \cap \partial\Omega$ has only a finite number of connected components. Let

$$\Lambda_P := \left\{ d\mu_p(z) \in M(\partial\Omega) \middle| \begin{array}{l} \exists \epsilon_k \longrightarrow 0 \text{ such that} \\ d\mu_P(z) = \lim_{\epsilon_k \to 0} \dfrac{e^{-\frac{|z-P|}{\epsilon_k}} dz}{\int_{\partial\Omega} e^{-\frac{|z-P|}{\epsilon_k}} dz} \end{array} \right\}$$

where $M(\partial\Omega)$ is the set of all bounded measures on $\partial\Omega$ and the convergence is weak convergence of measures. Certainly $\text{supp}(d\mu_P) \subset \bar{B}_{d(P,\partial\Omega)}(P) \cap \partial\Omega$.

A point $P \in \Omega$ is called a *nondegenerate peak* point if

(1) $\Lambda_P = \{d\mu_P(z)\}$, i.e. the set Λ_P contains exactly one element.

(2) $\exists a \in R^N$ such that

$$\int_{\partial\Omega} e^{<a,z-P>}(z-P)d\mu_P(z) = 0$$

and

$$\frac{\int_{\partial\Omega} e^{-\frac{|z-P|}{\epsilon}} e^{<a,z-P>}(z-P)\, dz}{\int_{\partial\Omega} e^{-\frac{|z-P|}{\epsilon}} dz} = O(\epsilon^{\alpha_0})$$

for some $\alpha_0 > 0$.

(3) The matrix $G(P) := \left(\int_{\partial\Omega} e^{<a,z-P>}(z_i - P_i)(z_j - P_j)d\mu_P(z)\right)$ is nondegenerate (where a is the same vector as in (2)).

Remarks: (1) The vector a in (2) and (3) is unique by [32].

(2) In [30] and [31], M.J. Ward has derived conditions similar to (2) for bubble-like solutions of singular perturbation problems. His approach is by asymptotic expansion and he does not give a rigorous construction of solutions.

The simplest example is when $\Omega = B_R(0), P = 0$. In this case, $d\mu_0(z) = \frac{1}{|B_R(0)|}dz$, $a = 0$ and

$$\int_{\partial B_R(0)} z d\mu_p(z) = 0,\ G(0) = (\int_{\partial B_R(0)} z_i z_j d\mu_p(z)) = \frac{1}{|B_R(0)|} I$$

where I is the identity matrix. Hence 0 is a "nondegenerate peak" point.

A more geometric characterization of a nondegenerate peak point is the following fact: P is a nondegenerate peak point if and only if $P \in$ int (conv(supp $(d\mu_P)$)) where int (conv(supp $(d\mu_P)$)) is the interior of the convex hull of the support of $d\mu_P$.

For example, let Ω be a convex domain. Let $P \in \Omega$ be such that $B_{d(P,\partial\Omega)}(P) \cap \partial\Omega$ contains at least three nondegenerate points (i.e., $B_{d(P,\partial\Omega)}(P)$ contacts at $\partial\Omega$ nondegenerately) then P is a nondegenerate peak point.

A nontrivial example of a nonconvex domain case is the following dumbell (see Figure 1).

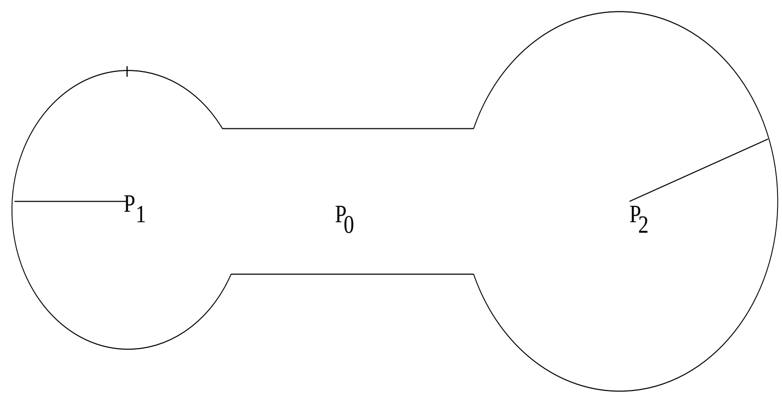

Figure 1. Dumbell Domain

The two centers are nondegenerate peak points as has been shown in [33].

To accommodate more general nonlinearities we assume that for all $\sigma > 0$ which are sufficiently small

(g1) $h_0 \in C^2(R^+)$ where h_0 satisfies

$$h_0(v) = O(|v|^{p_1}), h_0'(v) = O(|v|^{p_2-1}) \text{ as } |v| \to \infty$$

for some $1 < p_1, p_2 < \left(\frac{N+4}{N-4}\right)_+$ and there exists $1 < p_3 < \left(\frac{N+4}{N-4}\right)_+$ such that

$$|h'(v+\phi) - h'(v)| \le \begin{cases} C|\phi|^{p_3-1} \text{ if } p_3 > 2 \\ C(|\phi| + |\phi|^{p_3-1}) \quad \text{if } p_3 \le 2. \end{cases}$$

(g2) The equation

$$\begin{cases} \triangle V + g_\sigma(V) = 0 \quad \text{in } \mathbb{R}^N, \\ V > 0, V(0) = \max_{z \in R^n} \ V(z), \\ V \to 0 \quad \text{at } \infty \end{cases} \tag{1.3}$$

has a unique solution $V(y)$ (by the results of [13], V is radial, i.e., $V = V(r)$ and $V' < 0$ for $r = |y| \neq 0$) and V is nondegenerate. Namely the operator

$$L := \triangle + g_\sigma^{'}(V) \tag{1.4}$$

is invertible in the space $H_r^2(R^N) := \big\{u = u(|y|) \in H^2(R^N)\big\}$.

Our main result is

Theorem 1.1. *Assume that $P_0 \in \Omega$ is a "nondegenerate peak" point and m is in the metastable region, i.e., $f'(m) > 0$. Then there exists $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$ there is a spike solution v_ϵ of (1.2) where $v_\epsilon \longrightarrow 0$ in $C^1_{loc}(\bar{\Omega} \backslash P)$; v_ϵ has only one local (hence global) maximum point P_ϵ where $P_\epsilon \to P_0$ and $v_\epsilon(P_\epsilon) \to V(0) > 0$. Moreover,*

$$\epsilon^{-N} \left\{ \int_\Omega \epsilon^2 \left| \nabla v_\epsilon - \nabla V \left(\frac{x - P_\epsilon}{\epsilon} \right) \right|^2 + \int_\Omega \left| v_\epsilon - V \left(\frac{x - P_\epsilon}{\epsilon} \right) \right|^2 \right\} \to 0$$

as $\epsilon \to 0$ where $V(y)$ is the unique solution of

$$\begin{cases} \triangle V + g_0(V) = 0 \\ V(0) = \max_{y \in R^N} V(y), V > 0, \\ V(y) \to 0 \ \text{as} \ |y| \to \infty. \end{cases} \tag{1.5}$$

Remark : 1. A more detailed description of the convergence of v_ϵ as $\epsilon \to 0$ is obtained in the proof of Theorem 1.1 in Section 7 below.

2. The techniques here certainly work for a large class of nonlinearities, for example $g(v) = -v + v^r$, $g(v) = -v + v^r - av^s \, (1 < s < r < \frac{N+2}{N-2}, a \geq 0)$, $g(v) = v(v-a)(1-v)(0 < a < \frac{1}{2})$ (the bistable case

in population dynamics) and the class of nonlinearities introduced in Dancer [10].

3. We remark that for the following equation

$$\begin{cases} \epsilon^2 \triangle u - u + u^r = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

construction and characterization of interior spike solutions have been studied in [32]. Equation (1.6) is the stationary solution of the Keller-Segel model as well as the Gierer-Meinhardt system and the existence of boundary spike layer solutions has been studied in [23], [24] and [34]. Location and profile of the interior spike-layer solutions which are minimizers among positive functions for the Dirichlet problem corresponding to (1.6) have been studied in [25].

In [33], the first author gave both necessary and sufficient conditions for the existence of *single-peaked* solutions of the corresponding Dirichlet problem of (1.6). However, problem (1.6) and the corresponding Dirichlet problem do not have the volume constraint and the nonlinearity is simpler than the one considered in this paper. The method in [33] is variational and depends on the fact that equation (1.6) is homogeneous. There is recent work giving sufficient conditions for the existence of solutions with multiple interior spikes for related problems by Gui and Wei [16] and Kowalczyk [19].

4. In Theorem 1.1, we constructed single-peaked solutions under the condition that the spike point is a "nondegenerate peak" point. The existence of a "nondegenerate peak " point depends on the shape of the domain. There are domains which *do not* have any "nondegenerate peak" point, for example, cylindrical domains (see Figure 2). In a forthcoming paper [37], we will prove the existence of $K-$interior peak solutions for any positive integer K in any bounded domain. Therefore the existence of interior peak solutions is independent of the shape of the domain. The advantage of Theorem 1.1 is that it gives a detailed

description of the asymptotic behavior of solutions, in particular of the location of the peaks.

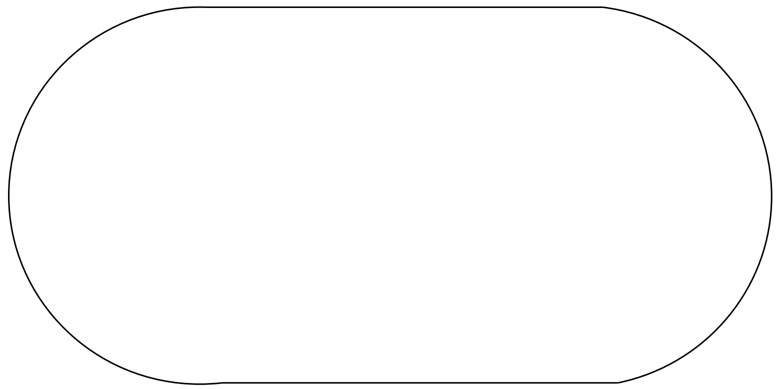

FIGURE 2. A Domain With No Nondegenerate Peak Point

5. The stability of spike solutions as constructed in Theorem 1.1 is unknown, though they are most likely to be unstable. In fact, we believe that one should be able to analyze the spectrum of the solutions by using their exact asymptotic behavior (the matrix $G(P_0)$ should play an important role). We conjecture that the solutions constructed in this paper should have an index of instability of $n+1$. It is an interesting question to characterize spike solutions with Morse index $< n+1$.

Our proof uses the Liapunov-Schmidt construction which was introduced in [12], [26], [27] and has been used in our earlier papers [35] and [36]. However, for the construction of boundary spike solutions, we just need an algebraic order estimate. Here for the interior peak case, the nonlocal term $\int_\Omega h(v)$ is of algebraic order ϵ^N, but the term that really governs the formation of interior spikes is exponentially small. Therefore we have to separate the algebraic small order from the exponentially small order. We use the method of viscosity solutions as introduced in [21] to estimate exponentially small terms.

The main points of the proof of Theorem 1.1 can be described as follows:

A)-Consider stationary solutions of the Cahn-Hilliard equation in the whole of R^n. This is equivalent to the problem

$$\Delta V + g_\sigma(V) = 0, \qquad y \in R^n$$

(with $u = m - \tau_\sigma - V$).

When m is in the metastable region and $\sigma > 0$ is sufficiently small, this equation has a unique ground state solution $V_\sigma(y)$. In fact, V_σ satisfies

$$\begin{cases} \triangle V + g_\sigma(V) = 0 & \text{in} \quad R^N, \\ V \geqslant 0, V(y) \longrightarrow 0 & \text{as } |y| \to +\infty \end{cases} \tag{1.7}$$

For V_σ we establish the asymptotic behavior at infinity as well as the continuous dependence on σ. This is done in Section 2.

B)-Spike solutions are expected to be perturbations of $V_\sigma(\frac{\cdot - P}{\epsilon})$ where $P \in \Omega$ is suitably chosen. However, $V_\sigma(\frac{\cdot - P}{\epsilon})$ does not satisfy the Neumann boundary condition on $\partial\Omega$. In order to correct this we define a new function $P_{\Omega_{\epsilon,P}} V_\sigma$ as the unique solution of

$$\begin{cases} \Delta v - p_\sigma v + h_\sigma(V_\sigma) = 0 \text{ in } \Omega_{\epsilon,P}, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_{\epsilon,P}. \end{cases}$$

where $\Omega_{\epsilon,P} := \{y | \epsilon y + P \in \Omega\}$.

This is done in Section 3.

C)- We choose σ such that

$$\sigma = \frac{1}{|\Omega|} \int_\Omega h\left(\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma\left(\frac{x - P}{\epsilon}\right)\right).$$

(This choice of σ will cancel terms of algebraic order in ϵ.)

We show that for ϵ sufficiently small there is a unique $\sigma_0 = O(\epsilon^N)$ which satisfies the above equation. We call $P_{\Omega_{\epsilon,P}} V_{\sigma_0} = w_{P,\epsilon}$. We use $\tau_{\sigma_0} + w_{P,\epsilon}$ as our approximate solution. This is done in Section 4.

D)-Let σ_0 be the value determined in C). The idea now is to look for a spike solution of the form $\tau_{\sigma_0} + w_{P,\epsilon} + \phi$ and, provided P is properly chosen, ϕ is expected to be insignificantly small. The equation determining ϕ is of the form

$$\Delta\phi - p_{\sigma_0}\phi + h^{'}_{\sigma_0}(w_{P,\epsilon})\phi + O(\phi^2) + E_{\epsilon,P} = 0 \ \text{ in } \Omega_{\epsilon,P}$$

$$\frac{\partial\phi}{\partial\nu} = 0 \text{ on } \partial\Omega_{\epsilon,P}$$

where $E_{\epsilon,P}$ is an error term. We will estimate this error term in Section 5 and show it to be exponentially small. It is then natural to try to solve the equation for ϕ by a contraction type argument. The problem is that the linearized operator $\Delta - p_{\sigma_0} + h'_{\sigma_0}(w_{P,\epsilon})$ is not uniformly invertible with respect to ϵ. Since $\Delta - p_{\sigma_0} + h'_{\sigma_0}(w_{P,\epsilon})$ is merely a perturbation of $\Delta - p_{\sigma_0} + h'(V(\frac{\cdot - P}{\epsilon}))$ which has an n-dimensional kernel (the span of $\frac{\partial V(\frac{\cdot - P}{\epsilon})}{\partial x_i}, i = 1, ..., N)$, we now replace the above equation by

(1.8)

$$\begin{cases} \Delta\phi - p_{\sigma_0}\phi + h^{'}_{\sigma_0}(w_{P,\epsilon})\phi + O(\phi^2) + E_{\epsilon,P} = v_\epsilon(P) \in C_{P,\epsilon} \ \text{ in } \Omega_{\epsilon,P}, \\ \phi \in K_{P,\epsilon} \\ \frac{\partial\phi}{\partial\nu} = 0 \text{ on } \partial\Omega_{\epsilon,P} \end{cases}$$

where

$$K_{P,\epsilon} = \text{span } \{\frac{\partial w_{p,\epsilon}}{\partial P_j}, j = 1, ..., N\}$$

and $C_{P,\epsilon} = K_{P,\epsilon}$ are the approximate kernel and approximate cokernel of $\Delta - p_{\sigma_0} + h'_{\sigma_0}(w_{P,\epsilon})$, respectively.

E)-We solve (1.8) for ϕ modulo the approximate kernel. To this end, we need a detailed analysis of the operator $\Delta - p_{\sigma_0} + h'_{\sigma_0}(w_{P,\epsilon})$. This together with the contraction argument is done in Section 6.

F)-In the last step, we study the vector field

$$P \to V_\epsilon(P) =: (\int_{\Omega_{\epsilon,P}} v_\epsilon(P)\frac{\partial w_{P,\epsilon}}{\partial P_1}, ..., \int_{\Omega_{\epsilon,P}} v_\epsilon(P)\frac{\partial w_{P,\epsilon}}{\partial P_N}).$$

The zeros of this vector field correspond to spike solutions of the Cahn-Hilliard equation. To discuss the zeros of $P \to V_\epsilon(P)$ we need very good estimates for the difference $P_{\Omega_{\epsilon,P}}V_\sigma - V_\sigma$. To this end, we let $P = P_0 + \epsilon(\frac{1}{2}ad(P, \partial\Omega) + \tilde{z})$. Much of Section 3 is devoted to this analysis. With a good estimate of $V_\epsilon(P)$, we discover that, in a small neighborhood of $P \in \Omega$ satisfying the geometric condition described in Theorem 1.1, there is a point P_ϵ such that $V_\epsilon(P_\epsilon) = 0$ and therefore the proof of Theorem 1.1 is completed. This is done in Section 7.

This paper is organized as follows. In Section 2, we study equation (1.7) in R^N, then we analyze the projection of the solution V_σ of (1.7) in Section 3. We choose σ in Section 4. In Section 5, we set up the technical framework and establish some error estimates. Problem (1.2), up to an approximate kernel and cokernel, is solved in Section 6 and thereby our problem is reduced to a finite dimensional one. In Section 7, we apply a degree-theoretic argument to solve the reduced problem (in which the nondegeneracy of the peak point P is essential) and complete the proof of Theorem 1.1. Throughout the paper C_N denotes constants which depend on the dimension N only.

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2. Equation in R^N

In this section, we study a parametrized semilinear elliptic equation in R^N.

As above, we let

$$g_\sigma(v) = f(m - \tau_\sigma - v) - f(m) - \sigma = -p_\sigma + h_\sigma(v)$$

where

$$p_\sigma = f'(m - \tau_\sigma),$$

$$h_\sigma(v) = f(m - \tau_\sigma - v) - f(m) - \sigma + f'(m - \tau_\sigma).$$

Recall that for σ small we introduced τ_σ to be the unique zero of

$$g_\sigma(\tau_\sigma) = 0$$

which is near 0. Then $\tau_\sigma = \tau(\sigma)$ is unique and continuously depends on σ. Moreover,

$$\tau(\sigma) = -\frac{\sigma}{f'(m)} + O(\sigma^2),$$
$$\tau'(\sigma) = -\frac{1}{f'(m)} + O(\sigma).$$

Recall that the following equation

$$\begin{cases} \triangle V + g_\sigma(V) = 0, \\ V \geqslant 0, V(0) = \max_{y \in R^N} V(y), \\ V(y) \longrightarrow 0 \qquad \text{as } |y| \longrightarrow \infty \end{cases}$$

has a unique solution V_σ and V_σ is radial.

Moreover, we have

Lemma 2.1.

$$(1) \quad \lim_{|y| \to \infty} |y|^{\frac{N-1}{2}} e^{\sqrt{p_\sigma}|y|} V_\sigma(y) = c_\sigma,$$

$$(2) \quad \lim_{|y| \to \infty} \frac{V_\sigma'(|y|)}{V_\sigma(|y|)} = -\sqrt{p_\sigma}$$

for some $c_\sigma > 0$.

Note that $p_\sigma = p_0 + O(\sigma)$. Hence it is not difficult to see that there are $R > 0, c > 0, C > 0$ independent of $\sigma > 0$ such that

$$|V_\sigma|, |V'{}_\sigma| \leqslant Ce^{-c|y|} \text{ for } |y| \geqslant R.$$

Let $\rho > 0$ be very small, $\sigma + \rho > 0$. Consider the function $w_\rho = \dfrac{V_{\sigma+\rho} - V_\sigma}{\rho}$. Then w_ρ satisfies

$$\triangle w_\rho + \frac{1}{\rho}[g_{\sigma+\rho}(V_{\sigma+\rho}) - g_\sigma(V_\sigma)] = 0.$$

Hence by the mean value theorem, we have

$$\frac{1}{\rho}[g_{\sigma+\rho}(V_{\sigma+\rho})-g_\sigma(V_\sigma)]=\frac{1}{\rho}[f(m-\tau_{\sigma+\rho}-V_{\sigma+\rho})-f(m-\tau_\sigma-V_\sigma)-\rho]$$

$$=f^{'}(m-\tau_{\sigma+t\rho}-V_{\sigma+t\rho})\left(-w_\rho-\frac{1}{\rho}(\tau_{\sigma+\rho}-\tau_\sigma)\right)-1)$$

where $0\leqslant t\leqslant 1$.

So

$$\triangle w_\rho+f^{'}(m-\tau_{\sigma+t\rho}-V_{\sigma+t\rho})\left(-w_\rho-\frac{1}{\rho}(\tau_{\sigma+\rho}-\tau_\sigma)\right)-1)=0.$$

Note that $w_\rho\longrightarrow 0$ as $|y|\to\infty$ (ρ fixed) and $|w_\rho|\leqslant C_R$ for $|y|\leqslant R$ where C_R is independent of ρ and depends on R only.

Since $|V_{\sigma+t\rho}|\leqslant Ce^{-c|y|}$ for somce $c>0$ when $|y|\geqslant R_0$ (R_0 large) we have $f'(m-\tau_{\sigma+\rho}-V_{\sigma+t\rho})\leqslant\frac{1}{2}f'(m)$ for $|y|\geq R_0$, provided that σ and ρ are small enough.

So $\max_{y\in R^N}|w_\rho|\leqslant C$, where C is independent of ρ.

Letting $\rho\to 0$, we have $|\frac{\partial V_\sigma}{\partial\sigma}|\leqslant C$ and

$$\triangle(\frac{\partial V_\sigma}{\partial\sigma})+f'(m-\tau_\sigma-V_\sigma)\left(-\frac{\partial V_\sigma}{\partial\sigma}-\frac{1}{f'(m)}\right)-1=0.$$

We have proved

Lemma 2.2. *For σ sufficiently small, $\frac{\partial V_\sigma}{\partial\sigma}$ exists and is continuous with respect to σ. It satisfies*

$$\triangle(\frac{\partial V_\sigma}{\partial\sigma})+f'(m-\tau_\sigma-V_\sigma)\left(-\frac{\partial V_\sigma}{\partial\sigma}-\frac{1}{f'(m)}\right)-1=0.$$

3. Projection of V_σ

In this section, we study properties of the function V_σ introduced in Section 2. In particular, we introduce a "projection" of V_σ in $H^1_N(\Omega)$, the linear subspace of $H^1(\Omega)$ of functions satisfying the Neumann boundary condition and prove some estimates.

Let U be any bounded smooth convex domain. (The condition of convexity will be removed below). We define $P_U V_\sigma$ as the unique solution of

$$\begin{cases} \triangle v - p_\sigma v + h_\sigma(V_\sigma) = 0 \text{ in } U, \\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial U \end{cases} \tag{3.1}$$

where

$$p_\sigma = -f'(m - \tau_\sigma),$$

$$h_\sigma(v) = f(m - \tau_\sigma - v) - f(m) - \sigma + f'(m - \tau_\sigma)v.$$

Set

$$\Omega_{\epsilon,P} := \{y | \epsilon y + P \in \Omega\},$$

$$\varphi_{\epsilon,P}(x) = V_\sigma(\frac{|x-P|}{\epsilon}) - P_{\Omega_{\epsilon,P}} V_\sigma(y), \ \epsilon y + P = x.$$

Then $\varphi_{\epsilon,P}(x)$ satisfies

$$\begin{cases} \epsilon^2 \triangle v - p_\sigma v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu} V_\sigma(\frac{|x-P|}{\epsilon}) & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

By Lemma 2.1 it is immediately seen that on $\partial\Omega$

$$\begin{aligned} \frac{\partial}{\partial \nu} V_\sigma(\frac{|x-P|}{\epsilon}) &= \frac{1}{\epsilon} V_\sigma'(\frac{|x-P|}{\epsilon}) \frac{< x-P, \nu >}{|x-P|} \\ = -\frac{1}{\epsilon}\Big(|x-P|^{-(N-1)/2} \cdot \epsilon^{+\frac{N-1}{2}} & e^{-\frac{\sqrt{p_\sigma}|x-P|}{\epsilon}} \sqrt{p_\sigma}\big(1 + O(\epsilon)\big)\Big) \frac{< x-P, \nu >}{|x-P|} \\ &= -\epsilon^{\frac{N-3}{2}} e^{-\frac{\sqrt{p_\sigma}|x-P|}{\epsilon}} \sqrt{p_\sigma}\big(1 + O(\epsilon)\big) \frac{< x-P, \nu >}{|x-P|^{\frac{N+1}{2}}}. \end{aligned} \tag{3.3}$$

Assume first that Ω is convex with respect to P. Namely, there is a constant $c_0 > 0$ such that

$$\langle x - P, \nu_x \rangle \geq c_0$$

for all $x \in \partial\Omega$, where ν_x is the unit outer normal at $x \in \partial\Omega$.

To analyze $P_{\Omega_{\epsilon,P}}V_\sigma$, we introduce another linear problem. Let $P^D_{\Omega_{\epsilon,P}}V_\sigma$ be the unique solution of

$$\begin{cases} \epsilon^2\triangle v - p_\sigma v + h_\sigma(V_\sigma) = 0 & \text{in } \Omega, \\ v = V_\sigma(\frac{|x-P|}{\epsilon}) & \text{on } \partial\Omega. \end{cases}$$

Set

$$\varphi^D_{\epsilon,P} = V_\sigma - P^D_{\Omega_{\epsilon,P}}V_\sigma, \psi_{\epsilon,P}(x) = -\epsilon\log\varphi^D_{\epsilon,P}(x).$$

Note that $\varphi_{\epsilon,P}$, $\varphi^D_{\epsilon,P}$ and $\psi_{\epsilon,P}$ depend on σ. Then $\psi_{\epsilon,P}$ satisfies

$$\begin{cases} \epsilon\triangle v - |\nabla v|^2 + p_\sigma = 0 & \text{in } \Omega, \\ v = -\epsilon\log(V_\sigma(\frac{|x-P|}{\epsilon})) & \text{on } \partial\Omega. \end{cases}$$

Note that for $x\in\partial\Omega$

$$\begin{aligned}\psi_{\epsilon,P}(x) &= -\epsilon\log\left((\frac{|x-P|}{\epsilon})^{-\frac{n-1}{2}}e^{-\frac{\sqrt{p_\sigma}|x-P|}{\epsilon}}(1+O(\epsilon))\right) \\ &= \sqrt{p_\sigma}|x-P| + \frac{n-1}{2}\epsilon\log(\frac{|x-P|}{\epsilon}) + O(\epsilon^2) \\ &= \sqrt{p}|x-P| + \frac{n-1}{2}\epsilon\log(\frac{|x-P|}{\epsilon}) + O(\sigma) + O(\epsilon^2)\end{aligned}$$

since $p_\sigma = p + O(\sigma)$.

By the results of Section 4 in [25], we have

Lemma 3.1. (1) $\frac{\partial\psi_{\epsilon,P}}{\partial\nu} = -(\sqrt{p}_\sigma + O(\epsilon))\frac{< x-P,\nu >}{|x-P|}$,

(2) $\psi_{\epsilon,P}(x) \longrightarrow \psi_0(x) = \inf_{z\in\partial\Omega}\sqrt{p_\sigma}(|z-x|+|z-P|)$ *as* $\epsilon\to 0$ *uniformly in* $\bar\Omega$. *In particular* $\psi_0(P) = 2\sqrt{p}_\sigma d(P,\partial\Omega)$.

Note that ψ_0 is a viscosity solution of the Hamilton-Jacobi equation $|\nabla u| = \sqrt{p_\sigma}$ in Ω (see [21]).

Let us now compare $\varphi_{\epsilon,P}(x)$ and $\varphi^D_{\epsilon,P}(x)$. In fact, we have

Lemma 3.2. *Assume that* Ω *is convex with respect to* P. *Then there exist* $\eta_0,\epsilon_0 > 0$ *such that for* $\epsilon\leqslant\epsilon_0$, *we have*

$$-(1+\eta_0\epsilon)\varphi^D_{\epsilon,P} \leqslant \varphi_{\epsilon,P} \leqslant -(1-\eta_0\epsilon)\varphi^D_{\epsilon,P}.$$

Proof. On $\partial\Omega$, we have

$$\begin{aligned}\frac{\partial \varphi_{\epsilon,P}^D}{\partial\nu} &= e^{-\frac{\psi_{\epsilon,P}(x)}{\epsilon}}(-\frac{1}{\epsilon})\frac{\partial\psi_{\epsilon,P}(x)}{\partial\nu}\\ &= -\frac{1}{\epsilon}(V_\sigma)\frac{\partial\psi_{\epsilon,P}(x)}{\partial\nu}\\ &= \frac{1}{\epsilon}V(\sqrt{p_\sigma}+O(\epsilon))\frac{<x-P,\nu>}{|x-P|}\\ &= \frac{1}{\epsilon}\frac{\partial V_\sigma}{\partial\nu}(1+O(\epsilon))\\ &= -(1+O(\epsilon))\frac{\partial\varphi_{\epsilon,P}}{\partial\nu}.\end{aligned}$$

Note that since Ω is convex, we have $\frac{\partial\varphi_{\epsilon,P}}{\partial\nu}<0$, hence by comparison principles

$$-(1+\eta_0\epsilon)\varphi_{\epsilon,P}^D \leqslant \varphi_{\epsilon,P} \leqslant -(1-\eta_0\epsilon)\varphi_{\epsilon,P}^D.$$

Lemma 3.2 is thus proved. □

Let

$$V_{\epsilon,P}(y)=\frac{1}{\varphi_{\epsilon,P}(P)}\cdot\varphi_{\epsilon,P}(x).$$

Then $V_{\epsilon,P}(0)=1, V_{\epsilon,P}>0$ and by Lemma 3.2 and Lemma 4.4 of [25], we have

Lemma 3.3. *For every sequence $\epsilon_k\to 0$ and σ fixed, there is a subsequence $\epsilon_{k\ell}\to 0$ such that $V_{\epsilon_{k\ell},P}\to\tilde V$ uniformly on every compact set of R^N, where $\tilde V$ is a positive solution of*

$$\begin{cases}\triangle u - p_\sigma u = 0 & \text{in } R^N,\\ u>0 & \text{in } R^N \text{ and } u(0)=1.\end{cases}$$

Moreover for any $c_1>0$, $\sup_{z\in\Omega_{\epsilon_{k\ell},P}} e^{-(\sqrt{p_\sigma}+c_1)|z|}\left|V_{\epsilon_{k\ell},P}(z)-\tilde V\right|\to 0$ as $\epsilon_{k\ell}\to 0$.

We have the following key computations.

Lemma 3.4. *Suppose* $P_\epsilon = P_0 + \epsilon(b + \tilde{z})$ *with* $|\tilde{z}| = O(\epsilon^\alpha), 0 < \alpha < \alpha_0, 2b = ad(P_0, \partial\Omega)$ *and* $\sigma = O(\epsilon^N)$ *as* $\epsilon \to 0$. *Then*

$$L_j(\epsilon, \tilde{z}) := \int_{\Omega_{\epsilon,P_\epsilon}} \Big[h_\sigma\big(P_{\Omega_{\epsilon,P_\epsilon}} V_\sigma\big) - h_\sigma\big(V_\sigma - \tau_\sigma\big)\Big] \frac{\partial V_\sigma}{\partial y_j}$$
$$= L_j(\tilde{z})\varphi_{\epsilon,P_\epsilon}(P_\epsilon) + O\big(\varphi_{\epsilon,P_\epsilon}(P_\epsilon)\epsilon^{\min(1,2\alpha,\alpha_0)}\big)$$

where $L(\tilde{z}) := (L_1(\tilde{z}), ..., L_N(\tilde{z}))$ *is a matrix and we have*

$$L_j(\tilde{z}) = \gamma \frac{\int_{\partial\Omega} e^{<z-P_0,b>} \langle z - P_0, \tilde{z}\rangle \big(z_j - P_{0,j}\big) d\mu_{P_0}(z)}{\int_{\partial\Omega} e^{<z-P_0,b>} d\mu_{P_0}(z)}$$

where $\gamma \neq 0$ *is a constant depending on* N *and* $d(P_0, \partial\Omega)$ *only.*

Proof:

Note that $\varphi^D_{\epsilon,P}(x)$ satisfies

$$\begin{cases} \epsilon^2 \Delta v - p_\sigma v = 0 \text{ in } \Omega, \\ v(x) = V_\sigma \left(\frac{|x-P|}{\epsilon}\right) \text{ on } \partial\Omega. \end{cases} \tag{3.4}$$

Recall that

$$\psi_{\epsilon,P}(x) = -\epsilon \log(\varphi^D_{\epsilon,P}(x)).$$

and that $\psi_{\epsilon,P}$ satisfies

$$\begin{cases} \epsilon \Delta v - |\nabla v|^2 + p_\sigma = 0 \text{ in } \Omega, \\ v = -\epsilon \log(V_\sigma) \text{ on } \partial\Omega. \end{cases}$$

If $\sigma = O(\epsilon^N)$ we have $p_\sigma = p_0 + O(\epsilon^N)$ and $\tau_\sigma = O(\epsilon^N)$ as $\epsilon \to 0$.

Hence $|\psi_{\epsilon,P} - \tilde{\psi}_{\epsilon,P}| \leq C\epsilon^N$ where $\tilde{\psi}_{\epsilon,P}$ is the unique solution of

$$\begin{cases} \epsilon \Delta v - |\nabla v|^2 + p_0 = 0 \text{ in } \Omega, \\ v = -\epsilon \log V \text{ on } \partial\Omega \end{cases}$$

and V is the unique solution of (1.3).

Let $G_\epsilon(x, y)$ be the Green's function of $-\epsilon^2\Delta + p_\sigma$ on $W^{1,2}_0(\Omega)$. Then we have by the standard representation formula,

Lemma 3.5.

$$\varphi^D_{\epsilon,P}(x) = \int_{\partial\Omega} V_\sigma(\frac{z-P}{\epsilon})\frac{\partial G_\epsilon(z,x)}{\partial \nu}dz. \tag{3.5}$$

By Lemma 3.5 and the estimates in [33], Section 3 we calculate

$$\varphi^D_{\epsilon,P}(x) = \frac{C_N + O(\epsilon)}{\epsilon^N}$$
$$\times \int_{\partial\Omega}\left\{e^{-\sqrt{p_\sigma}\frac{|z-P|+|z-x|}{\epsilon}}|z-P|^{-\frac{N-1}{2}}|z-x|^{-\frac{N-1}{2}}\frac{\langle z-x,\nu\rangle}{|z-x|}dz\right\} \tag{3.6}$$

as $\epsilon \longrightarrow 0$.

We immediately have

$$\varphi^D_{\epsilon,P}(P) = \frac{C_N + O(\epsilon)}{\epsilon^N}\int_{\partial\Omega}\left\{e^{-\sqrt{p_\sigma}\frac{2|z-P|}{\epsilon}}|z-P|^{-(N-1)}\frac{\langle z-P,\nu\rangle}{|z-P|}\right\}dz. \tag{3.7}$$

Let $\epsilon y + P = x$ and $|y| \le K$, then

$$|z-x| = |z-P-\epsilon y| = \epsilon|y - \frac{z-P}{\epsilon}|$$
$$= |z-P| - \langle y, \frac{z-P}{|z-P|}\rangle\epsilon + O(\epsilon^2). \tag{3.8}$$

By Lemma 3.2

$$\begin{aligned} L_j(\epsilon,\tilde z):\quad &= \int_{\Omega_{\epsilon,P_\epsilon}}\Big[h_\sigma\big(P_{\Omega_{\epsilon,P_\epsilon}}V_\sigma\big) - h_\sigma\big(V_\sigma\big)\Big]\frac{\partial V_\sigma}{\partial y_j} \\ &= -\int_{\Omega_{\epsilon,P_\epsilon}}\Big[h'_\sigma(V_\sigma)\varphi_{\epsilon,P_\epsilon}\Big]\frac{\partial V_\sigma}{\partial y_j} + O(\epsilon\varphi_{\epsilon,P}(P)) \\ &= \int_{\Omega_{\epsilon,P_\epsilon}}\Big[h'_\sigma(V_\sigma)\varphi^D_{\epsilon,P_\epsilon}\Big]\frac{\partial V_\sigma}{\partial y_j} + O(\epsilon\varphi^D_{\epsilon,P}(P)). \end{aligned}$$

Let $P = P_\epsilon = P_0 + \epsilon(b+\tilde z)$. Then

$$\frac{1}{\varphi^D_{\epsilon,P_\epsilon}(P_\epsilon)}L_j(\epsilon,\tilde z) =$$

$$= \int_{R^N} h'_\sigma(V_\sigma)V'_\sigma\frac{y_j}{|y|}\left\{\frac{\int_{\partial\Omega}\left\{e^{-\frac{2\sqrt{p_\sigma}|z-P_\epsilon|}{\epsilon}}e^{\sqrt{p_\sigma}\langle y,\frac{z-P_\epsilon}{|z-P_\epsilon|}\rangle}|z-P_\epsilon|^{-(N-1)}\frac{\langle z-P_\epsilon,\nu\rangle}{|z-P_\epsilon|}dz\right\}}{\int_{\partial\Omega}\left\{e^{-\frac{2\sqrt{p_\sigma}|z-P_\epsilon|}{\epsilon}}|z-P_\epsilon|^{-(N-1)}\frac{\langle z-P_\epsilon,\nu\rangle}{|z-P_\epsilon|}dz\right\}}\right\}$$
$$\times(1+O(\epsilon))$$

$$= \int_{\partial\Omega} \left\{ \frac{e^{-\frac{2\sqrt{p_\sigma}|z-P_\epsilon|}{\epsilon}} |z-P_0|^{-(N-1)} \frac{\langle z-P_0,\nu\rangle}{|z-P_0|}}{\int_{\partial\Omega} \left\{ e^{-\frac{2\sqrt{p_\sigma}|z-P_\epsilon|}{\epsilon}} |z-P_0|^{-(N-1)} \frac{\langle z-P_0,\nu\rangle}{|z-P_0|} dz \right\}} \right.$$

$$\left. \times \int_{R^N} h_\sigma'(V_\sigma) V_\sigma' \frac{y_j}{|y|} e^{\sqrt{p_\sigma}\langle y, \frac{z-P_\epsilon}{|z-P_\epsilon|}\rangle} dy \right\} dz (1+O(\epsilon))$$

$$= \int_{\partial\Omega} \left\{ \frac{e^{-\frac{2\sqrt{p_\sigma}|z-P_\epsilon|}{\epsilon}} |z-P_0|^{-(N-1)} \frac{\langle z-P_0,\nu\rangle}{|z-P_0|}}{\int_{\partial\Omega} \left\{ e^{-\frac{2\sqrt{p_\sigma}|z-P_\epsilon|}{\epsilon}} |z-P_0|^{-(N-1)} \frac{\langle z-P_0,\nu\rangle}{|z-P_0|} dz \right\}} \frac{z_j - P_{0,j}}{|z-P_0|} \right.$$

$$\left. \times \int_{R^N} h_\sigma'(V_\sigma) V_\sigma' \frac{1}{|y|} e^{y_1} dy \right\} dz (C_N + O(\epsilon))$$

$$= \int_{\partial\Omega} \left\{ \frac{e^{-\frac{2\sqrt{p_\sigma}|z-P_\epsilon|}{\epsilon}} |z-P_0|^{-(N-1)} \frac{\langle z-P_0,\nu\rangle}{|z-P_0|}}{\int_{\partial\Omega} \left\{ e^{-2\frac{\sqrt{p_\sigma}|z-P_\epsilon|}{\epsilon}} |z-P_0|^{-(N-1)} \frac{\langle z-P_0,\nu\rangle}{|z-P_0|} dz \right\}} \frac{z_j - P_{0,j}}{|z-P_0|} \right\} dz$$

$$\times (C_N + O(\epsilon)) \qquad \text{as } \epsilon \to 0.$$

Note that

$$e^{-2\sqrt{p_\sigma}\frac{|z-P_\epsilon|}{\epsilon}} = e^{-2\sqrt{p_\sigma}\frac{|z-P_0|}{\epsilon}} e^{2\sqrt{p_\sigma}\langle b, \frac{z-P_0}{|z-P_0|}\rangle} (1 + 2\langle \tilde{z}, \frac{z-P_0}{|z-P_0|}\rangle + O(\epsilon^{\min(1,2\alpha)})).$$

Hence

$$\frac{1}{\varphi^D_{\epsilon,P_\epsilon}(P_\epsilon)} L_j(\epsilon, \tilde{z})$$

$$= \int_{\partial\Omega} \frac{e^{-\frac{2\sqrt{p_\sigma}|z-P_0|}{\epsilon}} e^{\sqrt{p_\sigma}\langle a, z-P_0\rangle} |z-P_0|^{-(N-1)} \frac{\langle z-P_0,\nu\rangle}{|z-P_0|}}{\int_{\partial\Omega} \left\{ e^{-\frac{2\sqrt{p_\sigma}|z-P_0|}{\epsilon}} e^{\sqrt{p_\sigma}\langle a, z-P_0\rangle} |z-P_0|^{-(N-1)} \frac{\langle z-P_0,\nu\rangle}{|z-P_0|} dz \right\}} \frac{z_j - P_{0,j}}{|z-P_0|} dz$$

$$\times (C_N + O(\epsilon))$$

$$+ \int_{\partial\Omega} \frac{e^{-\frac{2\sqrt{p_\sigma}|z-P_0|}{\epsilon}} e^{\sqrt{p_\sigma}\langle a, z-P_0\rangle} |z-P_0|^{-(N-1)} \frac{\langle z-P_0,\nu\rangle}{|z-P_0|}}{\int_{\partial\Omega} \left\{ e^{-\frac{2\sqrt{p_\sigma}|z-P_0|}{\epsilon}} e^{\sqrt{p_\sigma}\langle a, z-P_0\rangle} |z-P_0|^{-(N-1)} \frac{\langle z-P_0,\nu\rangle}{|z-P_0|} dz \right\}}$$

$$\times \langle \tilde{z}, z-P_0\rangle \frac{z_j - P_{0,j}}{|z-P_0|} dz \, (2C_N + O(\epsilon))$$

$$= \int_{\partial\Omega} \frac{e^{-\frac{2\sqrt{p_\sigma}|z-P_0|}{\epsilon}} e^{\sqrt{p_\sigma}\langle a, z-P_0\rangle} |z-P_0|^{-(N-1)} \frac{\langle z-P_0,\nu\rangle}{|z-P_0|}}{\int_{\partial\Omega} \left\{ e^{-\frac{2\sqrt{p_\sigma}|z-P_0|}{\epsilon}} e^{\sqrt{p_\sigma}\langle a, z-P_0\rangle} |z-P_0|^{-(N-1)} \frac{\langle z-P_0,\nu\rangle}{|z-P_0|} dz \right\}}$$

$$\times \langle \tilde{z}, z-P_0\rangle (z_j - P_{0,j}) dz \, (C_N + O(\epsilon)) + O(\epsilon^{\min(1,2\alpha,\alpha_0)})$$

by the assumptions. (Note that we have included the factors 2 and powers of $|z-P_0|$ into the constant C_N.)

Hence

$$\frac{1}{\varphi_{\epsilon,P_\epsilon}(P_\epsilon)}L_j(\epsilon,\tilde{z})$$

$$=\gamma\frac{\int_{\partial\Omega}e^{<a,z-P_0>}\langle\tilde{z},z-P_0\rangle\big(z_j-P_{0,j}\big)d\mu_{P_0}(z)}{\int_{\partial\Omega}e^{<a,z-P_0>}d\mu_{P_0}(z)}+O(\epsilon^{\min(1,2\alpha,\alpha_0)}).$$

Lemma 3.4 is proved. □

Finally, we discuss the case when Ω is not convex with respect to P.

To this end, we introduce another function. Let U_ϵ be the solution of the following problem

$$\begin{cases}\epsilon^2\Delta U_\epsilon-p_\sigma U_\epsilon=0 \text{ in } \Omega,\\ U_\epsilon=1 \text{ on } \partial\Omega.\end{cases}$$

Set

$$\Psi_\epsilon=-\epsilon\log(U_\epsilon).$$

Then by Theorem 1 of [11], we have

$$\Psi_\epsilon(x)=\sqrt{p_\sigma}d(x,\partial\Omega)+O(\epsilon),\frac{\partial\Psi_\epsilon}{\partial\nu}=-\sqrt{p_\sigma}+O(\epsilon)$$

and

$$|U_\epsilon(x)|\leq Ce^{-\sqrt{p_\sigma}\frac{d(x,\partial\Omega)}{\epsilon}}.$$

Moreover, for any $c_0>0$ we have

$$\frac{U_\epsilon(\epsilon y+P)}{U_\epsilon(P)}\leq Ce^{(\sqrt{p_\sigma}+c_0)|y|}. \tag{3.9}$$

In this case we have

Lemma 3.6. *There exist $\eta_0,\alpha_0>0,\epsilon_0>0$ such that for $\epsilon\leq\epsilon_0$, we have*

$$-(1+\eta_0\epsilon)\varphi^D_{\epsilon,P}-Ce^{-\frac{\sqrt{p_\sigma}}{\varepsilon}(1+\alpha_0)d(P,\partial\Omega)}U_\epsilon<\varphi_{\epsilon,P}<-(1-\eta_0\epsilon)\varphi^D_{\epsilon,P}+Ce^{-\frac{\sqrt{p_\sigma}}{\varepsilon}(1+\alpha_0)d(P,\partial\Omega)}U_\epsilon$$

Proof: For any bounded smooth domain Ω we can choose a constant $R=(1+2\alpha_0)d(P,\partial\Omega)$ for some $\alpha_0>0$ such that $\Omega_1:=B_R(P)\cap\Omega$ is strictly convex with respect to P, i.e.

$$<x-P,\nu_x>\geq \nu_0>0,\quad x\in\partial\Omega_1.$$

Then on $\partial\Omega_1\cap\partial\Omega=\Gamma_1$, we have

$$\frac{\partial\varphi_{\epsilon,P}}{\partial\nu}\leq -(\sqrt{p_\sigma}+O(\epsilon))\frac{\partial\varphi^D_{\epsilon,P}}{\partial\nu}$$

On $\partial\Omega\backslash\Gamma_1$, we have

$$|\frac{\partial\varphi^D_{\epsilon,P}}{\partial\nu}|\leq Ce^{-(1+2\alpha_0)\frac{\sqrt{p_\sigma}}{\varepsilon}d(P,\partial\Omega)}$$

$$\frac{\partial\varphi_{\epsilon,P}}{\partial\nu}\leq Ce^{-(1+2\alpha_0)\frac{\sqrt{p_\sigma}}{\varepsilon}d(P,\partial\Omega)}\leq Ce^{-(1+\alpha_0)\frac{\sqrt{p_\sigma}}{\varepsilon}d(P,\partial\Omega)}\frac{\partial U_\epsilon}{\partial\nu}$$

for some $\alpha_0>0$. By comparison principles, we get the inequality.

Lemma 3.6 is thus proved. □

Since for ϵ small enough $e^{-\frac{\sqrt{p_\sigma}}{\varepsilon}(1+\alpha_0)d(P,\partial\Omega)}U_\epsilon\leq e^{-\frac{p_\sigma}{\epsilon}(2+\alpha_0)d(P,\partial\Omega)}$ which is smaller than $\varphi^D_{\epsilon,P}$ this term can be ignored. Hence Lemma 3.2 even holds for domains which are not convex with respect to a point $P\in\Omega$.

4. Choosing σ

In this section we choose σ. Let $P_{\Omega_{\epsilon,z}}V_\sigma$ be defined as in Section 3 and let P_0

The choice of σ is such that the algebraically and the exponentially small terms are separated in the equation. To explain how we choose σ, we plug the fucntion $v=\tau_\sigma+\tilde{w}$ into equation (1.2). We have after rescaling

$$\Delta\tilde{w}-p_\sigma\tilde{w}+h_\sigma(\tilde{w})-\frac{1}{|\Omega|}\int_\Omega[\sigma-h(\tau_\sigma+\tilde{w})=0.$$

In order to make the nonlocal term vanish, we let

$$\int_\Omega[\sigma-h(\tau_\sigma+\tilde{w})]\sim 0.$$

Since $v \sim P_{\Omega_{\epsilon,P}} V_\sigma$, this suggests that σ should satisfy

$$\int_\Omega [\sigma - h(\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma)] = 0.$$

We now solve the following equation

$$\sigma - \frac{1}{|\Omega|} \int_\Omega h(\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma(\frac{x-z}{\epsilon})) = 0$$

where $|P - P_0| \leq C\epsilon$.

Note that

$$\frac{1}{|\Omega|} \int_\Omega h(\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma(\frac{x-P}{\epsilon})) = \frac{\epsilon^N}{|\Omega|} \int_{\Omega_{\epsilon,P}} h(\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma).$$

Hence

$$\frac{\partial}{\partial \sigma} \frac{\epsilon^N}{|\Omega|} \int_{\Omega_{\epsilon,P}} h(\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma)$$

$$= \frac{\epsilon^N}{|\Omega|} \int_{\Omega_{\epsilon,P}} h^{'}(\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma) [\frac{\partial \tau_\sigma}{\partial \sigma} + \frac{\partial P_{\Omega_{\epsilon,P}} V_\sigma}{\partial \sigma}]$$

$$\leqslant C\epsilon^N$$

if ϵ and σ are small enough by the definitions of τ_σ and $P_{\Omega_{\epsilon,P}} V_\sigma$ amd by Lemma 2.2.

Hence by the Implicit Function Theorem, we have

Lemma 4.1. *For $\epsilon < \epsilon_0, \sigma < \sigma_1$, the following equation has a unique solution σ_0:*

$$\sigma = \frac{1}{|\Omega|} \int_\Omega h(\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma(\frac{x-z}{\epsilon})).$$

Note that

$$\sigma_0 = O(\epsilon^N).$$

5. Technical Framework

In this section, we set up the technical framework to solve equation (1.2). Without loss of generality, we assume that $P_0 = 0 \in \Omega$ is a nondegenerate peak point, i.e.

(1) $\Lambda_0 = \{d\mu_0(z)\}$.
(2) $\exists a \in R^N$ such that

$$\int_{\partial\Omega} e^{<a,z>} z d\mu_0(z) = 0$$

and

$$\int_{\partial\Omega} \left\{ \frac{e^{-\frac{|z|}{\epsilon}} e^{<a,z>}}{\int_{\partial\Omega} e^{-\frac{|z|}{\epsilon}} dz} \right\} z dz = O(\epsilon^{\alpha_0})$$

for some $\alpha_0 > 0$.
(3) The matrix $G(0) := \left(\int_{\partial\Omega} e^{<a,z>}(z_i z_j) d\mu_0(z)\right)$ is nondegenerate.

Let $z = \epsilon(\frac{a}{2} d(0,\partial\Omega) + \tilde{z})$ where $|\tilde{z}| < \epsilon^\alpha$ with $0 < \alpha < 1$ to be chosen later.

We assume that $\sigma = \sigma_0$ where σ_0 is defined by Lemma 4.1.

Define $H_\epsilon : H_N^2(\Omega_\epsilon) \to L^2(\Omega_\epsilon)$ by

$$H_\epsilon(v) := \Delta v - p_0 v + h(v) - \int_{\Omega_\epsilon} h(v) \tag{5.1}$$

where

$$H_N^2(\Omega) := \left\{ v \in H^2(\Omega_\epsilon) : \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega_\epsilon \right\}.$$

We are looking for a nontrivial zero of (5.1). It is easy to see that H_ϵ is a Fréchet differentiable map with its Fréchet derivative given by

$$H_\epsilon'(v)\phi = \triangle\phi - p_0\phi + h'(v)\phi - \frac{\epsilon^N}{|\Omega|} \int_{\Omega_\epsilon} h'(v)\phi$$

Set

$$w_{z,\epsilon}(y) := P_{\Omega_{\epsilon,z}} V_{\sigma_0}.$$

We are interested in finding the zeros of H_ϵ of the special form

$$w_{z,\epsilon} + \tau_{\sigma_0} + \phi$$

for sufficiently small $\epsilon > 0$ and sufficiently small $\phi \in H_N^2(\Omega)$. We shall see that solutions of this particular form correspond to single-peaked solutions of (1.2) with their peak concentrated near 0.

Equation (5.1) can also be written as

$$H_\epsilon^1(\tilde{w}) := \triangle \tilde{w} - p_\sigma \tilde{w} + h_\sigma(\tilde{w}) + \frac{1}{|\Omega|}\int_\Omega [\sigma - h(\tau_\sigma + \tilde{w})] = 0.$$

where

$$v = \tau_\sigma + \tilde{w}.$$

Set

$$\tilde{w} = P_{\Omega_{\epsilon,z}} V_\sigma + \phi_{\epsilon,z}.$$

Then we have

$$\begin{aligned} H_\epsilon^1(\tilde{w}) \quad := \quad & \triangle\phi_{\epsilon,z} - p_\sigma\phi_{\epsilon,z} + h_\sigma(P_{\Omega_{\epsilon,z}}V_\sigma + \phi_{\epsilon,z}) - h_\sigma(V_\sigma) \\ & + \frac{1}{|\Omega|}\int_\Omega [\sigma - h(\tau_\sigma + P_{\Omega_{\epsilon,z}}V_\sigma + \phi_{\epsilon,z})] \\ = \quad & \triangle\phi_{\epsilon,z} - p_\sigma\phi_{\epsilon,z} + h_\sigma'(P_{\Omega_{\epsilon,z}}V_\sigma)\phi_{\epsilon,z} \\ & + h_\sigma(P_{\Omega_{\epsilon,z}}V_\sigma + \phi_{\epsilon,z}) - h_\sigma(P_{\Omega_{\epsilon,z}}V_\sigma) - h_\sigma'(P_{\Omega_{\epsilon,z}}V_\sigma)\phi_{\epsilon,z} \\ & + h_\sigma(P_{\Omega_{\epsilon,z}}V_\sigma) - h_\sigma(V_\sigma) \\ & + \frac{1}{|\Omega|}\int_\Omega [h(\tau_\sigma + P_{\Omega_{\epsilon,z}}V_\sigma) - h(\tau_\sigma + P_{\Omega_{\epsilon,z}}V_\sigma + \phi_{\epsilon,z})] \\ & = F_\epsilon'(w_{z,\epsilon})\phi_{\epsilon,z} \\ & + N_{\epsilon,z}^1(\phi_{\epsilon,z}) \\ & + M_{\epsilon,z} \\ & + N_{\epsilon,z}^2(\phi_{\epsilon,z}) \end{aligned}$$

where

$$F_\epsilon'(w_{z,\epsilon})\phi_{\epsilon,z} := \triangle\phi_{\epsilon,z} - p_\sigma\phi_{\epsilon,z} + h_\sigma'(P_{\Omega_{\epsilon,z}}V_\sigma)\phi_{\epsilon,z},$$

$$N_{\epsilon,z}^1(\phi_{\epsilon,z}) := h_\sigma(P_{\Omega_{\epsilon,z}}V_\sigma + \phi_{\epsilon,z}) - h_\sigma(P_{\Omega_{\epsilon,z}}V_\sigma) - h_\sigma'(P_{\Omega_{\epsilon,z}}V_\sigma)\phi_{\epsilon,z},$$

$$N_{\epsilon,z}^2(\phi_{\epsilon,z}) = \frac{1}{|\Omega|}\int_\Omega [h(\tau_\sigma + P_{\Omega_{\epsilon,z}}V_\sigma) - h(\tau_\sigma + P_{\Omega_{\epsilon,z}}V_\sigma + \phi_{\epsilon,z})],$$

$$M_{\epsilon,z} = h_\sigma(P_{\Omega_{\epsilon,z}}V_\sigma) - h_\sigma(V_\sigma).$$

It is easy to see that

Lemma 5.1. *For ϵ sufficiently small*

$$\|N^1_{\epsilon,z}(\phi_{\epsilon,z})\|_{L^2(\Omega_{\epsilon,z})} \leqslant c\|\phi\|^2_{H^2(\Omega_{\epsilon,z})},$$
$$\|N^2_{\epsilon,z}(\phi_{\epsilon,z})\|_{L^2(\Omega_{\epsilon,z})} \leqslant c\epsilon^{\frac{N}{2}}\|\phi\|_{H^2(\Omega_{\epsilon,z})},$$
$$\|N^1_{\epsilon,z}(\phi_1) - N^1_{\epsilon,z}(\phi_2)\|_{L^2(\Omega_{\epsilon,z})} \leqslant c\big(\|\phi_1\|_{H^2(\Omega_{\epsilon,z})} + \|\phi_2\|_{H^2(\Omega_{\epsilon,z})}\big)$$
$$\|\phi_1 - \phi_2\|_{H^2(\Omega_{\epsilon,z})},$$
$$\|N^2_{\epsilon,z}(\phi_1) - N^2_{\epsilon,z}(\phi_2)\|_{L^2(\Omega_{\epsilon,z}} \leqslant c\epsilon^{\frac{N}{2}}\|\phi_1 - \phi_2\|_{H^2(\Omega_{\epsilon,z})}.$$

Moreover, we have the following error estimates.

Lemma 5.2.

$$\|M_{\epsilon,z}\|_{L^2(\Omega_{\epsilon,z})} \leqslant c\varphi_{\epsilon,z}^{\frac{1+\mu}{2}}(z) \quad \textit{for some } \mu > 0.$$

Proof. In fact

$$\begin{aligned}
\big|h_\sigma(P_{\Omega_{\epsilon,z}}V_\sigma) - h_\sigma(V_\sigma)\big|^2 &\leqslant c\big(h'_\sigma(V_\sigma)\big|P_{\Omega_{\epsilon,z}}V_\sigma - (V_\sigma - \tau_\sigma)\big|\big)^2 \\
&\leqslant c\big[V_\sigma^2 \cdot |V_{\epsilon,z}|^2\big] \cdot \varphi^2_{\epsilon,z}(z) \\
&\leq cV_\sigma^{2+\mu} \cdot V^2_{\epsilon,z} \cdot \varphi^{2-\mu}_{\epsilon,z}(z) \\
&\leq ce^{-\delta|y|}\varphi^{2-\mu}_{\epsilon,z}(z)
\end{aligned}$$

for some $\delta > 0$.

Hence

$$\|M_{\epsilon,z}\|^2_{L^2(\Omega_{\epsilon,z})} \leqslant c\varphi^{1+\mu}_{\epsilon,z}(z)$$

for some $\mu > 0$. □

6. Reduction to Finite Dimensions: Fredholm Inverses

In this section, we show that the linear operator $F'_\epsilon(w_{z,\epsilon}) = \Delta - p_\sigma + h_\sigma(P_{\Omega_{\epsilon,P}}V_\sigma)$ is invertible if the domain and the range are suitably restricted.

Set

$$K_{z,\epsilon} = \text{ span }\Big\{\frac{\partial w_{z,\epsilon}}{\partial z_i}\Big|i = 1, \cdots, N\Big\} \tag{6.1}$$

in $H^2_N(\Omega_\epsilon)$ and

$$C_{z,\epsilon} = \text{ span } \Big\{\frac{\partial w_{z,\epsilon}}{\partial z_i}\Big| i=1,\cdots,N\Big\} \cap L^2(\Omega_\epsilon).$$

$K_{z,\epsilon}$ is called the approximate kernel, while $C_{z,\epsilon}$ is called the approximate co-kernel.

Note that a function $\phi \in$ co-kernel of $F'_\epsilon(w_{z,\epsilon})$ if and only if for all $\psi \in H^2_N(\Omega_\epsilon)$ we have

$$\int_{\Omega_\epsilon} \phi F'{}_\epsilon(w_{z,\epsilon})\psi = 0.$$

Integrating by parts, we have

$$\int_{\partial\Omega} \psi \frac{\partial \phi}{\partial \nu} + \psi F'_\epsilon(w_{z,\epsilon})\phi = 0, \ \forall \psi \in H^2_N(\Omega_\epsilon).$$

Hence $\phi \in$ co-kernel of $F'{}_\epsilon(w_{z,\epsilon})$ if and only if

$$\begin{cases} F'_\epsilon(w_{z,\epsilon})\phi = 0 & \text{in } \Omega_\epsilon, \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

Therefore co-kernel of $F'_\epsilon(w_{z,\epsilon})$ = kernel of $F'_\epsilon(w_{z,\epsilon})$. Observe that span$\{\frac{\partial V}{\partial y_i}|i=1,\cdots,N\}$ is the kernel of L, where L is a linear operator defined as

$$L\phi := \Delta\phi - p_0\phi + h'(V)\phi, \qquad \phi \in H^2(R^N).$$

Our main result in this section can be stated as follows.

Proposition 6.1. *There exist positive constants ϵ_1, μ such that for all $\epsilon \in (0, \epsilon_1)$*

$$\|L_{z,\epsilon}\phi\|_{L^2(\Omega_\epsilon)} \geqslant \mu\|\phi\|_{H^2(\Omega_\epsilon)} \tag{6.4}$$

for all $|z| < \epsilon^\alpha$ and for all $\phi \in K^\perp_{z,\epsilon}$ where

$$L_{z,\epsilon} = \pi_{z,\epsilon} \circ F'_\epsilon(w_{z.\epsilon}) \tag{6.5}$$

and $\pi_{z,\epsilon}$ is the L^2-orthogonal projection from $L^2(\Omega_\epsilon)$ to $C^\perp_{z,\epsilon}$.

The next proposition gives the surjectivity of $L_{z,\epsilon}$.

Proposition 6.2. *There exists a positive constant* ϵ_2 *such that for all* $\epsilon \in (0, \epsilon_2)$ *and* $|z| < \epsilon^\alpha, \alpha > 1$ *the map*

$$L_{z,\epsilon} = \pi_{z,\epsilon} \circ F'_\epsilon(w_{z,\epsilon}) : K_{z,\epsilon}^\perp \longrightarrow C_{z,\epsilon}^\perp$$

is surjective.

Combining Propositions 6.1 and 6.2 gives us the invertibility of $L_{z,\epsilon}$.

Proposition 6.3.

$$L_{z,\epsilon} : K_{z,\epsilon}^\perp \longrightarrow C_{z,\epsilon}^\perp$$

is uniformly invertible, namely,

$$L_{z,\epsilon}^{-1} : C_{z,\epsilon}^\perp \longrightarrow K_{z,\epsilon}^\perp$$

exists bounded.

We now begin to prove Proposition 6.1.

Proof of Proposition 6.1: We follow the strategy used in [35].

Suppose (6.4) is false. Then there exist sequences $\{\epsilon_k\}, \{z_k\}$ and $\{\phi_k\}$, with $|z_k| \leqslant \epsilon_k^\alpha$ and $\epsilon_k \to 0$ as $k \to \infty$ such that

$$\phi_k \in K_{z_k,\epsilon_k}^\perp \quad \text{and}$$

$$\|L_{z_k,\epsilon_k}(\phi_k)\|_{L^2(\Omega_{\epsilon_k})} \to 0, \quad \|\phi_k\|_{H^2(\Omega_{\epsilon_k})} = 1. \tag{6.10}$$

We denote for $i = 1, \cdots, N$

$$e_{k,i} = \frac{\frac{\partial w_{z_k,\epsilon_k}}{\partial z_i}}{\|\frac{\partial w_{\epsilon_k,\epsilon_k}}{\partial z_i}\|}. \tag{6.11}$$

Note that as $\epsilon_k \to 0$

$$\|\frac{\partial w_{z_k,\epsilon_k}}{\partial z_i} - \frac{\partial V}{\partial y_i}\|_{H^2(\Omega_\epsilon)} \to 0. \tag{6.12}$$

Hence

$$\int_{\Omega_{\epsilon_k}} e_{k,i} e_{k,j} \longrightarrow c \int_{R^N} \frac{\partial V}{\partial y_i} \frac{\partial V}{\partial y_j} = 0 \ \text{ for } i \neq j.$$

Therefore after applying the Gram-Schmidt process to $\{e_{k,i}|i = 1, \cdots, N\}$ we obtain a family of orthonormal functions $\{e^*_{k,i}|i = 1, \cdots, N\}$ with

$$e^*_{k,i} = e_{k,i} + \delta_{k,i},\ \ i = 1, \cdots, N$$

where $\delta_{k,i} \longrightarrow 0$ in $L^2(\Omega_{\epsilon_k})$ as $k \to \infty$ for each $i = 1, \cdots, N$.

Hence

$$L_{z,\epsilon_k}\phi_k = F'_{\epsilon_k}(w_{z_k,\epsilon_k})\phi_k - \sum_{i=1}^{N-1}\Big(\int_{\Omega_{\epsilon_k}}[F'_{\epsilon_k}(w_{z_k,\epsilon_k})\phi_k]e_{k,i}\Big)e_{k,i} + E_k \tag{6.13}$$

where E_k is defined by (6.13) and it is easy to see that $\|E_k\|_{L^2(\Omega_{\epsilon_k})} \to 0$ as $k \to 0$.

Note that

$$\begin{aligned}\|L_{z,\epsilon_k}(\phi_k)\|^2_{L^2(\Omega_{\epsilon_k})} &= \|F'_{\epsilon_k}(w_{z_k,\epsilon_k})\phi_k\|^2_{L^2(\Omega_{\epsilon_k})} \\ &\quad - \sum_{i=1}^{n}\Big(\int_{\Omega_{\epsilon_k}}[F'_{\epsilon_k}(w_{z_k,\epsilon_k})\phi_k]e_{k,i}\Big)^2 + o(1)\end{aligned} \tag{6.14}$$

as $k \to \infty$.

Hence from (6.14), we have

$$\|F'_{\epsilon_k}(w_{z_k,\epsilon_k})\phi_k\|^2_{L^2(\Omega_{\epsilon_k})} - \sum_{i=1}^{N}\Big(\int_{\Omega_{\epsilon_k}}[F'_{\epsilon_k}(w_{z_k,\epsilon_k})\phi_k]e_{k,i}\Big)^2 \to 0 \tag{6.15}$$

as $k \to \infty$.

Let

$$\chi(x) = \begin{cases} 1, & \text{if } |x| < \frac{d(0,\partial\Omega)}{2}, \\ 0, & \text{if } |x| \geqslant \frac{2d(0,\partial\Omega)}{3} \end{cases}$$

and

$$\tilde{\phi_k} = \phi_k\chi(\epsilon_k y).$$

Then $\tilde{\phi_k}$ is well-defined in R^n and

$$\|\tilde{\phi_k}\|_{H^2(R^n)} \leqslant C\|\phi_k\|_{H^2(\Omega_{\epsilon_k})} \leqslant C.$$

Because of the decay of ϕ_k uniformly in k we have $\tilde{\phi_k} \to \phi_0$ weakly in $H^2(R^n)$ for some $\phi_0 \in H^2(R^n)$.

We now claim that $\phi_0 \equiv 0$.

In fact, since

$$\int \phi_k e_{k,i} = 0 \quad i = 1, \cdots, N$$

we have

$$\int_{R^N} \tilde{\phi}_k e_{k,i} = \int_{\Omega_{\epsilon_k}} \phi_k (1 - \chi(\epsilon_k y)) e_{k,i}$$
$$\leqslant \|\phi_k\|_{L^2(\Omega_{\epsilon_k})} \|(1 - \chi(\epsilon_k y)) e_{k,i}\|_{L^2(\Omega_{\epsilon_k})} \to 0.$$

Since $e_{k,i}$ has exponential decay outside $B^c(0, \delta)$ for any $\delta > 0$ it follows that

$$\int_{R^N} \phi_0 \frac{\partial V}{\partial y_i} = 0, \ i = 1, \cdots, N. \tag{6.16}$$

On the other hand, we now show that

$$F_0'(V)\phi_0 \in C_0 = \text{ span } \{\frac{\partial V}{\partial y_i} | i = 1, \cdots, N\}. \tag{6.17}$$

In fact, it is enough to show that

$$\int_{R^N} F_0'(V)\phi_0 g = 0, \qquad \text{for all } g \perp C_0. \tag{6.18}$$

To show (6.18), we note that by (6.15)

$$F_{\epsilon_k}'(w_{z_k,\epsilon_k})\phi_k = \psi_k^1 + \psi_k^2$$

where $\psi_k^1 \in C_{z_\epsilon,\epsilon_k}$, $\psi_k^2 \perp C_{z_k,\epsilon_k}$ and

$$\|\psi_k^2\|_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0.$$

Hence

$$\int_{R^N} \big(F_{\epsilon_k}'(w_{z_k,\epsilon_k})\big)\tilde{\phi}_k = \int_{R^N} F_{\epsilon_k}'(w_{z_k,\epsilon_k})\phi_k g - \int_{R^N} F_{\epsilon_k}'(w_{z_k,\epsilon_k})(1 - \chi(\epsilon y))\phi_k g$$
$$= \int_{R^N} \psi_k^1 g + \int_{R^N} \psi_k^2 g - \int_{R^N} F_{\epsilon_k}'(w_{z_k,\epsilon_k})(1 - \chi(\epsilon y))\phi_k g$$
$$\to 0$$

since $|\int_{R^N} F_{\epsilon_k}'(w_{z_k,\epsilon_k})(1-\chi(\epsilon y))\phi_k g| \leqslant C \int_{\{|y| \geqslant \frac{d(0,\partial\Omega)}{\epsilon_k}\}} g^2 \to 0$, $\big|\int_{R^n} \psi_k^2 g\big| \leqslant C\|\psi_k^2\|_{L^2(\Omega_{\epsilon_k})}\|g\|_{L^2(\Omega_{\epsilon_k})}$ and $\psi_k^1 \longrightarrow \psi_0$ in C_0.

Therefore (6.18) is proved and (6.17) is true.

But $F_0'(V)\phi_0 \perp C_0$ since

$$\int \big(F_0'(V)\phi_0\big)\frac{\partial V}{\partial y_i} = \int \phi_0 F_0'(V)\frac{\partial V}{\partial y_i} = 0, \quad i = 1, \cdots, N.$$

Hence $F_0'(V)\phi_0 = 0$. So $\phi_0 \in \ker(L)$ and $\phi_0 \perp$ span $\{\frac{\partial V}{\partial y_i} \big| i = 1, \cdots, N\}$.

By (6.16), this is impossible unless $\phi_0 \equiv 0$.

Therefore we obtain

$$\phi_0 \equiv 0. \tag{6.19}$$

Now we prove that $\|\phi_k\|_{H^2(\Omega_{\epsilon_k})} \longrightarrow 0$ which will give the desired contradiction. In fact

$$F'_{\epsilon_k}(w_{z_k,\epsilon_k})\phi_k = F'_{\epsilon_k}(w_{z_k,\epsilon_k})\tilde{\phi_k} + F'_{\epsilon_k}(w_{z_k,\epsilon_k})[1-\chi(\epsilon_k u)]\phi_k \tag{6.20}$$

and

$$\begin{aligned} \int F'_{\epsilon_k}(w_{z_k,\epsilon_k})\phi_k \frac{\partial w_{z_k,\epsilon_k}}{\partial z_i} &= \int F'_{\epsilon_k}(w_{z_k,\epsilon_k})\tilde{\phi_k}\frac{\partial w_{z_k,\epsilon_k}}{\partial z_i} \\ &\quad + \int F'_{\epsilon_k}(w_{z_k,\epsilon_k})(1-\chi\epsilon_k y))\phi_k \frac{\partial w_{z_k,\epsilon_k}}{\partial z_i} \\ &\longrightarrow 0 \end{aligned} \tag{6.21}$$

as $k \to \infty$.

By (6.16) we then have

$$\|F'_{\epsilon_k}(w_{z_k,\epsilon_k})\phi_k\|_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0.$$

Note that

$$\begin{aligned} F'_{\epsilon_k}(w_{z_k,\epsilon_k})\phi_k &= \triangle\phi_k - p_\sigma\phi_k + h_\sigma'(w_{z_k,\epsilon_k})\phi_k \\ &= \triangle\phi_k - p_\sigma\phi_k + h_\sigma'(w_{z_k,\epsilon_k})\tilde{\phi_k} + h_\sigma'(w_{z_k,\epsilon_k})(1-\chi(\epsilon_k y))\phi_k \end{aligned}$$

and

$$\|h'(w_{z_k,\epsilon_k})\tilde{\phi_k}\|_{L^2(\Omega_{\epsilon_k})} \to 0$$

$$\|h'(w_{z_k,\epsilon_k})(1-\chi(\epsilon_k y))\phi_k\|_{L^2(\Omega_{\epsilon_k})} \to 0.$$

So

$$\|\triangle\phi_k - p_\sigma\phi_k\|_{L^2(\Omega_{\epsilon_k})} \to 0.$$

On the other hand, we have

$$\|\phi_k\|_{H^2(\Omega_{\epsilon_k})} \leqslant C\|\triangle\phi_k - p_\sigma\phi_k\|_{L^2(\Omega_{\epsilon_k})}$$

(see [35], Appendix A).

Hence $\|\triangle\phi_k - p_\sigma\phi_k\|_{L^2(\Omega_{\epsilon_k})} \to 0$ as $k \longrightarrow \infty$, a contradiction to our hypothesis. Thus Proposition 6.1 is proved. □

Before proving Proposition 6.2, we now introduce a notion of "distance" between two closed subspaces E, F of a Hilbert space $H := L^2(\Omega_\epsilon)$. Following [17], we set

$$\vec{d}(E,F) = \sup\{d((x,F)|x \in E, \|x\|_H = 1\}$$

It is easy to see that $\vec{d}$ is non-symmetric, $\vec{d}(E,F) \leqslant 1$ and that

$$d(x,F) = 1 \text{ if and only if } x \perp F. \tag{6.24}$$

Moreover, it is not hard to show that

$$\vec{d}(E,F) = \vec{d}(F^\perp, E^\perp).$$

The following lemma will be needed in the proof of Proposition 6.2.

Lemma 6.5. *([17]; Lemma 1.3]) If $\vec{d}(E,F) < 1$, then $\pi_{F|E} : E \to F$ is injective and $\pi_{E|F} : F \to E$ has a bounded right inverse, where $\pi_E(\pi_F$, resp.) is the orthogonal projection from H to $E(F$, resp.). In particular, $\pi_{E|F} : F \to E$ is surjective.*

We are now ready to prove Proposition 6.2.

Proof of Proposition 6.2:

Let $Ck_{z,\epsilon} =$ co-kernel of $F'_\epsilon(w_{z,\epsilon})$. We first claim that

$$d(Ck_{z,\epsilon}, C_{z,\epsilon}) < 1 \tag{6.25}$$

for all $\epsilon > 0$ sufficiently small.

In fact, suppose (6.25) is not true. Then there exist $\epsilon_k \to 0$ and $\phi_k \in Ck_{z_k,\epsilon_k}$ such that

$$F'_{\epsilon_k}(w_{z_k,\epsilon_k})\phi_k = 0 \text{ in } \Omega_{\epsilon_k}, \frac{\partial\phi_k}{\partial\nu} = 0 \text{ on } \partial\Omega_{\epsilon_k}, \tag{6.26}$$

(6.27) $$\|\phi_k\|_{L^2(\Omega_{\epsilon_k})} = 1,$$

(6.28) $$\int \phi_k \frac{\partial(w_{z_k,\epsilon_k})}{\partial z_i} = 0, \quad i = 1, \cdots, N.$$

By (6.26) and (6.27), we have

$$\|\phi_k\|_{H^2(\Omega_{\epsilon_k})} \leqslant C.$$

Thus $\phi_k \to \phi_0$ weakly in $H^2(R^n)$ and ϕ_0 satisfies

$$\begin{cases} F_0'(V)\phi_0 = 0, \ \|\phi_0\|_{L^2(R^N)} = 1, \\ \int \phi_0 \frac{\partial V}{\partial y_i} = 0, \ i = 1, \cdots, N. \end{cases}$$

This is impossible.

Hence (6.25) is true.

Now by the fact that $d(E, F) = d(F^\perp, E^\perp)$, we have

$$d(\overline{C}^\perp_{z,\epsilon}, \overline{Ck}^\perp_{z,\epsilon}) < 1$$

where $\overline{C}^\perp_{z,\epsilon}(\overline{Ck}^\perp_{z,\epsilon}$, resp.)is the orthogonal complement of $C_{z,\epsilon}(Ck_{z,\epsilon}$, resp) in $L^2(\Omega_\epsilon)$.

Thus the map

(6.32) $$\pi_{\overline{C}^\perp_{z,\epsilon}}\Big|_{\overline{Ck}^\perp_{z,\epsilon}} : \overline{Ck}^\perp_{z,\epsilon} \to \overline{C}^\perp_{z,\epsilon}$$

is surjective, by Lemma 6.5.

Since $\overline{Ck}^\perp_{z,\epsilon}$ is the range of $F'_\epsilon(w_{z,\epsilon})$, it suffices to show that the map in (6.32) when restricted to $Ck^\perp_{z,\epsilon}$, which is $\pi_{z,\epsilon}$ is onto $C^\perp_{z,\epsilon}$. However, this follows easily from the expression

$$\pi_{\overline{C}^\perp_{z,\epsilon}}(\phi) = \phi - \pi_{C_{z,\epsilon}}\phi. \qquad \square$$

Finally in this section, we solve the following equation for $\phi \in K^\perp_{z,\epsilon}$.

$$\pi_{z,\epsilon} \circ H^1_\epsilon(w_{z,\epsilon})(w_{z,\epsilon} + \phi) = 0.$$

Since $L_{z,\epsilon}\Big|_{K^{\perp}_{z,\epsilon}}$ is invertible (and we shall denote its inverse by $L^{-1}_{z,\epsilon}$) by Proposition 6.3, this is equivalent to solving

$$\phi = -L^{-1}_{z,\epsilon}\circ\pi_{z,\epsilon}(F^{'}_{\epsilon}(w_{z,\epsilon})) - L^{-1}_{z,\epsilon}\circ\pi_{z,\epsilon}(N^1_{\epsilon,z}(\phi)+N^2_{\epsilon,z}(\phi)+M_{\epsilon,z}) :\equiv Q_{z,\epsilon}(\phi)$$

where $Q_{z,\epsilon}$ is defined in the last equality for every $\phi \in H^2_N(\Omega_\epsilon)$.

By Lemma 5.1, we have

(6.34)

$$\begin{aligned}\|Q_{z,\epsilon}(\phi_1) - Q_{z,\epsilon}(\phi_2)\| &\leqslant C\|N^1_{\epsilon,z}(\phi_1) - N^1_{\epsilon,z}(\phi_2)\|_{L^2(\Omega_\epsilon)} \\ &\quad + C\|N^2_{\epsilon,z}(\phi_1) - N^2_{\epsilon,z}(\phi_2)\|_{L^2(\Omega_\epsilon)} \\ &\leqslant C(\epsilon^{\frac{N}{2}} + c(\|\phi_1\|_{H^2(\Omega_\epsilon)}, \|\phi_2\|_{H^2(\Omega_\epsilon)}))\|\phi_1 - \phi_2\|_{H^2(\Omega_\epsilon)} \\ &\leqslant C(\delta, \epsilon_0)\|\phi_1 - \phi_2\|_{H^2(\Omega_\epsilon)}\end{aligned}$$

if $\|\phi_1\|_{H^2(\Omega_\epsilon)} \leqslant \delta, \|\phi_2\|_{H^2(\Omega_\epsilon)} \leqslant \delta, \epsilon \leq \epsilon_0$.

On the other hand, for $\|\phi\|_{H^2(\Omega_\epsilon)} < \delta$ we have

$$\begin{aligned}\|Q_{z,\epsilon}(\phi)\|_{H^2} &\leqslant \|F^{'}_{\epsilon,z}(w_z,\epsilon)\phi\|_{L^2} + \|N^1_{z,\epsilon}(\phi)\|_{L^2} \\ &\quad + \|N^2_{z,\epsilon}(\phi)\|_{L^2} + \|M_{z,\epsilon}\|_{L^2} \\ &\leqslant c(\varphi^{1+\eta(z)}_{\epsilon,z} + \delta\|\phi\|_{H^2(\Omega_\epsilon)}) \\ &\leqslant c(\varphi^{1+\tilde{\alpha}}_{\epsilon,z}(z) + \delta^{1+\tilde{\alpha}})\end{aligned}$$

for some $\tilde{\alpha} > 0$.

Take $\delta = \varphi_{\epsilon,z}(z)$. Then we have

$$\|Q_{z,\epsilon}(\phi)\|_{H^2} \leqslant C(\varphi^{1+\tilde{\alpha}}_{\epsilon,z}(z)). \tag{6.35}$$

Equation (6.35) says that $Q_{z,\epsilon}(\phi)$ is a continuous map mapping $B_\delta(0) \cap H^2_N(\Omega_\epsilon)$ into $\longrightarrow B_\delta(0) \cap K^{\perp}_{z,\epsilon}$.

Equation (6.34) says, $Q_{z,\epsilon}(\phi)$ is a contracting map if δ and ϵ_0 are small. Hence by the Contraction Mapping Principle we have

Proposition 6.6. *There exists $\epsilon > 0$ such that for $\epsilon < \epsilon_0, |z| < \epsilon^\alpha, 1 < \alpha < 2$ there is a unique $\phi_{\epsilon,z} \in K^{\perp}_{z,\epsilon}$ such that*

$$F_\epsilon(w_{z,\epsilon} + \phi_{\epsilon,z}) \in C_{z,\epsilon}. \tag{6.36}$$

Furthermore,

$$\|\phi_{\epsilon,z}\|_{H^2(\Omega_\epsilon)} \leq C\varphi_{\epsilon,z}^{\frac{1+\mu}{2}}(z). \tag{6.37}$$

7. The Reduced Problem

In this section, we shall prove our main result Theorem 1.1.

By Proposition 6.6, for $\epsilon \leqslant \epsilon_0$ and $|z| \leqslant \epsilon$, there exists a unique $\phi_{\epsilon,z} \in K_{z,\epsilon}^{\perp}$ such that

$$H_\epsilon^1(w_{z,\epsilon} + \phi_{\epsilon,z}) \in C_{\epsilon,z}. \tag{7.1}$$

Therefore it is enough to show that for some $|\tilde{z}| \leqslant \epsilon^\alpha$, we have

$$H_\epsilon^1(w_{z,\epsilon} + \phi_{z,\epsilon}) \perp C_{z,\epsilon}.$$

To this end, we now define a vector field

$$V_{\epsilon,j}(z) := \frac{1}{\epsilon^{\alpha-1}\varphi_{\epsilon,z}(z)}[\int_{\Omega_\epsilon} H_\epsilon^1(w_{z,\epsilon} + \phi_{\epsilon,z})\frac{\partial w_{\epsilon,z}}{\partial z_j}]$$

where $z = \epsilon\frac{a}{2}d(0,\partial\Omega) + \epsilon^{\alpha+1}\tilde{\tilde{z}}, |\tilde{\tilde{z}}| \leqslant 1$, where a is such that

$$\int_{\partial\Omega} e^{<x-P_0,a>} x_i d\mu_0(x) = 0.$$

The main estimate of this section is

Lemma 7.1. *For every $0 < \alpha < \alpha_0$, the vector field V_ϵ converges uniformly to $\bar{V}_0$ with $\tilde{\tilde{z}} \in B_1(0)$ as $\epsilon \to 0$, where*

$$\bar{V}_0 = (\bar{V}_{0,1}, \cdots, \bar{V}_{0,N}),$$
$$\bar{V}_{0,j} = \gamma(\int_{\partial\Omega} e^{<x-P_0,a>} x_i x_j d\mu_0(x))\tilde{\tilde{z}}_i$$

and γ is given by Lemma 3.4.

Once Lemma 7.1 is proved, then Theorem 1.1 follows easily. In fact, since 0 is a nondegenerate peak point, $\overline{V}_0$ has a nondegenerate zero at 0 (with degree different from 0). Then Lemma 7.1 and a simple degree theoretic argument imply that V_ϵ has a zero $z = \epsilon\frac{a}{2}d(0,\partial\Omega) + \epsilon^{\alpha+1}\tilde{\tilde{z}}$ with $\tilde{\tilde{z}}(\epsilon) \in B_{\frac{1}{2}}(0)$ for every ϵ sufficiently small. This solves the equation $H^1_\epsilon(w_{z,\epsilon}+\phi_{\epsilon,z})=0$ for every ϵ sufficiently small. Setting $z(\epsilon)=\epsilon\frac{a}{2}d(0,\partial\Omega)+\epsilon^{\alpha+1}\tilde{\tilde{z}}(\epsilon)$ and

$$v_\epsilon=\tau_{\sigma_0(\epsilon)}+w_{z(\epsilon),\epsilon}+\phi_{\epsilon,z(\epsilon)}$$

for $x\in\Omega$ and ϵ sufficiently small, it follows then

$$v_\epsilon\not\equiv 0 \text{ since } \phi_{\epsilon,z(\epsilon)}\to 0 \text{ in } H^2(\Omega_\epsilon) \text{ as } \epsilon\to 0$$

while $w_{z(\epsilon),\epsilon}$ remains bounded away from 0 in $H^2(\Omega_\epsilon)$ as $\epsilon\to 0$.

That is, v_ϵ is a non-trivial solution of (1.2). By the structure of v_ϵ, v_ϵ has all the properties of Theorem 1.1.

It remains to prove Lemma 7.1. To this end, we have

$$\begin{aligned}
&\int_{\Omega_{\epsilon,z}} H^1_\epsilon(w_{z,\epsilon}+\phi_{\epsilon,z})\frac{\partial w_{z,\epsilon}}{\partial z_j}\\
&=\int_{\Omega_{\epsilon,z}}[F_\epsilon'(w_{z,\epsilon})\phi_{\epsilon,z}]\frac{\partial w_{z,\epsilon}}{\partial z_j}\\
&+\int_{\Omega_{\epsilon,z}}[N^1_{\epsilon,z}(\phi_{\epsilon,z})]\frac{\partial w_{z,\epsilon}}{\partial z_j}\\
&+\int_{\Omega_{\epsilon,z}}[N^2_{\epsilon,z}(\phi_{\epsilon,z})]\frac{\partial w_{z,\epsilon}}{\partial z_j}\\
&+\int_{\Omega_{\epsilon,z}}M_{\epsilon,z}\frac{\partial w_{z,\epsilon}}{\partial z_j}\\
&=I_1+I_2+I_3+I_4
\end{aligned}$$

where $I_i, i=1,2,3,4$ are defined by the last equality.

Note that

$$\begin{aligned}
I_1=&\int_{\Omega_{\epsilon,z}}\big[h_\sigma'(P_{\Omega_{\epsilon,z}}V_\sigma)-h_\sigma'(V_\sigma)\big]\phi_{\epsilon,z}\frac{\partial w_{z,\epsilon}}{\partial z_j}\\
&+(\int_{\Omega_{\epsilon,z}}h_\sigma'(V_\sigma)\phi_{\epsilon,z}\frac{\partial \varphi_{\epsilon,z}}{\partial z_j}\\
&=O\big(e^{-\sqrt{p_\sigma}\frac{(2+\mu)d(z,\partial\Omega)}{\epsilon}}\big).
\end{aligned}$$

By Lemma 5.1 and Proposition 6.6 we have

$$|I_2|\le C|\varphi_{\epsilon,z}(z)|^{1+\mu}=O\big(e^{-\sqrt{p_\sigma}\frac{(2+\mu)d(z,\partial\Omega)}{\epsilon}}\big),$$

$$I_3 = [N^2_{\epsilon,z}(\phi_{\epsilon,z})] \int_{\Omega_{\epsilon,z}} \frac{\partial w_{z,\epsilon}}{\partial z_j}$$
$$= O\big(e^{-\sqrt{p_\sigma}\frac{(2+\mu)d(z,\partial\Omega)}{\epsilon}}\big)$$

for some $\mu > 0$.

So we just need to compute I_4.

In fact,

$$\begin{aligned} I_4 &= \int_{\Omega_{\epsilon,z}} \big[h_\sigma(P_{\Omega_{\epsilon,z}}V_\sigma) - h_\sigma(V_\sigma)\big] \frac{\partial P_{\Omega_{\epsilon,z}}V_\sigma}{\partial z_j} \\ &= \int_{\Omega_{\epsilon,z}} h'_\sigma(V_\sigma) \frac{\partial P_{\Omega_{\epsilon,z}}V_\sigma}{\partial z_j} \cdot \big(P_{\Omega_{\epsilon,z}}V_\sigma - (V_\sigma\big) \\ &+ O\big(e^{-\sqrt{p_\sigma}\frac{(2+\mu)d(z,\partial\Omega)}{\epsilon}}\big) \\ &= \epsilon \int_{\Omega_{\epsilon,z}} h'_\sigma(V_\sigma) \frac{\partial V_\sigma}{\partial y_j} \cdot \big(P_{\Omega_{\epsilon,z}}V_\sigma - (V_\sigma)\big) \\ &+ O\big(e^{-\sqrt{p_\sigma}\frac{(2+\mu)d(z,\partial\Omega)}{\epsilon}}\big). \end{aligned}$$

By Lemma 3.4, we conclude the proof of Lemma 7.1. □

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Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: wei@math.cuhk.edu.hk

Mathematisches Institut, Universität Stuttgart, D-70511 Stuttgart, Germany

E-mail address: winter@mathematik.uni-stuttgart.de