Exponential Stability of Delayed Recurrent Neural Networks with Markovian Jumping Parameters

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Abstract

In this paper, the global exponential stability analysis problem is considered for a class of recurrent neural networks (RNNs) with time delays and Markovian jumping parameters. The jumping parameters considered here are generated from a continuous-time discrete-state homogeneous Markov process, which are governed by a Markov process with discrete and finite state space. The purpose of the problem addressed is to derive some easy-to-test conditions such that the dynamics of the neural network is stochastically exponentially stable in the mean square, independent of the time delay. By employing a new Lyapunov-Krasovskii functional, a linear matrix inequality (LMI) approach is developed to establish the desired sufficient conditions, and therefore the global exponential stability in the mean square for the delayed RNNs can be easily checked by utilizing the numerically efficient Matlab LMI toolbox, and no tuning of parameters is required. A numerical example is exploited to show the usefulness of the derived LMI-based stability conditions.

Keywords

Recurrent neural networks; Markovian jumping parameters; Time delays; Stochastic systems; Global exponential stability in the mean square; Linear matrix inequality.

I. Introduction

In the past few decades, the mathematical properties of the recurrent neural networks (RNNs), such as the stability, the attractivity and the oscillation, have been intensively investigated. RNNs have been successfully applied in many areas, including image processing, pattern recognition, associative memory, and optimization problems. In fact, the stability analysis issue for RNNs with time delays has been an attractive subject of research in the past few years, where the time delays under consideration can be classified as constant delays, time-varying delays, and distributed delays. Various sufficient conditions, either delay-dependent or delay-independent, have been proposed to guarantee the global asymptotic or exponential stability for the RNNs with time-delays, see e.g. [4], [9], [12], [15] for some recent publications, where many methods have been exploited, such as the LMI approach and M-matrix approach.

Traditional RNNs assume that the continuous variables propagate from one processing unit to the next. However, RNNs sometimes have problems in catching long-term dependencies in the input stream. As the temporal sequences increase in length, the influence of early components of the sequence has less impact on the network output. Such a phenomenon is referred to as the problem of information latching [1]. A widely used approach to dealing with the information latching problem is to extract *finite state* representations (also called clusters, patterns, or modes) from trained networks [2], [5], [6], [7]. In other words, the RNNs may have finite modes, and the modes may switch (or jump) from one to another at different times. Recently, it has been shown in [14] that, the switching (or jumping) between different RNN modes can be governed by a

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Markovian chain. Hence, an RNN with such a "jumping" character may be modeled as a hybrid one; that is, the state space of the RNN contains both discrete and continuous states. For a specific mode, the dynamics of the RNN is continuous, but the parameter jumps among different modes may be seen as discrete events. Note that the concept of Markovian neural networks has already been used in some papers, see e.g. [11]. Therefore, RNNs with Markovian jumping parameters are of great significance in modeling a class of neural networks with finite network modes. It should be pointed out that, up to now, the stability analysis problem for RNNs with Markovian switching has received very little research attention, despite its practical importance.

In this paper, we are concerned with the analysis issue for the *global exponential stability* of RNNs with mixed time-delays and Markovian jumping parameters. To the best of the authors' knowledge, this is the first attempt to introduce and investigate the delayed RNNs with Markovian switching. It is worth mentioning that the control and filtering problems for Markovian jump systems (MJSs) have already been widely studied in the control and signal processing communities, see e.g. [10], [16]. The main purpose of this paper is to establish LMI-based stability criteria for testing whether the network dynamics is stochastically exponentially stable in the mean square, independent of the time delay. It is known that LMIs can be efficiently solved by utilizing the numerically attractive Matlab LMI toolbox, hence our proposed results would be practical. We will use a simple example to illustrate the usefulness of the derived LMI-based stability conditions.

II. Problem formulation

Notation. The notations in this paper are quite standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. We let $h > 0$ and $C([-h, 0]; \mathbb{R}^n)$ denote the family of continuous functions φ from $[-h, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-h\leq\theta\leq 0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{\max}(A^T A)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. $l_2[0, \infty)$ is the space of square integrable vector. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L^p_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C([-h, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h\leq\theta\leq 0} \mathbb{E}|\xi(\theta)|^p < \infty$ where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

In this paper, the recurrent neural network with time delays is described as follows:

$$\dot{u}(t) = -Au(t) + W_0 g_0(u(t)) + W_1 g_1(u(t-h)) + V \tag{1}$$

where $u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T \in \mathbb{R}^n$ is the state vector associated with the n neurons, the diagonal matrix $A = \mathrm{diag}(a_1, a_2, \cdots, a_n)$ has positive entries $a_i > 0$. The matrices $W_0 = (w^0_{ij})_{n\times n}$ and $W_1 = (w^1_{ij})_{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively. $g_k(u(t)) = [g_{k1}(u_1), g_{k2}(u_2), \cdots, g_{kn}(u_n)]^T$ $(k = 0, 1)$ denotes the neuron activation function with $g_k(0) = 0$, and $V = [V_1, V_2, \cdots, V_n]^T$ is a constant external input vector. The scalar $h > 0$, which may be unknown, denotes the time delay.

Assumption 1: The neuron activation functions in (1), $g_i(\cdot)$, are bounded and satisfy the following Lipschitz condition

$$|g_k(x) - g_k(y)| \leq |G_k(x - y)|, \quad \forall x, y \in \mathbb{R} \quad (k = 0, 1) \tag{2}$$

where $G_k \in \mathbb{R}^{n\times n}$ is a known constant matrix.

Remark 1: In the past, the activation functions have been required to be continuous, differentiable and monotonically increasing, such as the sigmoid-type of function. In this paper, these restrictions are removed, and only Lipschitz conditions and boundedness are needed in Assumption 1. Note that the type of activation functions in (2) have already been used in numerous papers, see [4] and references therein.

Let u^* be the equilibrium point of (1). For the purpose of simplicity, we can shift the intended equilibrium u^* to the origin by letting $x = u - u^*$, and then the system (1) can be transformed into:

$$\dot{x}(t) = -Ax(t) + W_0 l_0(x(t)) + W_1 l_1(x(t-h)), \tag{3}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector of the transformed system. It follows from (2) that the transformed neuron activation functions $l_k(x) = g_k(x + u^*) - g_k(u^*)$ $(k = 0, 1)$ satisfy

$$|l_k(x)| \le |G_k x|, \tag{4}$$

where $G_k \in \mathbb{R}^{n\times n}$ $(k = 0, 1)$ are specified in (2).

Now, based on the model (3), we are in a position to introduce the delayed recurrent neural networks with Markovian jumping parameters.

Let $\{r(t),\ t \ge 0\}$ be a right-continuous Markov process on the probability space which takes values in the finite space $\mathcal{S} = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})$ $(i, j \in \mathcal{S})$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } \ i \ne j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } \ i = j \end{cases}$$

where $\Delta > 0$ and $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$, $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ and $\gamma_{ii} = -\sum_{j\ne i}\gamma_{ij}$.

In this paper, we consider the following delayed recurrent neural network with Markovian jumping parameters, which is actually a modification of (3):

$$\dot{x}(t) = -A(r(t))x(t) + W_0(r(t))l_0(x(t)) + W_1(r(t))l_1(x(t-h)), \tag{5}$$

where $x(t)$, $l_0(x(t))$ and $l_1(x(t-h))$ have the same meanings as those in (3), and for a fixed system mode, $A(r(t))$, $W_0(r(t))$ and $W_1(r(t))$ are known constant matrices with appropriate dimensions.

Recall that the Markov process $\{r(t),\ t \ge 0\}$ takes values in the finite space $\mathcal{S} = \{1, 2, \ldots, N\}$. For the sake of simplicity, we write

$$A(i) := A_i, \quad W_0(i) := W_{0i}, \quad W_1(i) := W_{1i}. \tag{6}$$

Now we shall work on the network mode $r(t) = i,\ \ \forall i \in \mathcal{S}$. Observe the neural network (5) and let $x(t;\xi)$ denote the state trajectory from the initial data $x(\theta) = \xi(\theta)$ on $-h \le \theta \le 0$ in $L^2_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$. Clearly, the network (5) admits an equilibrium point (trivial solution) $x(t; 0) \equiv 0$ corresponding to the initial data $\xi = 0$.

The following stability concepts are needed in this paper.

Definition 1: For the delayed recurrent neural network (5) and every $\xi \in L^2_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$, the equilibrium point is asymptotically stable in the mean square if, for every network mode

$$\lim_{t\to\infty} \mathbb{E}|x(t;\xi)|^2 = 0; \tag{7}$$

and is exponentially stable in the mean square if, for every network mode, there exist scalars $\alpha > 0$ and $\beta > 0$ such that

$$\mathbb{E}|x(t;\xi)|^2 \le \alpha e^{-\beta t} \sup_{-h\le\theta\le 0} \mathbb{E}|\xi(\theta)|^2. \tag{8}$$

Our objective of this paper is to establish LMI-based stability criteria under which the network dynamics of (5) is exponentially stable in the mean square, independent of the time delay.

III. Main results and proofs

Let us first give the following lemmas which will be frequently used in the proofs of our main results in this paper.

Lemma 1: Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $\varepsilon > 0$. Then we have $x^T y + y^T x \leq \varepsilon x^T x + \varepsilon^{-1} y^T y$.

Proof: The proof follows from the inequality $(\varepsilon^{1/2}x - \varepsilon^{-1/2}y)^T(\varepsilon^{1/2}x - \varepsilon^{-1/2}y) \geq 0$ immediately. ■

Lemma 2: [3] Given constant matrices Ω_1, Ω_2, Ω_3 where $\Omega_1 = \Omega_1^T$ and $0 < \Omega_2 = \Omega_2^T$, then

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^T & \Omega_1 \end{bmatrix} < 0.$$

The main results of this paper are given as follows, which shows that the network dynamics of (5) is globally exponentially stable in the mean square if a set of linear matrix inequalities are feasible.

Theorem 1: If there exist two sequences of positive scalars $\{\mu_{0i} > 0,\ \mu_{1i} > 0,\ i \in \mathcal{S}\}$ and a sequence of positive definite matrices $P_i = P_i^T > 0$ $(i \in \mathcal{S})$ such that the following linear matrix inequalities

$$\begin{bmatrix} -A_iP_i - P_iA_i + \sum_{j=1}^N \gamma_{ij}P_j & \mu_{0i}G_0^T & P_iW_{0i} & \mu_{1i}G_1^T & P_iW_{1i} \\ \mu_{0i}G_0 & -\mu_{0i}I & 0 & 0 & 0 \\ W_{0i}^TP_i & 0 & -\mu_{0i}I & 0 & 0 \\ \mu_{1i}G_1 & 0 & 0 & -\mu_{1i}I & 0 \\ W_{1i}^TP_i & 0 & 0 & 0 & -\mu_{1i}I \end{bmatrix} < 0, \tag{9}$$

hold, then the dynamics of the neural network (5) is globally exponentially stable in the mean square.

Proof: Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$ denote the family of all nonnegative functions $Y(x,t,i)$ on $\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}$ which are continuously twice differentiable in x and differentiable in t. Denote

$$\varepsilon_{0i} = \mu_{0i}^{-1}, \quad \varepsilon_{1i} = \mu_{1i}^{-1}. \tag{10}$$

Fix $\xi \in L^2_{\mathcal{F}_0}([-h,0];\mathbb{R}^n)$ arbitrarily and write $x(t;\xi) = x(t)$. Define a Lyapunov functional candidate $Y(x,t,i) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$ by

$$Y(x(t), r(t) = i) := Y(x(t),t,i) = x^T(t)P_ix(t) + \int_{t-h}^t x^T(s)Qx(s)ds, \tag{11}$$

where $Q \geq 0$ is given as

$$Q = \varepsilon_{1i}^{-1} G_1^T G_1. \tag{12}$$

It is known (see [13]) that $\{x(t), r(t)\}$ $(t \geq 0)$ is a $C([-h,0];\mathbb{R}^n) \times \mathcal{S}$-valued Markov process. From (5), the weak infinitesimal operator $\mathcal{L}$ (see [10]) of the stochastic process $\{r(t), x(t)\}$ $(t \geq 0)$ is given by:

$$\begin{aligned} \mathcal{L}Y(x(t), r(t)) \; &:= \; \lim_{\Delta \to 0^+} \frac{1}{\Delta}\Big[\mathbb{E}\big\{Y(x(t+\Delta), r(t+\Delta))|x(t), r(t)=i\big\} - Y(x(t), r(t)=i)\Big] \\ &= \; x^T(t)\Big[-A_iP_i - P_iA_i + \sum_{j=1}^N \gamma_{ij}P_j + Q\Big]x(t) \\ &\quad + x^T(t)P_iW_{0i}l_0(x(t)) + l_0^T(x(t))W_{0i}^TP_ix(t) \\ &\quad + x^T(t)P_iW_{1i}l_1(x(t-h)) + l_1^T(x(t-h))W_{1i}^TP_ix(t) \\ &\quad - x^T(t-h)Qx(t-h) + \sum_{j=1}^N \gamma_{ij} \int_{t-h}^t x^T(s)Qx(s)ds. \end{aligned} \tag{13}$$

It follows from $\sum_{j=1}^{N}\gamma_{ij}=0$ that

$$\sum_{j=1}^{N}\gamma_{ij}\int_{t-h}^{t}x^T(s)Qx(s)ds=\left(\sum_{j=1}^{N}\gamma_{ij}\right)\left(\int_{t-h}^{t}x^T(s)Qx(s)ds\right)=0. \tag{14}$$

From Lemma 1 and (4), we have:

$$\begin{aligned} & x^T(t)P_iW_{0i}l_0(x(t))+l_0^T(x(t))W_{0i}^TP_ix(t)\\ \leq\; & \varepsilon_{0i}x^T(t)P_iW_{0i}W_{0i}^TP_ix(t)+\varepsilon_{0i}^{-1}l_0^T(x(t))l_0(x(t))\\ \leq\; & x^T(t)(\varepsilon_{0i}P_iW_{0i}W_{0i}^TP_i+\varepsilon_{0i}^{-1}G_0^TG_0)x(t) \end{aligned} \tag{15}$$

and

$$\begin{aligned} & x^T(t)P_iW_{1i}l_1(x(t-h))+l_1^T(x(t-h))W_{1i}^TP_ix(t)\\ \leq\; & \varepsilon_{1i}x^T(t)P_iW_{1i}W_{1i}^TP_ix(t)+\varepsilon_{1i}^{-1}l_1^T(x(t-h))l_1(x(t-h))\\ \leq\; & \varepsilon_{1i}x^T(t)P_iW_{1i}W_{1i}^TP_ix(t)+\varepsilon_{1i}^{-1}x^T(t-h)G_1^TG_1x(t-h). \end{aligned} \tag{16}$$

Define

$$\Pi:=-A_iP_i-P_iA_i+\sum_{j=1}^{N}\gamma_{ij}P_j+\varepsilon_{0i}^{-1}G_0^TG_0+\varepsilon_{0i}P_iW_{0i}W_{0i}^TP_i+\varepsilon_{1i}^{-1}G_1^TG_1+\varepsilon_{1i}P_iW_{1i}W_{1i}^TP_i. \tag{17}$$

In view of (12) and (14)-(17), it follows from (13) that

$$\mathcal{L}Y(x(t),i)\leq x^T(t)\Pi x(t). \tag{18}$$

Now, Pre- and post-multiplying the inequality (9) by the block-diagonal matrix

$$\mathrm{diag}\{I,\varepsilon_{0i}^{1/2}I,\varepsilon_{0i}^{1/2}I,\varepsilon_{1i}^{1/2}I,\varepsilon_{1i}^{1/2}I\}$$

yield

$$\begin{bmatrix} -A_iP_i-P_iA_i+\sum_{j=1}^{N}\gamma_{ij}P_j & \varepsilon_{0i}^{-1/2}G_0^T & \varepsilon_{0i}^{1/2}P_iW_{0i} & \varepsilon_{1i}^{-1/2}G_1^T & \varepsilon_{1i}^{1/2}P_iW_{1i}\\ \varepsilon_{0i}^{-1/2}G_0 & -I & 0 & 0 & 0\\ \varepsilon_{0i}^{1/2}W_{0i}^TP_i & 0 & -I & 0 & 0\\ \varepsilon_{1i}^{-1/2}G_1 & 0 & 0 & -I & 0\\ \varepsilon_{1i}^{1/2}W_{1i}^TP_i & 0 & 0 & 0 & -I \end{bmatrix}<0, \tag{19}$$

or

$$\begin{bmatrix} \Omega_1 & \Omega_3^T\\ \Omega_3 & -\Omega_2 \end{bmatrix}<0, \tag{20}$$

where

$$\Omega_1=-A_iP_i-P_iA_i+\sum_{j=1}^{N}\gamma_{ij}P_j,\quad \Omega_2=I,$$

$$\Omega_3=\begin{bmatrix}\varepsilon_{0i}^{-1/2}G_0^T & \varepsilon_{0i}^{1/2}P_iW_{0i} & \varepsilon_{1i}^{-1/2}G_1^T & \varepsilon_{1i}^{1/2}P_iW_{1i}\end{bmatrix}^T.$$

It follows from the Schur Complement Lemma (Lemma 2) that (20) holds if and only if

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0,$$

or

$$-A_i P_i - P_i A_i + \sum_{j=1}^{N} \gamma_{ij} P_j + \varepsilon_{0i}^{-1} G_0^T G_0 + \varepsilon_{0i} P_i W_{0i} W_{0i}^T P_i + \varepsilon_{1i}^{-1} G_1^T G_1 + \varepsilon_{1i} P_i W_{1i} W_{1i}^T P_i < 0, \tag{21}$$

which means $\Pi < 0$ where Π is defined in (17).

We are now ready to prove the exponential stability in the mean square for the neural network (5). Let $\beta > 0$ be the unique root of the equation

$$\lambda_{\min}(-\Pi) - \beta \lambda_{\max}(P_i) - \beta h \lambda_{\max}(Q) x^{\beta h} = 0, \tag{22}$$

where Q is defined in (12), P_i is the positive definite solution to (9) or (21), and h is the time-delay.

We can obtain from (11) that

$$\begin{aligned} \mathcal{L}\big[x^{\beta t} Y(x(t), r(t))\big] &= x^{\beta t}\big[\beta Y(x(t), r(t)) + \mathcal{L} Y(x(t), r(t))\big] \\ &\leq x^{\beta t}\Big(- \big[\lambda_{\min}(-\Pi) - \beta \lambda_{\max}(P_i)\big] |x(t)|^2 + \beta \lambda_{\max}(Q) \int_{t-h}^{t} |x(s)|^2 ds \Big). \end{aligned}$$

Then, integrating both sides from 0 to $T > 0$ gives

$$\begin{aligned} x^{\beta T} \mathbb{E} Y(x(T), r(T)) &\leq \big[\lambda_{\max}(P_i) + h \lambda_{\max}(Q)\big] \sup_{-h \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2 \\ &\quad - \big[\lambda_{\min}(-\Pi) - \beta \lambda_{\max}(P_i)\big] \mathbb{E} \int_0^T x^{\beta t} |x(t)|^2 dt \\ &\quad + \beta \lambda_{\max}(Q) \mathbb{E} \int_0^T x^{\beta t} \int_{t-h}^{t} |x(s)|^2 ds dt. \end{aligned}$$

Note that

$$\begin{aligned} &\int_0^T x^{\beta t} \int_{t-h}^{t} |x(s)|^2 ds dt \leq \int_{-h}^{T} \Big(\int_{\max(s,0)}^{\min(s+h,T)} x^{\beta t} dt \Big) |x(s)|^2 ds \\ &\leq \int_{-h}^{T} h x^{\beta(s+h)} |x(s)|^2 ds \leq h x^{\beta h} \int_0^T x^{\beta t} |x(t)|^2 dt + h x^{\beta h} \int_{-h}^{0} |\xi(\theta)|^2 d\theta. \end{aligned}$$

Then, considering the definition of β in (22), we have

$$\begin{aligned} x^{\beta T} \mathbb{E} Y(x(T), r(T)) &\leq \big[\lambda_{\max}(P_i) + h \lambda_{\max}(Q)\big] \sup_{-h \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2 \\ &\quad + \beta \lambda_{\max}(Q) h^2 x^{\beta h} \sup_{-h \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}|x(T)|^2 &\leq \lambda_{\min}^{-1}(P_i) \Big(\big[\lambda_{\max}(P_i) + h \lambda_{\max}(Q)\big] \sup_{-h \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2 \\ &\quad + \beta \lambda_{\max}(Q) h^2 x^{\beta h} \sup_{-h \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2 \Big) x^{-\beta T}. \end{aligned}$$

Notice that $(x^{\beta T} - 1) x^{-\beta T} < 1$ and let

$$\alpha := \lambda_{\min}^{-1}(P_i) \big[\lambda_{\max}(P_i) + h \lambda_{\max}(Q)(1 + \beta h x^{\beta h})\big].$$

Since $T > 0$ is arbitrary, the definition of mean square exponential stability is then satisfied, hence the proof of Theorem 1 is completed. ■

Remark 2: It is shown in Theorem 1 that, the exponential stability in the mean square of the neural network (5) can be checked if a set of coupled LMIs are feasible. Noting that the solvability of LMIs can be readily checked by the standard LMI Matlab toolbox [8] and no tuning of parameters is required, the main results presented in Theorem 1 is thus very practical.

It is worth mentioning that, our results can be easily extended to the multiple-delay case. Consider now the following Markovian jumping neural network with multiple time-varying delays:

$$\dot{x}(t) = -A(r(t))x(t) + W_0(r(t))l_0(x(t)) + \sum_{k=1}^{m} W_1(r(t))l_1(x(t - h_k(t))), \tag{23}$$

where $h_k(t)$ $(k = 1, 2, \cdots, m)$ is a time-varying scalar satisfying $0 \le h_k(t) \le h_k < \infty$ and $0 \le \dot{h}_k(t) \le \alpha_k$, and h_k and α_k are known constants.

In order to tackle the analysis problem for the mean square stability of the neural network (23), we just need to modify the Lyapunov-Krasovskii functional candidate in (11) as follows

$$Y(x(t), r(t) = i) := Y(x(t), t, i) = x^T(t)P_i x(t) + \sum_{k=1}^{m} \int_{t-h_k(t)}^{t} x^T(s)Qx(s)ds, \tag{24}$$

and then similar results can be obtained by following the same line of the proof of Theorem 1.

Remark 3: Finally, we point out that it is possible to generalize our main results to more complex neural networks, such as neural networks with distributed delays, parameter uncertainties and stochastic perturbations, and the corresponding results will appear in the near future.

IV. A numerical example

We present a simple example here in order to illustrate the usefulness of our main results. Our aim is to examine the global exponential stability of a given delayed neural network with Markovian jumping parameters.

Consider a two-neuron delayed neural network (5) with two modes. The network parameters are given as follows:

$$A_1 = \begin{bmatrix} 1.6 & 0 \\ 0 & 1.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad W_{01} = \begin{bmatrix} 1.2 & -1.5 \\ -1.7 & 1.2 \end{bmatrix}, \quad W_{02} = \begin{bmatrix} 0.6 & 0.1 \\ 0.1 & 0.2 \end{bmatrix},$$
$$W_{11} = \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 0.8 \end{bmatrix}, \quad W_{12} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -7 & 7 \\ 6 & -6 \end{bmatrix}.$$

By using the Matlab LMI toolbox, we solve the two LMIs in (9) for $\mu_{01} > 0$, $\mu_{11} > 0$, $\mu_{02} > 0$, $\mu_{12} > 0$, $P_1 > 0$ and $P_2 > 0$, and obtain

$$\mu_{01} = 1.1478, \quad \mu_{11} = 0.9373, \quad \mu_{02} = 1.0008, \quad \mu_{12} = 0.9108,$$
$$P_1 = \begin{bmatrix} 0.3996 & 0.0493 \\ 0.0493 & 0.4570 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.3982 & 0.0632 \\ 0.0632 & 0.4415 \end{bmatrix}.$$

Therefore, it follows from Theorem 1 that the Markovian jumping delayed neural network (5) is globally exponentially stable in the mean square.

V. Conclusions

In this paper, we have dealt with the problem of global exponential stability analysis for a class of general recurrent neural networks, which both time delays and Markovian jumping parameters. We have removed the traditional monotonicity and smoothness assumptions on the activation function. A linear matrix inequality (LMI) approach has been developed to solve the problem addressed. The conditions for the global exponential stability have been derived in terms of the positive definite solution to the LMIs, and a simple example has been used to demonstrate the usefulness of the main results.

References

[1] Bengio, Y., Frasconi, P. and Simard, P. (1993) The problem of learning long-term dependencies in recurrent networks. In *Proc. 1993 IEEE Int. Conf. Neural Networks*, vol. 3, 1183-1188.

[2] Bolle, D., Dupont, P. and Vinck, B. (1992) On the overlap dynamics of multi-state neural networks with a finite number of patterns. *J. Phys. A*, 25, 2859-2872.

[3] Boyd, S., Ghaoui, L. EI, Feron, E. and Balakrishnan, V. (1994) *Linear matrix inequalities in system and control theory.* Philadephia: SIAM.

[4] Cao, J. and Ho, D. W. C. (2005) A general framework for global asymptotic stability analysis of delayed neural networks based on LMI approach. *Chaos, Solitons & Fractals*, 24(5), 1317-1329.

[5] Casey, M. P. (1996) The dynamics of discrete-time computation with application to recurrent neural networks and finite state machine extraction. *Neural Comput.*, 8(6), 1135-1178.

[6] Cleeremans, A., Servan-Schreiber, D. and McClelland, J. L. (1989) Finite state automata and simple recurrent networks. *Neural Comput.*, 1(3), 372-381.

[7] Elman, J. L. (1990) Finding structure in time. *Cognitive Sci.*, 14, 179-211.

[8] Gahinet, P., Nemirovsky, A., Laub, A. J. and Chilali, M. (1995) *LMI control toolbox: for use with Matlab*, The MATH Works Inc.

[9] Huang, H. and Cao, J. (2003) On global asymptotic stability of recurrent neural networks with time-varying delays. *Applied Mathematics and Computation*, 142(1), 143-154.

[10] Ji, Y. and Chizeck, H. J. (1990) Controllability, stabilizability, and continuous-time Markovian jump linear quadratic control. *IEEE Trans. Automat. Control*, 35, 777-788.

[11] Kovacic, M. (1993) Timetable construction with Markovian neural network. *European J. Operational Research*, 69(1), 92-96.

[12] Liang, J. and Cao, J. (2003) Boundedness and stability for recurrent neural networks with variable coefficients and time-varying delays. *Physics Letters A*, 318(1-2), 53-64.

[13] Skorohod, A. V. (1989) *Asymptotic methods in the theory of stochastic differential equations*, Providence, RI: Amer. Math. Soc.

[14] Tino, P., Cernansky, M. and Benuskova, L. (2004) Markovian architectural bias of recurrent neural networks. *IEEE Trans. Neural Networks*, 15(1), 6-15.

[15] Wang, Z., Liu, Y. and Liu, X. (2005) On global asymptotic stability of neural networks with discrete and distributed delays, *Physics Letters A*, 345(4-6), 299-308.

[16] Wang, Z., Lam, J. and Liu, X. (2003) Nonlinear filtering for state delayed systems with Markovian switching. *IEEE Trans. Signal Processing*, 51(9), 2321-2328.