Robust Variance-Constrained H_∞ Control for Stochastic Systems with Multiplicative Noises

Zidong Wang,[a,b,*] Fuwen Yang,[a] Daniel W. C. Ho,[c] and Xiaohui Liu[a]

Abstract

In this paper, the robust variance-constrained H_∞ control problem is considered for uncertain stochastic systems with multiplicative noises. The norm-bounded parametric uncertainties enter into both the system and output matrices. The purpose of the problem is to design a state feedback controller such that, for all admissible parameter uncertainties, 1) the closed-loop system is exponentially mean-square quadratically stable; 2) the individual steady-state variance satisfies given upper bound constraints; and 3) the prescribed noise attenuation level is guaranteed in an H_∞ sense with respect to the additive noise disturbances. A general framework is established to solve the addressed multiobjective problem by using a linear matrix inequality (LMI) approach, where the required stability, the H_∞ characterization and variance constraints are all easily enforced. Within such a framework, two additional optimization problems are formulated, one is to optimize the H_∞ performance, and the other is to minimize the weighted sum of the system state variances. A numerical example is provided to illustrate the effectiveness of the proposed design algorithm.

Keywords

Stability; H_∞ performance; variance constraint; stochastic system; multiplicative noises; linear matrix inequality

I. Introduction

In many engineering control problems, the performance requirements are naturally expressed as the upper bounds on the steady-state variances [1], [9], [10], [13], [16]. The covariance control theory aims to solve the variance-constrained control problems while satisfying other performance indices, such as L_1, H_2, H_∞, pole placement, see e.g. [4], [8], [11], [16]. It has been shown that the covariance control approach is capable of solving multiobjective design problems, which has found applications in dealing with transient responses, round off errors in digital control, residence time/probability in aiming control problems, and stability robustness in the presence of parameter perturbations, see [16]. Such an advantage is based on the fact that several control design objectives, such as stability, time-domain and frequency-domain performance specifications, robustness and pole location, can be directly related to steady-state covariance of the closed-loop systems. Therefore, covariance control theory serves as a practical method for multiobjective control design as well as a foundation for linear system theory.

On the other hand, the control and filtering problems for stochastic systems with multiplicative noises (also called bilinear systems or systems with state-dependent noises) have recently received much attention, since many plants may be modelled by systems with multiplicative noises, and some

This work was supported in part by the Engineering and Physical Sciences Research Council (EPSRC) of the U.K. under Grant GR/S27658/01, the Nuffield Foundation of the U.K. under Grant NAL/00630/G, and the Alexander von Humboldt Foundation of Germany.

[a]Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex, UB8 3PH, United Kingdom.
[b]School of Information Sciences and Technology, Donghua University, Shanghai 200051, China.
[c]Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong.
[*]Corresponding author. Email: Zidong.Wang@brunel.ac.uk, Fax: ++44/1895 251686.

characteristics of nonlinear systems can be closely approximated by models with multiplicative noises rather than by linearized models. So far, there have been several approaches to dealing with the control and filtering problems for stochastic systems with multiplicative noises, which the linear matrix inequality (LMI) approach [18], the Riccati equation approach [20], [22], to name just a few. It is worth emphasizing that, the covariance control problem has been initially studied for stochastic systems with multiplicative noises in [3], [23]. In [3], Chung and Chang developed the coordinate transformation method to assign the state covariance for stochastic continuous-time systems with multiplicative noises. In [23], Yasuda et al. considered covariance control problem for stochastic continuous-time systems with multiplicative noises, where the solvability of an assignability condition and the robustness of covariance controllers were also discussed. However, when there exist modelling uncertainties and external disturbances, the issues of robust control and H_∞ disturbance rejection attenuation will need to be addressed, in addition to the expected steady-state variance constraints. Unfortunately, up to now, the robust H_∞ control problems with variance constraints have not yet been investigated for stochastic systems with multiplicative noises, and remains open and challenging, though similar problem has been studied in [19] for *linear* system.

It is our objective in this paper to propose an LMI approach to solving the robust variance-constrained H_∞ control problem for stochastic systems with both multiplicative noises and norm-bounded parameter uncertainties. We aim to design a state feedback controller such that, for all admissible parameter uncertainties, the closed-loop system is exponentially mean-square quadratically stable, the individual steady-state variance satisfies given upper bound constraints, and the prescribed noise attenuation level is guaranteed in an H_∞ sense with respect to the additive noise disturbances. We will show that all the three requirements can be ideally enforced within a unified LMI framework. In order to demonstrate the flexibility of the proposed framework, we will examine two types of the optimization problems that optimize either the H_∞ performance or the system state variances, and a numerical example is provided to illustrate the effectiveness of the proposed design algorithm.

It is worth pointing out that the work in this paper represents the first attempt to consider multiple performance objectives for stochastic systems with multiplicative noises. These objectives include individual variance constraints, performance robustness and H_∞ disturbance rejection attenuation level. A unified LMI approach is developed to deal with the multiobjective design problem, which is numerically more efficient than the traditional Riccati equation approach [20], [22]. On the other hand, the conditions obtained in this paper are sufficient, and how to reduce the conservatism in the design would be the issue for further research.

The remainder of this paper is organized as follows. In Section II, the robust variance-constrained H_∞ control problem for stochastic systems with both multiplicative noises and norm-bounded parameter uncertainties is formulated. The conditions for stability, H_∞ performance and state variance are expressed in terms of LMI in Section III, An LMI algorithm is developed in Section IV for designing robust variance-constrained H_∞ state feedback controllers with both multiplicative noises and deterministic norm-bounded parameter uncertainty. A numerical example is presented in Section V to show the applicability of the algorithm and some concluding remarks are provided in Section VI.

Notation. The notation used here is fairly standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices, and $\mathbb{I}^+$ stands for the set of nonnegative integers. The notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $\mathrm{Var}\{x_i\}$ means the variance of x_i. $\mathbb{E}\{x\}$ stands for the expectation of stochastic variable x and $\mathbb{E}\{x|y\}$ for the expectation of x conditional on y. The superscript "T" denotes the transpose. $\lambda_{\max}(M)$ stands for the maximum eigenvalue of matrix M. $\mathrm{diag}\{M_1, M_2, ...\}$ denotes a block diagonal matrix whose diagonal blocks are given by M_1, M_2, $\cdots$.

II. Problem Formulation

Consider the following class of *stochastic* discrete-time systems with both *multiplicative noises* and *deterministic* norm-bounded parameter uncertainties:

$$\begin{aligned} x_{k+1} &= (A + H_1 F E + A_s \eta_k) x_k + B_1 w_k + B_2 u_k, \\ z_k &= (C_1 + H_2 F E) x_k + D_{11} w_k + D_{12} u_k, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^r$ is the control input, $z_k \in \mathbb{R}^p$ is the controlled output. The process noise $w_k \in \mathbb{R}^q$ is a zero mean Gaussian white noise sequence with covariance $R > 0$, and the stochastic multiplicative noise $\eta_k \in \mathbb{R}$ is also a zero mean Gaussian white noise sequence but with unity covariance. The real matrices A, A_s, B_1, B_2, C_1, D_{11}, D_{12}, H_1, H_2 and E are known matrices with appropriate dimensions.

The real matrix $F \in \mathbb{R}^{i \times j}$, which could be time-varying, represents the deterministic norm-bounded parameter uncertainty and satisfies

$$FF^T \leq I. \tag{2}$$

The parameter uncertainty F is said to be admissible if it satisfies the condition (2).

Remark 1: The structure of the deterministic uncertainties in (2) has been used in many works concerning robust control and filtering problems, see e.g. [14], [15]. The intensity of the multiplicative noise η_k, which causes the bilinearities or stochastic uncertainties, can be scaled and absorbed in the matrix A_s. Hence, without loss of generality, we could assume that η_k is of unity covariance.

Applying the state feedback control law

$$u_k = K x_k, \tag{3}$$

to the system (1), we obtain the following closed-loop system:

$$\begin{aligned} x_{k+1} &= (A_K + A_s \eta_k) x_k + B_1 w_k, \\ z_k &= C_K x_k + D_{11} w_k, \end{aligned} \tag{4}$$

where K is the state feedback gain, and

$$A_K := A + B_2 K + H_1 F E, \tag{5}$$

$$C_K := C_1 + D_{12} K + H_2 F E. \tag{6}$$

Before giving our design goal, we introduce the following stability concept for the system (4).

Definition 1: The system (4) is said to be *exponentially mean-square quadratically stable* if, with $w_k = 0$, there exist constants $\alpha \geq 1$ and $\beta \in (0,1)$ such that

$$\mathbb{E}\{\|x_k\|^2\} \leq \alpha\beta^k \mathbb{E}\{\|x_0\|^2\}, \quad \forall x_0 \in \mathbb{R}^n, \quad k \in \mathbb{I}^+, \tag{7}$$

for all admissible uncertainties.

The aim in this paper is to design a state feedback controller of the form (3), such that for all admissible deterministic uncertainties, the following three requirements are simultaneously satisfied for the system (4):

(Q1) The system (4) is exponentially mean-square quadratically stable.

(Q2) For a given scalar $\gamma > 0$ and all nonzero w_k and zero initial condition $x_0 = 0$, the controlled output z_k satisfies

$$\sum_{k=0}^{N} \mathbb{E}\{\|z_k\|^2\} \leq \gamma^2 \sum_{k=0}^{N} \mathbb{E}\{\|w_k\|^2\}. \tag{8}$$

(Q3) The individual steady-state state variances satisfy the following constraints:

$$Var\{x_{i,k}\} := \lim_{k \longrightarrow \infty} \mathbb{E}\{x_{i,k} x_{i,k}^T\} < \sigma_i^2, \tag{9}$$

where $x_k = \begin{bmatrix} x_{1,k} & x_{2,k} & \ldots & x_{n,k} \end{bmatrix}^T$, and $\sigma_i^2 > 0$ $(i = 1, 2, ..., n)$ are given scalars specifying the acceptable variance upper bounds obtained from the engineering requirements.

The problem addressed above is referred to as the robust H_∞ control problem with variance constraints.

III. Stability, H_∞ Performance, Variance Analysis

In this section, the multiobjective (stability, H_∞ performance and variance analysis) will be considered for stochastic discrete-time systems with both multiplicative noises and deterministic norm-bounded parameter uncertainties.

A. Stability

Before deriving the stability conditions, two useful lemmas are given as follows.

Lemma 1: Let $V(x_k) = x_k^T P x_k$ be a Lyapunov functional where $P > 0$. If there exist real scalars $\lambda,\ \mu > 0,\ \nu > 0$, and $0 < \psi < 1$ such that both

$$\mu \|x_k\|^2 \leq V(x_k) \leq \nu \|x_k\|^2, \tag{10}$$

and

$$\mathbb{E}\{V(x_{k+1})|x_k\} - V(x_k) \leq \lambda - \psi V(x_k), \tag{11}$$

hold, then the process x_k satisfies that

$$\mathbb{E}\{\|x_k\|^2\} \leq \frac{\nu}{\mu}\|x_0\|^2(1-\psi)^k + \frac{\lambda}{\mu\psi}. \tag{12}$$

Proof: The proof follows a similar line of that of Theorem 2 of [17]. ∎

Lemma 2: Consider a system

$$\xi_{k+1} = \quad (M + N\eta_k)\xi_k \tag{13}$$

where η_k is a zero mean Gaussian white noise sequence, and M, N are constant matrices with appropriate dimensions. If the system (13) is exponentially mean-square stable, i.e., there exist constants $\alpha \geq 1$ and $\beta \in (0, 1)$ such that

$$\mathbb{E}\{\|\xi_k\|^2\} \leq \alpha\beta^k \mathbb{E}\{\|\xi_0\|^2\}, \quad \forall \xi_0 \in \mathbb{R}^n, \quad k \in \mathbb{I}^+, \tag{14}$$

and there exists a symmetric matrix Y satisfying

$$MYM^T - Y + NYN^T < 0, \tag{15}$$

then $Y \geq 0$.

Proof: It follows from (15) that

$$MYM^T - Y + NYN^T = -\Theta \tag{16}$$

for some $\Theta > 0$. Define a functional $W(\xi_k) = \xi_k^T Y \xi_k$. Applying super-Martingale property for the system (13) yields

$$\mathbb{E}\{W(\xi_{k+1})|\xi_k\} - W(\xi_k) = \xi_k^T(MYM^T - Y + NYN^T)\xi_k = -\xi_k^T \Theta \xi_k. \tag{17}$$

Summing (17) from 0 to n with respect to k, we obtain

$$\mathbb{E}(\xi_n^T Y \xi_n) - \xi_0^T Y \xi_0 = -\sum_{k=0}^{n} \xi_k^T \Theta \xi_k. \tag{18}$$

Let $n \to \infty$ in (18). It then follows from the exponential mean-square stability of the system (13) and the fact that

$$\lim_{n\to\infty} \mathbb{E}(\xi_n^T Y \xi_n) \leq \|Y\| \lim_{n\to\infty} \mathbb{E}(\xi_n^T \xi_n)$$

that $\lim_{n\to\infty} \mathbb{E}(\xi_n^T Y \xi_n) = 0$. Hence, we have from (18) that

$$\xi_0^T Y \xi_0 = \sum_{k=0}^{\infty} \xi_k^T \Theta \xi_k \geq 0. \tag{19}$$

Since (19) holds for any non-zero initial state ξ_0, we arrive at the conclusion that $Y \geq 0$. ■

According to Definition 1, we have the following theorem which provides the sufficient and necessary conditions for the exponential quadratic stability of the system (4).

Theorem 1: Given the feedback gain matrix K. The system (4) is exponentially mean-square quadratically stable if and only if, for all admissible uncertainties, there exists a positive definite matrix P satisfying

$$A_K^T P A_K - P + A_s^T P A_s < 0. \tag{20}$$

Proof: The proof of necessary follows directly from [12]. To prove the sufficiency, we define Lyapunov functional $V(x_k) = x_k^T P x_k$, where $P > 0$ is the solution to (20). By using super-Martingale property for the system (4) with $w_k = 0$, we obtain

$$\begin{aligned} & \mathbb{E}\{V(x_{k+1})|x_k\} - V(x_k) \\ = \; & x_k^T A_K^T P A_K x_k + x_k^T \mathbb{E}\{A_s^T P A_s \eta_k^2\} x_k - x_k^T P x_k \\ = \; & x_k^T (A_K^T P A_K - P + A_s^T P A_s) x_k. \end{aligned} \tag{21}$$

We know from (20) that, there must exist a sufficiently small scalar α satisfying $0 < \alpha < \lambda_{\max}(P)$ and

$$A_K^T P A_K - P + A_s^T P A_s < -\alpha I. \tag{22}$$

Therefore, it follows that

$$\mathbb{E}\{V(x_{k+1})|x_k\} - V(x_k) \leq -\alpha x_k^T x_k \leq -\frac{\alpha}{\lambda_{\max}(P)} V(x_k). \tag{23}$$

Then, the proof of the sufficiency follows immediately from Lemma 1. ■

Corollary 1: Given the feedback gain matrix K. The system (4) is exponentially mean-square quadratically stable if and only if, for all admissible uncertainties, there exists a positive definite matrix Q satisfying

$$A_K Q A_K^T - Q + A_s Q A_s^T < 0. \tag{24}$$

Proof: The proof follows easily from Theorem 1, the references [2] and the fact that $\rho(\Phi) = \rho(\Phi^T)$, where Φ is a square matrix and $\rho(\cdot)$ is the spectral radius. ■

B. H_∞ performance

Contrary to the standard H_∞ performance formulation, we shall use the expression (8) to describe the H_∞ performance of the stochastic system, where the expectation operator is utilized on both the controlled output and the disturbance input.

The following lemma, known as Schur Complement Lemma, will be essential in establishing our results in terms of LMIs.

Lemma 3: [2] Given constant matrices Ω_1, Ω_2, Ω_3 where $\Omega_1 = \Omega_1^T$ and $0 < \Omega_2 = \Omega_2^T$, then

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0,$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0,$$

or equivalently

$$\begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^T & \Omega_1 \end{bmatrix} < 0.$$

We are now ready to derive the sufficient conditions for establishing the H_∞-norm performance.

Theorem 2: For a given $\gamma > 0$ and a given feedback gain matrix K, the system (4) is exponentially mean-square quadratically stable and achieves the H_∞-norm constraint (8) for all nonzero w_k, if there exists a positive definite matrix P satisfying

$$\begin{bmatrix} A_K^T P A_K - P + A_s^T P A_s + C_K^T C_K & A_K^T P B_1 + C_K^T D_{11} \\ B_1^T P A_K + D_{11}^T C_K & B_1^T P B_1 - \gamma^2 I + D_{11}^T D_{11} \end{bmatrix} < 0, \tag{25}$$

for all admissible uncertainties.

Proof: It is obvious that (25) implies (20), hence it follows from Theorem 1 that the system (4) is exponentially mean-square quadratically stable.

Next, for any nonzero w_k, it follows from (25) that

$$\begin{aligned} & \mathbb{E}\{V(x_{k+1})|x_k\} - V(x_k) + \mathbb{E}\{z_k^T z_k\} - \gamma^2 \mathbb{E}\{w_k^T w_k\} \\ = \; & x_k^T (A_K^T P A_K - P + A_s^T P A_s) x_k + x_k^T A_K^T P B_1 w_k + w_k^T B_1^T P A_K x_k + w_k^T B_1^T P B_1 w_k \\ & + x_k^T C_K^T C_K x_k + x_k^T C_K^T D_{11} w_k + w_k^T D_{11}^T C_K x_k + w_k^T D_{11}^T D_{11} w_k - \gamma^2 w_k^T w_k \\ = \; & x_k^T (A_K^T P A_K - P + A_s^T P A_s + C_K^T C_K) x_k + x_k^T (A_K^T P B_1 + C_K^T D_{11}) w_k \\ & + w_k^T (B_1^T P A_K + D_{11}^T C_K) x_k + w_k^T (B_1^T P B_1 + D_{11}^T D_{11} - \gamma^2 I) w_k \\ = \; & \begin{bmatrix} x_k \\ w_k \end{bmatrix}^T \begin{bmatrix} A_K^T P A_K - P + A_s^T P A_s + C_K^T C_K & A_K^T P B_1 + C_K^T D_{11} \\ B_1^T P A_K + D_{11}^T C_K & B_1^T P B_1 - \gamma^2 I + D_{11}^T D_{11} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} < 0. \end{aligned} \tag{26}$$

Now, summing (26) from 0 to ∞ with respect to k leads to

$$\sum_{k=0}^{\infty} [\mathbb{E}\{V(x_{k+1})|x_k\} - V(x_k) + \mathbb{E}\{z_k^T z_k\} - \gamma^2 \mathbb{E}\{w_k^T w_k\}] < 0, \tag{27}$$

or

$$\sum_{k=0}^{\infty} \mathbb{E}\{\|z_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{\|w_k\|^2\} + V(x_0) - V(x_\infty). \tag{28}$$

Since $x_0 = 0$ and the system (4) is exponentially mean-square quadratically stable, it is straightforward to see that

$$\sum_{k=0}^{\infty} \mathbb{E}\{\|z_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{\|w_k\|^2\}, \tag{29}$$

which ends the proof. ∎

Note that the inequality (25) is not linear on the closed-loop matrix A_K. In the interest of establishing an LMI framework for the controller design, we now restate Theorem 2 in terms of an LMI as follows.

Theorem 3: For a given $\gamma > 0$ and a given feedback gain matrix K, the system (4) is exponentially mean-square quadratically stable and achieves the H_∞-norm constraint (8) for all nonzero w_k, if there exists a positive definite matrix Q satisfying

$$\begin{bmatrix} -Q & A_K Q & 0 & 0 & B_1 \\ Q A_K^T & -Q & Q A_s^T & Q C_K^T & 0 \\ 0 & A_s Q & -Q & 0 & 0 \\ 0 & C_K Q & 0 & -I & D_{11} \\ B_1^T & 0 & 0 & D_{11}^T & -\gamma^2 I \end{bmatrix} < 0. \tag{30}$$

Proof: Using the Schur Complement Lemma (Lemma 3) twice, we can see that (25) is equivalent to

$$\begin{bmatrix} -P + A_s^T P A_s & 0 & A_K^T & C_K^T \\ 0 & -\gamma^2 I & B_1^T & D_{11}^T \\ A_K & B_1 & -P^{-1} & 0 \\ C_K & D_{11} & 0 & -I \end{bmatrix} < 0, \tag{31}$$

or

$$\begin{bmatrix} -P & 0 & A_K^T & C_K^T & A_s^T \\ 0 & -\gamma^2 I & B_1^T & D_{11}^T & 0 \\ A_K & B_1 & -P^{-1} & 0 & 0 \\ C_K & D_{11} & 0 & -I & 0 \\ A_s & 0 & 0 & 0 & -P^{-1} \end{bmatrix} < 0. \tag{32}$$

Performing *twice* the congruence transformation to (32) by

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \tag{33}$$

we can see that (32) is equivalent to

$$\begin{bmatrix} -P^{-1} & A_K & 0 & 0 & B_1 \\ A_K^T & -P & A_s^T & C_K^T & 0 \\ 0 & A_s & -P^{-1} & 0 & 0 \\ 0 & C_K & 0 & -I & D_{11} \\ B_1^T & 0 & 0 & D_{11}^T & -\gamma^2 I \end{bmatrix} < 0. \tag{34}$$

Let $P = Q^{-1}$ in (34) and then again applying the congruence transformation by $\text{diag}\{I,\ Q,\ I,\ I,\ I\}$, we arrive at (30), and the proof is complete. ■

C. Variance analysis

Define the steady-state covariance by

$$\begin{aligned} \hat{Q} : &= \lim_{k\to\infty} \mathbb{E}\{x_k x_k^T\} \\ &= \lim_{k\to\infty} \mathbb{E}\{\begin{bmatrix} x_{1,k} & x_{2,k} & \dots & x_{n,k} \end{bmatrix} \begin{bmatrix} x_{1,k} & x_{2,k} & \dots & x_{n,k} \end{bmatrix}^T\}. \end{aligned} \tag{35}$$

Obviously, if the system (4) is exponentially mean-square quadratically stable, then in the steady-state, $\hat{Q}$ exists and satisfies the following equation

$$A_K \hat{Q} A_K^T - \hat{Q} + A_s \hat{Q} A_s^T + B_1 R B_1^T = 0. \tag{36}$$

Theorem 4: If there exists a positive definite matrix Q satisfying

$$\begin{bmatrix} -Q & A_K Q & A_s Q & B_1 \\ Q A_K^T & -Q & 0 & 0 \\ Q A_s^T & 0 & -Q & 0 \\ B_1^T & 0 & 0 & -R^{-1} \end{bmatrix} < 0, \tag{37}$$

then the system (4) is exponentially mean-square quadratically stable, and $\hat{Q} \leq Q$.

Proof: We first prove that (37) is equivalent to

$$A_K Q A_K^T - Q + A_s Q A_s^T + B_1 R B_1^T < 0. \tag{38}$$

By using Schur complement Lemma (Lemma 3) to (38), we have

$$\begin{bmatrix} -Q + A_s Q A_s^T + B_1 R B_1^T & A_K \\ A_K^T & -Q^{-1} \end{bmatrix} < 0, \tag{39}$$

$$\Longleftrightarrow \begin{bmatrix} -Q + B_1 R B_1^T & A_K & A_s \\ A_K^T & -Q^{-1} & 0 \\ A_s^T & 0 & -Q^{-1} \end{bmatrix} < 0, \tag{40}$$

$$\Longleftrightarrow \begin{bmatrix} -Q & A_K & A_s & B_1 \\ A_K^T & -Q^{-1} & 0 & 0 \\ A_s^T & 0 & -Q^{-1} & 0 \\ B_1^T & 0 & 0 & -R^{-1} \end{bmatrix} < 0. \tag{41}$$

Performing the congruence transformation to (41) by diag$\{I,\ Q,\ Q,\ I\}$ yields (37). Hence, there exists a matrix $Q > 0$ satisfying (37) if and only if there exists a matrix $Q > 0$ satisfying (38).

Next, it follows directly from (38) and Theorem 1 that the system (4) is exponentially mean-square quadratically stable. Hence, $\hat{Q}$ exists and meets (36).

Subtracting (36) from (38) gives

$$A_K(Q - \hat{Q})A_K^T - (Q - \hat{Q}) + A_s(Q - \hat{Q})A_s^T < 0, \tag{42}$$

which indicates from Lemma 2 that $Q - \hat{Q} \geq 0$. The proof is now completed. ∎

The results provided in the above theorem will be essential for designing the controllers, which guarantee the stability, H_∞ performance and variance constraints for the uncertain stochastic systems with multiplicative noises in the next section.

IV. Robust State Feedback Controller Design

In this section, we will present the solution to the robust H_∞ state feedback controller design problem with variance constraints for the *stochastic* discrete-time systems with both multiplicative noises and *deterministic* norm-bounded parameter uncertainty. That is, we will design the controller that achieves the requirements (Q1), (Q2) and (Q3) described in Section II.

Prior to giving our main results, we recall the following useful lemma.

Lemma 4: (S-procedure) [21] Let $M = M^T$, H and E be real matrices of appropriate dimensions, and F satisfying (2), then

$$M + HFE + E^T F^T H^T < 0, \tag{43}$$

if and only if, there exists a positive scalar ε such that

$$M + \varepsilon H H^T + \frac{1}{\varepsilon} E^T E < 0, \tag{44}$$

or equivalently

$$\begin{bmatrix} M & \varepsilon H & E^T \\ \varepsilon H^T & -\varepsilon I & 0 \\ E & 0 & -\varepsilon I \end{bmatrix} < 0. \tag{45}$$

The following theorem provides an LMI approach to the addressed multiobjective (stability, H_∞ performance and variance constraints) design problem for the uncertain stochastic discrete-time systems with multiplicative noises.

Theorem 5: Given $\gamma > 0$ and $\sigma_i^2 > 0$ $(i = 1, 2, \cdots, n)$. If there exist a positive definite matrix $Q > 0$, a real matrix G and positive scalars ε_1 and ε_2 such that the following set of linear matrix inequalities (LMIs)

$$\begin{bmatrix} -Q & AQ + B_2G & 0 & 0 & B_1 & \varepsilon_1 H_1 & 0 \\ QA^T + G^T B_2^T & -Q & QA_s^T & QC_1^T + G^T D_{12}^T & 0 & 0 & QE^T \\ 0 & A_sQ & -Q & 0 & 0 & 0 & 0 \\ 0 & C_1Q + D_{12}G & 0 & -I & D_{11} & \varepsilon_1 H_2 & 0 \\ B_1^T & 0 & 0 & D_{11}^T & -\gamma^2 I & 0 & 0 \\ \varepsilon_1 H_1^T & 0 & 0 & \varepsilon_1 H_2^T & 0 & -\varepsilon_1 I & 0 \\ 0 & EQ & 0 & 0 & 0 & 0 & -\varepsilon_1 I \end{bmatrix} < 0, \tag{46}$$

$$\begin{bmatrix} -Q & AQ + B_2G & A_sQ & B_1 & \varepsilon_2 H_1 & 0 \\ QA^T + G^T B_2^T & -Q & 0 & 0 & 0 & QE^T \\ QA_s^T & 0 & -Q & 0 & 0 & 0 \\ B_1^T & 0 & 0 & -R^{-1} & 0 & 0 \\ \varepsilon_2 H_1^T & 0 & 0 & 0 & -\varepsilon_2 I & 0 \\ 0 & EQ & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0, \tag{47}$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} Q \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T < \sigma_1^2, \tag{48}$$

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix} Q \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T < \sigma_2^2, \tag{49}$$

$$\vdots$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} Q \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T < \sigma_n^2, \tag{50}$$

are feasible, then there exists a state feedback controller of the form (3) such that three requirements (Q1), (Q2) and (Q3) are satisfied for all admissible deterministic uncertainties. Moreover, the desired controller (3) can be determined by

$$K = GQ^{-1}. \tag{51}$$

Proof: We first prove that (30) holds if and only if (46) holds, and (37) is true if and only if (47) is true. To do this, we rewrite (30) in the form of (43) as follows:

$$\begin{bmatrix} -Q & (A+B_2K)Q & 0 & 0 & B_1 \\ Q(A+B_2K)^T & -Q & QA_s^T & Q(C_1+D_{12}K)^T & 0 \\ 0 & A_sQ & -Q & 0 & 0 \\ 0 & (C_1+D_{12}K)Q & 0 & -I & D_{11} \\ B_1^T & 0 & 0 & D_{11}^T & -\gamma^2 I \end{bmatrix}$$

$$+\begin{bmatrix} H_1 \\ 0 \\ 0 \\ H_2 \\ 0 \end{bmatrix} F \begin{bmatrix} 0 & EQ & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & EQ & 0 & 0 & 0 \end{bmatrix}^T F^T \begin{bmatrix} H_1 \\ 0 \\ 0 \\ H_2 \\ 0 \end{bmatrix}^T < 0. \tag{52}$$

In order to cope with the uncertainty factor F, we apply Lemma 4 to (52) and then have the conclusion that, (52) holds if and only if there exists a positive scalar ε_1 such that the following LMI holds:

$$\begin{bmatrix} -Q & (A+B_2K)Q & 0 & 0 & B_1 & \varepsilon_1 H_1 & 0 \\ Q(A+B_2K)^T & -Q & QA_s^T & Q(C_1+D_{12}K)^T & 0 & 0 & QE^T \\ 0 & A_sQ & -Q & 0 & 0 & 0 & 0 \\ 0 & (C_1+D_{12}K)Q & 0 & -I & D_{11} & \varepsilon_1 H_2 & 0 \\ B_1^T & 0 & 0 & D_{11}^T & -\gamma^2 I & 0 & 0 \\ \varepsilon_1 H_1^T & 0 & 0 & \varepsilon_2 H_1^T & 0 & -\varepsilon_1 I & 0 \\ 0 & EQ & 0 & 0 & 0 & 0 & -\varepsilon_1 I \end{bmatrix} < 0. \tag{53}$$

Similarly, we rewrite (37) in the form of (43) as follows:

$$\begin{bmatrix} -Q & (A+B_2K)Q & A_sQ & B_1 \\ Q(A+B_2K)^T & -Q & 0 & 0 \\ QA_s^T & 0 & -Q & 0 \\ B_1^T & 0 & 0 & -R^{-1} \end{bmatrix} +$$

$$\begin{bmatrix} H_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} F \begin{bmatrix} 0 & EQ & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & EQ & 0 & 0 \end{bmatrix}^T F^T \begin{bmatrix} H_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0, \tag{54}$$

and apply Lemma 4 again to (54), we know that (54) holds if and only if there exists a positive scalar

ε_2 such that the following LMI holds:

$$\begin{bmatrix} -Q & (A+B_2K)Q & A_sQ & B_1 & \varepsilon_2 H_1 & 0 \\ Q(A+B_2K)^T & -Q & 0 & 0 & 0 & QE^T \\ QA_s^T & 0 & -Q & 0 & 0 & 0 \\ B_1^T & 0 & 0 & -R^{-1} & 0 & 0 \\ \varepsilon_2 H_1^T & 0 & 0 & 0 & -\varepsilon_2 I & 0 \\ 0 & EQ & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0. \tag{55}$$

Let

$$G = KQ, \tag{56}$$

it is straightforward to see that (53) is identical to (46), and (55) is identical to (47).

To this end, it follows immediately from Theorem 3 and Theorem 4 that, with the feedback gain matrix K given in (56) (or (51)), the closed-loop system (4) is exponentially mean-square quadratically stable, the H_∞-norm constraint (8) is achieved for all nonzero w_k, and the steady-state covariance $\hat{Q}$ exists and satisfies $\hat{Q} \leq Q$. In other words, the requirements (Q1) and (Q2) are met. Next, considering the definitions (9) and (35), we can obtain that

$$\begin{aligned} Var\{x_{i,k}\} &= \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \hat{Q} \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T \\ &\leq \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} Q \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T. \end{aligned} \tag{57}$$

Therefore, the n LMIs given in (48)-(50) indicates that the requirement (Q3) is also met. This completes the proof. ■

Remark 2: The robust H_∞ controller with variance constraints can be obtained by solving the $n+2$ LMIs described in (46)-(50) in Theorem 5. Such a set of LMIs can be solved efficiently via the interior point method [2]. Note that LMIs (46)-(50) are affine in the scalar positive parameters ε_1, ε_2, the positive definite matrix Q and a real matrix G. Hence, they can be defined as LMI variables in order to increase the solvability while reducing the conservatism with respect to the parameter uncertainties.

Up to now, by means of an LMI approach, we have proposed the controller design procedure which guarantees the simultaneous satisfaction of the requirements (Q1), (Q2) and (Q3). In order to show the flexibility of the proposed LMI framework, we now discuss the following two optimization problems:

(P1) *The optimal variance-constrained H_∞ control problem for uncertain stochastic systems with multiplicative noises*:

$$\min_{Q>0,\ G,\ \varepsilon_1>0,\ \varepsilon_2>0} \gamma \quad \text{subject to} \ \ (46)-(50) \ \text{for given } \sigma_1^2,\ \sigma_2^2,\ \cdots,\ \sigma_n^2 \tag{58}$$

(P2) *The minimum weighted variance H_∞ control problem for uncertain stochastic systems with multiplicative noises*:

$$\min_{Q>0,\ G,\ \varepsilon_1>0,\ \varepsilon_2>0} \alpha_1\sigma_1^2 + \alpha_2\sigma_2^2 + \cdots + \alpha_n\sigma_n^2 \quad \text{subject to} \ \ (46)-(50) \ \text{for given } \gamma \tag{59}$$

where α_i $(i = 1, 2, ..., n)$ are given weighting coefficients for variances and satisfy $\sum_{i=1}^{n} \alpha_i = 1$.

Remark 3: In many engineering applications, the performance constraints on the steady-state variances are often specified *a priori*. That is, the upper bounds σ_1^2, σ_2^2, $\cdots$, σ_n^2 can be prescribed. Hence, in addition to the individual variance constraints, the problem (P1) will help exploit the design freedom to meet the optimal H_∞ performance. This is certainly attractive because the addressed multiobjective problem can be solved while a local optimal performance can also be achieved, and the computation is efficient by using the Matlab LMI toolbox.

Remark 4: In the problem (P2), the variances are weighted against their importance in the real engineering systems, and then the feedback gain is sought so as to minimize the weighted sum of the variance. We could, of course, optimize the variances of individual system states by setting the weighting coefficients of certain variances to zeros. Therefore, the problem (P2) is flexible in terms of both the engineering requirements and the computational efficiency.

V. A Numerical Example

Consider an uncertain stochastic discrete-time system with multiplicative noises described by (1) with the model parameters given as follows:

$$A = \begin{bmatrix} -0.1 & 0.3 & -0.2 \\ 0 & -0.25 & 0.1 \\ 0.1 & 0 & 0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.3 \\ 0 \\ 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}, \quad D_{11} = 1, \quad D_{12} = 2, \quad A_s = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.3 \\ 0.2 \\ 0 \end{bmatrix}, \quad H_2 = 0; \quad E = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad R = 1.$$

Now, let us examine the following three cases.

Case 1: $\gamma^2 = 1.8$, $\sigma_1^2 = 0.5$, $\sigma_2^2 = 0.5$, $\sigma_3^2 = 0.2$.

This case is exactly concerned with the addressed robust H_∞ control problem with specified variance constraints, hence can be tackled by using Theorem 5 with $n = 3$. By employing the Matlab LMI toolbox, the solution is given by

$$Q = \begin{bmatrix} 0.2699 & -0.1592 & -0.0710 \\ -0.1592 & 0.4995 & 0.1866 \\ -0.0710 & 0.1866 & 0.1376 \end{bmatrix}, \quad G = \begin{bmatrix} -0.1348 & 0.0911 & 0.0215 \end{bmatrix},$$

$$\varepsilon_1 = 0.5729, \quad \varepsilon_2 = 0.5601, \quad K = \begin{bmatrix} -0.4968 & 0.1243 & -0.2683 \end{bmatrix}.$$

Case 2: $\sigma_1^2 = 0.5$, $\sigma_2^2 = 0.5$, $\sigma_3^2 = 0.2$.

In this case, we wish to design the controller which minimizes the H_∞ performance under the variance constraints specified above. That is, we want to solve the problem (P1). Solving the optimization problem (58) using LMI toolbox yields the optimal value $\gamma_{\text{opt}} = 1.6583$ and

$$Q = \begin{bmatrix} 0.4624 & -0.1193 & -0.1075 \\ -0.1193 & 0.4982 & 0.1597 \\ -0.1075 & 0.1597 & 0.1552 \end{bmatrix}, \quad G = \begin{bmatrix} -0.1745 & 0.1041 & 0.0223 \end{bmatrix},$$
$$\varepsilon_1 = 1.6594, \quad \varepsilon_2 = 1.2316, \quad K = \begin{bmatrix} -0.4047 & 0.2325 & -0.3757 \end{bmatrix}.$$

Case 3: $\gamma^2 = 1.8$, $\alpha_1 = 0.3$; $\alpha_2 = 0.4$; $\alpha_3 = 0.3$.

We now deal with the problem (P2). Solving the optimization problem (59), we obtain the minimum individual variance values $\sigma^2_{1\text{min}} = 0.3020$, $\sigma^2_{2\text{min}} = 0.3410$, $\sigma^2_{3\text{min}} = 0.0636$, and

$$Q = \begin{bmatrix} 0.3013 & -0.0830 & -0.0232 \\ -0.0830 & 0.3405 & 0.1059 \\ -0.0232 & 0.1059 & 0.0630 \end{bmatrix}, \quad G = \begin{bmatrix} -0.1215 & 0.0667 & 0.0168 \end{bmatrix},$$
$$\varepsilon_1 = 1.1133, \quad \varepsilon_2 = 0.9002, \quad K = \begin{bmatrix} -0.3734 & 0.1360 & -0.0997 \end{bmatrix}.$$

Remark 5: Within the LMI framework developed in this paper, we can show that there is some trade-off that can be used for satisfying specific performance requirements. For example, the H_∞ performance will be improved if the variances constraints become more relaxed (larger). Also, if the value of the H_∞ performance constraint is allowed to be increased, then the steady-state variances can be further reduced. Hence, the proposed approach allows much flexibility in making compromise between the variances and the H_∞ performance, while the essential multiple objectives can all be achieved simultaneously.

VI. Conclusions

In this paper, a robust H_∞ controller with variance constraints has been designed for a class of stochastic systems with both multiplicative noises and norm-bounded parameter uncertainties. A general framework for solving this problem is established using an LMI approach in conjunction with stability, H_∞ optimization characterization and variance constraints. Two types of the optimization problems have been proposed by either optimizing H_∞ performance or the system state variances. Sufficient conditions have been derived in terms of a set of feasible LMIs. We point out that our method can be extended to the output feedback case, and different representations of uncertainties can also be considered such as those in [5], [6], [7]. These are possibly the topics of our future research.

References

[1] K. Boukas, Stabilization of stochastic nonlinear hybrid systems, *Int. J. Innovative Computing, Information and Control*, vol. 1, no. 1, pp. 131-141, 2005.

[2] S. Boyd, L. E. Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics, Philadelphia, 1994.

[3] H. -Y. Chung and W. -J. Chang, Constrainted variance design for bilinear stochastic continuous systems, *IEE Proceedings-D: Control Theory and Applications*, vol. 138, no. 12, pp. 145-150, 1991.

[4] E. G. Jr. Collins and R. E. Skelton, A theory of state covariance assignment for discrete systems, *IEEE Trans. Automat. Control*, vol. 32, no. 1, pp. 35-41, 1987.

[5] H. Gao and C. Wang, A delay-dependent approach to robust H_∞ filtering for uncertain discrete-time state-delayed systems, *IEEE Trans. Signal Processing*, vol. 52, no. 6, pp. 1631-1640, 2004.

[6] H. Gao, J. Lam, L. Xie and C. Wang, New approach to mixed H_2/H_∞ filtering for polytopic discrete-time systems, *IEEE Trans. Signal Processing*, vol. 53, no. 8, pp. 3183-3192, 2005.

[7] H. Gao, J. Lam, C. Wang and Y. Wang, Delay-dependent output-feedback stabilisation of discrete-time systems with time-varying state delay, *IEE Proc. Control Theory and Applications*, vol. 151, no. 6, pp. 691-698, 2004.

[8] A. F. Hotz and R. E. Skelton, A covariance control theory, *Int. J. Control*, vol. 46, no. 1, pp. 13-32, 1987.

[9] B. Huang, S. Shah and E. Kwok, Good, bad or optimal? Performance assessment of multivariable processes, *Automatica*, vol. 33, no. 6, pp. 1175-1183, 1997.

[10] B. Huang, Minimum variance control and performance assessment of time-variant processes, *J. Process Control*, vol. 12, no. 6, pp. 707-719, 2002.

[11] Y. S. Hung and F. Yang, Robust H_∞ filtering with error variance constraints for uncertain discrete time-varying systems with uncertainty, *Automatica*, vol. 39, no. 7, pp. 1185-1194, 2003.

[12] T. Morozan, Stabilization of some stochastic discrete-time control systems, *Stochastic Anal. Appl.*, vol. 1, pp. 89-116, 1983.

[13] H. Ruan, E. Yaz and T. Zhai, Current output filter for state estimation of nonlinear stochastic systems with linear measurement, *Int. J. Innovative Computing, Information and Control*, vol. 1, no. 2, pp. 277-287, 2005.

[14] P. Shi, E. K. Boukas, Y. Shi and R. K. Agarwal, Optimal guaranteed cost control of uncertain discrete time-delay systems, *J. Computational and Applied Mathematics*, vol. 157, no. 2, pp. 435-451, 2003

[15] P. Shi, M. Karan, C. Kaya, Robust Kalman filter design for Markovian jump linear systems with norm-bounded unknown nonlinearities, *Circuits, Systems & Signal Processing*, vol. 24, no. 2, pp. 135-150, 2005.

[16] R. E. Skelton, T. Iwasaki and K. Grigoriadis, *A unified approach to linear control design*, Taylor & Francis, London, 1997.

[17] T. J. Tarn and Y. Rasis, Observers for nonlinear stochastic systems, *IEEE Trans. Automat. Control*, vol. 21, no. 6, pp. 441-447, 1976.

[18] F. Wang and V. Balakrishnan, Robust steady-state filtering for systems with deterministic and stochastic uncertainties, *IEEE Trans. Signal Processing*, vol. 51, no. 10, pp. 2550-2558, 2003.

[19] Z. Wang and H. Unbehauen, Robust H_∞ control for systems with time-varying parameter uncertainty and variance constraints, *Cybernetics and Systems: An International Journal*, vol. 31, no. 2, pp. 175-191, 2000.

[20] Z. Wang and H. Qiao, Robust filtering for bilinear uncertain stochastic discrete-time systems, *IEEE Trans. Signal Processing*, vol. 50, no. 3, pp. 560-567, 2002.

[21] L. Xie, Output feedback H_∞ control of systems with parameter uncertainty, *Int. J. Control*, vol. 63, no. 4, pp. 741-750, Mar. 1996.

[22] F. Yang, Z. Wang and Y. S. Hung, Robust Kalman filtering for discrete time-varying uncertain systems with multiplicative noise, *IEEE Trans. Automat. Control*, vol. 47, no. 7, pp. 1179-1183, 2002.

[23] K. Yasuda, S. Kherat, R. E. Skelton and E. Yaz, Covariance control and robustness of bilinear systems, in *Proc. IEEE Conf. Decision Contr.*, Honolulu, Hawaii, pp. 1421-1425, 1990.