Robust H_∞ Filtering for Networked Systems with Multiple State Delays

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Abstract

In this paper, a new robust H_∞ filter design problem is studied for a class of networked systems with multiple state-delays. Two kinds of incomplete measurements, namely, measurements with random delays and measurements with stochastic missing phenomenon, are simultaneously considered. Such incomplete measurements are induced by the limited bandwidth of communication networks, and are modeled as a linear function of a certain set of indicator functions that depend on the same stochastic variable. Attention is focused on the analysis and design problems of a full-order robust H_∞ filter such that, for all admissible parameter uncertainties and all possible incomplete measurements, the filtering error dynamics is exponentially mean-square stable and a prescribed H_∞ attenuation level is guaranteed. Some recently reported methodologies, such as delay-dependent and parameter-dependent stability analysis approaches, are employed to obtain less conservative results. Sufficient conditions, which are dependent on the occurrence probability of both the random sensor delay and missing measurement, are established for the existence of the desired filters in terms of certain linear matrix inequalities (LMIs). When these LMIs are feasible, the explicit expression of the desired filter can also be characterized. Finally, numerical examples are given to illustrate the effectiveness and applicability of the proposed design method.

Keywords

Robust H_∞ filtering; networked systems; random communication delays; measurement missing; delay-probability-dependent; parameter-dependent; LMIs

I. Introduction

The past few years have witnessed rapid developments in network technologies. As a result, the feedback control loop for more and more control systems is based on a network. This kind of control systems are called networked control systems (NCSs) [3, 30]. In networked systems, serial communication networks are used to exchange information (reference input, plant output, control input, etc.) among control system components (sensors, controller, actuators, etc.). Networked systems have many advantages, such as low cost, reduced weight and power requirements, simple installation and maintenance, and high reliability. Therefore, increasing attention has been paid to the study of networked systems [3, 14, 15, 21, 30]. It should be pointed out that, in NCSs, since the signals are transmitted over the communication network of limited bandwidth, network-induced delays and data dropout are always inevitable, which makes the analysis and design of networked systems complicated. Conventional control theories with many ideal assumptions, such as synchronized control and non-delayed sensing and actuation, must be re-evaluated before they can be applied to networked systems.

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It has now been well known that the existence of time delays is commonly encountered in many dynamic systems, and time delay has become an important source of instability and performance degradation [9,16,17]. Up to now, many researchers have studied the stability and controller design problems for networked systems in the presence of network-induced delays [10,27,28]. In [12], by introducing buffers at the controller nodes and actuator nodes and making the buffers longer than the worst-case delay, the networked-induced delays have been made time-invariant. In [30], the stability of the NCSs has been analyzed by a hybrid system approach when the network-induced delay is deterministic. Since network delays are usually random and time-varying by nature, recently, the network-induced delays have been modeled in various *probabilistic* ways. For example, in [15], time delays have been assumed to be varying in a random fashion and have statistically mutually independent transfer-to-transfer probability distribution. In [29], network-induced delays have been modeled as two Markov chains, and the resulting closed-loop systems are jump linear systems with two modes. In [20], the random communication delays have been considered as white in nature with known probability distributions. It should be pointed out that the binary random delay has gained particular research interests because of its simplicity and practicality in describing network-induced delays [26], where the binary switching sequence is viewed as a Bernoulli distributed white sequence taking on values of 0 and 1, see [23,25].

Another problem in networked systems is the data missing (dropout) phenomenon [11]. When there is a network connecting the sensor and the controller or filter in a system, the measurement missing phenomenon should be taken into consideration. So far, there are generally three approaches to describe the missing measurement or data dropout. The first approach is to describe the data missing as a binary switching sequence that is specified by a conditional probability distribution in measurement equation, and the binary switching sequence is viewed as a Bernoulli distributed white sequence taking on values of 0 and 1 [24]. The second approach is to use a discrete-time linear system with Markovian jumping parameter to represent random packet-loss model for the network [19]. The third method is to replace the missing data by zeros and then construct an incompleteness matrix in the measurement [18].

Most of the aforementioned references have dealt with the stability or stabilization problems when the effects of network-induced delay and/or data dropouts being taken into account, see [10,10,11,21,28–30] and the references therein. On the other hand, as a branch of state estimation, the H_∞ filtering problem has gained persistent attention even since it was first introduced in late 80's [4]. Furthermore, recognizing that modeling errors are inevitable and time delays are commonly encountered in a variety of dynamical systems, up to now, many researchers have investigated the robust H_∞ filtering problems for systems with various parameter uncertainties and/or delays [5–8,16] by means of the linear matrix inequality (LMI) as well as Riccati-like equation approaches. Even though, little work has been done on the filtering problem for NCSs. Recently, H_∞ filter problem has been studied with the data missing or measurement delay [23,24]. So far, to the best of the authors' knowledge, the H_∞ filtering problems for NCSs with simultaneous data missing and measurement delays have not been fully investigated and very few corresponding results have been available in the literature, which motivates the present study.

In this paper, we are concerned with the design problem of robust H_∞ filters for a class of networked systems. Fig. 1 illustrates a typical diagram for the information flows of networked filtering, where multiple sensors send data to the filter over a common network. For simplicity, we will only deal with the single sensor case, but we declare that the extension to multiple sensor case is not difficult. The observed data is transmitted over communication network. We consider both the network-induced delays and the data missing phenomenon in the measurement equation. The system has multiple state-delays and the uncertain parameters are assumed to reside in a convex polytope. Indicator functions are employed to provide a unified representation to describe stochastic data missing and random measurement delays, which is simple yet efficient. A sufficient condition for the existence of a feasible solution to the problem is derived, which guarantees that the filtering error system is exponentially mean-square stable and a prescribed H_∞ attenuation level is achieved, for all possible missing measurements, all possible measurement delays and all admissible parameter uncertainties. A novel Lyapunov functional is proposed to provide delay-probability-dependent stability criteria, and slack matrices are introduced to make Lyapunov matrices depend on uncertain parameters. These two manipulations significantly reduce the possible conservatism caused in the filter design. An LMI approach is developed to design the expected filters, and the filter parameters are then determined. Numerical examples are provided to demonstrate the usefulness of the present methods.

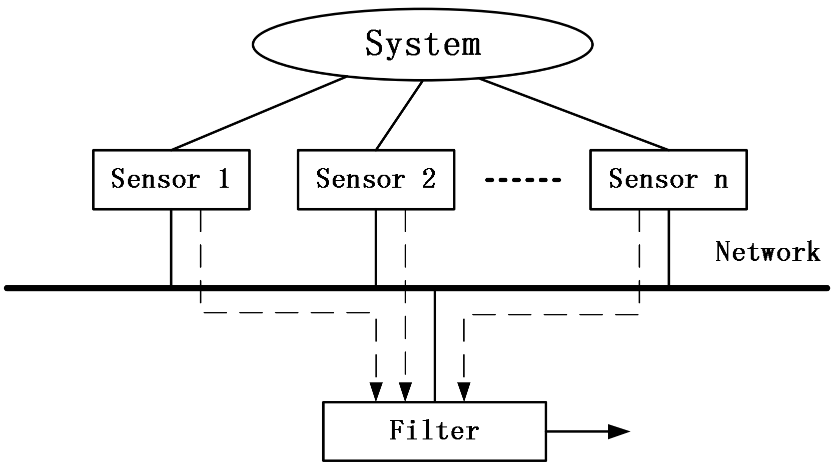

Fig. 1. Filtering for networked systems.

The contribution of this paper can be summarized as follows: (i) a new unified representation describing data missing and measurement delays simultaneously is proposed, which is simple yet efficient; (ii) robust H_∞ filtering for a class of networked systems with multiple state-delays is considered; (iii) some recently appeared results are incorporated to obtain parameter-dependent and delay-probability-dependent filter design results, which are less conservative than traditional ones. The remainder of this paper is organized as follows. In the next section, the robust H_∞ filtering problem for a class of NCSs with multiple state-delays is formulated, and a new representation describing data missing and measurement delays simultaneously is provided. In Section III, the robust H_∞ filter analysis result is proposed, and the robust H_∞ filter design problem is dealt with in Section IV. Numerical examples are given in Section V and some concluding remarks are provided in Section VI.

Notation. The notations used throughout the paper are fairly standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. $P > 0$ (respectively, $P < 0$) means that P is real symmetric and positive definite (respectively, negative definite). The subscript "T" denotes the matrix transpose. $Pr\{\cdot\}$ represents the occurrence probability of the event "$\cdot$", and when x and y are both stochastic variables, $\mathbb{E}\{x\}$ and $\mathbb{E}\{x|y\}$ stand for the expectation of x and the expectation of x conditional on y, respectively. $I_{\{\diamond\}}$ stands for the indicator function, i.e., $I_{\{\diamond\}} = 1$ if and only if $\diamond$ is true, otherwise, $I_{\{\diamond\}} = 0$. $l_2[0,\infty)$ is the space of all square-summable vector functions over $[0,\infty)$, and $\|x\|$ is the standard l_2 norm of x, i.e., $\|x\| = (x^T x)^{1/2}$, and $\mathbb{Z}^+$ stands for the set of nonnegative integers. In symmetric block matrices, we use "$*$" to represent a term that is induced by symmetry, and $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix.

The notation $\text{diag}_q\{\star\}$ is employed to stand for $\text{diag}\{\underbrace{\star\ \cdots\ \star}_{q}\}$, Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations and, sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. Problem Formulation and Preliminaries

Consider the following class of uncertain discrete-time linear system with multiple delays in the state:

$$\begin{cases} x_{k+1} = A_0 x_k + \sum_{j=1}^{q} A_j x_{k-d_j} + B w_k, \\ \quad z_k = H_0 x_k + \sum_{j=1}^{q} H_j x_{k-d_j} + E w_k, \\ \quad x_k = \varphi_k, \quad k = -d_q, -d_q+1, \ldots, 0, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector; $z_k \in \mathbb{R}^m$ is the signal to be estimated; $w_k \in \mathbb{R}^s$ is the disturbance noise belonging to $l_2[0,\ \infty)$; $d_j \in \mathbb{Z}^+$ $(j = 1, \ldots, q)$ are known constant time delays. Without losing generality, we assume that $d_1 < d_2 < \ldots < d_q$ for simplicity. φ_k is a given real initial sequence on $[-d_q, 0]$.

The measurements, which may contain random communication delays and stochastic data missing, are described by

$$y_k = I_{\{\tau_k=0\}} C_0 x_k + \sum_{j=1}^{q} I_{\{\tau_k=d_j\}} C_j x_{k-d_j} + D w_k, \tag{2}$$

where $y_k \in \mathbb{R}^l$ is the measured output vector and w_k is defined in (1). $I_{\{\tau_k=0\}}$ and $I_{\{\tau_k=d_j\}}$ are the indicator functions with $\mathbb{E}\{I_{\{\tau_k=0\}}\} = Pr\{\tau_k = 0\} = p_0$ and $\mathbb{E}\{I_{\{\tau_k=d_j\}}\} = Pr\{\tau_k = d_j\} = p_j$, where p_j $(0 \le j \le q)$ are known positive scalars and $\sum_{j=0}^{q} p_j \le 1$. τ_k is a stochastic variable used to determine, at time k, how large the occurred delay could be and the possibility of data missing. Similar to [23,24,26], we assume the sequence of τ_k is mutually independent.

Assumption 1: All the delays, which including the state delays in (1) and the random communication delays in (2), are assumed to be finite and have an upper bound.

Remark 1: Assumption 1 is natural since, 1) for state-delays, there are always bounds in real systems; 2) for network-induced delays, the network will drop the data that fail to arrive the filter in a finite time [30], which, in this paper, can be considered as the measurement missing.

Remark 2: Note that the filter node of the networked system is supposed to be time-driven. This means that the filter starts the calculation periodically at the sampling time of the system, so the network-induced delays can be regarded as integers. The delays described in (1) and those in (2) are essentially different. The former are the inherent state-delays in the system, and are not affected by the communication channel, while the latter are the network-induced *random* delay via a communication channel and are dependent on the network load [14,15]. Since these two kinds of delays may belong to different subsets of the set of natural numbers, by defining $\{d_j\}$ $(j = 1, 2, \ldots, q)$ as the union of the two subsets and selecting some of A_j or C_j $(j = 1, 2, \ldots, q)$ as 0 properly, we can always assume that these two kinds of delays take values from the same set.

Remark 3: The measurement described in (2) is new, which can be regarded as a substantial extension of the delayed sensor model in [23]. The main advantage of the proposed description (2) is that it can provide a unified representation to account for both the random communication delays and stochastic data missing. Specifically, if $\sum_{j=0}^{q} p_j < 1$, we can confirm that the measurements arrive at a certain time with a probability $\sum_{j=0}^{q} p_j$ irrespective of the existence of communication delays, and the measurements are missing with the probability $1 - \sum_{j=0}^{q} p_j$. If $\sum_{j=0}^{q} p_j = 1$, there will be no missing phenomenon. Furthermore, (2) can also easily describe the delays longer than 1, which is hard to be considered using the measurement mode in [23]. Our results cover the measurement mode in [23] by imposing $p_0 + p_1 = 1$, and $p_j = 0$, $\forall 2 \leq j \leq q$.

The system matrices have appropriate dimensions and are assumed to be uncertain but belong to a known convex compact set of a polytopic type, i.e.,

$$\Omega := (A_0, \ldots, A_q, B, C_0, \ldots, C_q, D, H_0, \ldots, H_q, E,) \in \mathfrak{R}, \tag{3}$$

where $\mathfrak{R}$ is a given convex polyhedral domain described by v vertices:

$$\mathfrak{R} := \left\{ \Omega(\lambda) | \Omega(\lambda) = \sum_{i=1}^{v} \lambda_i \Omega_i; \sum_{i=1}^{v} \lambda_i = 1, \lambda_i \geq 0 \right\}, \tag{4}$$

and $\Omega_i := (A_{0i}, \ldots, A_{qi}, B_i, C_{0i}, \ldots, C_{qi}, D_i, H_{0i}, \ldots, H_{qi}, E_i)$ denotes the ith vertex of the polytope.

Consider a full-order filter of the form

$$\begin{cases} \hat{x}_{k+1} = G\hat{x}_k + Ky_k, \\ \hat{z}_k = L\hat{x}_k, \end{cases} \tag{5}$$

where $\hat{x}_k$ is the filter state vector, $\hat{z}_k$ is an estimation for z_k, and G, K, L are filter parameters to be determined.

By defining

$$\eta_k = \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}, \quad \tilde{z}_k = z_k - \hat{z}_k, \tag{6}$$

we have the filtering error system as follows:

$$\begin{cases} \eta_{k+1} = \tilde{A}_0 \eta_k + (I_{\{\tau_k = 0\}} - p_0) \bar{A}_0 \eta_k + \sum_{j=1}^{q} \tilde{A}_j Z \eta_{k-d_j} + \sum_{j=1}^{q} (I_{\{\tau_k = d_j\}} - p_j) \bar{A}_j Z \eta_{k-d_j} + \tilde{B} w_k, \\ \tilde{z}_k = \tilde{C}_0 \eta_k + \sum_{j=1}^{q} \tilde{C}_j Z \eta_{k-d_j} + \tilde{D} w_k, \end{cases} \tag{7}$$

where

$$\tilde{A}_0 := \begin{bmatrix} A_0 & 0 \\ p_0 K C_0 & G \end{bmatrix}, \quad \bar{A}_0 := \begin{bmatrix} 0 & 0 \\ K C_0 & 0 \end{bmatrix}, \quad \tilde{A}_j := \begin{bmatrix} A_j \\ p_j K C_j \end{bmatrix}, \tag{8}$$

$$\bar{A}_j := \begin{bmatrix} 0 \\ K C_j \end{bmatrix}, \quad \tilde{B} := \begin{bmatrix} B \\ K D \end{bmatrix}, \quad \tilde{C}_0 := \begin{bmatrix} H_0 & -L \end{bmatrix}, \tag{9}$$

$$\tilde{C}_j := H_j, \quad \tilde{D} := E, \quad Z := \begin{bmatrix} I & 0 \end{bmatrix}. \tag{10}$$

Considering the existence of the stochastic variables τ_k, we introduce the definition of stochastic stability in the mean-square sense for the filtering error system.

Definition 1: [22] The filtering error system (7) is said to be exponentially mean-square stable if, with $w_k = 0$, for any initial conditions, there exist constants $\alpha > 0$ and $\kappa \in (0, 1)$ such that

$$\mathbb{E}\{\|\eta_k\|^2\} \leqslant \alpha \kappa^k \sup_{-d_q \leq i \leq 0} \mathbb{E}\{\|\eta_i\|^2\}, \quad k \in \mathbb{Z}^+. \tag{11}$$

Assumption 2: The system (1) is assumed to be exponentially mean-square stable for the whole uncertain domain (4).

Remark 4: *Assumption 2* is a prerequisite for the filtering error system (7) to be exponentially mean-square stable. Since the filter (5) can't affect the state of the original system and x_k is one part of η_k, so the exponentially mean-square stable of x_k is a necessary condition of the exponentially mean-square stable of η_k.

With Definition 1, we aim to design the filter of the form (5) for the system (1) such that for all admissible parameter uncertainties and all possible sensor delays as well as data missing, the filtering error system (7) satisfies the following requirements (R1) and (R2) simultaneously:
(R1) The filtering error system (7) is exponentially mean-square stable.
(R2) Under the zero-initial condition, the filtering error $\tilde{z}_k$ satisfies

$$\sum_{k=0}^{\infty}\mathbb{E}\{\|\tilde{z}_k\|^2\} \leqslant \gamma^2\sum_{k=0}^{\infty}\mathbb{E}\{\|w_k\|^2\} \tag{12}$$

for all nonzero w_k, where $\gamma > 0$ is a prescribed scalar.

III. Robust H_∞ Filtering Analysis

In this section, we shall discuss the robust H_∞ performance analysis result for the filtering error system (7) with parameter-dependent and delay-probability-dependent approach, in the light of which, the filter design problem can be dealt with in the next section. The following Lemma will be useful in deriving our main results in the sequel.

Lemma 1: [13] Assume that $a \in \mathbb{R}^{n_a}$, $b \in \mathbb{R}^{n_b}$ and $N \in \mathbb{R}^{n_a\times n_b}$. Then, for any matrices $X \in \mathbb{R}^{n_a\times n_a}$, $Y \in \mathbb{R}^{n_a\times n_b}$, and $Z \in \mathbb{R}^{n_b\times n_b}$ satisfying

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \geq 0,$$

the following holds:

$$-2a^T Nb \leq \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X & Y-N \\ Y^T-N^T & Z \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}. \tag{13}$$

Lemma 2: Let $\{\tau_k, k \geq 0\}$ be a random sequence taking value in the finite set $\{0, d_1, d_2, \ldots, d_q, d_{lost}\}$. Suppose that, for any k, the distribution law of τ_k is given as $Pr\{\tau_k = 0\} = p_0$, $Pr\{\tau_k = d_j\} = p_j$ $(1 \leq j \leq q)$, and $Pr\{\tau_k = d_{lost}\} = 1 - \sum_{j=0}^{q} p_j$. Then, for any $0 \leq i \leq q$ and $0 \leq j \leq q$, we have

$$\mathbb{E}\left\{(I_{\{\tau_k=d_i\}} - p_i)(I_{\{\tau_k=d_j\}} - p_j)\right\} = \begin{cases} p_i(1-p_i), & i = j \\ -p_ip_j, & i \neq j \end{cases} \tag{14}$$

Proof: It can be calculated that

$$\begin{aligned}\mathbb{E}\left\{(I_{\{\tau_k=d_i\}} - p_i)(I_{\{\tau_k=d_j\}} - p_j)\right\} &= \mathbb{E}\left\{I_{\{\tau_k=d_i\}}I_{\{\tau_k=d_j\}} - I_{\{\tau_k=d_i\}}p_j - p_iI_{\{\tau_k=d_j\}} + p_ip_j\right\} \\ &= \mathbb{E}\left\{I_{\{\tau_k=d_i\}}I_{\{\tau_k=d_j\}}\right\} - p_ip_j - p_ip_j + p_ip_j \\ &= \mathbb{E}\left\{I_{\{\tau_k=d_i\}}I_{\{\tau_k=d_j\}}\right\} - p_ip_j \end{aligned} \tag{15}$$

Since $\mathbb{E}\left\{I_{\{\tau_k=d_i\}}I_{\{\tau_k=d_j\}}\right\} = p_i = p_j$ $(i = j)$ and $\mathbb{E}\left\{I_{\{\tau_k=d_i\}}I_{\{\tau_k=d_j\}}\right\} = 0$ $(i \neq j)$, the expression (14) follows directly. ■

For the easy exposition of our results, we first consider the case when no uncertainties appear in the system parameters, i.e., $\Omega \in \mathfrak{R}$ is arbitrary but fixed. The following theorem gives an H_∞ performance analysis condition for the filtering error system (7).

Theorem 1: Consider system (1) and assume that $\Omega \in \mathfrak{R}$ is arbitrary but fixed. Given a full order filter of the form (5) and a prescribed H_∞ attenuation level $\gamma > 0$. If there exist matrices $0 < P^T = P \in \mathbb{R}^{2n\times 2n}$,

$0 < Q_j^T = Q_j \in \mathbb{R}^{n\times n}$, $0 < X_j^T = X_j \in \mathbb{R}^{2n\times 2n}$, $Y_j \in \mathbb{R}^{2n\times n}$, $0 < Z_j^T = Z_j \in \mathbb{R}^{n\times n}$, $0 < R_j^T = R_j \in \mathbb{R}^{s\times s}$, $S_j \in \mathbb{R}^{s\times n}$, $0 < T_j^T = T_j \in \mathbb{R}^{n\times n}$ $(j = 1,\dots,q)$ such that the following LMIs

$$\begin{bmatrix} -I & 0 & 0 & 0 & \tilde{C}_0 & \tilde{C}_d & \tilde{D} & 0 \\ * & -P_d & 0 & 0 & 0 & \rho_d P_d \hat{A}_d & 0 & 0 \\ * & * & -P & 0 & \rho_0 P \bar{A}_0 & 0 & 0 & 0 \\ * & * & * & -P & P\tilde{A}_0 & P\tilde{A}_d & P\tilde{B} & 0 \\ * & * & * & * & -P+\Psi & -Y_d & \sum_{j=1}^q Z^T S_j^T & (\tilde{A}_0 - I)^T Z^T \Pi \\ * & * & * & * & * & -Q_d & -S_d^T & \tilde{A}_d^T Z^T \Pi \\ * & * & * & * & * & * & \sum_{j=1}^q d_j R_j - \gamma^2 I & \tilde{B}^T Z^T \Pi \\ * & * & * & * & * & * & * & -\Pi \end{bmatrix} < 0 \tag{16}$$

$$\begin{bmatrix} X_j & Y_j \\ * & Z_j \end{bmatrix} \geq 0, \quad \begin{bmatrix} R_j & S_j \\ * & T_j \end{bmatrix} \geq 0, \quad \forall j = 1,\dots,q \tag{17}$$

hold, where

$$\begin{aligned} \Psi &:= \sum_{j=1}^q (d_j X_j + Z^T Y_j^T + Y_j Z + Z^T Q_j Z), \\ \Pi &:= \sum_{j=1}^q d_j (Z_j + T_j), \quad \tilde{A}_d := \begin{bmatrix} \tilde{A}_1 & \cdots & \tilde{A}_q \end{bmatrix}, \\ \tilde{C}_d &:= \begin{bmatrix} \tilde{C}_1 & \cdots & \tilde{C}_q \end{bmatrix}, \quad \rho_j := \sqrt{p_j} \quad (0 \leq j \leq q), \\ \rho_d &:= \mathrm{diag}\{\rho_1,\cdots,\rho_q\}, \quad P_d := \mathrm{diag}_q\{P\}, \\ Q_d &:= \mathrm{diag}\{Q_1,\cdots,Q_q\}, \quad \hat{A}_d := \mathrm{diag}\{\bar{A}_1,\cdots,\bar{A}_q\}, \\ Y_d &:= \begin{bmatrix} Y_1 & \cdots & Y_q \end{bmatrix}, \quad S_d := \begin{bmatrix} S_1 & \cdots & S_q \end{bmatrix}, \end{aligned}$$

then the filtering error system (7) is exponentially mean-square stable with the prescribed H_∞ attenuation level bound γ given in (12).

Proof: From system (7), it is easy to see that

$$\eta_{k-d_j} = \eta_k - \sum_{m=k-d_j}^{k-1} (\eta_{m+1} - \eta_m) = \eta_k - \sum_{m=k-d_j}^{k-1} \xi_m \tag{18}$$

where

$$\begin{aligned} \xi_m := \eta_{m+1} - \eta_m \quad &= \quad (\tilde{A}_0 - I)\eta_m + (I_{\{\tau_k = 0\}} - p_0)\bar{A}_0 \eta_m + \sum_{j=1}^q \tilde{A}_j Z \eta_{m-d_j} \\ &\quad + \sum_{j=1}^q (I_{\{\tau_k = d_j\}} - p_j)\bar{A}_j Z \eta_{m-d_j} + \tilde{B} w_m \end{aligned} \tag{19}$$

By substituting (18) into (7), we can obtain

$$\begin{aligned}\eta_{k+1} &= \left[\tilde{A}_0 + \sum_{i=1}^{q} \tilde{A}_i Z\right] \eta_k - \sum_{j=1}^{q} \sum_{m=k-d_j}^{k-1} \tilde{A}_j Z \xi_m \\ &\quad + \big[(I_{\{\tau_k=0\}} - p_0)\bar{A}_0 + \sum_{j=1}^{q} (I_{\{\tau_k=d_j\}} - p_j)\bar{A}_j Z\big]\eta_k \\ &\quad - \sum_{j=1}^{q} \sum_{m=k-d_j}^{k-1} (I_{\{\tau_k=d_j\}} - p_j)\bar{A}_j Z \xi_m + \tilde{B} w_k. \end{aligned} \tag{20}$$

Let $\Theta_k := \left[\eta_k^T,\ \eta_{k-1}^T, \cdots, \eta_0^T\right]^T$ where η_k is defined in (6). Consider the following Lyapunov functional:

$$V_k(\Theta_k) = V_{1k}(\Theta_k) + V_{2k}(\Theta_k) + V_{3k}(\Theta_k) + V_{4k}(\Theta_k) \tag{21}$$

where

$$\begin{aligned} V_{1k}(\Theta_k) &= \eta_k^T P \eta_k, \\ V_{2k}(\Theta_k) &= \sum_{j=1}^{q} \sum_{i=k-d_j}^{k-1} \eta_i^T Z^T Q_j Z \eta_i, \\ V_{3k}(\Theta_k) &= \sum_{j=1}^{q} \sum_{i=-d_j}^{-1} \sum_{m=k+i}^{k-1} \xi_m^T Z^T Z_j Z \xi_m, \\ V_{4k}(\Theta_k) &= \sum_{j=1}^{q} \sum_{i=-d_j}^{-1} \sum_{m=k+i}^{k-1} \xi_m^T Z^T T_j Z \xi_m, \end{aligned}$$

with $P > 0$ and $Q_j > 0$, $Z_j > 0$, $T_j > 0$ $(j = 1, \ldots, q)$ satisfying (17) and Z being defined in (10).

Defining

$$\Delta V_k := \mathbb{E}\left\{V_{k+1}(\Theta_{k+1}) | \Theta_k\right\} - V_k(\Theta_k),$$

the difference of the Lyapunov functional with $w_k = 0$ can be calculated from (20) as follows:

$$
\begin{aligned}
\Delta V_{1k} = {} & \eta_k^T \left[(\tilde{A}_0 + \sum_{i=1}^{q} \tilde{A}_i Z)^T P (\tilde{A}_0 + \sum_{i=1}^{q} \tilde{A}_i Z) - P \right] \eta_k \\
& + \mathbb{E}\Bigg\{ \eta_k^T \left[(I_{\{\tau_k=0\}} - p_0)\bar{A}_0 + \sum_{j=1}^{q} (I_{\{\tau_k=d_j\}} - p_j)\bar{A}_j Z \right]^T P \\
& \quad \times \left[(I_{\{\tau_k=0\}} - p_0)\bar{A}_0 + \sum_{j=1}^{q} (I_{\{\tau_k=d_j\}} - p_j)\bar{A}_j Z \right] \eta_k \Bigg\} \\
& + \left(\sum_{j=1}^{q} \sum_{m=k-d_j}^{k-1} \tilde{A}_j Z \xi_m \right)^T P \left(\sum_{j=1}^{q} \sum_{m=k-d_j}^{k-1} \tilde{A}_j Z \xi_m \right) \\
& + \mathbb{E}\left\{ \left[\sum_{j=1}^{q} \sum_{m=k-d_j}^{k-1} (I_{\{\tau_k=d_j\}} - p_j)\bar{A}_j Z \xi_m \right]^T P \left[\sum_{j=1}^{q} \sum_{m=k-d_j}^{k-1} (I_{\{\tau_k=d_j\}} - p_j)\bar{A}_j Z \xi_m \right] \right\} \\
& - 2 \sum_{j=1}^{q} \sum_{m=k-d_j}^{k-1} \eta_k^T (\tilde{A}_0 + \sum_{i=1}^{q} \tilde{A}_i Z)^T P \tilde{A}_j Z \xi_m \\
& - 2\mathbb{E}\Bigg\{ \Big[\sum_{j=1}^{q} \sum_{m=k-d_j}^{k-1} (I_{\{\tau_k=d_j\}} - p_j)\bar{A}_j Z \xi_m \Big]^T P \\
& \quad \times \Big[(I_{\{\tau_k=0\}} - p_0)\bar{A}_0 + \sum_{i=1}^{q} (I_{\{\tau_k=d_i\}} - p_i)\bar{A}_i Z \Big] \eta_k \Bigg\}
\end{aligned}
$$

From Lemma 1, we can obtain:

$$
\begin{aligned}
& -2\eta_k^T \left(\tilde{A}_0 + \sum_{i=1}^{q} \tilde{A}_i Z \right)^T P \tilde{A}_j Z \xi_m \\
& \leq \eta_k^T X_j \eta_k + 2\xi_m^T Z^T Y_j^T \eta_k \\
& - 2\xi_m^T Z^T \tilde{A}_j^T P \left(\tilde{A}_0 + \sum_{i=1}^{q} \tilde{A}_i Z \right) \eta_k + \xi_m^T Z^T Z_j Z \xi_m
\end{aligned} \tag{22}
$$

with $0 < X_j^T = X_j \in \mathbb{R}^{2n \times 2n}$, $Y_j \in \mathbb{R}^{2n \times n}$, $0 < Z_j^T = Z_j \in \mathbb{R}^{n \times n}$ satisfying (17).

According to Lemma 2, it follows from tedious but straightforward manipulations that

$$
\begin{aligned}
\Delta V_{1k} \quad \leq \quad & \eta_k^T \left(\tilde{A}_0^T P \tilde{A}_0 - P + p_0 \bar{A}_0^T P \bar{A}_0 \right) \eta_k + \left(\sum_{j=1}^{q} \tilde{A}_j Z \eta_{k-d_j} \right)^T P \left(\sum_{j=1}^{q} \tilde{A}_j Z \eta_{k-d_j} \right) \\
& + \sum_{j=1}^{q} p_j \left(\bar{A}_j Z \eta_{k-d_j} \right)^T P \left(\bar{A}_j Z \eta_{k-d_j} \right) + \sum_{j=1}^{q} d_j \eta_k^T X_j \eta_k + 2 \sum_{j=1}^{q} \eta_k^T Z^T Y_j^T \eta_k \\
& -2 \sum_{j=1}^{q} \eta_k^T Y_j Z \eta_{k-d_j} + 2 \left(\sum_{j=1}^{q} \tilde{A}_j Z \eta_{k-d_j} \right)^T P \tilde{A}_0 \eta_k + \sum_{j=1}^{q} \sum_{i=k-d_j}^{k-1} \xi_i^T Z^T Z_j Z \xi_i \\
& - \left(p_0 \bar{A}_0 \eta_k + \sum_{j=1}^{q} p_j \bar{A}_j Z \eta_{k-d_j} \right)^T P \left(p_0 \bar{A}_0 \eta_k + \sum_{j=1}^{q} p_j \bar{A}_j Z \eta_{k-d_j} \right)
\end{aligned}
\tag{23}
$$

In addition, we have

$$
\begin{aligned}
\Delta V_{2k} &= \sum_{j=1}^{q} \eta_k^T Z^T Q_j Z \eta_k - \sum_{j=1}^{q} \eta_{k-d_j}^T Z^T Q_j Z \eta_{k-d_j} && (24) \\
\Delta V_{3k} &= \sum_{j=1}^{q} \sum_{i=-d_j}^{-1} \left[\xi_k^T Z^T Z_j Z \xi_k - \xi_{k+i}^T Z^T Z_j Z \xi_{k+i} \right] \\
&= \sum_{j=1}^{q} d_j \xi_k^T Z^T Z_j Z \xi_k - \sum_{j=1}^{q} \sum_{i=k-d_j}^{k-1} \xi_i^T Z^T Z_j Z \xi_i && (25) \\
\Delta V_{4k} &= \sum_{j=1}^{q} d_j \xi_k^T Z^T T_j Z \xi_k - \sum_{j=1}^{q} \sum_{i=k-d_j}^{k-1} \xi_i^T Z^T T_j Z \xi_i && (26)
\end{aligned}
$$

Then, we obtain from (23)-(26) and (19) that:

$$
\begin{aligned}
\Delta V_k &= \Delta V_{1k} + \Delta V_{2k} + \Delta V_{3k} + \Delta V_{4k} \\
&\leq \zeta_k^T \begin{bmatrix} M_1 & M_2 \\ * & M_3 \end{bmatrix} \zeta_k
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
\zeta_k &:= [\eta_k^T \quad \eta_{k-d_1}^T Z^T \quad \cdots \quad \eta_{k-d_q}^T Z^T]^T \\
M_1 &:= \tilde{A}_0^T P \tilde{A}_0 + \rho_0^2 \bar{A}_0^T P \bar{A}_0 - P + \Psi + (\tilde{A}_0 - I)^T Z^T \Pi Z (\tilde{A}_0 - I) \\
M_2 &:= -Y_d + \tilde{A}_0^T P \tilde{A}_d + (\tilde{A}_0 - I)^T Z^T \Pi Z \tilde{A}_d \\
M_3 &:= \tilde{A}_d^T P \tilde{A}_d + \rho_d^2 \hat{A}_d^T P_d \hat{A}_d - Q_d + \tilde{A}_d^T Z^T \Pi Z \tilde{A}_d
\end{aligned}
\tag{28}
$$

By the Schur Complement [2], LMI (16) implies $\Delta V_k(\Theta_k) < 0$ for all nonzero ζ_k, so we can always find a positive scalar $\vartheta > 0$ such that

$$
\begin{bmatrix} M_1 & M_2 \\ * & M_3 \end{bmatrix} < \begin{bmatrix} -\vartheta I & 0 \\ 0 & 0 \end{bmatrix}
\tag{29}
$$

and subsequently

$$
\mathbb{E} \left\{ V_{k+1}(\Theta_{k+1}) | \Theta_k \right\} - V_k(\Theta_k) < -\vartheta \|\eta_k\|^2.
\tag{30}
$$

Furthermore, from Lemma 1 of [22], we can confirm that the filtering error system (7) is exponentially mean-square stable.

Next, for any nonzero w_k, it follows from (7) and (21) that

$$\mathbb{E}\left\{V_{k+1}(\Theta_{k+1})|\Theta_k\right\} - V_k(\Theta_k) + \mathbb{E}\left\{\tilde{z}_k^T\tilde{z}_k\right\} - \gamma^2\mathbb{E}\left\{w_k^T w_k\right\} \leq \chi_k^T \begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 \\ * & \Omega_4 & \Omega_5 \\ * & * & \Omega_6 \end{bmatrix} \chi_k \tag{31}$$

where

$$\begin{aligned}
\chi_k &:= [\eta_k^T \quad \eta_{k-d_1}^T Z^T \quad \cdots \quad \eta_{k-d_q}^T Z^T \quad w_k^T]^T \\
\Omega_1 &:= \tilde{A}_0^T P\tilde{A}_0 + \rho_0^2\bar{A}_0^T P\bar{A}_0 - P + \Psi + (\tilde{A}_0 - I)^T Z^T \Pi Z(\tilde{A}_0 - I) + \tilde{C}_0^T\tilde{C}_0 \\
\Omega_2 &:= -Y_d + \tilde{A}_0^T P\tilde{A}_d + (\tilde{A}_0 - I)^T Z^T \Pi Z\tilde{A}_d + \tilde{C}_0^T\tilde{C}_d \\
\Omega_3 &:= \tilde{C}_0^T\tilde{D} + \tilde{A}_0^T P\tilde{B} + \sum_{j=1}^{q} Z^T S_j^T + (\tilde{A}_0 - I)^T Z^T \Pi Z\tilde{B} \\
\Omega_4 &:= \tilde{A}_d^T P\tilde{A}_d + \rho_d^2\hat{A}_d^T P_d\hat{A}_d - Q_d + \tilde{A}_d^T Z^T \Pi Z\tilde{A}_d + \tilde{C}_d^T\tilde{C}_d \\
\Omega_5 &:= \tilde{C}_d^T\tilde{D} - S_d^T + \tilde{A}_d^T P\tilde{B} + \tilde{A}_d^T Z^T \Pi Z\tilde{B} \\
\Omega_6 &:= \tilde{D}^T\tilde{D} + \tilde{B}^T P\tilde{B} + \sum_{j=1}^{q} d_j R_j + \tilde{B}^T Z^T \Pi Z\tilde{B} - \gamma^2 I
\end{aligned} \tag{32}$$

Again, using Schur complement [2], it can be observed from (16) and (31) that for any χ_k and w_k that are not all zero,

$$\mathbb{E}\left\{V_{k+1}(\Theta_{k+1})|\Theta_k\right\} - V_k(\Theta_k) + \mathbb{E}\left\{\tilde{z}_k^T\tilde{z}_k\right\} - \gamma^2\mathbb{E}\left\{w_k^T w_k\right\} < 0 \tag{33}$$

Now, summing up (33) from 0 to ∞ with respect to k yields

$$\sum_{k=0}^{\infty}\left\{\|\tilde{z}_k\|^2\right\} < \gamma^2\sum_{k=0}^{\infty}\left\{\|w_k\|^2\right\} + \mathbb{E}\left\{V_0\right\} - \mathbb{E}\left\{V_\infty\right\} \tag{34}$$

Since the system (7) is exponentially mean-square stable, it is straightforward to see that (12) holds under the zero initial condition. This concludes the proof. ■

Remark 5: Theorem 1 provides an efficient way for filtering analysis for fixed parameter systems, and this can be easily extended to the polytopic uncertain systems with the concept of quadratic stability. That is, for all admissible uncertain parameters, there exits a fixed Lyapunov functional for all vertices of the polytope, which is also called parameter-independent approach. This will, however, inevitably introduce overdesign. Recently, many robust H_∞ filtering results using parameter-dependent approach have been reported in the literature [5,13], most of which can provide a decoupling between the system matrices and the positive definite matrices. In the following, we give the parameter-dependent results for filtering problem of polytopic uncertain systems.

Corollary 1: Consider system (1) with fixed and known parameters and a given filter of the form (5). For a prescribed H_∞ attenuation level $\gamma > 0$, if there exist matrices $0 < P^T = P \in \mathbb{R}^{2n\times 2n}$, $M \in \mathbb{R}^{2n\times 2n}$, $0 < Q_j^T = Q_j \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$, $0 < X_j^T = X_j \in \mathbb{R}^{2n\times 2n}$, $Y_j \in \mathbb{R}^{2n\times n}$, $0 < Z_j^T = Z_j \in \mathbb{R}^{n\times n}$,

$0 < R_j^T = R_j \in \mathbb{R}^{s\times s}$, $S_j \in \mathbb{R}^{s\times n}$, $0 < T_j^T = T_j \in \mathbb{R}^{n\times n}$ $(j = 1, \ldots, q)$ such that (17) and the following LMI

$$\begin{bmatrix} -I & 0 & 0 & 0 & \tilde{C}_0 & \tilde{C}_d & \tilde{D} & 0 \\ * & \Delta_d & 0 & 0 & 0 & \rho_d M_d^T \hat{A}_d & 0 & 0 \\ * & * & \Delta & 0 & \rho_0 M^T \bar{A}_0 & 0 & 0 & 0 \\ * & * & * & \Delta & M^T \tilde{A}_0 & M^T \tilde{A}_d & M^T \tilde{B} & 0 \\ * & * & * & * & -P + \Psi & -Y_d & \sum_{j=1}^q Z^T S_j^T & (\tilde{A}_0 - I)^T Z^T N \\ * & * & * & * & * & -Q_d & -S_d^T & \tilde{A}_d^T Z^T N \\ * & * & * & * & * & * & \sum_{j=1}^q d_j R_j - \gamma^2 I & \tilde{B}^T Z^T N \\ * & * & * & * & * & * & * & \Pi - N^T - N \end{bmatrix} < 0 \tag{35}$$

hold, where Ψ, Π, $\tilde{A}_d$, $\tilde{C}_d$, ρ_j, $0 \leq j \leq q$, ρ_d, P_d, Q_d, $\hat{A}_d$, Y_d and S_d are the same as defined in Theorem 1, and $M_d := \mathrm{diag}_q\{M\}$, $\Delta := P - M^T - M$, $\Delta_d := \mathrm{diag}_q\{\Delta\}$, then the filtering error system (7) is exponentially mean-square stable and satisfies the prescribed H_∞ attenuation level given in (12).

Proof: We only need to show that (35) leads to (16). From (35), we have $M^T + M - P > 0$, $M_d^T + M_d - P_d > 0$ and $N^T + N - \Pi > 0$. Noting that P, P_d and Π are positive definite, we can confirm that M, M_d and N are nonsingular [5]. From $(M - P)^T P^{-1}(M - P) \geq 0$, $(M_d - P_d)^T P_d^{-1}(M_d - P_d) \geq 0$ and $(N - \Pi)^T \Pi^{-1}(N - \Pi) \geq 0$, we obtain $M^T P^{-1} M \geq M + M^T - P$, $M_d^T P_d^{-1} M_d \geq M_d + M_d^T - P_d$ and $N^T \Pi^{-1} N \geq N + N^T - \Pi$, respectively. Together with (35), we arrive at

$$\begin{bmatrix} -I & 0 & 0 & 0 & \tilde{C}_0 & \tilde{C}_d & \tilde{D} & 0 \\ * & \tilde{\Delta}_d & 0 & 0 & 0 & \rho_d M_d^T \hat{A}_d & 0 & 0 \\ * & * & \tilde{\Delta} & 0 & \rho_0 M^T \bar{A}_0 & 0 & 0 & 0 \\ * & * & * & \tilde{\Delta} & M^T \tilde{A}_0 & M^T \tilde{A}_d & M^T \tilde{B} & 0 \\ * & * & * & * & -P + \Psi & -Y_d & \sum_{j=1}^q Z^T S_j^T & (\tilde{A}_0 - I)^T Z^T N \\ * & * & * & * & * & -Q_d & -S_d^T & \tilde{A}_d^T Z^T N \\ * & * & * & * & * & * & \sum_{j=1}^q d_j R_j - \gamma^2 I & \tilde{B}^T Z^T N \\ * & * & * & * & * & * & * & -N^T \Pi^{-1} N \end{bmatrix} < 0 \tag{36}$$

where $\tilde{\Delta} := -M^T P^{-1} M$ and $\tilde{\Delta}_d := M_d^T P_d^{-1} M_d$.

Performing the congruence transformation to (36) by $\mathrm{diag}\{I, M_d^{-1} P_d, M^{-1} P, M^{-1} P, I, I, I, N^{-1}\Pi\}$, we obtain (16), and the proof is completed. ■

By introducing new additional matrices M and N, which are not constrained to be symmetric or positive definite, LMI (35) contains no product term between the system matrices and the positive definite matrices. Therefore, Corollary 1 can be directly extended to polytopic uncertain system to obtain a robust H_∞ filtering performance analysis result with the idea of parameter-dependent approach. In other words, for each vertex, an individual Lyapunov function will be used, which will certainly provide a less conservative result.

Corollary 2: Consider system (1) with uncertain parameters satisfying (3). For a given full-order filter of the form (5) and a prescribed H_∞ attenuation level $\gamma > 0$, if there exist matrices $0 < P_i^T = P_i \in \mathbb{R}^{2n\times 2n}$, $0 < Q_{ji}^T = Q_{ji} \in \mathbb{R}^{n\times n}$, $0 < X_{ji}^T = X_{ji} \in \mathbb{R}^{2n\times 2n}$, $Y_{ji} \in \mathbb{R}^{2n\times n}$, $0 < Z_{ji}^T = Z_{ji} \in \mathbb{R}^{n\times n}$, $0 < R_{ji}^T = R_{ji} \in \mathbb{R}^{s\times s}$, $S_{ji} \in \mathbb{R}^{s\times n}$, $0 < T_{ji}^T = T_{ji} \in \mathbb{R}^{n\times n}$, $j = 1, \ldots, q$, $i = 1, \ldots, v$, and $M \in \mathbb{R}^{2n\times 2n}$, $N \in \mathbb{R}^{n\times n}$ such that for all

$i = 1, \ldots, v$, the following LMIs hold:

$$\begin{bmatrix} -I & 0 & 0 & 0 & \tilde{C}_{0i} & \tilde{C}_{di} & \tilde{D}_i & 0 \\ * & \Gamma_{di} & 0 & 0 & 0 & \rho_d M_d^T \hat{A}_{di} & 0 & 0 \\ * & * & \Gamma_i & 0 & \rho_0 M^T \bar{A}_{0i} & 0 & 0 & 0 \\ * & * & * & \Gamma_i & M^T \tilde{A}_{0i} & M^T \tilde{A}_{di} & M^T \tilde{B}_i & 0 \\ * & * & * & * & -P_i + \Psi_i & -Y_{di} & \sum_{j=1}^q Z^T S_{ji}^T & (\tilde{A}_{0i} - I)^T Z^T N \\ * & * & * & * & * & -Q_{di} & -S_{di}^T & \tilde{A}_{di}^T Z^T N \\ * & * & * & * & * & * & \sum_{j=1}^q d_j R_{ji} - \gamma^2 I & \tilde{B}_i^T Z^T N \\ * & * & * & * & * & * & * & \Pi_i - N^T - N \end{bmatrix} < 0 \tag{37}$$

$$\begin{bmatrix} X_{ji} & Y_{ji} \\ * & Z_{ji} \end{bmatrix} \geq 0, \quad \begin{bmatrix} R_{ji} & S_{ji} \\ * & T_{ji} \end{bmatrix} \geq 0, \quad \forall j = 1, \ldots, q \tag{38}$$

where $\Gamma_i := P_i - M^T - M$, $\Gamma_{di} := P_{di} - M_d^T - M_d$, $\Psi_i := \sum_{j=1}^q (d_j X_{ji} + Z^T Y_{ji}^T + Y_{ji} Z + Z^T Q_{ji} Z)$, $\Pi_i := \sum_{j=1}^q d_j (Z_{ji} + T_{ji})$, $\tilde{A}_{di} := \begin{bmatrix} \tilde{A}_{1i} & \cdots & \tilde{A}_{qi} \end{bmatrix}$, $\tilde{C}_{di} := \begin{bmatrix} \tilde{C}_{1i} & \cdots & \tilde{C}_{qi} \end{bmatrix}$, $\rho_i := \sqrt{p_i}$, $0 \leq j \leq q$. $\rho_d := \mathrm{diag}\{\rho_1 \ \cdots \ \rho_q\}$, $P_{di} := \mathrm{diag}_q\{P_i\}$, $Q_{di} := \mathrm{diag}\{Q_{1i} \ \cdots \ Q_{qi}\}$, $\hat{A}_{di} := \mathrm{diag}\{\tilde{A}_{1i} \ \cdots \ \tilde{A}_{qi}\}$, $Y_{di} := \begin{bmatrix} Y_{1i} & \cdots & Y_{qi} \end{bmatrix}$, $S_{di} := \begin{bmatrix} S_{1i} & \cdots & S_{qi} \end{bmatrix}$, then the filtering error system (7) is robust exponentially mean-square stable and satisfies the prescribed H_∞ attenuation level given in (12).

Proof: For any system with parameters satisfying (3), one can always find coefficients λ_i ($i = 1, \ldots, v$), such that $\Omega = \sum_{i=1}^v \lambda_i \Omega_i$, $\sum_{i=1}^v \lambda_i = 1$, $\lambda_i \geq 0$. If (37) holds for all $i = 1, \ldots, v$, we consider the convex combination of inequalities (37) and then obtain

$$\begin{bmatrix} -I & 0 & 0 & 0 & \tilde{C}_0 & \tilde{C}_d & \tilde{D} & 0 \\ * & \tilde{\Gamma}_d(\lambda) & 0 & 0 & 0 & \rho_d M_d^T \hat{A}_d & 0 & 0 \\ * & * & \tilde{\Gamma}(\lambda) & 0 & \rho_0 M^T \bar{A}_0 & 0 & 0 & 0 \\ * & * & * & \tilde{\Gamma}(\lambda) & M^T \tilde{A}_0 & M^T \tilde{A}_d & M^T \tilde{B} & 0 \\ * & * & * & * & \tilde{\Gamma}_{55}(\lambda) & -Y_d(\lambda) & \tilde{\Gamma}_{57}(\lambda) & \tilde{\Gamma}_{58} \\ * & * & * & * & * & -Q_d(\lambda) & -S_d(\lambda)^T & \tilde{A}_d^T Z^T N \\ * & * & * & * & * & * & \tilde{\Gamma}_{77}(\lambda) & \tilde{B}^T Z^T N \\ * & * & * & * & * & * & * & \tilde{\Gamma}_{88} \end{bmatrix} < 0 \tag{39}$$

where $\tilde{\Gamma}(\lambda) := P(\lambda) - M^T - M$, $\tilde{\Gamma}_d(\lambda) := P_d(\lambda) - M_d^T - M_d$, $\tilde{\Gamma}_{55}(\lambda) := -P(\lambda) + \Psi(\lambda)$, $\tilde{\Gamma}_{57}(\lambda) := \sum_{j=1}^q Z^T S_j(\lambda)^T$, $\tilde{\Gamma}_{58} := (\tilde{A}_0 - I)^T Z^T N$, $\tilde{\Gamma}_{77}(\lambda) := \sum_{j=1}^q d_j R_j(\lambda) - \gamma^2 I$, $\tilde{\Gamma}_{88}(\lambda) := \Pi(\lambda) - N^T - N$.

Similarly, we get

$$\begin{bmatrix} X_j(\lambda) & Y_j(\lambda) \\ * & Z_j(\lambda) \end{bmatrix} \geq 0, \quad \begin{bmatrix} R_j(\lambda) & S_j(\lambda) \\ * & T_j(\lambda) \end{bmatrix} \geq 0, \quad \forall j = 1, \ldots, q \tag{40}$$

where $P(\lambda) = \sum_{i=1}^v \lambda_i P_i$, $Q(\lambda) = \sum_{i=1}^v \lambda_i Q_i$, $X(\lambda) = \sum_{i=1}^v \lambda_i X_i$, $Y(\lambda) = \sum_{i=1}^v \lambda_i Y_i$, $Z(\lambda) = \sum_{i=1}^v \lambda_i Z_i$, $R(\lambda) = \sum_{i=1}^v \lambda_i R_i$, $S(\lambda) = \sum_{i=1}^v \lambda_i S_i$, $T(\lambda) = \sum_{i=1}^v \lambda_i T_i$ are parameter-dependent Lyapunov matrices. So we can conclude from Theorem 1 that the filtering error system (7) is exponentially mean-square stable and the filtering error $\tilde{z}_k$ satisfies (12), and the proof is completed. ■

Remark 6: Corollary 2 provides a delay-probability-dependent and parameter-dependent robust H_∞ performance criterion for the filtering error system (7). if we impose $X_{ji} = 0$, $Y_{ji} = 0$, $Z_{ji} = 0$, $R_{ji} = 0$, $S_{ji} = 0$, $T_{ji} = 0$, (37) implies corresponding result which is not dependent on the size of the delays $d_j > 0$, $j = 1, \ldots, q$.

IV. Robust H_∞ Filter Design

In the previous section, robust H_∞ filter analysis problems have been studied. In Corollary 2, there exist products of unknown matrices M and N with filter parameters G, K, L, so they cannot directly be used for filter design with the existing LMI toolbox. In this section, we shall focus on the robust H_∞ filter design problems for a class of state-delay networked systems with missing and delay measurements by using parameter-dependent and delay-probability-dependent approaches.

Theorem 2: Consider system (1) with uncertain parameters $\Omega \in \mathfrak{R}$ and let $\gamma > 0$ be a given H_∞ attenuation level. Then, an admissible full-order robust H_∞ filter of the form (5) exists if there exist matrices $V \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times n}$, $U \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$, $\bar{G} \in \mathbb{R}^{n\times n}$, $\bar{K} \in \mathbb{R}^{n\times l}$, $\bar{L} \in \mathbb{R}^{m\times n}$, and $0 < \bar{P}_{1i}^T = \bar{P}_{1i} \in \mathbb{R}^{n\times n}$, $\bar{P}_{2i} \in \mathbb{R}^{n\times n}$, $0 < \bar{P}_{3i}^T = \bar{P}_{3i} \in \mathbb{R}^{n\times n}$, $0 < Q_{ji}^T = Q_{ji} \in \mathbb{R}^{n\times n}$, $0 < \bar{X}_{1ji}^T = \bar{X}_{1ji} \in \mathbb{R}^{n\times n}$, $\bar{X}_{2ji} \in \mathbb{R}^{n\times n}$, $0 < \bar{X}_{3ji}^T = \bar{X}_{3ji} \in \mathbb{R}^{n\times n}$, $\bar{Y}_{1ji} \in \mathbb{R}^{n\times n}$, $\bar{Y}_{2ji} \in \mathbb{R}^{n\times n}$, $0 < Z_{ji}^T = Z_{ji} \in \mathbb{R}^{n\times n}$, $0 < R_{ji}^T = R_{ji} \in \mathbb{R}^{s\times s}$, $S_{ji} \in \mathbb{R}^{s\times n}$, $0 < T_{ji}^T = T_{ji} \in \mathbb{R}^{n\times n}$ $(j = 1, \ldots, q,\ i = 1, \ldots, v)$ such that for all $i = 1, \ldots, v$, the following LMIs

$$\begin{bmatrix} -I & 0 & 0 & 0 & \bar{\Upsilon}_{15i} & H_{di} & E_i & 0 \\ * & \bar{\Upsilon}_{22i} & 0 & 0 & 0 & \bar{\Upsilon}_{26i} & 0 & 0 \\ * & * & \bar{\Upsilon}_{33i} & 0 & \bar{\Upsilon}_{35i} & 0 & 0 & 0 \\ * & * & * & \bar{\Upsilon}_{44i} & \bar{\Upsilon}_{45i} & \bar{\Upsilon}_{46i} & \bar{\Upsilon}_{47i} & 0 \\ * & * & * & * & \bar{\Upsilon}_{55i} & \bar{\Upsilon}_{56i} & \bar{\Upsilon}_{57i} & \bar{\Upsilon}_{58i} \\ * & * & * & * & * & -Q_{di} & -S_{di}^T & A_{di}^T N \\ * & * & * & * & * & * & \bar{\Upsilon}_{77i} & B_i^T N \\ * & * & * & * & * & * & * & \bar{\Upsilon}_{88i} \end{bmatrix} < 0 \tag{41}$$

$$\begin{bmatrix} \begin{bmatrix} \bar{X}_{1ji} & \bar{X}_{2ji} \\ * & \bar{X}_{3ji} \end{bmatrix} & \begin{bmatrix} \bar{Y}_{1ji} \\ \bar{Y}_{2ji} \end{bmatrix} \\ * & Z_{ji} \end{bmatrix} \geq 0, \quad \begin{bmatrix} R_{ji} & S_{ji} \\ * & T_{ji} \end{bmatrix} \geq 0, \quad \forall j = 1, \ldots, q \tag{42}$$

$$\begin{bmatrix} \bar{P}_{1i} & \bar{P}_{2i} \\ * & \bar{P}_{3i} \end{bmatrix} > 0 \tag{43}$$

hold, where

$$\begin{aligned}
\bar{\Upsilon}_{15i} &:= \begin{bmatrix} H_{0i} & H_{0i} - \bar{L} \end{bmatrix}, \\
\bar{\Upsilon}_{22i} &:= \begin{bmatrix} \bar{P}_{1di} - V_d^T - V_d & \bar{P}_{2di} - F_d - V_d^T - U_d \\ * & \bar{P}_{3di} - F_d - F_d^T \end{bmatrix}, \\
\bar{\Upsilon}_{26i} &:= \begin{bmatrix} \rho_d \bar{K}_d \hat{C}_{di} \\ 0 \end{bmatrix}, \\
\bar{\Upsilon}_{33i} &:= \bar{\Upsilon}_{44i} := \begin{bmatrix} \bar{P}_{1i} - V^T - V & \bar{P}_{2i} - F - V^T - U \\ * & \bar{P}_{3i} - F - F^T \end{bmatrix}, \\
\bar{\Upsilon}_{35i} &:= \begin{bmatrix} \rho_0 \bar{K} C_{0i} & \rho_0 \bar{K} C_{0i} \\ 0 & 0 \end{bmatrix}, \\
\bar{\Upsilon}_{45i} &:= \begin{bmatrix} V^T A_{0i} + p_0 \bar{K} C_{0i} & V^T A_{0i} + p_0 \bar{K} C_{0i} + \bar{G} \\ F^T A_{0i} & F^T A_{0i} \end{bmatrix}, \\
\bar{\Upsilon}_{46i} &:= \begin{bmatrix} V^T A_{di} + \bar{K} C_{di} p_d \\ F^T A_{di} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\bar{\Upsilon}_{47i} &:= \begin{bmatrix} V^T B_i + \bar{K} D_i \\ F^T B_i \end{bmatrix}, \\
\bar{\Upsilon}_{55i} &:= \begin{bmatrix} -\bar{P}_{1i} + \bar{\Phi}_{1i} & -\bar{P}_{2i} + \bar{\Phi}_{2i} \\ * & -\bar{P}_{3i} + \bar{\Phi}_{3i} \end{bmatrix}, \\
\bar{\Upsilon}_{56i} &:= \begin{bmatrix} -\bar{Y}_{1di} \\ -\bar{Y}_{2di} \end{bmatrix}, \\
\bar{\Upsilon}_{57i} &:= \begin{bmatrix} \sum_{j=1}^{q} S_{ji}^T \\ \sum_{j=1}^{q} S_{ji}^T \end{bmatrix}, \\
\bar{\Upsilon}_{58i} &:= \begin{bmatrix} A_{0i}^T N - N \\ A_{0i}^T N - N \end{bmatrix}, \\
\bar{\Upsilon}_{77i} &:= \sum_{j=1}^{q} d_j R_{ji} - \gamma^2 I, \\
\bar{\Upsilon}_{88i} &:= \Pi_i - N^T - N,
\end{aligned}$$

and $p_d := \mathrm{diag}\{p_1, \cdots, p_q\}$, $\rho_j := \sqrt{p_j}$ $(0 \le j \le q)$, $\rho_d := \mathrm{diag}\{\rho_1, \cdots, \rho_q\}$, $\bar{P}_{1di} := \mathrm{diag}_q\{\bar{P}_{1i}\}$, $\bar{P}_{2di} := \mathrm{diag}_q\{\bar{P}_{2i}\}$, $\bar{P}_{3di} := \mathrm{diag}_q\{\bar{P}_{3i}\}$, $V_d := \mathrm{diag}_q\{V\}$, $F_d := \mathrm{diag}_q\{F\}$, $U_d := \mathrm{diag}_q\{U\}$, $\bar{K}_d := \mathrm{diag}_q\{\bar{K}\}$, $\bar{Y}_{1di} := \left[\bar{Y}_{11i} \ldots \bar{Y}_{1qi}\right]$, $\bar{Y}_{2di} := \left[\bar{Y}_{21i} \ldots \bar{Y}_{2qi}\right]$, $\bar{\Phi}_{1i} := \sum_{j=1}^{q} (d_j \bar{X}_{1ji} + \bar{Y}_{1ji} + \bar{Y}_{1ji}^T + Q_{ji})$, $\bar{\Phi}_{2i} := \sum_{j=1}^{q} (d_j \bar{X}_{2ji} + \bar{Y}_{1ji} + \bar{Y}_{2ji}^T + Q_{ji})$, $\bar{\Phi}_{3i} := \sum_{j=1}^{q} (d_j \bar{X}_{3ji} + \bar{Y}_{2ji} + \bar{Y}_{2ji}^T + Q_{ji})$, $H_{di} := [H_{1i} \ldots H_{qi}]$, $\hat{C}_{di} := \mathrm{diag}\{C_{1i}, \ldots, C_{qi}\}$, $A_{di} := \left[A_{1i} \ldots A_{qi}\right]$, $C_{di} := \left[C_{1i} \ldots C_{qi}\right]$. Moreover, if (39)-(43) are true, the desired filter parameters can be given by

$$G = U^{-1}\bar{G}, \quad K = U^{-1}\bar{K}, \quad L = \bar{L}. \tag{44}$$

Proof: Corollary 2 provides a sufficient condition for the filtering error system (7) to be exponentially mean-square stable and also achieve the H_∞-norm constraint (12). Our goal here is to derive the expression of the filter parameters from (5). From (37), for all $i = 1, \ldots, v$, we have $M + M^T - P_i > 0$. Considering that P_i is positive definite, it can be further confirmed that M is nonsingular [5]. We partition M, M^{-1}, and P_i as follows:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad P_i = \begin{bmatrix} P_{1i} & P_{2i} \\ * & P_{3i} \end{bmatrix}, \tag{45}$$

where the partitioning of the above three matrices is compatible with that of $\tilde{A}_0$ defined in (8).

Introduce the following matrices:

$$T_1 = \begin{bmatrix} M_{11} & I \\ M_{21} & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} I & S_{11} \\ 0 & S_{21} \end{bmatrix}, \tag{46}$$

which imply that $M^{-1}T_1 = T_2$ and $MT_2 = T_1$. Define $T_{2d} := \mathrm{diag}_q\{T_2\}$, and let $\hat{P}_i := \begin{bmatrix} \hat{P}_{1i} & \hat{P}_{2i} \\ * & \hat{P}_{3i} \end{bmatrix} = T_2^T P_i T_2$, $\hat{X}_{ji} := \begin{bmatrix} \hat{X}_{1ji} & \hat{X}_{2ji} \\ * & \hat{X}_{3ji} \end{bmatrix} = T_2^T X_{ji} T_2$, $\hat{Y}_{ji} := \begin{bmatrix} \hat{Y}_{1ji} \\ \hat{Y}_{2ji} \end{bmatrix} = T_2^T Y_{ji}$. Then, performing congruence transformation to (37) by $\mathrm{diag}\{I, T_{2d}, T_2, T_2, T_2, I, I, I\}$, we can obtain

$$\begin{bmatrix} -I & 0 & 0 & 0 & \hat{\Upsilon}_{15i} & \hat{\Upsilon}_{16i} & E_i & 0 \\ * & \hat{\Upsilon}_{22i} & 0 & 0 & 0 & \hat{\Upsilon}_{26i} & 0 & 0 \\ * & * & \hat{\Upsilon}_{33i} & 0 & \hat{\Upsilon}_{35i} & 0 & 0 & 0 \\ * & * & * & \hat{\Upsilon}_{44i} & \hat{\Upsilon}_{45i} & \hat{\Upsilon}_{46i} & \hat{\Upsilon}_{47i} & 0 \\ * & * & * & * & \hat{\Upsilon}_{55i} & \hat{\Upsilon}_{56i} & \hat{\Upsilon}_{57i} & \hat{\Upsilon}_{58i} \\ * & * & * & * & * & -Q_{di} & -S_{di}^T & A_{di}^T N \\ * & * & * & * & * & * & \hat{\Upsilon}_{77i} & B_i^T N \\ * & * & * & * & * & * & * & \hat{\Upsilon}_{88i} \end{bmatrix} < 0 \tag{47}$$

where

$$\begin{aligned}
\hat{\Upsilon}_{15i} &:= \begin{bmatrix} H_{0i} & H_{0i}S_{11} - LS_{21} \end{bmatrix},\ \hat{\Upsilon}_{16i} := \begin{bmatrix} H_{1i} & \cdots & H_{qi} \end{bmatrix},\ \hat{\Upsilon}_{22i} := diag_q\{\hat{\Upsilon}_{33i}\}, \\
\hat{\Upsilon}_{26i} &:= diag\left\{ \begin{bmatrix} \rho_1 M_{21}^T K C_{1i} \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} \rho_q M_{21}^T K C_{qi} \\ 0 \end{bmatrix} \right\}, \\
\hat{\Upsilon}_{33i} &= \hat{\Upsilon}_{44i} := \begin{bmatrix} \hat{P}_{1i} - M_{11}^T - M_{11} & \hat{P}_{2i} - I - M_{11}^T S_{11} - M_{21}^T S_{21} \\ * & \hat{P}_{3i} - S_{11} - S_{11}^T \end{bmatrix}, \\
\hat{\Upsilon}_{35i} &:= \begin{bmatrix} \rho_0 M_{21}^T K C_{0i} & \rho_0 M_{21}^T K C_{0i} S_{11} \\ 0 & 0 \end{bmatrix}, \\
\hat{\Upsilon}_{45i} &:= \begin{bmatrix} M_{11}^T A_{0i} + p_0 M_{21}^T K C_{0i} & M_{11}^T A_{0i} S_{11} + p_0 M_{21}^T K C_{0i} S_{11} + M_{21}^T G S_{21} \\ A_{0i} & A_{0i} S_{11} \end{bmatrix}, \\
\hat{\Upsilon}_{46i} &:= \begin{bmatrix} M_{11}^T A_{di} + M_{21}^T K C_{di} p_d \\ A_{di} \end{bmatrix},\ \hat{\Upsilon}_{47i} := \begin{bmatrix} M_{11}^T B_i + M_{21}^T K D_i \\ B_i \end{bmatrix}, \\
\hat{\Upsilon}_{55i} &:= \begin{bmatrix} -\hat{P}_{1i} + \hat{\Phi}_{1i} & -\hat{P}_{2i} + \hat{\Phi}_{2i} \\ * & -\hat{P}_{3i} + \hat{\Phi}_{3i} \end{bmatrix},\ \hat{\Upsilon}_{56i} := \begin{bmatrix} -\hat{Y}_{1di} \\ -\hat{Y}_{2di} \end{bmatrix},\ \hat{\Upsilon}_{57i} := \begin{bmatrix} \sum_{j=1}^q S_{ji}^T \\ \sum_{j=1}^q S_{11}^T S_{ji}^T \end{bmatrix}, \\
\hat{\Upsilon}_{58i} &:= \begin{bmatrix} (A_{0i} - I)^T N \\ S_{11}^T (A_{0i} - I)^T N \end{bmatrix},\ \hat{\Upsilon}_{77i} := \sum_{j=1}^q d_j R_{ji} - \gamma^2 I,\ \hat{\Upsilon}_{88i} := \Pi_i - N^T - N, \\
\hat{Y}_{1di} &:= \left[\hat{Y}_{11i} \dots \hat{Y}_{1qi}\right],\ \hat{Y}_{2di} := \left[\hat{Y}_{21i} \dots \hat{Y}_{2qi}\right], \\
\hat{\Phi}_{1i} &:= \sum\nolimits_{j=1}^q (d_j \hat{X}_{1ji} + \hat{Y}_{1ji} + \hat{Y}_{1ji}^T + Q_{ji}), \\
\hat{\Phi}_{2i} &:= \sum\nolimits_{j=1}^q (d_j \hat{X}_{2ji} + \hat{Y}_{1ji} S_{11} + \hat{Y}_{2ji}^T + Q_{ji} S_{11}), \\
\hat{\Phi}_{3i} &:= \sum\nolimits_{j=1}^q (d_j \hat{X}_{3ji} + \hat{Y}_{2ji} S_{11} + S_{11}^T \hat{Y}_{2ji}^T + S_{11}^T Q_{ji} S_{11}).
\end{aligned}$$

Define a new matrix $\Lambda \in \mathbb{R}^{2qn \times 2qn}$ with its entries being $\Lambda_{\alpha\beta,\ (2\alpha-1)\beta} = \Lambda_{(\alpha+q)\beta,\ 2\alpha\beta} = 1$ for all $1 \le \alpha \le q$ and $1 \le \beta \le n$, and other entries being all zero. Once again, performing congruence transformation to (47) by diag$\{I,\ \Lambda,\ I,\ I,\ I,\ I,\ I,\ I\}$, it can be inferred that (47) is equivalent to the following:

$$\begin{bmatrix} -I & 0 & 0 & 0 & \hat{\Upsilon}_{15i} & \hat{\Upsilon}_{16i} & E_i & 0 \\ * & \tilde{\Upsilon}_{22i} & 0 & 0 & 0 & \tilde{\Upsilon}_{26i} & 0 & 0 \\ * & * & \hat{\Upsilon}_{33i} & 0 & \hat{\Upsilon}_{35i} & 0 & 0 & 0 \\ * & * & * & \hat{\Upsilon}_{44i} & \hat{\Upsilon}_{45i} & \hat{\Upsilon}_{46i} & \hat{\Upsilon}_{47i} & 0 \\ * & * & * & * & \hat{\Upsilon}_{55i} & \hat{\Upsilon}_{56i} & \hat{\Upsilon}_{57i} & \hat{\Upsilon}_{58i} \\ * & * & * & * & * & -Q_{di} & -S_{di}^T & A_{di}^T N \\ * & * & * & * & * & * & \hat{\Upsilon}_{77i} & B_i^T N \\ * & * & * & * & * & * & * & \hat{\Upsilon}_{88i} \end{bmatrix} < 0 \tag{48}$$

where

$$\tilde{\Upsilon}_{22i} := \begin{bmatrix} \mathrm{diag}_q\{\hat{P}_{1i} - M_{11}^T - M_{11}\} & \mathrm{diag}_q\{\hat{P}_{2i} - I - M_{11}^T S_{11} - M_{21}^T S_{21}\} \\ * & \mathrm{diag}_q\{\hat{P}_{3i} - S_{11} - S_{11}^T\} \end{bmatrix},$$

$$\tilde{\Upsilon}_{26i} := \begin{bmatrix} \rho_d M_{21d}^T K_d \hat{C}_{di} \\ 0 \end{bmatrix},\ M_{21d} := \mathrm{diag}_q\{M_{21}\},\ K_d := \mathrm{diag}_q\{K\}.$$

Furthermore, we let $\bar{P}_i := \begin{bmatrix} \bar{P}_{1i} & \bar{P}_{2i} \\ * & \bar{P}_{3i} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & S_{11}^{-1} \end{bmatrix}^T \begin{bmatrix} \hat{P}_{1i} & \hat{P}_{2i} \\ * & \hat{P}_{3i} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S_{11}^{-1} \end{bmatrix}$, and for all $1 \le j \le q$, $\bar{X}_{ji} := \begin{bmatrix} \bar{X}_{1ji} & \bar{X}_{2ji} \\ * & \bar{X}_{3ji} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & S_{11}^{-1} \end{bmatrix}^T \hat{X}_{ji} \begin{bmatrix} I & 0 \\ 0 & S_{11}^{-1} \end{bmatrix}$, $\bar{Y}_{ji} := \begin{bmatrix} \bar{Y}_{1ji} \\ \bar{Y}_{2ji} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & S_{11}^{-1} \end{bmatrix}^T \hat{Y}_{ji}$. Performing congruence transformations to (48) by diag$\{I, \mathrm{diag}_q\{I\}, \mathrm{diag}_q\{S_{11}^{-1}\}, I, S_{11}^{-1}, I, S_{11}^{-1}, I, S_{11}^{-1}, I, I, I\}$ and defining the following matrix variables:

$$V = M_{11},\ F = S^{-1},\ U = M_{21}^T S_{21} S_{11}^{-1},\ \bar{G} = M_{21}^T G S_{21} S_{11}^{-1},\ \bar{K} = M_{21}^T K,\ \bar{L} = L S_{21} S_{11}^{-1} \tag{49}$$

we can easily derive the result (39) in Theorem 2.

After similar manipulations, we can also get (42) and (43). Noting that every step in our derivation is an equivalent transformation, it then follows from Corollary 2 that (39)-(43) are sufficient conditions guaranteeing that the system (7) is exponentially mean-square stable and the H_∞ norm constraint (12) is achieved.

Furthermore, we know from (39) that V, F, U are all nonsingular matrices, so we can always find square and nonsingular matrices M_{21} and S_{21} satisfying $UF^{-1} = M_{21}^T S_{21}$ [22]. Therefore, it results from (49) that:

$$G_0 = M_{21}^{-T} \bar{G} F^{-1} S_{21}^{-1},\ K_0 = M_{21}^{-T} \bar{K}\ L_0 = \bar{L} F^{-1} S_{21}^{-1} \tag{50}$$

By substituting the parameters in (50) into the transfer function of the filter and considering the relationship $U = M_{21}^T S_{21} F$, we obtain

$$T(z) = \bar{L} F^{-1} S_{21}^{-1} (zI - M_{21}^{-T} \bar{G} F^{-1} S_{21}^{-1})^{-1} M_{21}^{-T} \bar{K} = \bar{L}(zI - U^{-1}\bar{G})^{-1} U^{-1} \bar{K}, \tag{51}$$

which means that the desired filter parameters can also be given by (44). This ends the proof. ■

Remark 7: By introducing slack matrices, Theorem 2 provides a parameter dependent H_∞ filter design result. When we impose on the inequalities (39) the following additional constraints

$$\begin{gathered} \bar{P}_{1i} = V > 0,\ \bar{P}_{2i} = F > 0,\ \bar{P}_{3i} = F,\ Q_{ij} = Q_j,\ \bar{X}_{1ji} = \bar{X}_{1j}, \\ \bar{X}_{2ji} = \bar{X}_{2j},\ \bar{X}_{3ji} = \bar{X}_{3j},\ \bar{Y}_{1ji} = \bar{Y}_{1j},\ \bar{Y}_{2ji} = \bar{Y}_{2j},\ Z_{ji} = Z_j, \\ R_{ji} = R_j,\ S_{ji} = S_j,\ T_{ji} = T_j,\ \forall i = 1, \ldots, v,\ \forall j = 1, \ldots, q,\ N = \Pi,\ \text{and } U = F - V, \end{gathered} \tag{52}$$

the corresponding (more conservative) result using a single Lyapunov matrix over the whole parameter uncertainty is given. We can observe from such a simplification process that the extra degree of freedom greatly reduces the conservatism in the filter design.

Remark 8: Compared with the result in [16], we have changed the Lyapunov functional (21) to include more entries, which are relative to the size of delays as well as the probability of the stochastic variable τ_k. Consequently, Theorem 2 provided a delay-probability-dependent approach to robust H_∞ filter design problem. If we set the corresponding entries to be zero, i.e.,

$$X_j = 0,\ Y_j = 0,\ Z_j = 0,\ R_j = 0,\ S_j = 0,\ T_j = 0,\ \forall j = 1, \ldots, q, \tag{53}$$

LMI (39) reduces to be a delay-independent result. This also shows that Theorem 2 is powerful in the sense that it provides sufficient conditions for both the delay-probability-dependent and the delay-independent cases. The comparison between the filter designs aforementioned two methods is given in the next section.

Remark 9: In Theorem 2, there are no products of unknown matrix M with filter parameters G, K and L, so the full-order robust H_∞ filter can be obtained from the solution of convex optimization problems in terms of linear matrix inequalities, which can be solved via efficient interior-point algorithms [2].

Note that (39)-(43) are LMIs over both the matrix variables and the prescribed scalar γ^2. This implies that the scalar γ^2 can be included as one of the optimization variables for LMIs (39)-(43), which makes it possible to obtain the minimum noise attenuation level bound. Then, the minimum guaranteed cost of robust full-order H_∞ filter can be readily found by solving the following convex optimization problems:

Problem 1: The sub-optimal robust H_∞ filtering problem for networked systems with multiple state-delays using the parameter-dependent and delay-probability-dependent approach can be brought forward as follows

$$\min_{\substack{V,F,U,N,\bar{G},\bar{K},\bar{L},\bar{P}_{1i}>0,\bar{P}_{2i}>0,\bar{P}_{3i}>0,Q_{ji}>0,\bar{X}_{1ji}>0,\\ \bar{X}_{2ji},\bar{X}_{3ji}>0,\bar{Y}_{1ji},\bar{Y}_{2ji},Z_{ji}>0,R_{ji}>0,S_{ji},T_{ji}>0,\forall i=1,\ldots,v}} \gamma^2, \quad \text{s.t. } (39)-(43). \tag{54}$$

Problem 2: The sub-optimal robust H_∞ filtering problem for networked systems with multiple state-delays using the parameter-independent and delay-probability-dependent approach can be described as follows

$$\min_{\substack{V,F,U,N,\bar{G},\bar{K},\bar{L},\bar{P}_{1i}>0,\bar{P}_{2i}>0,\bar{P}_{3i}>0,Q_{ji}>0,\bar{X}_{1ji}>0,\\ \bar{X}_{2ji},\bar{X}_{3ji}>0,\bar{Y}_{1ji},\bar{Y}_{2ji},Z_{ji}>0,R_{ji}>0,S_{ji},T_{ji}>0,\forall i=1,\ldots,v}} \gamma^2, \quad \text{s.t. } (39)-(43) \text{ and } (52). \tag{55}$$

Problem 3: The sub-optimal robust H_∞ filtering problem for networked systems with multiple state-delays using the parameter-dependent and delay-independent approach can be stated as follows

$$\min_{\substack{V,F,U,N,\bar{G},\bar{K},\bar{L},\bar{P}_{1i}>0,\bar{P}_{2i}>0,\bar{P}_{3i}>0,Q_{ji}>0,\bar{X}_{1ji}>0,\\ \bar{X}_{2ji},\bar{X}_{3ji}>0,\bar{Y}_{1ji},\bar{Y}_{2ji},Z_{ji}>0,R_{ji}>0,S_{ji},T_{ji}>0,\forall i=1,\ldots,v}} \gamma^2, \quad \text{s.t. } (39)-(43) \text{ and } (53). \tag{56}$$

Problem 4: The sub-optimal robust H_∞ filtering problem for networked systems with multiple state-delays using the parameter-independent and delay-independent approach can be represented as follows

$$\min_{\substack{V,F,U,N,\bar{G},\bar{K},\bar{L},\bar{P}_{1i}>0,\bar{P}_{2i}>0,\bar{P}_{3i}>0,Q_{ji}>0,\bar{X}_{1ji}>0,\\ \bar{X}_{2ji},\bar{X}_{3ji}>0,\bar{Y}_{1ji},\bar{Y}_{2ji},Z_{ji}>0,R_{ji}>0,S_{ji},T_{ji}>0,\forall i=1,\ldots,v}} \gamma^2, \quad \text{s.t. } (39)-(43), (52) \text{ and } (53). \tag{57}$$

For the four problems mentioned above, the parameters of the sub-optimal filters can be determined by (49), and the sub-optimal H_∞ attenuation levels are given by $\gamma^* = \sqrt{\gamma_{opt}^2}$, where γ_{opt}^2 are the sub-optimal solutions of the corresponding convex optimization problems.

V. A Numerical Example

Consider the system (1) with the following matrices borrowed from [5] with some modifications:

$$A_0 = \begin{bmatrix} 0 & 0.3 \\ -0.2 & \theta \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$H_0 = [\ 1 \quad 2\ ], \quad H_1 = [\ 0 \quad 0\ ], \quad H_2 = [\ 0 \quad 0\ ], \quad E = 0,$$

$$C_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix},$$

where θ is an uncertain real parameter satisfying $0.2 \le \theta \le 0.4$. In addition, the constant delays are assumed to be $d_1 = 1$ and $d_2 = 2$.

Case 1: Let $p_0 = 0.6$, $p_1 = 0.2$ and $p_2 = 0.1$. In other words, the measurements can be ideally transmitted over the network with probability 0.6, one-step measurement delay can occur with probability 0.2, two-step measurement delay can occur with probability 0.1, and the measurements are missing with probability 0.1. With the prescribed parameters, Problem 1 can be solved by using the Matlab LMI toolbox [2]. As a result, the minimum noise attenuation level bound is $\gamma_{opt} = 3.2086$, and the parameters of the robust H_∞ filter are given by

$$G = \begin{bmatrix} -0.0461 & 0.1991 \\ -0.0848 & 0.0511 \end{bmatrix}, \quad K = \begin{bmatrix} 0.0359 & 0.1523 \\ 0.0912 & 0.4262 \end{bmatrix}, \quad L = \begin{bmatrix} 1.0044 & 1.9988 \end{bmatrix}.$$

The time-domain simulation of the above filter is shown in Fig. 2~Fig. 3. Here, the disturbance noise is taken as

$$w_k = \exp(-k/30) \times n_k,$$

where, at time k, n_k is a random variable of uniform distribution on $[-1, 1]$. The uncertain parameter θ is randomly set to be $\theta = 0.3192$. The dashed line in Fig. 2 is z_k and its estimation $\hat{z}_k$ is given by the solid line. In Fig. 3, the plot of actual disturbance attenuation level γ_k versus time k is provided, from which we can see that γ_k is always less than the worst case disturbance attenuation level γ^2.

Case 2: In this case, we will show how the probabilities in the measurement equation affect the H_∞ performance of the filtering process, and the advantages of the parameter-dependent and the delay-probability-dependent approaches. To this end, for simplicity, we fix $p_1 = p_2 = 0.1$ and let p_0 vary from 0 to 0.8 with the interval 0.1. We solve Problems 1~4 by using Matlab LMI toolbox and show the relationship of p_0 versus γ_{opt} in Fig. 4.

In Fig. 4, the solid line stands for the result from parameter-dependent and delay-probability-dependent approach (PdDPd); the dotted line stands for the result from parameter-independent and delay-probability-dependent way (PiDPd); the dash-dotted line stands for the parameter-dependent and delay-independent method (PdDi); and the dashed line stands for the parameter-independent and delay-independent result (PiDi). From Fig. 4, we can intuitively obtain the following two relations. One is the relation between H_∞ performance of the filtering process and the measurement missing probability, that is, a better performance can be achieved with less measurements missing. Another relation we can get from Fig. 4 is that for all possible

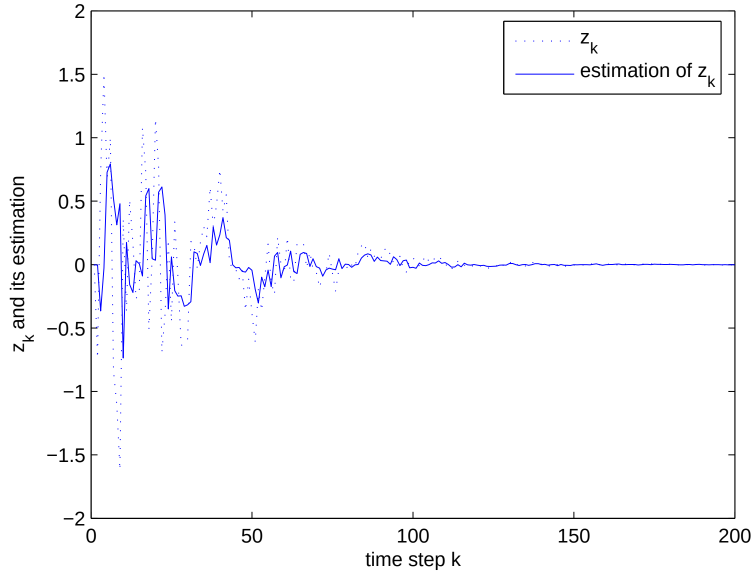

Fig. 2. z_k and its estimation.

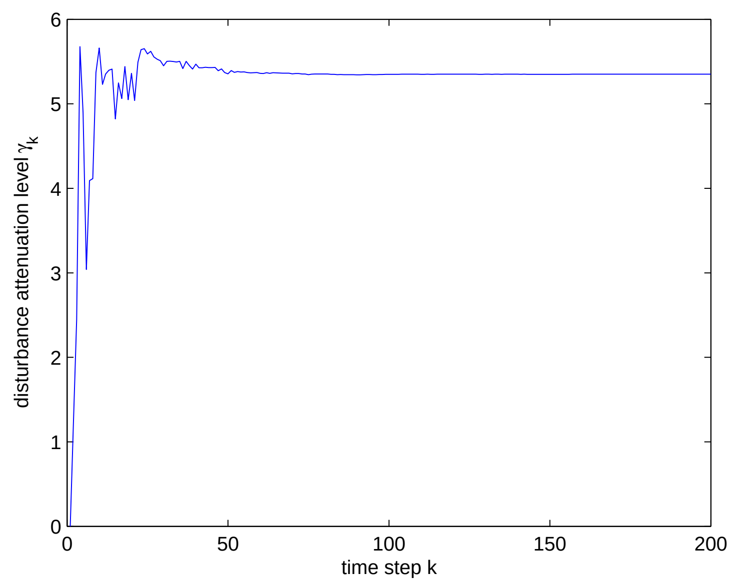

Fig. 3. The actual disturbance attenuation level γ_k versus time k.

measurements, the parameter-dependent and delay-probability-dependent approaches can achieve the best H_∞ performance in the four approaches, while the parameter independent and delay independent algorithms correspond to the worst result. This clearly demonstrates the less conservatism of the parameter-dependent and delay-probability-dependent approaches.

Case 3: In this case, we discuss the conservatism of the delay-probability-dependent and delay-independent approaches. Again, we let $p_0 = 0.6$, $p_1 = 0.2$ and $p_2 = 0.1$, and for simplicity, we impose that $d_1 = d_2 = d$, and d varies from 1 to 10. Simulation results are shown in Fig. 5, which indicates that in the case of a small delay,

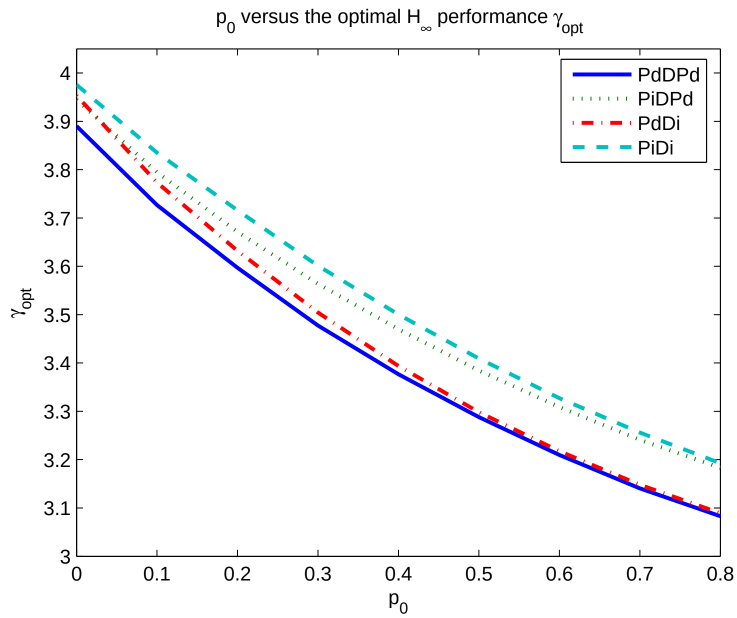

Fig. 4. p_0 versus the sub-optimal H_∞ performance γ_{opt}.

the delay-probability-dependent result (PdDPd) is observably less conservative than the delay-independent one (PdDPi), whereas the conservatism of these two approaches are about at the same level in the case of a large delay.

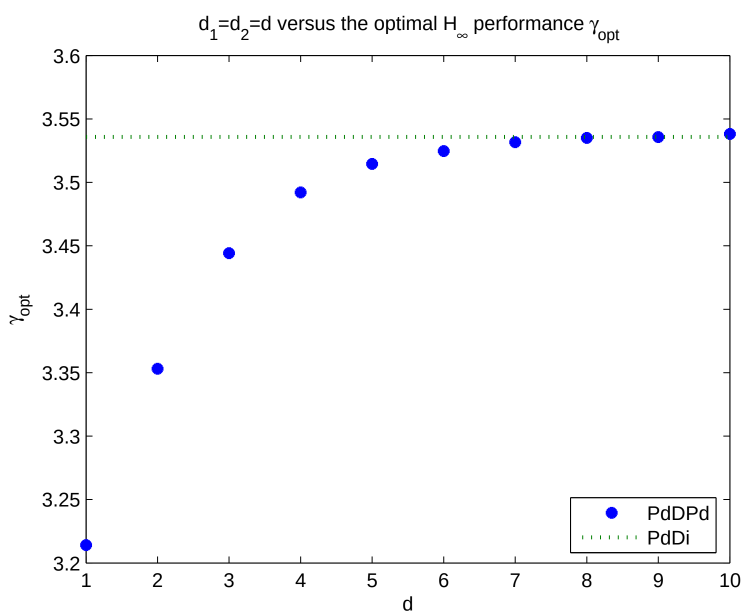

Fig. 5. $d_1 = d_2 = d$ versus the sub-optimal H_∞ performance γ_{opt}.

VI. Conclusions

The problem of robust H_∞ filtering for a class of networked systems with multiple state delays has been considered in this paper. A unified representation simultaneously describing data missing and measurement delay has been proposed. A parameter-dependent and delay-probability-dependent approach has been used to get a less conservative result. The robust H_∞ filter has been designed in terms of feasible LMIs, which guarantees the exponentially mean-square stability of the filtering error system as well as a prescribed H_∞ performance requirement for all possible observations and all admissible parameter uncertainties. Sub-optimal filter design problems are also provided by optimizing the H_∞ filtering performances. Moreover, our method can be extended to deal with the corresponding robust H_∞ control problem of networked systems.

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