Exponential Synchronization of Complex Networks with Markovian Jump and Mixed Delays

Yurong Liu, Zidong Wang* and Xiaohui Liu

Abstract

In this paper, we investigate the exponential synchronization problem for an array of N linearly coupled complex networks with Markovian jump and mixed time-delays. The complex network consists of m modes and the network switches from one mode to another according to a Markovian chain with known transition probability. The mixed time-delays are composed of discrete and distributed delays, both of which are mode-dependent. The nonlinearities imbedded with the complex networks are assumed to satisfy the sector condition that is more general than the commonly used Lipschitz condition. By making use of the Kronecker product and the stochastic analysis tool, we propose a novel Lyapunov-Krasovskii functional suitable for handling distributed delays and then show that the addressed synchronization problem is solvable if a set of linear matrix inequalities (LMIs) are feasible. Therefore, a unified LMI approach is developed to establish sufficient conditions for the coupled complex network to be globally exponentially synchronized in the mean square. Note that the LMIs can be easily solved by using the Matlab LMI toolbox and no tuning of parameters is required. A simulation example is provided to demonstrate the usefulness of the main results obtained.

Keywords

Synchronization; Complex network; Markovian Jumping; Discrete time-delay; Distributed time-delay; Kronecker product; Linear matrix inequality.

I. Introduction

The last decade has witnessed rapidly growing research interests on the dynamics analysis of complex networks since the pioneering work of Watts and Strogatz [37]. The main reason is twofold: 1) complex network exists in our daily life with examples including the Internet, the World Wide Web (WWW), the World Trade Web, linguistic webs and food webs etc; 2) dynamical behaviors of complex networks have found numerous applications in various fields such as physics, technology, and the life sciences. Synchronization has proven to be one of the most important controlling activities to excite the collective behavior of complex dynamical networks, and has received increasing research attention in, for example, the large-scale and complex networks of chaotic oscillators [1,15,19,32], the coupled systems exhibiting spatio-temporal chaos and autowaves [27,39], and the array of coupled neural networks [18,26,38].

On the other hand, time delays often occur in complex networks because of the limited speed of signals travelling through the links [2,10,11,25] and the frequently delayed couplings in biological neural networks, gene regulatory networks, communication networks and electrical power grids [11,17,19]. It has been well known that time delays can cause complex dynamics such as periodic or quasi-periodic motions, Hopf bifurcation and higher-dimensional chaos. According to the way it arises, the time-delay can be generally categorized as two classes: discrete and distributed. Note that continuously distributed delays have gained particular

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attention since a network usually has a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths [34, 35]. Recently, synchronization problems for various networks with discrete and/or distributed time-delays have extensively studied, see e.g. [11,17,19,23,24,38] and the references cited therein.

In reality, complex networks may exhibit a special characteristic called network mode switching. For example, in [4, 30, 36], it has been revealed that a neural network sometimes has finite modes that switch from one to another at different times, and such a switching (or jumping) can be governed by a Markovian chain. In [3], the bufferless packet *switching* of trees and leveled networks has been illustrated to be achievable with certain network topologies. In [5], a sensor network has been shown to have *jumping* behavior due to the network's working environment (normal or hazardous) and the mobility of sensor node. In [9], it has been concluded that the shuffle-exchange networks can model practical interconnection systems due to their size of its *switching* (*jumping*) elements and uncomplicated configuration. In [31], the associative memory of a stochastic Hopfield-like neural automata has been reported to *jump* between pattern attractors. It is worth mentioning that the control and filtering problems for dynamical systems with Markovian jumping parameters have already been widely studied, see e.g. [13, 14, 29, 36] and the references therein. Although complex networks with Markovian jumping parameters have great application potential in a variety of areas, there has been very little existing literature on the synchronization problem for Markovian jumping complex networks with or without mixed time-delays, and the purpose of this paper is therefore to shorten such a gap.

In this paper, we deal with the synchronization problem for an array of coupled complex networks with simultaneous presence of both the discrete and distributed time-delays. The addressed complex network consists of m modes and the network switches from one mode to another according to a Markovian chain with known transition probability. By utilizing a novel Lyapunov-Krasovskii functional and the Kronecker product, we show that the addressed synchronization problem is solvable if a set of linear matrix inequalities (LMIs) are feasible. The main novelty of this paper can be summarized as follows: 1) a new class of complex networks is proposed that contain Markovian jumping parameters; 2) both the discrete and distributed time-delays are considered that are dependent on the jumping mode; 3) rather than the commonly used Lipschitz-type function, a more general sector-like nonlinear function is employed; and 4) a new Lyapunov-Krasovskii functional is exploited to cater the mode-dependent distributed delays. A simulation example is provided to show the usefulness of the proposed global synchronization condition.

Notations: The notations are quite standard. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes matrix transposition and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). Let I_n be the $n \times n$ identity matrix, and $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$. If A is a square matrix, denote by $\lambda_{\max}(A)$ (respectively, $\lambda_{\min}(A)$) means the largest (respectively, smallest) eigenvalue of A. The notation $A \otimes B$ stands for the Kronecker product of matrices A and B. For $h > 0$, $C([-h, 0]; \mathbb{R}^n)$ denotes the family of continuous functions φ from $[-h, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-h \leq \theta \leq 0} |\varphi(\theta)|$, $l_2[0, \infty]$ is the space of square integrable vector. Moreover, let $(\boldsymbol{\Omega}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L_{\mathcal{F}_0}^p([-h, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C([-h, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^p < \infty$ where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathcal{P}$. The asterisk $*$ in a symmetric matrix is used to denote term that is induced by symmetry. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. PROBLEM FORMULATION

Let $r(t)$ $(t \geq 0)$ be a right-continuous Markovian chain on a probability space $(\boldsymbol{\Omega}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ taking values in a finite state space $\mathcal{N} = \{1, 2, ..., m\}$ with generator $\Pi = \{\pi_{ij}\}$ given by

$$P\{r(t + \Delta) = j \mid r(t) = i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \pi_{ij}\Delta + o(\Delta), & \text{if } i = j. \end{cases}$$

Here $\Delta > 0$, and $\pi_{ij} \geq 0$ is the transition rate from i to j if $j \neq i$ while

$$\pi_{ii} = -\sum_{j \neq i} \pi_{ij}.$$

Consider the following complex dynamical network coupled by N identical nodes with Markovian jumping parameters and mixed time-delays:

$$\begin{aligned} \frac{dx_k(t)}{dt} &= f(x_k(t), r(t)) + g(x_k(t - \tau_{1,r(t)})) + \int_{t-\tau_{2,r(t)}}^{t} h(x_k(s))ds \\ &\quad + \sum_{l=1}^{N} w_{kl}\Gamma_{r(t)}x_l(t), \quad k = 1, 2, ..., N, \end{aligned} \tag{1}$$

where $x_k(t) = [x_{k1}(t), x_{k2}(t), \cdots, x_{kn}(t)]^T$ is the state vector of the kth node; $\{r(t), t > 0\}$ is the continuous-time Markov process which describes the evolution of the mode at time t; $f(x_k(t), i) = (f_1(x_{k1}(t), i), ..., f_n(x_{kn}(t), i))^T$, $g(x_k(t-\tau_{1,i})) = (g_1(x_{k1}(t-\tau_{1,i})), ..., g_n(x_{kn}(t-\tau_{1,i})))^T$, $h(x_k(s)) = (h_1(x_{k1}(s)), ..., h_n(x_{kn}(s)))^T$ with $f(\cdot, i)$ $(i = 1, 2, ..., m)$, $g(\cdot)$ and $h(\cdot)$ being continuous nonlinear vector functions; $\tau_{1,r(t)}$ denotes the discrete time delay of the network in the mode $r(t)$, and $\tau_{2,r(t)}$ stands for the distributed time delay in the mode $r(t)$; $\Gamma_{r(t)} = \text{diag}\{\gamma_{1,r(t)}, \gamma_{2,r(t)}, .., \gamma_{n,r(t)}\} \geq 0$ is a matrix linking the lth state variable of each node in the network mode $r(t)$ if $\gamma_{l,r(t)} \neq 0$; and $W = (w_{kl}) \in \mathbb{R}^{N \times N}$ is the coupled configuration matrix of the network with $w_{kl} \geq 0$ $(k \neq l)$ but not all zero.

For notation simplicity, we let

$$\begin{aligned} x(t) &= (x_1^T(t), x_2^T(t), ..., x_N^T(t))^T, \\ F(x(t), r(t)) &= (f^T(x_1(t), r(t)), f^T(x_2(t), r(t)), ..., f^T(x_N(t)), r(t))^T, \\ G(x(t)) &= (g^T(x_1(t)), g^T(x_2(t)), ..., g^T(x_N(t)))^T, \\ H(x(t)) &= (h^T(x_1(t)), h^T(x_2(t)), ..., h^T(x_N(t)))^T. \end{aligned}$$

With the matrix Kronecker product, we can rewrite the network (1) in the following compact form:

$$\frac{dx(t)}{dt} = F(x(t), r(t)) + G(x(t - \tau_{1,r(t)})) + \int_{t-\tau_{2,r(t)}}^{t} H(x(s))ds + (W \otimes \Gamma_{r(t)})x(t). \tag{2}$$

Remark 1: The complex network model (1) or (2) is quite general, which contains the Markovian jumping parameters and the mixed time-delays. Furthermore, the nonlinear function $F(\cdot, \cdot)$, the discrete time-delay and the distributed time delay are all dependent on the Markovian jumping modes. It should be pointed out that the synchronization problem for complex networks with *both* discrete *and* distributed time-delays has not received much attention in the literature, not to mention the case when the Markovian mode jumps are also involved.

Throughout this paper, we make the following assumptions:

(A1) [11, 17] The coupling configuration matrix $W = (w_{ij})$ is symmetric, i.e., $W = W^T$, and satisfies

$$\sum_{l=1}^{N} w_{kl} = \sum_{l=1}^{N} w_{lk} = 0, \quad i = 1, 2, ..., N. \tag{3}$$

(A2) [20, 34, 35] For $\forall x, \ y \in \mathbb{R}^n, \ i \in \{1, 2, ..., m\}$, the nonlinear functions $f(\cdot, i)$, $g(\cdot)$ and $h(\cdot)$ satisfy

$$[f(x, i) - f(y, i) - B_{1,i}(x - y)]^T [f(x, i) - f(y, i) - B_{2,i}(x - y)] \leq 0, \tag{4}$$

$$[g(x) - g(y) - D_1(x - y)]^T [g(x) - g(y) - D_2(x - y)] \leq 0, \tag{5}$$

$$[h(x) - h(y) - V_1(x - y)]^T [h(x) - h(y) - V_2(x - y)] \leq 0, \tag{6}$$

where $B_{1,i}, B_{2,i}, D_1, D_2, V_1, V_2$ are constant matrices with $B_{1,i} - B_{2,i} \geq 0$, $D_1 - D_2 \geq 0$ and $D_1 - D_2 \geq 0$.

Remark 2: The conditions (4)-(6) are known as *sector-like* description of the nonlinearities, which are in a more general form than the usual Lipschitz functions. By adopting such a presentation, it would be possible to reduce the conservatism of the main results caused by quantifying the nonlinear functions via an LMI technique.

Definition 1: The complex network (1) or (2) is said to be globally exponentially synchronized in the mean square if, for any solution $x(t)$, there exist constants $\mu > 0$ and $\alpha > 0$ such that

$$\mathbb{E}\left\{|x_k(t) - x_l(t)|^2\right\} \leq \mu e^{-\alpha t}$$

holds for any $k, l \in \{1, 2, ..., N\}$.

In this paper, our main aim is to deal with the exponential mean-square synchronization problem of the network (1) or (2), and derive LMI-based sufficient conditions that guarantee the network to be exponentially synchronized in the mean square.

III. MAIN RESULTS AND PROOFS

Before stating our main results, we introduce the following lemmas.

Lemma 1: Let $\mathcal{U} = (\alpha_{ij})_{N \times N}, \ P \in \mathbb{R}^{n \times n}, \ x = (x_1^T, x_2^T, ..., x_N^T)^T$ and $y = (y_1^T, y_2^T, ..., y_N^T)^T$ with $x_k, \ y_k \in \mathbb{R}^n (k = 1, 2, ..., N)$. If $\mathcal{U} = \mathcal{U}^T$ and each row sum of $\mathcal{U}$ is zero, then

$$x^T(\mathcal{U} \otimes P)y = -\sum_{1 \leq i < j \leq N} \alpha_{ij}(x_i - x_j)^T P(y_i - y_j).$$

Lemma 2 ([12]) Let $a < b, M \in \mathbb{R}^{n \times n}$ be a positive semi-definite matrix, and $\omega(\cdot) : [a, b] \to \mathbb{R}^n$ be a vector function. If the integrations concerned are well defined, then the following inequality holds:

$$\left(\int_a^b \omega(s)ds\right)^T M \left(\int_a^b \omega(s)ds\right) \leq (b - a)\int_a^b \omega^T(s)M\omega(s)ds. \tag{7}$$

Denote

$$\bar{\pi} = \max_{1 \leq i \leq m}\{|\pi_{ii}|\}, \ \bar{\tau}_1 = \max_{1 \leq i \leq m}\{\tau_{1,i}\}, \ \bar{\tau}_2 = \max_{1 \leq i \leq m}\{\tau_{2,i}\},$$

$$\underline{\tau}_1 = \min_{1 \leq i \leq m}\{\tau_{1,i}\}, \ \underline{\tau}_2 = \min_{1 \leq i \leq m}\{\tau_{2,i}\}, \ \tau = \max\{\bar{\tau}_1, \ \bar{\tau}_2\}.$$

We are now in a position to give the main results of this paper as follows.

Theorem 1: Under Assumptions (A1) and (A2), the network (1) or (2) is globally exponentially synchronized in the mean square if there exist two positive definite matrices Q and R, a set of positive definite matrices P_i $(i = 1, 2, ..., m)$ and three positive scalar constants $\delta_1, \ \delta_2$ and δ_3 such that the following LMIs hold:

$$\Omega(k,l,i) \;=\; \begin{bmatrix} -Nw_{kl}\hat{\Gamma}_i + \bar{P}_i + \Xi_i & P_i + \delta_1\breve{B}_{2,i} & \delta_2\breve{D}_2 & P_i & \delta_3\breve{V}_2 & P_i \\ * & -\delta_1 I & 0 & 0 & 0 & 0 \\ * & * & c_1 Q - \delta_2 I & 0 & 0 & 0 \\ * & * & * & -Q & 0 & 0 \\ * & * & * & * & c_{2i}R - \delta_3 I & 0 \\ * & * & * & * & * & -\frac{1}{\tau_{2,i}}R \end{bmatrix} < 0, \qquad (8)$$

$$(1 \le k < l \le N, \;\; 1 \le i \le m)$$

where

$$\breve{B}_{1,i} \;=\; (B_{1,i}^T B_{2,i} + B_{2,i}^T B_{1,i})/2; \;\; \breve{B}_{2,i} = (B_{1,i}^T + B_{2,i}^T)/2; \tag{9}$$

$$\breve{D}_1 \;=\; (D_1^T D_2 + D_2^T D_1)/2; \;\; \breve{D}_2 = (D_1^T + D_2^T)/2; \tag{10}$$

$$\breve{V}_1 \;=\; (V_1^T V_2 + V_2^T V_1)/2; \;\; \breve{V}_2 = (V_1^T + V_2^T)/2; \tag{11}$$

$$c_1 \;=\; 1 + \bar{\pi}(\bar{\tau}_1 - \underline{\tau}_1), \quad c_{2i} = \tau_{2,i} + \frac{1}{2}\bar{\pi}(\bar{\tau}_2{}^2 - \underline{\tau}_2{}^2); \tag{12}$$

$$\hat{\Gamma}_i \;=\; P_i\Gamma_i + \Gamma_i P_i, \;\; \bar{P}_i = \sum_{1 \le j \le m} \pi_{ij} P_j, \;\; \Xi_i = -\delta_1\breve{B}_{1,i} - \delta_2\breve{D}_1 - \delta_3\breve{V}_1. \tag{13}$$

Proof: Firstly, it can be inferred from (8) that there exists a constant $\vartheta > 0$ such that

$$\Omega_\vartheta(k,l,i) := \Omega(k,l,i) + \vartheta \operatorname{diag}\{\overbrace{1,1,...,1}^{n}, \overbrace{0,0,...,0}^{5n}\} < 0. \tag{14}$$

Since $\{(x(t), r(t)), t \ge 0\}$ is not a Markov process, in order to cast our model into the framework for a Markov system, let us define a new Markov process $\{x_t, r(t), t \ge 0\}$ with

$$x_t(s) = x(t+s), \quad -\tau \le s \le 0.$$

Consider the following stochastic Lyapunov functional candidate for the complex network (1) or (2):

$$V(x_t, t, r(t)) = x^T(t)(U \otimes P_{r(t)})x(t) + \int_{t-\tau_{1,r(t)}}^{t} G^T(x(s))(U \otimes Q)G(x(s))ds$$

$$+ \bar{\pi} \int_{\underline{\tau}_1}^{\bar{\tau}_1} \int_{t-s}^{t} G^T(x(\theta)))(U \otimes Q)G(x(\theta))d\theta ds + \int_{0}^{\tau_{2,r(t)}} \int_{t-s}^{t} H^T(x(\theta))(U \otimes R)H(x(\theta))d\theta ds$$

$$+ \bar{\pi} \int_{\underline{\tau}_2}^{\bar{\tau}_2} \int_{0}^{u} \int_{t-s}^{t} H^T(x(\theta))(U \otimes R)H(x(\theta))d\theta ds du + \varepsilon_0 \int_{0}^{\tau} \int_{t-s}^{t} x^T(\theta)(U \otimes I_n)x(\theta)d\theta ds, (15)$$

where $\varepsilon_0 = \vartheta/\tau$ and $U = \begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & N-1 \end{bmatrix}_{N \times N}.$

Let $\mathscr{L}$ be the weak infinitesimal generator of the random process $\{(x_t, r(t)),\ t \geq 0\}$ along the network (1) or (2). Then,

$$
\begin{aligned}
&\mathscr{L}V(x_t, t, i) \\
&= 2x^T(t)(U \otimes P_i)\left(F(x(t), i) + G(x(t - \tau_{1,i})) + \int_{t-\tau_{2,i}}^{t} H(x(s))ds + (W \otimes \Gamma_i)x(t)\right) \\
&\quad + \sum_{j=1}^{m} \pi_{ij} x^T(t)(U \otimes P_j)x(t) + G^T(x(t))(U \otimes Q)G(x(t)) - G^T(x(t - \tau_{1,i}))(U \otimes Q)G(x(t - \tau_{1,i})) \\
&\quad + \sum_{j=1}^{m} \pi_{ij} \int_{t-\tau_{1,j}}^{t} G^T(x(s))(U \otimes Q)G(x(s))ds + \bar{\pi}(\bar{\tau}_1 - \underline{\tau}_1)G^T(x(t)))(U \otimes Q)G(x(t)) \\
&\quad - \bar{\pi} \int_{\underline{\tau}_1}^{\bar{\tau}_1} G^T(x(t - s)))(U \otimes Q)G(x(t - s))ds + \tau_{2,i}H^T(x(t))(U \otimes R)H(x(t)) \\
&\quad - \int_0^{\tau_{2,i}} H^T(x(t - s))(U \otimes R)H(x(t - s))ds + \sum_{j=1}^{m} \pi_{i,j} \int_0^{\tau_{2,j}} \int_{t-s}^{t} H^T(x(\theta))(U \otimes R)H(x(\theta))d\theta ds \\
&\quad + \bar{\pi} \int_{\underline{\tau}_2}^{\bar{\tau}_2} \int_0^u H^T(x(t))(U \otimes R)H(x(t))ds\,du - \bar{\pi} \int_{\underline{\tau}_2}^{\bar{\tau}_2} \int_0^u H^T(x(t - s))(U \otimes R)H(x(t - s))ds\,du \\
&\quad + \varepsilon_0 \tau x^T(t)(U \otimes I_n)x(t) - \varepsilon_0 \int_0^{\tau} x^T(t - s)(U \otimes I_n)x(t - s)ds.
\end{aligned}
\tag{16}
$$

It is obvious that

$$
-\bar{\pi} \int_{\underline{\tau}_1}^{\bar{\tau}_1} G^T(x(t - s)))(U \otimes Q)G(x(t - s))ds = -\bar{\pi} \int_{t-\bar{\tau}_1}^{t-\underline{\tau}_1} G^T(x(s)))(U \otimes Q)G(x(s))ds,
\tag{17}
$$

$$
- \int_0^{\tau_{2,i}} H^T(x(t - s))(U \otimes R)H(x(t - s))ds = - \int_{t-\tau_{2,i}}^{t} H^T(x(s))(U \otimes R)H(x(s))ds,
\tag{18}
$$

$$
\bar{\pi} \int_{\underline{\tau}_2}^{\bar{\tau}_2} \int_0^u H^T(x(t))(U \otimes R)H(x(t))ds\,du = \frac{1}{2}\bar{\pi}(\bar{\tau}_2^{\,2} - \underline{\tau}_2^{\,2})H^T(x(t))(U \otimes R)H(x(t)),
\tag{19}
$$

$$
-\bar{\pi} \int_{\underline{\tau}_2}^{\bar{\tau}_2} \int_0^u H^T(x(t - s))(U \otimes R)H(x(t - s))ds\,du = -\bar{\pi} \int_{\underline{\tau}_2}^{\bar{\tau}_2} \int_{t-s}^{t} H^T(x(\theta))(U \otimes R)H(x(\theta))d\theta ds,
\tag{20}
$$

$$
-\varepsilon_0 \int_0^{\tau} x^T(t - s)(U \otimes I_n)x(t - s)ds = -\varepsilon_0 \int_{t-\tau}^{t} x^T(s)(U \otimes I_n)x(s)ds.
\tag{21}
$$

Also, it is not difficult to verify that

$$
\begin{aligned}
&\sum_{j=1}^{m} \pi_{ij} \int_{t-\tau_{1,j}}^{t} G^T(x(s))(U \otimes Q)G(x(s))ds \\
&= \sum_{j \neq i}^{m} \pi_{ij} \int_{t-\tau_{1,j}}^{t} G^T(x(s))(U \otimes Q)G(x(s))ds + \pi_{ii} \int_{t-\tau_{1,i}}^{t} G^T(x(s))(U \otimes Q)G(x(s))ds \\
&\leq \sum_{j \neq i}^{m} \pi_{ij} \int_{t-\bar{\tau}_1}^{t} G^T(x(s))(U \otimes Q)G(x(s))ds + \pi_{ii} \int_{t-\underline{\tau}_1}^{t} G^T(x(s))(U \otimes Q)G(x(s))ds \\
&= -\pi_{ii} \int_{t-\bar{\tau}_1}^{t} G^T(x(s))(U \otimes Q)G(x(s))ds + \pi_{ii} \int_{t-\underline{\tau}_1}^{t} G^T(x(s))(U \otimes Q)G(x(s))ds \\
&\leq \bar{\pi} \int_{t-\bar{\tau}_1}^{t-\underline{\tau}_1} G^T(x(s))(U \otimes Q)G(x(s))ds.
\end{aligned}
\tag{22}
$$

Similarly, we have

$$\sum_{j=1}^{m} \pi_{ij} \int_{0}^{\tau_{2,j}} \int_{t-s}^{t} H^T(x(\theta))(U \otimes R)H(x(\theta))d\theta ds \leq \bar{\pi} \int_{\underline{\tau}_2}^{\overline{\tau}_2} \int_{t-s}^{t} H^T(x(\theta))(U \otimes R)H(x(\theta))d\theta ds. \quad (23)$$

Substituting (17)-(23) into (16) yields that

$$
\begin{aligned}
&\mathscr{L}V(x_t, t, i) \\
=~& 2x^T(t)(U \otimes P_i)\left(F(x(t), i) + G(x(t - \tau_{1,i})) + \int_{t-\tau_{2,i}}^{t} H(x(s))ds + (W \otimes \Gamma_i)x(t) \right) \\
&+ \sum_{j=1}^{m} \pi_{ij}x^T(t)(U \otimes P_j)x(t) + \varepsilon_0\tau x^T(t)(U \otimes I_N)x(t) + (1 + \bar{\pi}(\overline{\tau}_1 - \underline{\tau}_1))G^T(x(t)))(U \otimes Q)G(x(t)) \\
&- G^T(x(t - \tau_{1,i}))(U \otimes Q)G(x(t - \tau_{1,i})) + (\tau_{2,i} + \frac{1}{2}\bar{\pi}(\overline{\tau}_2{}^2 - \underline{\tau}_2{}^2))H^T(x(t))(U \otimes R)H(x(t)) \\
&- \int_{t-\tau_{2,i}}^{t} H^T(x(s))(U \otimes R)H(x(s))ds - \varepsilon_0 \int_{t-\tau}^{t} x^T(s)(U \otimes I_N)x(s)ds. \quad (24)
\end{aligned}
$$

Noting that $UW = NW$, we have

$$(U \otimes P_i)(W \otimes \Gamma_i) = (UW) \otimes (P_i\Gamma_i) = (NW) \otimes (P_i\Gamma_i). \quad (25)$$

With Lemma 1 and (25), we can deduce from (24) that

$$
\begin{aligned}
&\mathscr{L}V(x_t, t, i) \\
=~& \sum_{1 \leq k < l \leq N} \Bigg[2(x_k(t) - x_l(t))^T \Big(P_i\big(f(x_k(t), i) - f(x_l(t), i)\big) + P_i\big(g(x_k(t - \tau_{1,i})) - g(x_l(t - \tau_{1,i}))\big) \\
&+ P_i \int_{t-\tau_{2,i}}^{t} \big(h(x_k(s)) - h(x_l(s))\big)ds - Nw_{kl}P_i\Gamma_i(x_k(t) - x_l(t)) \Big) + (x_k(t) - x_l(t))^T \bar{P}_i(x_k(t) - x_l(t)) \\
&+ \big(g(x_k(t)) - g(x_l(t))\big)^T c_1 Q\big(g(x_k(t)) - g(x_l(t))\big) + \big(h(x_k(t)) - h(x_l(t))\big)^T c_{2i} R\big(h(x_k(t)) - h(x_l(t))\big) \\
&- \int_{t-\tau_{2,i}}^{t} \big(h(x_k(s)) - h(x_l(s))\big)^T R\big(h(x_k(s)) - h(x_l(s))\big)ds + \varepsilon_0\tau(x_k(t) - x_l(t))^T(x_k(t) - x_l(t)) \\
&- \big(g(x_k(t - \tau_{1,i})) - g(x_l(t - \tau_{1,i}))\big)^T Q\big(g(x_k(t - \tau_{1,i})) - g(x_l(t - \tau_{1,i}))\big) \\
&- \varepsilon_0 \int_{t-\tau}^{t} (x_k(s) - x_l(s))^T(x_k(s) - x_l(s))ds \Bigg]. \quad (26)
\end{aligned}
$$

From Lemma 2, it follows that

$$
\begin{aligned}
&- \int_{t-\tau_{2,i}}^{t} \big(h(x_k(s)) - h(x_l(s))\big)^T R\big(h(x_k(s)) - h(x_l(s))\big)ds \\
\leq~& -\frac{1}{\tau_{2,i}} \left[\int_{t-\tau_{2,i}}^{t} \big(h(x_k(s)) - h(x_l(s))\big)^T ds \right] R \left[\int_{t-\tau_{2,i}}^{t} \big(h(x_k(s)) - h(x_l(s))\big)ds \right]. \quad (27)
\end{aligned}
$$

With the relationships (26) and (27), we have

$$\mathscr{L}V(x_t, t, i) \leq \sum_{1 \leq k < l \leq N} \left[\xi_{k,l,i}^T(t)\Omega_1(k, l, i)\xi_{k,l,i}(t) - \varepsilon_0 \int_{t-\tau}^{t} (x_k(s) - x_l(s))^T(x_k(s) - x_l(s))ds \right], \quad (28)$$

where

$$\xi_{k,l,i}(t) = \left[(x_k(t) - x_l(t))^T \quad (f(x_k(t),i) - f(x_l(t),i))^T \quad (g(x_k(t)) - g(x_l(t)))^T \right.$$
$$(g(x_k(t - \tau_{1,i})) - g(x_l(t - \tau_{1,i})))^T \quad (h(x_k(t)) - h(x_l(t)))^T$$
$$\left. \int_{t-\tau_{2,i}}^{t} (h(x_k(s)) - h(x_l(s)))^T ds \right]^T, \tag{29}$$

$$\Omega_1(k,l,i) = \begin{bmatrix} -Nw_{kl}\hat{\Gamma}_i + \bar{P}_i + \varepsilon_0 \tau I_n & P_i & 0 & P_i & 0 & P_i \\ * & 0 & 0 & 0 & 0 & 0 \\ * & * & c_1 Q & 0 & 0 & 0 \\ * & * & * & -Q & 0 & 0 \\ * & * & * & * & c_{2i}R & 0 \\ * & * & * & * & * & -\frac{1}{\tau_{2,i}}R \end{bmatrix} \tag{30}$$

with $\hat{\Gamma}_i$ being defined in (13).

Notice that (4)-(6) implies

$$\begin{bmatrix} x_k(t) - x_l(t) \\ f(x_k(t),i) - f(x_l(t),i) \end{bmatrix}^T \begin{bmatrix} \breve{B}_{1,i} & -\breve{B}_{2,i} \\ -\breve{B}_{2,i}^T & I \end{bmatrix} \begin{bmatrix} x_k(t) - x_l(t) \\ f(x_k(t),i) - f(x_l(t),i) \end{bmatrix} \leq 0, \tag{31}$$

$$\begin{bmatrix} x_k(t) - x_l(t) \\ g(x_k(t)) - g(x_l(t)) \end{bmatrix}^T \begin{bmatrix} \breve{D}_1 & -\breve{D}_2 \\ -\breve{D}_2^T & I \end{bmatrix} \begin{bmatrix} x_k(t) - x_l(t) \\ g(x_k(t)) - g(x_l(t)) \end{bmatrix} \leq 0, \tag{32}$$

$$\begin{bmatrix} x_k(t) - x_l(t) \\ h(x_k(t)) - h(x_l(t)) \end{bmatrix}^T \begin{bmatrix} \breve{V}_{1,i} & -\breve{V}_{2,i} \\ -\breve{V}_{2,i}^T & I \end{bmatrix} \begin{bmatrix} x_k(t) - x_l(t) \\ h(x_k(t)) - h(x_l(t)) \end{bmatrix} \leq 0, \tag{33}$$

where $\breve{B}_{1,i}, \breve{B}_{2,i}, \breve{D}_1, \breve{D}_2, \breve{V}_1, \breve{V}_2$ are defined in (9)-(11). Then, from (28) and (31)-(33), we can infer that

$$\mathscr{L}V(x_t,t,i) \leq \sum_{1 \leq k < l \leq N} \left\{ \xi_{k,l,i}^T(t)\Omega_1(k,l,i)\xi_{k,l,i}(t) - \varepsilon_0 \int_{t-\tau}^{t} (x_k(s) - x_l(s))^T(x_k(s) - x_l(s))ds \right.$$

$$+ \begin{bmatrix} x_k(t) - x_l(t) \\ f(x_k(t),i) - f(x_l(t),i) \end{bmatrix}^T \begin{bmatrix} -\delta_1 \breve{B}_{1,i} & \delta_1 \breve{B}_{2,i} \\ \delta_1 \breve{B}_{2,i} & -\delta_1 I \end{bmatrix} \begin{bmatrix} x_k(t) - x_l(t) \\ f(x_k(t),i) - f(x_l(t),i) \end{bmatrix}$$

$$+ \begin{bmatrix} x_k(t) - x_l(t) \\ g(x_k(t)) - g(x_l(t)) \end{bmatrix}^T \begin{bmatrix} -\delta_2 \breve{D}_1 & \delta_2 \breve{D}_2 \\ \delta_2 \breve{D}_2 & -\delta_2 I \end{bmatrix} \begin{bmatrix} x_k(t) - x_l(t) \\ g(x_k(t)) - g(x_l(t)) \end{bmatrix}$$

$$\left. + \begin{bmatrix} x_k(t) - x_l(t) \\ h(x_k(t)) - h(x_l(t)) \end{bmatrix}^T \begin{bmatrix} -\delta_3 \breve{V}_1 & \delta_3 \breve{V}_2 \\ \delta_3 \breve{V}_2 & -\delta_3 I \end{bmatrix} \begin{bmatrix} x_k(t) - x_l(t) \\ h(x_k(t)) - h(x_l(t)) \end{bmatrix} \right\}$$

$$= \sum_{1 \leq k < l \leq N} \left[\xi_{k,l,i}^T(t)\Omega_\vartheta(k,l,i)\xi_{k,l,i}(t) - \varepsilon_0 \int_{t-\tau}^{t} (x_k(s) - x_l(s))^T(x_k(s) - x_l(s))ds \right], \tag{34}$$

where $\Omega_\vartheta(k,l,i)$ is defined in (14).

We are now ready to deal with the exponential synchronization of the network (1) or (2). For this purpose, we consider the following functional

$$\mathscr{V}(x(t),t,r(t)) = e^{\varepsilon t}V(x(t),t,r(t)), \tag{35}$$

where $\varepsilon > 0$ is a constant to be determined. Obviously, we can see that

$$\mathscr{L}\mathscr{V}(x(t),t,i) = e^{\varepsilon t}\mathscr{L}V(x(t),t,i) + \varepsilon e^{\varepsilon t}V(x(t),t,i). \tag{36}$$

From (5) and (6), it can be inferred that there exists a constant $\alpha > 0$ such that

$$|g(x) - g(y)| \leq \alpha|x - y|, \text{ and } |h(x) - h(y)| \leq \alpha|x - y|, \ \forall x, \ y \in \mathbb{R}^n.$$

Set

$$c_2 = \max_{1 \leq i \leq m} \{c_{2i}\}, \lambda_0 = \max_{1 \leq i \leq m} \{\lambda_{\max}(P_i)\}, \quad c_0 = c_1\lambda_{\max}(Q)\alpha^2 + c_2\lambda_{\max}(R)\alpha^2 + \varepsilon_0\tau.$$

Then, for $V(x(t), t, i)$, we have

$$V(x(t), t, i)$$

$$\leq x^T(t)(U \otimes P_i)x(t) + \int_{t-\tau}^{t} G^T(x(s))(U \otimes Q)G(x(s))ds$$

$$+\bar{\pi}\int_{\underline{\tau}_1}^{\bar{\tau}_1}\int_{t-\tau}^{t} G^T(x(\theta)))(U \otimes Q)G(x(\theta))d\theta ds + \int_{0}^{\tau_{2,i}}\int_{t-\tau}^{t} H^T(x(\theta))(U \otimes R)H(x(\theta))d\theta ds$$

$$+\bar{\pi}\int_{\underline{\tau}_2}^{\bar{\tau}_2}\int_{0}^{u}\int_{t-\tau}^{t} H^T(x(\theta))(U \otimes R)H(x(\theta))d\theta ds du + \varepsilon_0\int_{0}^{\tau}\int_{t-\tau}^{t} x^T(\theta)(U \otimes I_n)x(\theta)d\theta ds$$

$$= x^T(t)(U \otimes P_i)x(t) + c_1\int_{t-\tau}^{t} G^T(x(s))(U \otimes Q)G(x(s))ds$$

$$+c_{2i}\int_{t-\tau}^{t} H^T(x(s))(U \otimes R)H(x(s))ds + \varepsilon_0\tau\int_{t-\tau}^{t} x^T(s)(U \otimes I_n)x(s)ds$$

$$= \sum_{1 \leq k < l \leq N}\left[(x_k(t) - x_l(t))^T P_i(x_k(t) - x_l(t)) + c_1\int_{t-\tau}^{t}\left(g(x_k(s)) - g(x_l(s))\right)^T Q\left(g(x_k(s)) - g(x_l(s))\right)ds\right.$$

$$\left. + c_{2i}\int_{t-\tau}^{t}\left(h(x_k(s)) - h(x_l(s))\right)^T R\left(h(x_k(s)) - h(x_l(s))\right)ds + \varepsilon_0\tau\int_{t-\tau}^{t}(x_k(s) - x_l(s))^T(x_k(s) - x_l(s))ds\right]$$

$$\leq \sum_{1 \leq k < l \leq N}\left[\lambda_{\max}(P_i)|x_k(t) - x_l(t)|^2 + c_1\lambda_{\max}(Q)\int_{t-\tau}^{t}\left|g(x_k(s)) - g(x_l(s))\right|^2 ds\right.$$

$$\left. + c_{2i}\lambda_{\max}(R)\int_{t-\tau}^{t}\left|h(x_k(s)) - h(x_l(s))\right|^2 ds + \varepsilon_0\tau\int_{t-\tau}^{t}\left|x_k(s) - x_l(s)\right|^2 ds\right]$$

$$\leq \sum_{1 \leq k < l \leq N}\left[\lambda_0|x_k(t) - x_l(t)|^2 + c_0\int_{t-\tau}^{t}\left|x_k(s) - x_l(s)\right|^2 ds\right]. \tag{37}$$

Now, choose ε such that the following inequalities holds:

$$\varepsilon\lambda_0 + \lambda_{\max}(\Omega_\vartheta(k, l, i)) \leq 0 \quad (1 \leq k < l \leq N, \ 1 \leq i \leq m), \quad \varepsilon c_0 - \varepsilon_0 \leq 0. \tag{38}$$

Then, it follows readily from (34), (36) and (37) that

$$\mathscr{L}\mathscr{V}(x(t), t, i) \leq 0.$$

By the generalized Itô formula (cf. [16, 28]), we have

$$\mathbb{E}\mathscr{V}(x(t), t, r(t)) = \mathbb{E}\mathscr{V}(x(0), 0, r(0)) + \mathbb{E}\int_{0}^{t}\mathscr{L}\mathscr{V}(x(s), s, r(s))ds$$

$$\leq \mathbb{E}\mathscr{V}(x(0), 0, r(0))$$

which implies that

$$e^{\varepsilon t}\mathbb{E}V(x(t), t, r(t)) \leq \mathbb{E}V(x(0), 0, r(0)). \tag{39}$$

Furthermore, it follows from (37) that

$$\mathbb{E}V(x(0), 0, r(0)) \leq \mu_0,\qquad(40)$$

where

$$\mu_0 = \sum_{1 \leq k < l \leq N} \left[\lambda_0 \mathbb{E}\left\{|x_k(0) - x_l(0)|^2\right\} + c_0 \mathbb{E} \int_{-\tau}^0 |x_k(s) - x_l(s)|^2 ds \right].$$

Letting $\rho = \min_{1 \leq i \leq m} \lambda_{\min}(P_i)$, we have

$$\begin{aligned}
\mathbb{E}V(x(t), t, r(t)) &\geq \mathbb{E}\left\{x^T(t)(U \otimes P_i)x(t)\right\} \\
&= \sum_{1 \leq k < l \leq N} \mathbb{E}\left\{(x_k(t) - x_l(t))^T P_i(x_k(t) - x_l(t))\right\} \\
&\geq \rho \sum_{1 \leq k < l \leq N} \mathbb{E}\left\{|x_k(t) - x_l(t)|^2\right\}.
\end{aligned}\qquad(41)$$

This, together with (39) and (40), implies that

$$\sum_{1 \leq k < l \leq N} \mathbb{E}\left\{|x_k(t) - x_l(t)|^2\right\} \leq \frac{\mu_0}{\rho} e^{-\varepsilon t},$$

and therefore the network (1) or (2) is exponentially synchronized in the mean square. ∎

Remark 3: In Theorem 1, the criteria are established that ensures that the coupled Markovian jumping complex network with mixed delays is globally synchronized in the mean square for all admissible parameter jumps and mode-dependent time-delays, and the criteria are expressed in terms of the solution to a set of LMIs. Note that LMIs can be effectively solved and checked by the algorithms such as the interior-point method from Matlab toolbox.

In order to show the generality of the main results obtained in Theorem 1, we consider two special cases for the complex network (1) or (2).

Case 1. In the network (1) or (2), let's assume that the network evolves at one mode only. In other words, there is no Markovian mode jumping. To be precise, in Eq. (2), let $F(x(t), r(t))$, $\tau_{1,(r)}$, $\tau_{2,(r)}$ and $\Gamma_{r(t)}$ be replaced by $F(x(t))$, τ_1, τ_2 and Γ, respectively. Then, Eq. (2) can be changed to

$$\frac{dx(t)}{dt} = F(x(t)) + G(x(t - \tau_1)) + \int_{t - \tau_2}^t H(x(s))ds + (W \otimes \Gamma)x(t).\qquad(42)$$

Furthermore, in Assumption (A2), the condition (4) is replaced by

$$[f(x) - f(y) - B_1(x - y)]^T [f(x) - f(y) - B_2(x - y)] \leq 0,\qquad(43)$$

where B_1 and B_2 are constant matrices with $B_1 - B_2 \geq 0$.

For the synchronization problem of the network (42), we have following results without proof.

Corollary 1: Under Assumptions (A1)-(A2) except with the condition (4) replaced by (43), the complex network (42) is globally exponentially synchronized if there exist three positive definite matrices $P > 0, Q > 0$ and $R > 0$, and three positive scalar constants δ_1, δ_2 and δ_3 such that the following LMIs hold:

$$\Phi(k, l) = \begin{bmatrix} -Nw_{kl}(P\Gamma + \Gamma P) + \Xi & P + \delta_1 \breve{B}_2 & \delta_2 \breve{D}_2 & P & \delta_3 \breve{V}_2 & P \\ * & -\delta_1 I & 0 & 0 & 0 & 0 \\ * & * & Q - \delta_2 I & 0 & 0 & 0 \\ * & * & * & -Q & 0 & 0 \\ * & * & * & * & \tau_2 R - \delta_3 I & 0 \\ * & * & * & * & * & -\frac{1}{\tau_2} R \end{bmatrix} < 0,$$

$$(1 \leq k < l \leq N)$$

where $\breve{B}_1 = (B_1^T B_2 + B_2^T B_1)/2$, $\breve{B}_2 = (B_1^T + B_2^T)/2$, $\Xi = -\delta_1 \breve{B}_1 - \delta_2 \breve{D}_1 - \delta_3 \breve{V}_1$, and $\breve{D}_1$, $\breve{D}_2$, $\breve{V}_1$ and $\breve{V}_2$ are defined as in (10) and (11).

Case 2. In this case, we consider the same array of complex networks with discrete time-delay only. Then, the complex network (1) reduces to

$$\frac{dx(t)}{dt} = F(x(t), r(t)) + G(x(t - \tau_{1,r(t)})) + (W \otimes \Gamma_{r(t)})x(t). \tag{44}$$

For the network (44), the following result follows readily.

Corollary 2: Under Assumptions (A1)-(A2), the network (44) is globally exponentially synchronized in the mean square if there exists a positive definite matrices $Q > 0$, a set of positive definite matrices P_i $(i = 1, 2, ..., m)$, and two positive scalar constants δ_1 and δ_2 such that the following LMIs hold:

$$\Psi(k, l, i) = \begin{bmatrix} -Nw_{kl}(P\Gamma_i + \Gamma_i P) + \bar{P}_i + \Theta_i & P_i + \delta_1 \breve{B}_{2,i} & \delta_2 \breve{D}_2 & P_i \\ * & -\delta_1 I & 0 & 0 \\ * & * & c_1 Q - \delta_2 I & 0 \\ * & * & * & -Q \end{bmatrix} < 0,$$

$$(1 \le k < l \le N, \quad 1 \le i \le m)$$

where $\Theta_i = -\delta_1 \breve{B}_{1,i} - \delta_2 \breve{D}_1$, and $\breve{B}_{1,i}$, $\breve{B}_{2,i}$, $\breve{D}_1$, $\breve{D}_2$, $\bar{P}_i$ and c_1 are defined as in Theorem 1.

Remark 4: In this paper, the complex network consists of m modes and the network switches from one mode to another according to a Markovian chain with known transition probability. In our future work, we could further consider two independent Markovian jumping modes for *node switching* as well as *network topology switching*, which would reflect more features of the complexity. On the other hand, many networks in real world are directed and the study about directed dynamical networks is therefore quite interesting and useful. One of our future research topics would be the extension of our results to directed dynamical networks by referring to the techniques used in [21, 22].

IV. A NUMERICAL EXAMPLE

In this section, we present a simulation example so as to illustrate the usefulness of our main results.

For the sake of simplification, we consider a 3-node network with Markovian switching between two modes. The *uncoupled* dynamical subsystem of the kth node $(k = 1, 2, 3)$ in the network is given by

$$\begin{aligned} \dot{x}_{k1} &= \alpha_{r(t)}(x_{k2} - x_{k1} - \sigma(x_{k1})) + \int_{t-\tau_{2,r(t)}}^{t} \tanh(0.6 x_{k1}(s)) ds, \\ \dot{x}_{k2} &= x_{k1} - x_{k2} + x_{k3} + \int_{t-\tau_{2,r(t)}}^{t} 0.2 \cos(2 x_{k2}(s)) ds, \\ \dot{x}_{k3} &= -25.58 x_{k2} - 0.8 \sin(2 x_{k1}(t - \tau_{1,r(t)})) + \int_{t-\tau_{2,r(t)}}^{t} 0.5 \tanh(0.6 x_{k3}(s)) ds, \end{aligned} \tag{45}$$

where

$$\begin{aligned} \alpha_1 &= 15, \ \alpha_2 = 14, \\ \tau_{1,1} &= 4, \ \tau_{1,2} = 5, \\ \tau_{2,1} &= 0.5, \ \tau_{2,2} = 0.4, \\ \sigma(s) &= bs + \frac{1}{2}(a - b)(|s + 1| - |s - 1|) \end{aligned}$$

with $a = -5/7$ and $b = -8/7$. The phase portraits of system (45) with $r(t) = 1$ and $r(t) = 2$ are given in Fig. 1.

Remark 5: System (45) without time delay terms is just the well-known Chua's circuit which possesses a chaotic attractor [6]. The dynamics analysis problems for system (45) with discrete time-delay but without distributed time-delay terms has been investigated in the literature, see, e.g., [8, 33].

Now let us turn to the coupled complex network (1) or (2). Besides the parameters specified already, we further take

$$\Pi = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}, \ W = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \ \Gamma_1 = \text{diag}\{15, 15, 15\}, \ \Gamma_2 = \text{diag}\{10, 10, 10\}.$$

Obviously, in (45), we have

$$\begin{aligned} f(x_k, r(t)) &= [\alpha_{r(t)}(x_{k2} - x_{k1} - \sigma(x_{k1})), \ x_{k1} - x_{k2} + x_{k3}, \ -25.58 x_{k2}]^T, \\ g(x_k) &= [0, \ 0, \ -0.8\sin(2x_{k1})]^T, \\ h(x_k) &= [\tanh(0.6x_{k1}), \ 0.2\cos(2x_{k2}), 0.5\tanh(0.6x_{k3})]^T. \end{aligned}$$

With the parameters given above, it can be verified that

$$B_{1,1} = \begin{bmatrix} -15(1 - 8/7) & 15 & 0 \\ 1 & -1 & 1 \\ 0 & -25.58 & 0 \end{bmatrix}, \ B_{2,1} = \begin{bmatrix} -15(1 - 5/7) & 15 & 0 \\ 1 & -1 & 1 \\ 0 & -25.58 & 0 \end{bmatrix},$$

$$B_{1,2} = \begin{bmatrix} -14(1 - 8/7) & 15 & 0 \\ 1 & -1 & 1 \\ 0 & -25.58 & 0 \end{bmatrix}, \ B_{2,2} = \begin{bmatrix} -14(1 - 5/7) & 15 & 0 \\ 1 & -1 & 1 \\ 0 & -25.58 & 0 \end{bmatrix},$$

$D_1 = \text{diag}\{1.6, \ 0, \ 0\}, \ D_2 = \text{diag}\{-1.6, \ 0, \ 0\}, \ V_1 = \text{diag}\{0.6, \ 0.4, \ 0.3\}, \ V_2 = \text{diag}\{0, \ -0.4, \ 0\},$

$\bar{\tau}_1 = 5, \ \underline{\tau}_1 = 4, \ \bar{\tau}_2 = 0.5, \ \underline{\tau}_2 = 0.4, \ \tau = 5, \ \bar{\pi} = 4.$

By using the Matlab LMI toolbox, we solve the LMI (8) and obtain the feasible solutions as follows:

$$P_1 = \begin{bmatrix} 0.1819 & -0.0198 & 0.0360 \\ -0.0198 & 0.0746 & 0.0390 \\ 0.0360 & 0.0390 & 0.1777 \end{bmatrix}, \ P_2 = \begin{bmatrix} 0.1428 & -0.0090 & 0.0490 \\ -0.0090 & 0.1581 & 0.0209 \\ 0.0490 & 0.0209 & 0.1538 \end{bmatrix},$$

$$Q = 10^{-3} \begin{bmatrix} 0.0057 & 0.0000 & 0.0022 \\ 0.0000 & 0.0066 & -0.0000 \\ 0.0022 & -0.0000 & 0.0031 \end{bmatrix}, \ R = \begin{bmatrix} 60.8122 & -0.0001 & 0.0912 \\ -0.0001 & 123.5552 & 0.0001 \\ 0.0912 & 0.0001 & 115.9344 \end{bmatrix},$$

$\delta_1 = 0.0755, \ \delta_2 = 0.0577, \ \delta_3 = 186.6805.$

Therefore, it follows from Theorem 1 that the network (1) or (2) with given parameters is globally exponentially synchronized in the mean square.

To confirm the theoretical results via simulation, we let $err_{kl}(t) = |x_k(t) - x_l(t)|$, and

$$[x_1^T(s) \quad x_2^T(s) \quad x_3^T(s)]^T = [-1.6; 0; 1.6; -1.4; -0.2; 1.8; -1.8; 0.2; 1.4]^T, \ -5 \leq s \leq 0.$$

Then, it can be seen from Fig. 2 that the synchronization errors between different nodes indeed approach zero, and our objective of synchronization is well achieved.
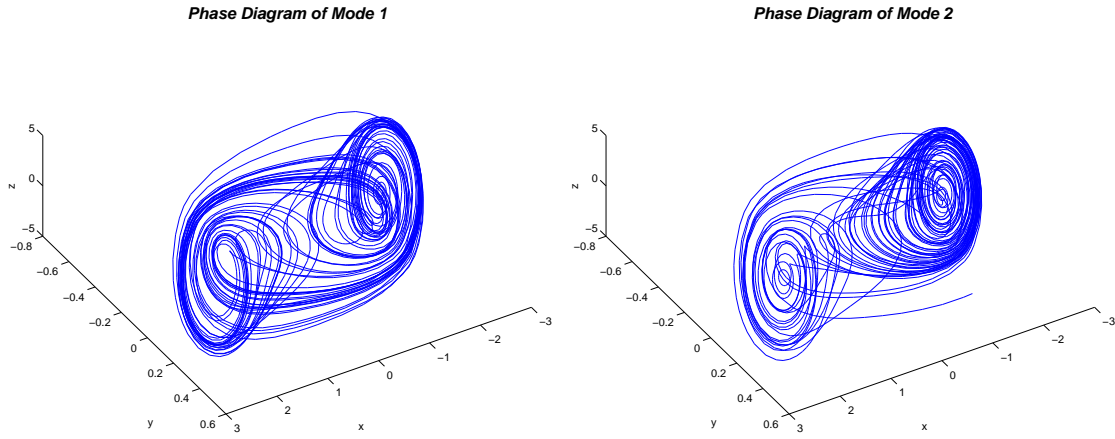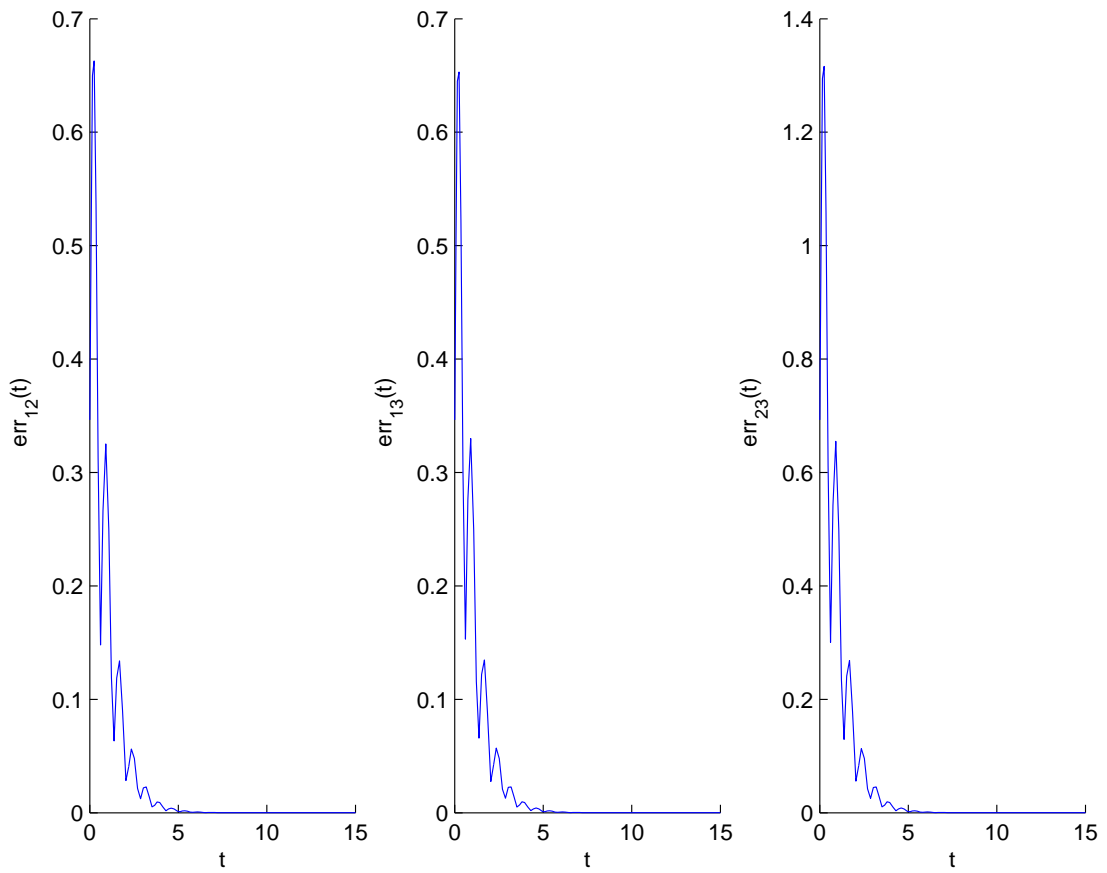
Fig. 1.   Phase Diagrams of the Two Modes.

Fig. 2.   State trajectory of the synchronization error.

V. Conclusions

In this paper, we have investigated the exponential synchronization problem for an array of linearly coupled complex networks with Markovian jumping and mixed time-delays. Both the discrete and distributed time-delays are dependent on network modes. By utilizing a novel Lyapunov-Krasovskii functional and the Kronecker product, we have shown that the addressed synchronization problem is solvable if a set of linear matrix inequalities (LMIs) are feasible. A unified LMI approach has been developed to establish sufficient conditions for the coupled complex networks to be globally exponentially synchronized in the mean square,

and a simulation example has been provided to show the usefulness of the proposed global exponential synchronization condition.

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