Observer-based Networked Control for Continuous-time Systems with Random Sensor Delays*

Chong Lin†, Zidong Wang‡ and Fuwen Yang§

Abstract - This paper is concerned with the networked control system design for continuous-time systems with random measurement, where the measurement channel is assumed to subject to random sensor-delay. A design scheme for the observer-based output feedback controller is proposed to render the closed-loop networked system exponentially mean-square stable with H_∞ performance requirement. The technique employed is based on appropriate delay systems approach combined with a matrix variable decoupling technique. The design method is fulfilled through solving linear matrix inequalities. A numerical example is used to verify the effectiveness and the merits of the present results.

Keywords: Networked control, random measurement, observer, output feedback control, exponentially mean-square stability, H_∞ control.

1 Introduction

The integrated study on networks and control is a very challenging and promising research area [31]. Networked control systems (NCSs), connected over networked media, have received increasing research interest. Wide applications of NCSs include manufacturing plants, vehicles and crafts, communication networks, and internet-based control owing to NCSs' advantages such as low cost, reduced weight and power requirements, simple installation and maintenance as well as high reliability [3, 23, 26]. So far, there has been considerable research work appeared to address modelling, stability analysis, control and filtering problems for NCSs, see [1–5, 11, 22, 31–34] and the references therein. Most of studies on NCSs have concentrated on state feedbacks [3, 11, 27, 32–36], and the commonly investigated systems has been discrete-time models, sampled-data models, continuous-time models through sampled-data feedback controls. Upon unavailable state information, observer-based feedbacks have to be performed to achieve control purposes [21, 36].

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†C. Lin is with the Institute of Complexity Science, Qingdao University, Qingdao 266071, P. R. China. Email: linchong_2004@hotmail.com

‡Z. Wang is with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex, UB8 3PH, U.K. Email: Zidong.Wang@brunel.ac.uk

§F. Yang is with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex, UB8 3PH, U.K. Email: Fuwen.Yang@brunel.ac.uk

It is well-known that, due to connections over communication channels, the network induced delay and the "random" phenomenon are two common and important features in a NCS. By "random" phenomenon we mean the occurrence of random communication delay [24, 29, 35], random measurement [22, 25], random packet losses [20, 26, 27]. The random measurement was studied as early as in [19] for estimation problems, and this trend recently attracted much attention in filtering [22, 24, 25] and control designs [29, 30]. In such a random case, the overall NCS is no longer a deterministic model, and this evidently bring difficulties to the control design, as the difficulties are not only entailed in control designs for deterministic models but also encountered in stochastic parts. Recently, the observer-based feedback controls have been further studied for discrete-time NCSs with random measurements and time delays [29, 30]. In [29], the closed-loop system was transformed into a delay-free model, and an observer-based H_∞ control design scheme was given in terms of a linear matrix inequality (LMI) to render the closed-loop systems exponentially mean-square stable. In [30], an LMI-based robust H_∞ dynamic output feedback control design was provided using discrete time-delay system approach. However, for continuous-time NCSs with random measurements and time delays, there has been little theoretical work appeared on effective observer-based control designs using continuous-time domain approach. It has been revealed through our investigation that the extension to continuous-time settings involves much difficulty which needs more restrictive steps to obtain effective LMI design schemes.

Motivated by the above observations, in this paper, we study the dynamic output feedback control problem for NCSs with random measurements and time delays directly from continuous-time systems approach. This problem is not solved in the NCS literature. It should be mentioned that some recent techniques in the design of dynamic controls and filters adopting Lyapunov functionals with partial information of the closed-loop states could simplify the designs but unavoidably bring conservatism in solving related problems [25,30]. To overcome this drawback, we will use appropriate Lyapunov functionals with full information of the closed-loop states to increase the solvability of the control design. We will present sufficient conditions in terms of LMIs for control designs such that the closed-loop system is exponentially mean-square stable and meanwhile satisfies the desired disturbance attenuation level. The technique used is the stochastic theory [14,16] combined with appropriate variable decoupling method. We also adopt an appropriate free-weighting matrix method [9] suitable for the derivation of the main results for our considered problem. Moreover, we will point out the conservative steps in deriving our results and pay considerable efforts in reducing the possible conservatism. The contribution of this work mainly lies in that it fills in the gap of observer-based feedback control for NCSs in continuous-time system settings with random measurements and time delays. Furthermore, when reduced to special cases of the stability study for NCSs, the present work also implies a less conservative method for the stability test. We will give a numerical example to illustrate the effectiveness and the merits of the present results.

Notation: R^n denotes the n-dimensional real Euclidean space; $\mathcal{L}_2[0,\infty)$ is the space of square integrable vectors; I_p and $0_{p\times q}$ are, respectively, the $p \times p$ dimensional identity matrix and the $p \times q$ dimensional zero matrix (In case of no confusing, we also use I and 0 to denote, respectively, the identity matrix and the zero matrix with compatible dimensions.); the superscripts 'T' and '-1' stand for the matrix transpose and inverse, respectively; $W > 0$ ($W \geq 0$, $W < 0$ and

$W \leq 0$, respectively) means that W is a real symmetric positive definite (positive semi-definite, negative definite, negative semi-definite, respectively) matrix; $\|\cdot\|$ is the spectral norm; $\mathcal{E}\{\cdot\}$ denotes the expectation and $Prob\{\cdot\}$ means the probability; $\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$ denote, respectively, the maximum eigenvalue and the minimum eigenvalue of a matrix.

2 Problem Formulation

Consider a continuous-time system given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_{xw}w(t), \\ z(t) &= C_z x(t) + B_z u(t) + B_{zw}w(t), \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $z \in R^p$ and $w \in R^q$ are the state, the control input, the controlled output and the disturbance input belonging to $\mathcal{L}_2[0, \infty)$, respectively; A, B, B_{xw}, B_z, B_{zw} and C_z are constant real matrices with appropriate dimensions. The measured output $y(t) \in R^r$ may or may not experience sensor delay, and it can be described by two random events:

$$\begin{cases} \text{Event 1}: & y(t) \text{ does not experience sensor delay,} \\ \text{Event 2}: & y(t) \text{ experiences sensor delay.} \end{cases}$$

Let the stochastic variable $\delta(t)$ be defined as

$$\delta(t) = \begin{cases} 1, & \text{if Event 1 occurs,} \\ 0, & \text{if Event 2 occurs.} \end{cases}$$

As remarked in [35], $\delta(t)$ is a Markovian process and can be assumed to follow an exponential distribution of switchings, which satisfies

$$\begin{aligned} &Prob\{\delta(t) = 1\} = \mathcal{E}\{\delta(t)\} = \bar{\delta}, \\ &Prob\{\delta(t) = 0\} = 1 - \mathcal{E}\{\delta(t)\} = 1 - \bar{\delta}, \end{aligned} \tag{2}$$

where the constant $\bar{\delta} \in [0, 1]$ reflects the occurrence probability of the event of no sensor delays. We assume for a more general case that the measurement is described by

$$y(t) = \delta(t)Cx(t) + (1 - \delta(t))Dx(t - \tau_t) + B_{yw}w(t), \tag{3}$$

where C, D and B_{yw} are constant matrices with appropriate dimensions, τ_t stands for the time-varying delay in the measurement channel which is piecewise continuous and may be unknown. We remark that the random measurement mode in (3) can also be interpreted as that the sensor has two random measurement channels.

In this paper, we consider two cases for the time delay τ_t:

- *Case 1:* There exist scalars τ_1, τ_2 and h with $\tau_2 > \tau_1 \geq 0$ such that

$$\tau_1 \leq \tau_t \leq \tau_2, \quad \dot{\tau}_t \leq h. \tag{4}$$

- *Case 2:* There exist scalars τ_1 and τ_2 with $\tau_2 > \tau_1 \geq 0$ such that

$$\tau_1 \leq \tau_t \leq \tau_2. \tag{5}$$

Case 1 means that the time delay is a smooth function of t and its derivative is known to be upper bounded by h, while Case 2 implies that the information of the derivative of τ_t is unknown. Note that for most of NSCs the communication delay can be converted to be piecewise continuous but its derivative is unavailable [3, 34], in which situation only Case 2 is effective. Anyway, in the sequel, we will focus on Case 1 unless specified since the results under Case 2 are straightforward from those under Case 1 with special treatments (see Corollaries 1 and 3 later).

Let the full-order dynamic observer-based feedback control be

$$\begin{cases} \dot{\hat{x}}(t) = K_a\hat{x}(t) + K_c y(t) \\ u(t) = K_b\hat{x}(t), \end{cases} \tag{6}$$

where $\hat{x} \in R^n$ is the observer state, and the feedback gains K_a, K_b and K_c are to be designed. Denote $\xi(t) = [x(t)^T \;\; \hat{x}(t)^T]^T$. Then the closed-loop system of (1) with (2)-(6) is described by

$$\begin{aligned} \dot{\xi}(t) &= M\xi(t) + M_\tau\xi(t-\tau_t) + (\delta(t)-\bar{\delta})[N\xi(t) + N_\tau\xi(t-\tau_t)] + B_{\xi w}w(t), \\ z(t) &= M_z\xi(t) + B_{zw}w(t), \end{aligned} \tag{7}$$

where

$$\begin{aligned} M &= \begin{bmatrix} A & BK_b \\ \bar{\delta}K_cC & K_a \end{bmatrix}, \quad M_\tau = \begin{bmatrix} 0 & 0 \\ (1-\bar{\delta})K_cD & 0 \end{bmatrix}, \quad B_{\xi w} = \begin{bmatrix} B_{xw} \\ K_cB_{yw} \end{bmatrix}, \\ N &= \begin{bmatrix} 0 & 0 \\ K_cC & 0 \end{bmatrix}, \quad N_\tau = \begin{bmatrix} 0 & 0 \\ -K_cD & 0 \end{bmatrix}, \quad M_z = [C_z \;\; B_zK_b]. \end{aligned} \tag{8}$$

Here, although the dynamic of the closed-loop system requires only initial values of $\hat{x}(0)$, $w(0)$ and $x(t) = \phi(t)$ ($t \in [-\tau_2, 0]$), for later convenience, we extend the range of the definition of $\phi(t)$ from $[-\tau_2, 0]$ to $[-2\tau_2, 0]$ and define a continuous function $\hat{\phi}(t)$ on $[-2\tau_2, 0]$ such that $\hat{\phi}(0) = \hat{x}(0)$. So, we have $\xi(t) = [\phi(t)^T, \hat{\phi}(t)^T]^T$ for $t \in [-2\tau_2, 0]$. We also define $w(t) = 0$ for $t \in [-\tau_0, 0)$. It is seen that system (7) is a stochastic system with

$$\mathcal{E}\{(\delta(t)-\bar{\delta})\} = 0, \quad \mathcal{E}\{(\delta(t)-\bar{\delta})^2\} = \bar{\delta}(1-\bar{\delta}). \tag{9}$$

Stochastic theory has had a wide of applications in both theory and practice, and many results have appeared in recent years tackling various problems ranging from stochastic stabilization, filtering and control, see for instance [7, 8, 16, 17, 28]. Let

$$\begin{aligned} f(\xi, t) &:= M\xi(t) + M_\tau\xi(t-\tau_t) + B_{\xi w}w(t), \\ g(\xi, t) &:= N\xi(t) + N_\tau\xi(t-\tau_t). \end{aligned} \tag{10}$$

Since $f(\xi, t)$ and $g(\xi, t)$ in (7) satisfy the local Lipschitz condition and the linear growth condition, the existence and uniqueness of solution to (7) is guaranteed [15]. Moreover, under $w(t) = 0$, it admits a trivial solution (equilibrium) $\xi \equiv 0$. We will use the following definitions of stochastic stability and H_∞ performance requirements throughout the paper.

Definition 1. System (7) is said to be exponentially mean-square stable (EMSS) if, under $w(t) = 0$, there exist two constants $a > 0$ and $b > 0$ such that for all $t \geq 0$, the following inequality holds

$$\mathcal{E}\{\|\xi(t)\|^2\} \leq ae^{-bt} \sup_{\sigma \in [-2\tau_2, 0]} \mathcal{E}\{\|\xi(\sigma)\|^2\}. \tag{11}$$

Definition 2. Given $\gamma > 0$, system (7) is said to be EMSS with H_∞ performance γ (EMSS-γ) if it is EMSS and the following requirement is satisfied under zero-initial conditions:

$$\int_0^\infty \mathcal{E}\{\|z(t)\|^2\}dt \leq \gamma^2 \int_0^\infty \mathcal{E}\{\|w(t)\|^2\}dt. \tag{12}$$

The main purpose of this paper is to design a controller of the form (6) such that the closed-loop system (7) is EMSS-γ, namely, to render the closed-loop system not only EMSS but also meet the disturbance attenuation level γ. More specifically, we first establish delay-dependent matrix inequality conditions for system (7) to be EMSS-γ, and then seek LMI-based scheme for the design of the parameters K_a, K_b and K_c.

3 Main Results

Due to the special structure of matrices M_τ and N_τ in system (7), one may choose $[I_n \ \ 0]\xi = x$ to construct certain terms of Lyapunov functionals in order to establish stability conditions (see [25, 30]). Although this simplifies the derivation procedure and leads to simpler matrix inequalities, such a treatment obviously decreases the solvability for the obtained conditions. In this work, we use the full information of ξ to construct a suitable functional $J(\xi_t, t)$ and a similar type Lyapunov functional $V(\xi_t, t)$ in our study. In details, motivated by recent construction type for retarded systems in [9], we suggest the following type of functionals suitable for system (7) to investigate the H_∞ performance analysis:

$$J(\xi_t, t) = J_1(\xi_t, t) + J_2(\xi_t, t) + J_3(\xi_t, t), \tag{13}$$

where $\xi_t = \xi(t + \sigma)$, $\sigma \in [-2\tau_2, 0]$ and

$$\begin{aligned}
J_1(\xi_t, t) &= \xi(t)^T P \xi(t), \\
J_2(\xi_t, t) &= \int_{t-\tau_t}^{t} \xi(s)^T Q \xi(s) ds + \sum_{i=1}^{2} \int_{t-\tau_i}^{t} \xi(s)^T Q_i \xi(s) ds, \\
J_3(\xi_t, t) &= \int_{-\tau_2}^{0} \int_{t+\theta}^{t} \begin{bmatrix} f(\xi, s) \\ \delta_0 g(\xi, s) \end{bmatrix}^T Z \begin{bmatrix} f(\xi, s) \\ \delta_0 g(\xi, s) \end{bmatrix} ds d\theta \\
&\quad + \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^{t} \begin{bmatrix} f(\xi, s) \\ \delta_0 g(\xi, s) \end{bmatrix}^T Z_1 \begin{bmatrix} f(\xi, s) \\ \delta_0 g(\xi, s) \end{bmatrix} ds d\theta,
\end{aligned}$$

in which $\delta_0 = \sqrt{\bar{\delta}(1-\bar{\delta})}$, and $P > 0$, $Q > 0$, $Q_i > 0$, $Z > 0$ and $Z_1 > 0$ are to be determined. For system (7) with $w(t) = 0$, we use the following Lyapunov functional to obtain EMSS conditions:

$$V(\xi_t, t) = V_1(\xi_t, t) + V_2(\xi_t, t) + V_3(\xi_t, t), \tag{14}$$

where $V_i(\xi_t, t) = J_i(\xi_t, t)$ with $w(t) = 0$, $i = 1, 2, 3$. As will be seen later in the results, the introduction of the terms J_3 and V_3 could lead to desired conditions not only dependent on the maximum delay size τ_2 but also dependent on the delay range $\tau_2 - \tau_1$. We use $\mathcal{L}V$ to denote the infinitesimal operator of V [14, 16, 35], which is defined as

$$\mathcal{L}V(\xi_t, t) := \lim_{\Delta \to 0^+} \Delta^{-1}[\mathcal{E}\{V(\xi_{t+\Delta}, t + \Delta|(\xi_t, t)\} - V(\xi_t, t)]. \tag{15}$$

The following lemma is useful in the development, which verifies that $V(\xi_t, t)$ is a Lyapunov functional and meanwhile shows that certain condition could ensure system (7) to be EMSS.

Lemma 1: Suppose that K_a, K_b, K_c, $P > 0$, $Q > 0$, $Q_i > 0$, $Z > 0$ and $Z_1 > 0$ are given, and $V(\xi_t, t)$ is chosen as in (14). If there exists a constant $c > 0$ such that

$$\mathcal{E}\{\mathcal{L}V(\xi_t, t)\} \leq -c\mathcal{E}\{\|\xi(t)\|^2\} \tag{16}$$

holds for all $t \geq 0$, then system (7) is EMSS.

Proof: By Definition 1, the proof is analogous to those in [14, 18, 28] with slight modifications, and thus is omitted here. □

The next lemma, which converts a matrix inequality with interval variables into the form with vertices only, will be used to establish the analysis result for EMSS-γ. It is a variation of Lemma 2 in [35] and here we provide a simpler version for the proof.

Lemma 2: Let $\Omega, \Omega_1 \in R^{p\times p}$ be symmetric constant matrices. Then,

$$\Omega + \tau_t\Omega_1 < 0, \tag{17}$$

holds for all $\tau_t \in [\tau_1, \tau_2]$ if and only if the following two inequalities hold:

$$\begin{aligned} &\Omega + \tau_1\Omega_1 < 0, \\ &\Omega + \tau_2\Omega_1 < 0. \end{aligned} \tag{18}$$

If this is the case, for any $z(t) \in R^p$, the following is true

$$z(t)^T(\Omega + \tau_t\Omega_1)z(t) \leq \max\{\lambda_{max}(\Omega + \tau_1\Omega_1), \lambda_{max}(\Omega + \tau_2\Omega_1)\}\|z(t)\|^2. \tag{19}$$

Proof: The necessity is obvious. To prove the sufficiency, we know that for any $\tau_t \in [\tau_1, \tau_2]$, there exists an $\alpha_t \in [0, 1]$ such that $\tau_t = \alpha_t\tau_1 + (1 - \alpha_t)\tau_2$. This gives $\Omega + \tau_t\Omega_1 = \alpha_t(\Omega + \tau_1\Omega_1) + (1 - \alpha_t)(\Omega + \tau_2\Omega_1) < 0$. If this is the case, then $z(t)^T(\Omega + \tau_t\Omega_1)z(t) \leq \alpha_t\lambda_{max}(\Omega + \tau_1\Omega_1)\|z(t)\|^2 + (1 - \alpha_t)\lambda_{max}(\Omega + \tau_2\Omega_1)\|z(t)\|^2 \leq \max\{\lambda_{max}(\Omega + \tau_1\Omega_1), \lambda_{max}(\Omega + \tau_2\Omega_1)\}\|z(t)\|^2$. □

With the aid of Lemmas 1 and 2, we now present the analysis result for system (7) to be EMSS-γ.

Theorem 1: Given $\gamma > 0$, the closed-loop system (7) is EMSS-γ if there exist $2n \times 2n$ matrices $P > 0$, $Q > 0$, $Q_1 > 0$ and $Q_2 > 0$, $4n \times 4n$ matrices $Z > 0$, $Z_1 > 0$, $L_1 > 0$, $L_2 > 0$ and $L_3 > 0$, $(8n+q) \times 2n$ matrices F, G and H, such that

$$\begin{bmatrix} \Phi + \Phi_0 & \sqrt{\tau_1} F[I, I] & \sqrt{\tau_2 - \tau_1} H[I, I] \\ \star & -L_1 & 0 \\ \star & \star & -L_3 \end{bmatrix} < 0, \tag{20}$$

$$\begin{bmatrix} \Phi + \Phi_0 & \sqrt{\tau_2} F[I, I] & \sqrt{\tau_2 - \tau_1} G[I, I] \\ \star & -L_1 & 0 \\ \star & \star & -L_2 \end{bmatrix} < 0, \tag{21}$$

$$E_u L_1 E_u + E_l L_1 E_l - Z \leq 0, \tag{22}$$

$$E_u L_2 E_u + E_l L_2 E_l - Z_1 \leq 0, \tag{23}$$

$$E_u L_3 E_u + E_l L_3 E_l - Z - Z_1 \leq 0, \tag{24}$$

where each ellipsis $\star$ denotes a block induced by symmetry, and

$$\begin{aligned}
\Phi &= [I_{2n}, 0_{2n\times(6n+q)}]^T P\bar{M} + \bar{M}^T P[I_{2n}, 0_{2n\times(6n+q)}] + \bar{M}_z^T \bar{M}_z \\
&\quad + \mathrm{diag}\{Q + Q_1 + Q_2, (h-1)Q, -Q_1, -Q_2, -\gamma^2 I_q\} \\
&\quad + F[I_{2n}, -I_{2n}, 0_{2n\times(4n+q)}] + [I_{2n}, -I_{2n}, 0_{2n\times(4n+q)}]^T F^T \\
&\quad + G[0_{2n}, -I_{2n}, I_{2n}, 0_{2n\times(2n+q)}] + [0_{2n}, -I_{2n}, I_{2n}, 0_{2n\times(2n+q)}]^T G^T \\
&\quad + H[0_{2n}, I_{2n}, 0_{2n}, -I_{2n}, 0_{2n\times q}] + [0_{2n}, I_{2n}, 0_{2n}, -I_{2n}, 0_{2n\times q}]^T H^T, \\
\Phi_0 &= [\bar{M}^T, \delta_0 \bar{N}^T](\tau_2 Z + (\tau_2 - \tau_1) Z_1)[\bar{M}^T, \delta_0 \bar{N}^T]^T, \\
\bar{M} &= [M,\ M_\tau,\ 0_{2n\times 4n},\ B_{\xi w}], \quad \bar{N} = [N,\ N_\tau,\ 0_{2n\times(4n+q)}], \\
\bar{M}_z &= [M_z,\ 0_{p\times 6n},\ B_{zw}], \quad E_u = \mathrm{diag}\{I_{2n}, 0_{2n}\}, \quad E_l = \mathrm{diag}\{0_{2n}, I_{2n}\}.
\end{aligned}$$

Proof: The proof is twofold: we first choose a functional J of the form (13) to show that the H_∞ performance requirement (12) is satisfied, and then use the Lyapunov functional V of the form (14) to prove the EMSS property.

Denote

$$\eta(t) := [\xi(t)^T, \xi_\tau^T, \xi_1^T, \xi_2^T, w(t)^T]^T, \quad \xi_\tau := \xi(t - \tau_t), \quad \xi_i := \xi(t - \tau_i), \quad i = 1, 2. \tag{25}$$

From the Newton-Leibniz formula $0 = \xi(t) - \xi_\tau - \int_{t-\tau_t}^t \dot{\xi}(s)ds$, we have that

$$\begin{aligned}
\varphi_1(t) &:= 2\eta(t)^T F \left[\xi(t) - \xi_\tau - \int_{t-\tau_t}^t \dot{\xi}(s)ds \right] = 0, \\
\varphi_2(t) &:= 2\eta(t)^T G \left[\xi_1 - \xi_\tau - \int_{t-\tau_t}^{t-\tau_1} \dot{\xi}(s)ds \right] = 0, \\
\varphi_3(t) &:= 2\eta(t)^T H \left[\xi_\tau - \xi_2 - \int_{t-\tau_2}^{t-\tau_t} \dot{\xi}(s)ds \right] = 0,
\end{aligned} \tag{26}$$

hold for any $(8n+q) \times 2n$ matrices F, G and H. Let the functional $J(\xi_t, t)$ be chosen as in (13).

Then, from (15), $\mathcal{L}J$ for the evolution of J is given by (see [16,35])

$$\begin{aligned}
\mathcal{L}J(\xi_t,t) &= 2\xi(t)^TPf(\xi,t)+\xi(t)^T(Q+Q_1+Q_2)\xi(t)-(1-\dot{\tau}_t)\xi_\tau^TQ\xi_\tau-\sum_{i=1}^{2}\xi_i^TQ_i\xi_i \\
&\quad+\left[\begin{array}{c} f(\xi,t) \\ \delta_0 g(\xi,t)\end{array}\right]^T(\tau_2Z+(\tau_2-\tau_1)Z_1)\left[\begin{array}{c} f(\xi,t) \\ \delta_0 g(\xi,t)\end{array}\right] \\
&\quad-\int_{t-\tau_2}^{t}\left[\begin{array}{c} f(\xi,s) \\ \delta_0 g(\xi,s)\end{array}\right]^T Z\left[\begin{array}{c} f(\xi,s) \\ \delta_0 g(\xi,s)\end{array}\right]ds \\
&\quad-\int_{t-\tau_2}^{t-\tau_1}\left[\begin{array}{c} f(\xi,s) \\ \delta_0 g(\xi,s)\end{array}\right]^T Z_1\left[\begin{array}{c} f(\xi,s) \\ \delta_0 g(\xi,s)\end{array}\right]ds \\
&= 2\xi(t)^TPf(\xi,t)+\xi(t)^T(Q+Q_1+Q_2)\xi(t)-(1-\dot{\tau}_t)\xi_\tau^TQ\xi_\tau-\sum_{i=1}^{2}\xi_i^TQ_i\xi_i \\
&\quad+\left[\begin{array}{c} f(\xi,t) \\ \delta_0 g(\xi,t)\end{array}\right]^T(\tau_2Z+(\tau_2-\tau_1)Z_1)\left[\begin{array}{c} f(\xi,t) \\ \delta_0 g(\xi,t)\end{array}\right] \\
&\quad-\int_{t-\tau_t}^{t}\left[\begin{array}{c} f(\xi,s) \\ \delta_0 g(\xi,s)\end{array}\right]^T Z\left[\begin{array}{c} f(\xi,s) \\ \delta_0 g(\xi,s)\end{array}\right]ds \\
&\quad-\int_{t-\tau_t}^{t-\tau_1}\left[\begin{array}{c} f(\xi,s) \\ \delta_0 g(\xi,s)\end{array}\right]^T Z_1\left[\begin{array}{c} f(\xi,s) \\ \delta_0 g(\xi,s)\end{array}\right]ds \\
&\quad-\int_{t-\tau_2}^{t-\tau_t}\left[\begin{array}{c} f(\xi,s) \\ \delta_0 g(\xi,s)\end{array}\right]^T (Z+Z_1)\left[\begin{array}{c} f(\xi,s) \\ \delta_0 g(\xi,s)\end{array}\right]ds \\
&\quad+\varphi_1(t)+\varphi_2(t)+\varphi_3(t).
\end{aligned} \tag{27}$$

Note that, in $\varphi_1(t)$, the following inequality holds for any $4n\times 4n$ matrix $L>0$:

$$\begin{aligned}
&-2\eta(t)^TF\int_{t-\tau_t}^{t}\dot{\xi}(s)ds \\
&= -2\eta(t)^TF[I_{2n},I_{2n}]\int_{t-\tau_t}^{t}\left[\begin{array}{c} f(\xi,s) \\ (\delta(s)-\bar{\delta})g(\xi,s)\end{array}\right]ds, \\
&\le \tau_t\eta(t)^TF[I_{2n},I_{2n}]L_1^{-1}[I_{2n},I_{2n}]^TF^T\eta(t) \\
&\quad+\int_{t-\tau_t}^{t}\left[\begin{array}{c} f(\xi,s) \\ (\delta(s)-\bar{\delta})g(\xi,s)\end{array}\right]^T L_1\left[\begin{array}{c} f(\xi,s) \\ (\delta(s)-\bar{\delta})g(\xi,s)\end{array}\right]ds
\end{aligned}$$

and, similarly, in $\varphi_i(t)$ $(i=2,3)$, the following inequalities hold for any $4n\times 4n$ matrices $L_i>0$:

$$\begin{aligned}
&-2\eta(t)^TG\int_{t-\tau_t}^{t-\tau_1}\dot{\xi}(s)ds \\
&\le (\tau_t-\tau_1)\eta(t)^TG[I_{2n},I_{2n}]L_2^{-1}[I_{2n},I_{2n}]^TG^T\eta(t) \\
&\quad+\int_{t-\tau_t}^{t-\tau_1}\left[\begin{array}{c} f(\xi,s) \\ (\delta(s)-\bar{\delta})g(\xi,s)\end{array}\right]^T L_2\left[\begin{array}{c} f(\xi,s) \\ (\delta(s)-\bar{\delta})g(\xi,s)\end{array}\right]ds,
\end{aligned}$$

and

$$
\begin{aligned}
& -2\eta(t)^T H \int_{t-\tau_2}^{t-\tau_t} \dot{\xi}(s)ds \\
\leq\ & (\tau_2-\tau_t)\eta(t)^T H[I_{2n}, I_{2n}]L_3^{-1}[I_{2n}, I_{2n}]^T H^T \eta(t) \\
& + \int_{t-\tau_2}^{t-\tau_t} \begin{bmatrix} f(\xi,s) \\ (\delta(s)-\bar{\delta})g(\xi,s) \end{bmatrix}^T L_3 \begin{bmatrix} f(\xi,s) \\ (\delta(s)-\bar{\delta})g(\xi,s) \end{bmatrix} ds.
\end{aligned}
$$

Considering (4), (7), (9), (10) and taking the expectation on (27), we have

$$
\begin{aligned}
& \mathcal{E}\{\mathcal{L}V(\xi_t,t) + \|z(t)\|^2 - \gamma^2\|w(t)\|^2\} \\
\leq\ & \mathcal{E}\{\eta(t)^T(\Phi + \Phi_0 + \tau_t\Phi_1 + (\tau_t-\tau_1)\Phi_2 + (\tau_2-\tau_t)\Phi_3)\eta(t)\} + \varphi_4(t),
\end{aligned} \tag{28}
$$

where notations are as before, and

$$
\begin{aligned}
\Phi_1 &= F[I_{2n}, I_{2n}]L_1^{-1}[I_{2n}, I_{2n}]^T F^T, \\
\Phi_2 &= G[I_{2n}, I_{2n}]L_2^{-1}[I_{2n}, I_{2n}]^T G^T, \\
\Phi_3 &= H[I_{2n}, I_{2n}]L_3^{-1}[I_{2n}, I_{2n}]^T H^T, \\
\varphi_4(t) &= \int_{t-\tau_t}^{t} \begin{bmatrix} f(\xi,s) \\ \delta_0 g(\xi,s) \end{bmatrix}^T (E_u L_1 E_u + E_l L_1 E_l - Z) \begin{bmatrix} f(\xi,s) \\ \delta_0 g(\xi,s) \end{bmatrix} ds \\
&\quad + \int_{t-\tau_t}^{t-\tau_1} \begin{bmatrix} f(\xi,s) \\ \delta_0 g(\xi,s) \end{bmatrix}^T (E_u L_2 E_u + E_l L_2 E_l - Z_1) \begin{bmatrix} f(\xi,s) \\ \delta_0 g(\xi,s) \end{bmatrix} ds \\
&\quad + \int_{t-\tau_2}^{t-\tau_t} \begin{bmatrix} f(\xi,s) \\ \delta_0 g(\xi,s) \end{bmatrix}^T (E_u L_3 E_u + E_l L_3 E_l - Z - Z_1) \begin{bmatrix} f(\xi,s) \\ \delta_0 g(\xi,s) \end{bmatrix} ds.
\end{aligned}
$$

Now, applying the Schur complement, conditions (20)-(21) are equivalent to

$$
\bar{\Phi}_1 = \Phi + \Phi_0 + \tau_1\Phi_1 + (\tau_2-\tau_1)\Phi_3 < 0, \tag{29}
$$

$$
\bar{\Phi}_2 = \Phi + \Phi_0 + \tau_2\Phi_1 + (\tau_2-\tau_1)\Phi_2 < 0. \tag{30}
$$

From (29)-(30), (22)-(24) and Lemma 2, we continue (28) as

$$
\mathcal{E}\{\mathcal{L}J(\xi_t,t) + \|z(t)\|^2 - \gamma^2\|w(t)\|^2\} \leq \max\{\lambda_{max}(\bar{\Phi}_1), \lambda_{max}(\bar{\Phi}_2)\}\mathcal{E}\{\|\eta\|^2\} \leq 0. \tag{31}
$$

Under zero-initial conditions and noticing $J(\xi_T, T) \geq 0$ for any $T > 0$, integrating (31) from 0 to ∞ yields that the H_∞ performance requirement (12) is satisfied.

Next we show the EMSS property using the Lyapunov functional V of the form (14). With a procedure similar to the above, we can arrive under the given conditions and by virtue of Lemma 2 that,

$$
\mathcal{E}\{\mathcal{L}V(\xi_t,t)\} \leq \max\{\lambda_{max}(\bar{\Phi}_1), \lambda_{max}(\bar{\Phi}_2)\}\mathcal{E}\{\|\xi(t)\|^2\}. \tag{32}
$$

Hence, system (7) is EMSS from Lemma 1. This completes the proof. $\square$

Given K_a, K_b, K_c and $\gamma > 0$, the conditions of Theorem 1 are in terms of strict LMIs which could be easily solved using existing LMI solvers. The maximum tolerant delay bound for τ_2 can be searched. As a by-product, the minimum level of γ can be computed simultaneously. Note that our purpose is to design LMI schemes to seek these feedback gains K_a, K_b and K_c. This task is difficult even for deterministic time-delay systems as the difficulty of solving such matrix inequalities is NP-hard. In the sequel, we will first develop an LMI design method using appropriate decoupling technique, and then provide discussions on how to reduce conservatism while increasing the solvability of the matrix inequality conditions (20)-(24).

Theorem 2: Given $\gamma > 0$, the closed-loop system (7) is EMSS-γ if there exist $n \times n$ matrices $X > 0$ and $Y > 0$, $2n \times 2n$ matrices $\bar{Q} > 0$, $\bar{Q}_1 > 0$ and $\bar{Q}_2 > 0$, $4n \times 4n$ matrices $\bar{Z} > 0$, $\bar{Z}_1 > 0$, $\bar{L}_1 > 0$, $\bar{L}_2 > 0$ and $\bar{L}_3 > 0$, $(8n+q) \times 2n$ matrices $\bar{F}$, $\bar{G}$ and $\bar{H}$, $n \times n$ matrix Γ_a, $m \times n$ matrix Γ_b and $n \times r$ matrix Γ_c, such that the following LMIs hold for some scalars $\beta_1 > 0$ and $\beta_2 > 0$:

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \sqrt{\tau_1}\bar{F}[I,I] & \sqrt{\tau_2-\tau_1}\bar{H}[I,I] & \Pi_{15} & \Pi_{16} & \Pi_{17} \\ \star & \Pi_{22} & 0 & 0 & \Pi_{25} & 0 & 0 \\ \star & \star & -\bar{L}_1 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\bar{L}_3 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\beta_1 I_n & 0 & 0 \\ \star & \star & \star & \star & \star & -\beta_1^{-1} I_n & 0 \\ \star & \star & \star & \star & \star & \star & -I_p \end{bmatrix} < 0, \tag{33}$$

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \sqrt{\tau_2}\bar{F}[I,I] & \sqrt{\tau_2-\tau_1}\bar{G}[I,I] & \Pi_{15} & \Pi_{16} & \Pi_{17} \\ \star & \Pi_{22} & 0 & 0 & \Pi_{25} & 0 & 0 \\ \star & \star & -\bar{L}_1 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\bar{L}_2 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\beta_2 I_n & 0 & 0 \\ \star & \star & \star & \star & \star & -\beta_2^{-1} I_n & 0 \\ \star & \star & \star & \star & \star & \star & -I_p \end{bmatrix} < 0, \tag{34}$$

$$E_u\bar{L}_1E_u + E_l\bar{L}_1E_l - \bar{Z} \le 0, \tag{35}$$

$$E_u\bar{L}_2E_u + E_l\bar{L}_2E_l - \bar{Z}_1 \le 0, \tag{36}$$

$$E_u\bar{L}_3E_u + E_l\bar{L}_3E_l - \bar{Z} - \bar{Z}_1 \le 0, \tag{37}$$

where E_u and E_l are as in Theorem 1, and

$$
\begin{aligned}
\Pi_{11} &= \begin{bmatrix} YA + A^TY & \begin{pmatrix} A^TX + YA \\ +\bar{\delta}C^T\Gamma_c^T + \Gamma_a^T \end{pmatrix} & 0 & 0 & 0 & YB_{xw} \\ \star & \begin{pmatrix} XA + A^TX \\ +\bar{\delta}\Gamma_c C + \bar{\delta}C^T\Gamma_c^T \end{pmatrix} & (1-\bar{\delta})\Gamma_c D & (1-\bar{\delta})\Gamma_c D & 0 & XB_{xw} + \Gamma_c B_{yw} \\ \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0_{4n} & 0 \\ \star & \star & \star & \star & \star & 0_q \end{bmatrix} \\
&\quad + \mathrm{diag}\,\{\bar{Q} + \bar{Q}_1 + \bar{Q}_2, (h-1)\bar{Q}, -\bar{Q}_1, -\bar{Q}_2, -\gamma^2 I_q\} \\
&\quad + \bar{F}[I_{2n}, -I_{2n}, 0_{2n\times(4n+q)}] + [I_{2n}, -I_{2n}, 0_{2n\times(4n+q)}]^T\bar{F}^T \\
&\quad + \bar{G}[0_{2n}, -I_{2n}, I_{2n}, 0_{2n\times(2n+q)}] + [0_{2n}, -I_{2n}, I_{2n}, 0_{2n\times(2n+q)}]^T\bar{G}^T \\
&\quad + \bar{H}[0_{2n}, I_{2n}, 0_{2n}, -I_{2n}, 0_{2n\times q}] + [0_{2n}, I_{2n}, 0_{2n}, -I_{2n}, 0_{2n\times q}]^T\bar{H}^T, \\
\Pi_{12} &= \begin{bmatrix} A^TY & A^TX + \bar{\delta}C^T\Gamma_c^T + \Gamma_a^T & 0 & \delta_0 C^T\Gamma_c^T \\ A^TY & A^TX + \bar{\delta}C^T\Gamma_c^T & 0 & \delta_0 C^T\Gamma_c^T \\ 0 & (1-\bar{\delta})D^T\Gamma_c^T & 0 & -\delta_0 D^T\Gamma_c^T \\ 0 & (1-\bar{\delta})D^T\Gamma_c^T & 0 & -\delta_0 D^T\Gamma_c^T \\ 0_{4n\times n} & 0_{4n\times n} & 0_{4n\times n} & 0_{4n\times n} \\ B_{xw}^TY & B_{xw}^TX + B_{yw}^T\Gamma_c^T & 0 & 0 \end{bmatrix}, \\
\Pi_{22} &= -2\mathrm{diag}\,\{\begin{bmatrix} Y & Y \\ Y & X \end{bmatrix}, \begin{bmatrix} Y & Y \\ Y & X \end{bmatrix}\} + \tau_2\bar{Z} + (\tau_2 - \tau_1)\bar{Z}_1, \\
\Pi_{15} &= \begin{bmatrix} Y \\ 0_{(7n+q)\times n} \end{bmatrix}, \quad \Pi_{25} = \begin{bmatrix} Y \\ 0_{3n\times n} \end{bmatrix}, \quad \Pi_{16} = \begin{bmatrix} \Gamma_b^T B^T \\ 0_{(7n+q)\times n} \end{bmatrix}, \quad \Pi_{17} = \begin{bmatrix} C_z^T + \Gamma_b^T B_z^T \\ C_z^T \\ 0_{6n\times p} \\ B_{zw}^T \end{bmatrix}.
\end{aligned}
$$

If this is the case, the feedback gains K_a, K_b and K_c are given by

$$K_a = U^{-1}(\Gamma_a - XB\Gamma_b)Y^{-1}W^{-T}, \quad K_b = \Gamma_b Y^{-1}W^{-T}, \quad K_c = U^{-1}\Gamma_c, \tag{38}$$

where U and W are two invertible matrices satisfying $UW^T = I - XY^{-1}$.

Proof: It is seen from (33) or (34) that $\begin{bmatrix} Y & Y \\ Y & X \end{bmatrix} > 0$, which gives $X - Y > 0$, implying that $I - XY^{-1}$ is invertible. Let U and W be any invertible matrices satisfying $UW^T = I - XY^{-1}$. Choose

$$P = \begin{bmatrix} X & U \\ U^T & * \end{bmatrix} > 0, \quad P^{-1} = \begin{bmatrix} Y^{-1} & W \\ W^T & * \end{bmatrix} > 0, \tag{39}$$

where each ellipsis $*$ denotes a positive definite matrix block that will not influence the subsequent development (of course it makes $PP^{-1} = I$). In the sequel, we show that if (33)-(37) are satisfied, then (20)-(24) hold with $P > 0$ chosen as in (39), and thus the result follows immediately from Theorem 1. To this end, let us define

$$S = \begin{bmatrix} I & I \\ W^TY & 0 \end{bmatrix}, \tag{40}$$

which is invertible and produces

$$S^T P = \begin{bmatrix} Y & 0 \\ X & U \end{bmatrix}, \quad S^T P S = \begin{bmatrix} Y & Y \\ Y & X \end{bmatrix}. \tag{41}$$

We first show that (33) implies (20). By Schur complement, the matrix inequality (20) holds if and only if

$$\begin{bmatrix} \Phi & [\bar{M}^T, \delta_0 \bar{N}^T]\mathrm{diag}\{P, P\} & \sqrt{\tau_1} F[I, I] & \sqrt{\tau_2 - \tau_1} H[I, I] \\ \star & -\mathrm{diag}\{P, P\}(\tau_2 Z + (\tau_2 - \tau_1) Z_1)^{-1}\mathrm{diag}\{P, P\} & 0 & 0 \\ \star & \star & -L_1 & 0 \\ \star & \star & \star & -L_3 \end{bmatrix} < 0. \tag{42}$$

Due to

$$[\tau_2 Z + (\tau_2 - \tau_1) Z_1 - \mathrm{diag}\{P, P\}](\tau_2 Z + (\tau_2 - \tau_1) Z_1)^{-1}[\tau_2 Z + (\tau_2 - \tau_1) Z_1 - \mathrm{diag}\{P, P\}] \geq 0, \tag{43}$$

which gives

$$-\mathrm{diag}\{P, P\}(\tau_2 Z + (\tau_2 - \tau_1) Z_1)^{-1}\mathrm{diag}\{P, P\} \leq -2\mathrm{diag}\{P, P\} + \tau_2 Z + (\tau_2 - \tau_1) Z_1, \tag{44}$$

we have that (42) holds if

$$\begin{bmatrix} \Phi & [\bar{M}^T, \delta_0 \bar{N}^T]\mathrm{diag}\{P, P\} & \sqrt{\tau_1} F[I, I] & \sqrt{\tau_2 - \tau_1} H[I, I] \\ \star & -2\mathrm{diag}\{P, P\} + \tau_2 Z + (\tau_2 - \tau_1) Z_1 & 0 & 0 \\ \star & \star & -L_1 & 0 \\ \star & \star & \star & -L_3 \end{bmatrix} < 0. \tag{45}$$

Now, applying the congruence transformation $\mathrm{diag}\{S, S, S, S, I_q, S, S, S, S, S, S\}$ to (45) and setting

$$\begin{aligned}
&\bar{Q} = S^T Q S, \quad \bar{Q}_1 = S^T Q_1 S, \quad \bar{Q}_2 = S^T Q_2 S, \\
&\bar{Z} = \mathrm{diag}\{S, S\}^T Z \mathrm{diag}\{S, S\}, \quad \bar{Z}_1 = \mathrm{diag}\{S, S\}^T Z_1 \mathrm{diag}\{S, S\}, \\
&\bar{L}_i = \mathrm{diag}\{S, S\}^T L_i \mathrm{diag}\{S, S\}, \quad i = 1, 2, 3, \\
&\bar{F} = \mathrm{diag}\{S, S, S, S\}^T F S, \quad \bar{G} = \mathrm{diag}\{S, S, S, S\}^T G S, \quad \bar{H} = \mathrm{diag}\{S, S, S, S\}^T H S, \\
&\Gamma_a = X B K_b W^T Y + U K_a W^T Y, \quad \Gamma_b = K_b W^T Y, \quad \Gamma_c = U K_c,
\end{aligned} \tag{46}$$

we obtain that (45) is equivalent to

$$\bar{\Pi} + \bar{Y}\bar{K} + \bar{K}^T \bar{Y}^T < 0, \tag{47}$$

where

$$\begin{aligned}
&\bar{\Pi} = \begin{bmatrix} \Pi_{11} + \Pi_{17}\Pi_{17}^T & \Pi_{12} & \sqrt{\tau_1}\bar{F}[I, I] & \sqrt{\tau_2 - \tau_1}\bar{H}[I, I] \\ \star & \Pi_{22} & 0 & 0 \\ \star & \star & -\bar{L}_1 & 0 \\ \star & \star & \star & -\bar{L}_3 \end{bmatrix}, \\
&\bar{Y} = \left[\Pi_{15}^T, \ \Pi_{25}^T, \ 0_{n \times 8n}\right]^T, \\
&\bar{K} = \left[\Pi_{16}^T, \ 0_{n \times 12n}\right].
\end{aligned}$$

Inequality (47) holds if the following is true for any $\beta_1 > 0$,

$$\bar{\Pi} + \beta_1^{-1}\bar{Y}\bar{Y}^T + \beta_1\bar{K}^T\bar{K} < 0, \tag{48}$$

which is equivalent to

$$\begin{bmatrix} \bar{\Pi} & \bar{Y} & \bar{K}^T \\ \bar{Y}^T & -\beta_1 I_n & 0 \\ \bar{K} & 0 & -\beta_1^{-1} I_n \end{bmatrix} < 0. \tag{49}$$

The above inequality is, by Schur complement again, exactly that of (33), and we conclude that this implies (20).

Next we show that (34) implies (21). This can be done by using a procedure analogous to the above. As for the verification of other inequalities, applying the congruence transformation diag $\{S, S\}$ to (22)-(24) and setting matrix variables as in (46), it is seen that (22)-(24) are equivalent to (35)-(37). So far, we have proven that (33)-(37) ensure (20)-(24) and thus the closed-loop system (7) is EMSS-γ. In this case, from (46), the feedback gains are computed as in (38). This completes the proof. □

Theorem 2 provides an LMI method towards solving the matrix inequalities in (20)-(24), and hence presents controller designs of the form (6) to make the closed-loop system (7) EMSS-γ. The novelty of the result mainly lies in that an LMI design scheme is proposed for NCSs in continuous-time system settings with random measurements and time delays, which is not solved in the literature. Furthermore, the derivation is proceeded using appropriate Lyapunov functionals and matrix decoupling techniques. There is still much room in reducing the conservatism entailed in solving (20)-(24). Next, we offer some discussions on how to increase the solvability.

In the literature of control of delay type systems, such inequalities as that of (20) cannot be equivalently transformed into LMIs as the difficulty involved in its optimization is HP-hard. Even for linear deterministic delay systems, it is very hard to obtain control designs from stability analysis results without any constraint. In Theorem 2, we have encountered two conservative steps, i.e., the first one is that (48) implies (47), and the second one is in (45) to bound the term $-P(\tau_2 Z + (\tau_2 - \tau_1)Z_1)^{-1}P$. We give two remarks to address respectively.

Remark 1: The step of (48)⇒(47) can be improved by specifying a matrix $K_0 \in R^{m\times n}$ *a priori*. That is,

$$\bar{\Pi} + \bar{Y}\bar{K}_0 + \bar{K}_0^T\bar{Y}^T + \beta_1^{-1}\bar{Y}\bar{Y}^T + \beta_1(\bar{K} - \bar{K}_0)^T(\bar{K} - \bar{K}_0) < 0 \implies (47), \tag{50}$$

where $\bar{K}_0 = [BK_0,\ 0_{n\times(19n+q)}]$. As a result, the conditions (33) and (34) in Theorem 2 are replaced by similar ones with Π_{11}, Π_{12}, and Π_{16} replaced by Π'_{11}, Π'_{12} and Π'_{16}, respectively, where

$$\begin{aligned} \Pi'_{11} &= \Pi_{11} + \mathrm{diag}\,\{YBK_0 + K_0^T B^T Y,\ 0_{7n+q}\}, \\ \Pi'_{12} &= \Pi_{12} + [I_n,\ 0_{n\times(7n+q)}]^T K_0^T B^T Y [I_n,\ 0_{3n}], \\ \Pi'_{16} &= \Pi_{16} - [BK_0,\ 0_{n\times(7n+q)}]^T. \end{aligned} \tag{51}$$

The reason of the resultant improvement with above replacement lies in that when K_0 is chosen close to a computed Γ_b the deduction step of (50) involves no conservatism, and moreover, when $K_0 = 0$ the conditions of (33) and (34) are recovered. However, how to choose such a matrix K_0 involves much difficulty. In case of stabilizable pair (A, B) (which is not a very restrictive constraint), we could select K_0 such that $A + BK_0$ is Hurwitz.

Remark 2: The other conservative step is in (45)⇒(42) where the inequality (44) is used to bound the term $-\text{diag}\{P, P\}(\tau_2 Z + (\tau_2 - \tau_1)Z_1)^{-1}\text{diag}\{P, P\}$. This step can be improved by adopting the cone complementary algorithm [6], which is popular in recent control designs. To avoid using algorithms, we can introduce two scaling parameters $\epsilon_1 > 0$ and $\epsilon_2 > 0$ to improve the LMI conditions in Theorem 2. That is, we replace (44) by

$$\begin{aligned} & -\text{diag}\{P, P\}(\tau_2 Z + (\tau_2 - \tau_1)Z_1)^{-1}\text{diag}\{P, P\} \\ \leq\ & -2\text{diag}\{\epsilon_1 P, \epsilon_2 P\} + \text{diag}\{\epsilon_1 I_{2n}, \epsilon_2 I_{2n}\}[\tau_2 Z + (\tau_2 - \tau_1)Z_1]\text{diag}\{\epsilon_1 I_{2n}, \epsilon_2 I_{2n}\}. \end{aligned} \tag{52}$$

As a result, the conditions (33) and (34) in Theorem 2 are replaced by similar ones with Π_{22} replaced by Π'_{22} where

$$\begin{aligned} \Pi'_{22} &= -2\text{diag}\{\epsilon_1 \begin{bmatrix} Y & Y \\ Y & X \end{bmatrix}, \epsilon_2 \begin{bmatrix} Y & Y \\ Y & X \end{bmatrix}\} \\ &\quad + \text{diag}\{\epsilon_1 I_{2n}, \epsilon_2 I_{2n}\}[\tau_2 \bar{Z} + (\tau_2 - \tau_1)\bar{Z}_1]\text{diag}\{\epsilon_1 I_{2n}, \epsilon_2 I_{2n}\}. \end{aligned} \tag{53}$$

It is seen that the resulting conditions with this replacement cover those in Theorem 2 as the special choice of $\epsilon_1 = \epsilon_2 = 1$ is not required.

Here we state that all the above results are for delay satisfying Case 1. As for delay satisfying Case 2, the resulting conditions are straightforward from those for Case 1, and we merely list them as a corollary below.

Corollary 1: Given $\gamma > 0$, the closed-loop system (7) with delay satisfying Case 2 is EMSS-γ if the respective conditions in Theorem 1, Theorem 2, Remark 1 and Remark 2 hold by removing all terms with matrix Q (i.e., by simply setting $Q = 0$).

Proof: It follows immediately by choosing the functionals of (13) and (14) with $Q = 0$. □

We stress that the present work in this paper starts from the analysis result in Theorem 1. Recently, for linear retarded type systems, some new developments have been reported for stability analysis using free-weighting matrix method or slack variable approach, see [9, 10, 12, 13] and the references therein. These developments can be adopted to reduce conservatism in Theorem 1 and thus could lead to better control designs. However, meanwhile, the corresponding control design results not only need more conservative steps but also bring in much computational burden due to introduction of more matrix variables in solving LMIs. Anyway, we point out that these recent stability analysis methods for retarded systems may be a good try to further improve the design method presented in this paper.

Finally, we would like to remark that the main results in this section imply several results for stability analysis and stabilization via state feedback. We now list two corollaries below for stability tests which are straightforward from Theorem 1. Consider the following system of the form (7) with $w(t) = 0$,

$$\dot{x}(t) = M_1 x(t) + M_{1\tau} x(t-\tau_t) + (\delta(t) - \bar{\delta})[N_1 x(t) + N_{1\tau} x(t-\tau_t)], \tag{54}$$

where $x \in R^n$, $M_1, M_{1\tau}, N_1$ and $N_{1\tau}$ are constant matrices with appropriate dimensions.

Corollary 2 (For Case 1): System (54) with delay satisfying Case 1 is EMSS if there exist $n \times n$ matrices $\tilde{P} > 0$, $\tilde{Q} > 0$, $\tilde{Q}_1 > 0$ and $\tilde{Q}_2 > 0$, $2n \times 2n$ matrices $\tilde{Z} > 0$, $\tilde{Z}_1 > 0$, $\tilde{L}_1 > 0$, $\tilde{L}_2 > 0$ and $\tilde{L}_3 > 0$, $4n \times 2n$ matrices $\tilde{F}$, $\tilde{G}$ and $\tilde{H}$, such that

$$\begin{bmatrix} \Psi & \sqrt{\tau_1}\tilde{F}[I, I] & \sqrt{\tau_2 - \tau_1}\tilde{H}[I, I] \\ \star & -\tilde{L}_1 & 0 \\ \star & \star & -\tilde{L}_3 \end{bmatrix} < 0, \tag{55}$$

$$\begin{bmatrix} \Psi & \sqrt{\tau_2}\tilde{F}[I, I] & \sqrt{\tau_2 - \tau_1}\tilde{G}[I, I] \\ \star & -\tilde{L}_1 & 0 \\ \star & \star & -\tilde{L}_2 \end{bmatrix} < 0, \tag{56}$$

$$\tilde{E}_u \tilde{L}_1 \tilde{E}_u + \tilde{E}_l \tilde{L}_1 \tilde{E}_l - \tilde{Z} \le 0, \tag{57}$$

$$\tilde{E}_u \tilde{L}_2 \tilde{E}_u + \tilde{E}_l \tilde{L}_2 \tilde{E}_l - \tilde{Z}_1 \le 0, \tag{58}$$

$$\tilde{E}_u \tilde{L}_3 \tilde{E}_u + \tilde{E}_l \tilde{L}_3 \tilde{E}_l - \tilde{Z} - \tilde{Z}_1 \le 0, \tag{59}$$

where

$$\begin{aligned} \Psi &= [I_n, 0_{n\times 3n}]^T \tilde{P}\tilde{M} + \tilde{M}^T \tilde{P}[I_n, 0_{n\times 3n}] \\ &\quad + \mathrm{diag}\,\{\tilde{Q} + \tilde{Q}_1 + \tilde{Q}_2, (h-1)\tilde{Q}, -\tilde{Q}_1, -\tilde{Q}_2\} \\ &\quad + \tilde{F}[I_n, -I_n, 0_{n\times 2n}] + [I_n, -I_n, 0_{n\times 2n}]^T \tilde{F}^T \\ &\quad + \tilde{G}[0_n, -I_n, I_n, 0_n] + [0_n, -I_n, I_n, 0_n]^T \tilde{G}^T \\ &\quad + \tilde{H}[0_n, I_n, 0_n, -I_n] + [0_n, I_n, 0_n, -I_n]^T \tilde{H}^T \\ &\quad + [\tilde{M}^T, \delta_0 \tilde{N}^T](\tau_2 Z + (\tau_2 - \tau_1) Z_1)[\tilde{M}^T, \delta_0 \tilde{N}^T]^T, \\ \tilde{M} &= [M_1,\ M_{1\tau},\ 0_{n\times 2n}], \quad \tilde{N} = [N_1,\ N_{1\tau},\ 0_{n\times 2n}], \\ \tilde{E}_u &= \mathrm{diag}\,\{I_n, 0_n\}, \quad \tilde{E}_l = \mathrm{diag}\,\{0_n, I_n\}. \end{aligned}$$

Corollary 3 (For Case 2): System (54) with delay satisfying Case 2 is EMSS if the conditions in Corollary 2 hold by removing all terms with matrix $\tilde{Q}$ (i.e., by simply setting $\tilde{Q} = 0$).

The above Corollaries 2 and 3 can be used for the stability tests of NCSs with random communication delays, which extend those in [11, 33, 34] and some other references therein. The merits and the less conservativeness will be shown in the next section.

So far, we have provided delay-dependent LMI conditions for our considered problems. If the maximum delay bound τ_2 is unavailable, we have to use the delay-independent conditions which can be easily obtained from the present delay-dependent results in this work. This can be done by setting $Z = 0$ and $Z_1 = 0$ in the functionals of J and V. Due to space limitation and for conciseness, this treatment is omitted.

4 An Illustrative Example

In this section, a well-studied example is used to illustrate the the effectiveness of the present results. When reduced to the special case without random variables, it is also provides merits of our stability method in Corollary 3 compared with existing work.

Consider the NCS studied in [11, 33, 34] and the references therein, which is governed by $\dot{x}(t) = Ax(t) + Bu(t)$ with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \tag{60}$$

The state feedback gain is $K = [-3.75 \quad -11.5]$.

Under consideration of random communication delay, i.e., $u(t) = \delta Kx(t) + (1-\delta)Kx(t-\tau_t)$, the closed-loop control system is of the form (54) with

$$M_1 = A + \bar{\delta}BK, \quad M_{1\tau} = (1-\bar{\delta})BK, \quad N_1 = BK, \quad N_{1\tau} = -BK. \tag{61}$$

When no random variable (i.e., $\delta(t) \equiv 0$) and $0 = \tau_1 \leq \tau_t \leq \tau_2$, this closed-loop system reduced to the one studied in [11,33,34] and the references therein. In this case, the computed maximum delay bounds τ_{max} of τ_2 for maintaining the system stability using Corollary 3 and other methods are listed in Table 1. It is seen that Corollary 3 produces the least conservative results. The computation is performed using a Pentium M1.66-GHz computer with 512-MB RAM. The computational times using Corollary 3 and the result in [33] are less than 1 second while the times using the results in [34] and [11] are, respectively, within 1-2 seconds and longer than 2 seconds. This can be expected as the number of total decision variables in Corollary 3 is close to that of [33] but quite smaller than those of [34] and [11]. In details, Corollary 3 (when removing redundant variables) involves $7.5n^2 + 1.5n$ decision variables and those of [33], [34] and [11] are, respectively, $7n^2 + n$, $14.5n^2 + 1.5n$ and $70n^2 + 7n$. Moreover, Corollary 3 is also applicable for the case of $\delta(t) \not\equiv 0$, and the results for different $\bar{\delta} \in [0, 1]$ are provided in Table 1. It also deserves mentioning that Corollary 3 is applicable for $\tau_1 \neq 0$ as well.

Table 1: Comparison results for computed maximum τ_{max}

Methods	$\bar{\delta} = 0$	$\bar{\delta} = 0.2$	$\bar{\delta} = 0.5$	$\bar{\delta} = 0.8$
[11]	0.699	—	—	—
[33]	0.8695	—	—	—
[34]	0.8871	—	—	—
Corollary 3	1.0432	1.3413	1.5748	1.7425

Now let us consider the observer-based feedback control for this example. Assume that there exists disturbance, and the measurement output (which may experience random sensor delay) and the controlled output are, respectively, $y(t) = \delta(t)Cx(t) + (1-\delta(t))Dx(t-\tau_t) + B_{yw}w(t)$ and

$z(t) = C_z x(t) + B_z u(t) + B_{zw} w(t)$, where

$$C = D = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad B_{yw} = 0, \quad B_{xw} = \begin{bmatrix} 0.3 & 0 \end{bmatrix}^T,$$
$$C_z = \begin{bmatrix} 0 & 0.5 \end{bmatrix}, \quad B_z = 0.2, \quad B_{zw} = 0.2.$$

It is seen that the above model is open-loop unstable as A is not Hurwitz. Our purpose is to provide a dynamic observer-based control (upon unavailable of state information) such that the closed-loop system of the above model is EMSS-γ. This problem is not solved in the literature. Let $\tau_t \in [0, 1.2]$ with the probability of random delay measurement being $\mathcal{E}\{1 - \delta(t)\} = 0.5$ (i.e., $\bar{\delta} = 0.5$). For $\beta_1 = \beta_2 = 1$ and $K_0 = K$ as above, the method of Theorem 2 (in fact, Corollary 1, due to the consideration of unavailable derivative of the time delay) and Remark 1 produces a set of feasible solutions to corresponding LMIs with a lower bound of H_∞ performance level $\gamma = 0.42$. The matrix variables are computed as

$$X = \begin{bmatrix} 0.3429 & 0.8459 \\ 0.8459 & 1.0670 \times 10^7 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.1265 & 0.2860 \\ 0.2860 & 0.8289 \end{bmatrix},$$
$$\Gamma_a = \begin{bmatrix} 0.0601 & -0.2633 \\ 0.4191 & 1.0670 \times 10^6 \end{bmatrix}, \quad \Gamma_b = \begin{bmatrix} -0.7467 & -4.5497 \end{bmatrix}, \quad \Gamma_c = \begin{bmatrix} -0.0672 \\ -0.4602 \end{bmatrix}.$$

Hence, the corresponding feedback gains are computed from (38), with choice of $W = I$ and $U = I - XY^{-1}$, as

$$K_a = \begin{bmatrix} -1.8825 & 0.5916 \\ -3.4902 & 0.9511 \end{bmatrix}, \quad K_b = \begin{bmatrix} 29.5779 & -15.6947 \end{bmatrix}, \quad K_c = \begin{bmatrix} 0.0393 \\ 0.0889 \end{bmatrix}.$$

For simulations, we select the time delay $\tau_t = 0.6(1 + \sin(t)) \in [0, 1.2]$ and the disturbance input as

$$w(t) = \begin{cases} \cos(t), & 0 \le t \le 20, \\ 0, & \text{otherwise.} \end{cases}$$

Figure 1 shows the response of $\int_0^t \mathcal{E}\{\|z(s)\|^2\}dt / \int_0^t \mathcal{E}\{\|w(s)\|^2\}dt$ under zero-initial conditions. It is seen that the ratio is less than 0.045, which reveals that the H_∞ disturbance attenuation level is less than the required $\gamma = 0.42$, i.e., $\sqrt{0.045} = 0.2121 < 0.42$. Figure 2 shows the response of the closed-loop states under $w(t) = 0$ and initial condition $[x(0)^T, \hat{x}(0)^T]^T = [0.5, -0.8, 0, -0.2]$, which reveals that the closed-loop system is EMSS. These two figures verify that under the above control the closed-loop system is EMSS-γ with $\gamma = 0.42$.

The above illustrates the computation of minimum γ and corresponding feedback gains such that the closed-loop system is EMSS-γ. As mentioned previously, the method in this paper also provides the search of maximum tolerant delay bound for τ_2 with fixed γ and τ_1 such that the design method is still applicable. We also mention here that in case of available derivative of τ_t, we can use the corresponding LMI conditions in Corollary 2 and Theorem 2 (with Remarks 1 and 2) to increase the solvability. This is due to the apparent fact that the inclusion of the terms containing matrix Q could reduce the conservatism in solving LMIs.

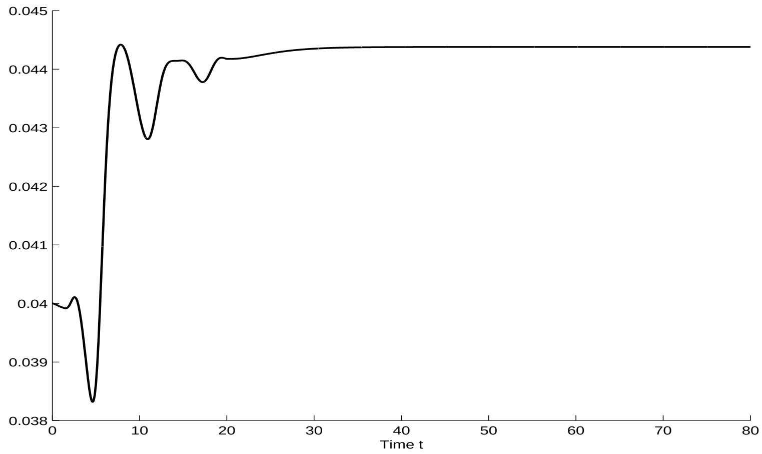

Figure 1: Ratio of energy of $\mathcal{E}\{\|z(s)\|^2\}$ to energy of $\mathcal{E}\{\|w(s)\|^2\}$

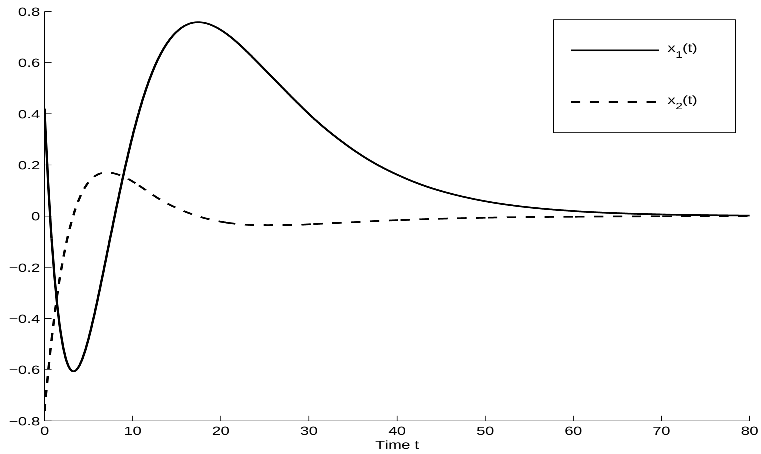

Figure 2: Response of closed-loop system state $x(t)$

5 Conclusion

An LMI method has been presented for observer-based H_∞ control of NCSs in continuous-time system settings with random measurements and time delays. Improved schemes have also been provided for the design method. When reduced to the stability test of NCSs, the merits and less conservatism of our method have been illustrated in comparison with existing work. This work is an important complement to recent developments in discrete-time settings. The present method is useful for solving related problems (such as guaranteed cost control, H_∞ filtering, etc.) in the literature of continuous-time NCSs with random measurements, packet losses and time delays. However, if both random measurement and random packet loss are taken into account, the considered problem in this paper is quite involved and this is one of our future research works. Furthermore, although we merely consider the case of a single delay, we believe our results could be extended to the case of multiple delays and/or distribution delays.

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