MEASURING DISTANCE BETWEEN SYSTEMS UNDER BOUNDED POWER EXCITATION

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Abstract. This work suggests a way of measuring distance between two linear systems under a given bounded power excitation. The measure introduced can be used to bound from above and below the difference in closed loop behaviour of two plants with the same controller for a specified reference or disturbance spectrum. Given an unknown, single input 'real' plant and its identified model, an upper bound on the distance between the plant and its model as expressed by this measure can be obtained from time domain data.

Key words. ν-gap metric, uncertainty in linear models, persistent excitation.

1. Introduction. Robust control theory is often motivated based on square summable or square integrable (bounded energy) signals. In many practical applications, the signals are better modelled as persistent disturbances. Motivating robust control design or analysis from a persistent signals viewpoint is complicated by the fact that the set of 'quasi-stationary' bounded power signals which induces the infinity norm is *not* a linear vector space [1]. Nevertheless, interpreting and extending standard robust control results in a persistent signals set-up has been an active area of research for the last few years; see [2], [1] and references therein.

Given a controller which robustly stabilises two plants (or a plant and its model), it is known that the difference in closed loop frequency response of the two plants with the same controller can be bounded from below and above using pointwise chordal distance between the frequency responses of the two plants. A natural question to ask is whether the difference in closed loop response of two systems for a given reference and disturbance spectrum can be bounded from above and below using an appropriate, *signal dependent* notion of distance. This work provides an affirmative answer to this question.

Specifically, the problem considered is as follows. Consider two closed loops (P_1, C_1, C_2) and (P_2, C_1, C_2), each with the same controller $C = C_1C_2$, the same controller configuration (C_1 in the forward path, C_2 in the feedback path) and (possibly) different plants P_1, P_2. These two loops may represent the 'achieved' closed loop (*i.e.* with the real plant) and the 'designed' closed loop (*i.e.* with the model). For a given bounded power excitation (which could be a reference or a disturbance signal), the difference in the closed loop behaviour will be small if the two plants P_1 and P_2 are close in an appropriate sense. It is known that the difference in closed loop behaviour would be small (at least, for any square summable excitation) if the distance between the two systems as measured by the $\nu-$gap metric (discussed in more details in section 3) or by the gap metric is small. However, *even if* the $\nu-$gap is large, it is possible that the difference in the closed loop response is small for a specific range of spectra of interest provided we find a controller which stabilises both systems with adequate stability margins. Here, a measure of distance over a subset of linear shift invariant systems is introduced which characterises the difference in closed loop response for a given range of signal spectra. Upper and lower bounds on this difference are established in terms of this new measure. For a plant and its candidate model, bounds on this measure are given in terms of time domain data.

The rest of the paper is organised as follows. Section 2 outlines the notation and defines the sets of signals and systems used in this paper. Section 3 introduces the $\nu-$gap metric. A new function δ_x for measuring distance between two systems is introduced in section 4. In section 5, this function δ_x is used to bound from above and below the difference in closed loop performance for a given range of signal spectrum. Bounds on δ_x from time domain data are introduced in section 6. Section 7 illustrates the use of this function with examples and finally section 8 briefly summarises the contribution of this paper.

2. Preliminaries. Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers respectively. $\mathbb{C}^{m\times n}$ denotes the space of $m \times n$ complex matrices. $\mathbb{Z}$ denotes set of integers. $l_\infty(\mathbb{Z})$ denotes the space of bounded sequences indexed by integers. For $A \in \mathbb{C}^{m\times n}$, $\sigma_i(A)$ denotes the $i-$th largest singular value of A. Maximum and minimum singular values of a matrix A are denoted by $\overline{\sigma}(A)$ and $\underline{\sigma}(A)$ respectively.

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- **Signals:** Let $R_u(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} u(t-\tau)u^T(t)$. Define

$$\mathcal{S}^n = \{u \,|\, u \in l_\infty(\mathbb{Z}), u(t) = 0\,\forall t < 0,\; R_u(\tau) \text{ exists}\,\forall\,\tau,\; \phi_u(\omega) := \sum_{\tau=-\infty}^{\infty} R_u(\tau)e^{-j\tau\omega} \text{ exists}\,\forall\,\omega\,\}. \tag{2.1}$$

Here, *power spectrum* $\phi_u(\omega)$ need not be bounded and may contain impulses in general. This set is called as a set of *quasi-stationary* signals in [3]. Define semi-norm $\|f\|_{\mathcal{S}} := \sqrt{\text{trace}\, R_f(0)}$. For signals with continuous spectrum, the equality $\|f\|_{\mathcal{S}} = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{trace}\, \phi_f(\omega)\, d\omega}$ also holds.
For two signals $v,\, w \in \mathcal{S}^n$, define

$$R_{wv}(\tau) := \lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} v(t-\tau)w^T(t)$$
$$\text{and } \phi_{wv}(\omega) := \sum_{\tau=-\infty}^{\infty} R_{wv}(\tau)e^{-j\tau\omega}$$

provided the limits exist for each τ and each ω. For $v,\, w \in \mathcal{S}^n$, note that $v + w \in \mathcal{S}^n$ *provided* the cross-correlation function $R_{wv}(\tau)$ exists for all τ and the cross power spectrum $\phi_{wv}(\omega)$ exists for all ω. This set is obviously not a linear space and may be embedded in a linear space which includes non-stationary signals [1]. However, the use of $\mathcal{S}^n$ here to model persistent signals is motivated by two reasons. First, the correlation or spetrum based description is deemed as natural to describe persistent disturbances, even in a non-probabilistic setting. Secondly, the spectral content of a signal in $\mathcal{S}^n$ proves useful in assessing the performance of feedback systems in terms of their graph symbols restricted to imaginary axis. This point will become apparent in sections 4 and 5.
- **Systems:** Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\partial\mathbb{D}$ denote the boundary of $\mathbb{D}$. $\mathcal{L}_\infty$ denotes the normed space of all functions essentially bounded on $\mathbb{D}$ and having norm $\|f\|_{\mathcal{L}_\infty} := ess\ \sup_\omega \overline{\sigma}(f(e^{j\omega}))$, where $\overline{\sigma}(\cdot)$ represents the maximum singular value. $\mathcal{H}_\infty$ denotes the normed space of functions analytic in $\mathbb{D}$ and having norm $\|f\|_\infty := \sup_{z\in\mathbb{D}} \overline{\sigma}(f(z)) < \infty$.
Consider a linear, shift invariant discrete time system P, which can be expressed as $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ with
  1. N and M are right coprime and $G = \begin{bmatrix} N \\ M \end{bmatrix}$ inner; and
  2. $\tilde{N}$ and $\tilde{M}$ are left coprime and $\tilde{G} = \begin{bmatrix} -\tilde{M} & \tilde{N} \end{bmatrix}$ co-inner.

  G (resp. $\tilde{G}$) is called the normalised right (resp. normalised left) graph symbol of plant P. The set of systems of interest here are those with continuous normalised graph symbols:

$$\mathcal{P}^{m\times n} = \{P : G, \tilde{G} \text{ exist and are continuous on } \partial\mathbb{D}\}.$$

The superscript $m \times n$ is dropped when it is clear from context. The normalised graph symbols of a plant P_i will be denoted as $\tilde{G}_i$ and G_i. The set $\mathcal{P}^{m\times n}$ includes all systems whose normalised graph symbols may be uniformly approximated by real rational transfer functions; this follows from ([4], theorem 7.12). However, it is worth mentioning that there are systems which have a continuous frequency response on the unit circle but whose normalised graph symbols are *not* continuous on the unit circle; see [5] for an example.
The set of all real rational transfer functions with n inputs and m outputs, denoted by $\mathcal{R}^{m\times n}$, is a subset of $\mathcal{P}^{m\times n}$.
The controller is denoted by C and its normalised right (left) inverse graph symbol is denoted by $K = \begin{bmatrix} V \\ U \end{bmatrix}$ $\left(\tilde{K} = \begin{bmatrix} -\tilde{U} & \tilde{V} \end{bmatrix}\right)$.

3. The $\nu-$gap metric. The notion of measuring distance between linear systems in terms of distance between their graph spaces was introduced by Vidyasagar in [6]. In this paper, a metric called the graph metric was introduced which is characterised by the smallest distance, in a certain sense, between the coprime factors of two systems. This work was followed by a number of advances in characterisation and computation of similar metrics under which feedback stability is a robust property. The pointwise gap metric [7], the gap metric [8] and the chordal metric [9] induce the same topology as the graph metric. [10] discusses the properties of gap metric related to robustness for normalised coprime factor perturbations.

In [11], a metric called the $\nu-$gap metric was defined. It is closely related to the gap metric but has a nicer frequency response interpretation and leads to less conservative robustness results in general. Specifically, the $\nu-$*gap* between two plants P_1 and P_2 is defined as [11]

$$\begin{aligned}\delta_\nu(P_1,P_2) &= \inf_{Q,Q^{-1}\in\mathcal{L}_\infty} \| G_1 - G_2 Q \|_\infty \text{ if } I(P_1,P_2)=0 \\ &= \quad 1 \quad \text{otherwise}\end{aligned} \tag{3.1}$$

where $I(P_1,P_2) :=$ wno det $(G_2^* G_1) =$ wno det $(\tilde{G}_1 \tilde{G}_2^*)$ and wno (g) denotes the winding number of $g(z)$ evaluated on the standard Nyquist contour indented around any poles and zeros on $\partial\mathbb{D}$. For a real rational transfer matrix X such that $X, X^{-1} \in \mathcal{L}_\infty$ the winding number wno det $(X) = \eta(X^{-1}) - \eta(X)$ where $\eta(f)$ denotes the number of unstable poles of f. Thus $\nu-$gap is seen to be the infinity norm of the smallest perturbation of the normalised coprime factorisation G_1 of P_1 which yields a - not necessarily coprime - factorisation $G_2 Q$ of P_2. The choice of factorisation of P_2 , *i.e.* the choice of Q is constrained by the winding number condition. If $G_2 Q$ is constrained to be coprime instead, one gets the gap (instead of $\nu-$gap) between P_1 and P_2. See *e.g.* section 9.3 in [13] for details.

When the winding number condition is satisfied, $\delta_\nu(P_1,P_2)$ equals the $\mathcal{L}_2$*-gap*,

$$\delta_{\mathcal{L}_2}(P_1,P_2) := \|\tilde{G}_2 G_1\|_\infty = \sup_\omega \kappa(P_1,P_2)(e^{j\omega}). \tag{3.2}$$

$\kappa(P_1,P_2)(e^{j\omega})$ is the pointwise *chordal* distance defined by

$$\kappa(P_1,P_2)(e^{j\omega}) := \overline{\sigma}\left((I+P_2P_2^*)^{-\frac{1}{2}}(P_1-P_2)(I+P_1^*P_1)^{-\frac{1}{2}}\right)(e^{j\omega}). \tag{3.3}$$

$\delta_\nu(P_1,P_2)$ is a measure of difference in the closed loop performance of P_1 in feedback with a controller C and P_2 in feedback with the same controller C. Given a nominal controller C that stabilises a (possibly frequency weighted) plant P_i, a useful closed loop performance measure is

$$\begin{aligned}b(P_i,C) &= \|H(P_i,C)\|_\infty^{-1} = \inf_\omega \underline{\sigma}(\tilde{K}G_i)(e^{j\omega}) \\ &= \inf_\omega \underline{\sigma}(\tilde{G}_i K)(e^{j\omega})\end{aligned} \tag{3.4}$$

where $\underline{\sigma}(\cdot)$ denotes the minimum singular value and the closed loop transfer function $H(P_i,C)$ is defined by

$$H(P_i,C) = \begin{bmatrix} P_i \\ I \end{bmatrix} (I - CP_i)^{-1} \begin{bmatrix} -C & I \end{bmatrix}.$$

It is known that any controller stabilising a plant P_1 and achieving $b(P_1,C) > \alpha$ stabilises the plant set $\{P_2 : \delta_\nu(P_1,P_2) \le \alpha\}$ [11]. More importantly, the *pointwise* difference in the closed loop performance of nominal plant P_1 and a perturbed plant P_2 for the same controller C can be quantified in terms of $\kappa(P_1,P_2)$ as [11]:

$$\begin{aligned}&\kappa(P_1,P_2)(e^{j\omega}) \le \overline{\sigma}(H(P_1,C) - H(P_2,C))(e^{j\omega}) \\ &\le \kappa(P_1,P_2)(e^{j\omega})\,\overline{\sigma}(H(P_1,C))(e^{j\omega})\,\overline{\sigma}(H(P_2,C))(e^{j\omega}).\end{aligned} \tag{3.5}$$

The upper bound in (3.5) is useful only if C stabilises both P_1 and P_2.

The aim here is to characterise the difference in closed loop behaviour, in a fashion similar to (3.5), for signals belonging to the set $\mathcal{S}^n$ as defined in (2.1). The next section defines a way of measuring distance between systems under a specific bounded power excitation.

4. A New Measure of Distance. Let Φ be a set of functions defined by

$$\Phi := \{X \,|\, X : [-\pi, \pi] \to \mathbb{R},\ X(\omega) \geq 0 \\ X(\omega) \text{ is monotonic non-decreasing and bounded}\}. \tag{4.1}$$

For $X \in \Phi$, define a semi-norm

$$\|X\|_{\Phi} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dX(\omega). \tag{4.2}$$

The definition (4.2) may be related to the the set $\mathcal{S}^n$ as follows. Let $r \in \mathcal{S}^n$ be such that $\phi_r = xI^{n\times n}$, where $x(\omega) \geq 0$ is a continuous, scalar and bounded real function over $[-\pi, \pi]$. Let

$$X(\omega) = \int_{-\pi}^{\omega} x(\tau)d\tau$$

Then $X \in \Phi$ and $\frac{dX}{d\omega} = x$ ([4], theorem 6.20). Also, the equality

$$\|r\|_{\mathcal{S}}^2 = n\|X\|_{\Phi} \tag{4.3}$$

follows from the definitions of semi-norms $\|\cdot\|_{\mathcal{S}}$ and $\|\cdot\|_{\Phi}$. Functions belonging to the set Φ will later be used in section 5 to express bounds on the range of spectra of interest.

Now define a function $\delta_x : \mathcal{P}^{m\times n} \times \mathcal{P}^{m\times n} \times \Phi \to \mathbb{R}_+$,

$$\delta_x(P_1, P_2, X) := \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{trace}\left((\tilde{G}_2G_1)^*(\tilde{G}_2G_1)(e^{j\omega})\right) dX(\omega)\right\}^{\frac{1}{2}}. \tag{4.4}$$

Thus $\delta_x^2(P_1, P_2, X)$ is seen to be a Stieltjes integral of trace $\left((\tilde{G}_2G_1)^*(\tilde{G}_2G_1)\right)$ with respect to a 'weight' $X(\omega)$[1]. The choice of this weight will be determined by the shape of spectrum of interest. For a given pair $P_1, P_2 \in \mathcal{P}^{m\times n}$, trace $\left((\tilde{G}_2G_1)^*(\tilde{G}_2G_1)\right)$ may be easily shown to be a continuous function mapping $[-\pi, \pi]$ to $\mathbb{R}$. Also, X is monotonic from the definition of Φ. From these two facts, it follows that $\delta_x(P_1, P_2, X)$ is well defined for any $P_1, P_2 \in \mathcal{P}^{m\times n}$ ([4], theorem 6.8).

The following lemma sums up properties of δ_x as a measure of distance between systems in $\mathcal{P}^{m\times n}$:

LEMMA 1.

- *For a given $X_0 \in \Phi$ and $P_1, P_2 \in \mathcal{P}^{m\times n}$,*

$$0 \leq \delta_x^2(P_1, P_2, X_0) \leq n\|X_0\|_{\Phi} \text{ and} \tag{4.5}$$

$$\delta_x(P_1, P_2, X_0) = \delta_x(P_2, P_1, X_0). \tag{4.6}$$

- *For given $X_0 \in \Phi$ and $P_1, P_2, P_3 \in \mathcal{P}^{m\times n}$,*

$$\delta_x(P_1, P_2, X_0) \leq \delta_x(P_1, P_3, X_0) + \delta_x(P_3, P_2, X_0). \tag{4.7}$$

- *Suppose $X_0 \in \Phi$ is continuously differentiable, with $\frac{dX_0}{d\omega} > 0$ over an interval $[\omega_0, \omega_p] \subseteq [-\pi, \pi]$. Then*

$$\delta_x(P_1, P_2, X_0) = 0 \Leftrightarrow \kappa(P_1, P_2)(e^{j\omega}) = 0\, \forall\, \omega \in [\omega_0, \omega_p] \tag{4.8}$$

where $\kappa(\cdot,\cdot)$ is chordal distance as defined in (3.3).

- *Suppose $X_0(\omega) = \sum_{i=1}^{\infty} a_i h(\omega - \omega_i)$ where $\{\omega_i\}$, $i = 1, 2, 3, \ldots$ is a sequence of distinct points in $[-\pi, \pi]$ and $\{a_i\}$ is such that $a_i > 0\, \forall\, i$ and the sequence of partial sums $\sum a_i$ is convergent. Here $h(\cdot)$ represents the unit step function. Then*

$$\delta_x(P_1, P_2, X_0) = 0 \Leftrightarrow \kappa(P_1, P_2)(e^{j\omega_i}) = 0\, \forall\, \omega_i. \tag{4.9}$$

[1] Wherever square roots of positive numbers are used, it will be assumed that positive square root is considered.

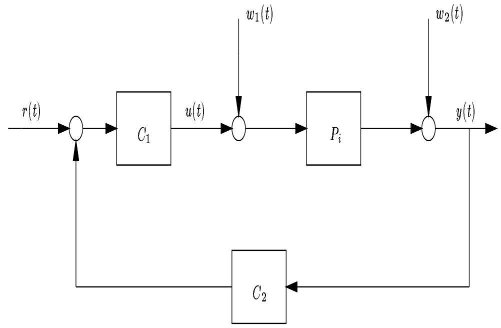

FIG. 5.1. *Closed Loop System*

Proof : See Appendix.

For any continuously differentiable $X \in \Phi$ such that $\frac{dX}{d\omega} > 0 \, \forall \, \omega \in [-\pi, \pi]$, the above result shows that $\delta_x(P_1, P_2, X)$ is a metric over $\mathcal{P}^{m \times n}$. Further, for $P_1, P_2 \in \mathcal{P}^{m \times n}$ and a scalar transfer function x, $x^{-1} \in \mathcal{H}_\infty \cap \mathcal{P}^{1 \times 1}$, it may be easily shown that

$$\begin{aligned} \text{if} \quad X(\omega) &= \int_{-\pi}^{\omega} |x(e^{j\tau})|^2 \, d\tau, \\ \text{then} \quad \delta_x(P_1, P_2, X) &= \|\tilde{G}_2 G_1 x\|_2. \end{aligned}$$

In general, however, X need not even be continuous for $\delta_x(P_1, P_2, X)$ to be well defined.

5. Closed Loop Error Bounds. Consider the closed loop in figure 5.1. Let $\begin{bmatrix} -\tilde{U} & \tilde{V} \end{bmatrix}$ be the normalised left inverse graph symbol of controller $C = C_1 C_2$ (*i.e.* $C = \tilde{V}^{-1}\tilde{U}$ and $\tilde{U}$ and $\tilde{V}$ are left coprime). C_1 is square transfer function matrix, and is chosen such that $\tilde{V}C_1 \in \mathcal{H}_\infty$, $\inf_\omega \underline{\sigma}(\tilde{V}C_1)(e^{j\omega}) > 0$. The transfer function from r to $\begin{bmatrix} y \\ u \end{bmatrix}$ in figure 5.1 with $P = P_i$ can easily be shown to be

$$T(P_i, C_1, C_2) := G_i(\tilde{K}G_i)^{-1}\tilde{V}C_1. \tag{5.1}$$

Define two constants dependent on controller configuration:

$$\begin{aligned} \alpha_c &= \inf_\omega \underline{\sigma}(\tilde{V}C_1)(e^{j\omega}) && (5.2) \\ &= 1 \text{ if } C_1 = \tilde{V}^{-1},\, C_2 = \tilde{U} \\ &= \frac{1}{\sqrt{1 + \|C\|_\infty^2}} \text{ if } C_1 = I,\, C_2 = \tilde{V}^{-1}\tilde{U}. && (5.3) \\ \text{and} \quad \beta_c &= \sup_\omega \overline{\sigma}(\tilde{V}C_1)(e^{j\omega}) \\ &= 1 \text{ if } C_1 = \tilde{V}^{-1},\, C_2 = \tilde{U} \\ &\leq 1 \text{ if } C_1 = I,\, C_2 = \tilde{V}^{-1}\tilde{U} \text{ or if } C_1 = \tilde{V}^{-1}\tilde{U},\, C_2 = I. && (5.4) \end{aligned}$$

Note that these definitions do not exclude 'open loop' case; $C_1 = I$, $C_2 = 0$.

Suppose that upper and lower bounds on the spectrum of disturbance or excitation r of interest are known. This information can be used to bound the difference in closed loop response of two plants to r, as the next theorem shows.

THEOREM 1. *Suppose, $T(P_i, C_1, C_2)$ as defined in (5.1) are exponentially stable, with P_1, $P_2 \in \mathcal{P}^{m \times n}$. Let $r \in \mathcal{S}^n$ be such that $\phi_r(\omega)$ is continuous, where $\mathcal{S}^n$ is the set defined in (2.1). Let x_1, x_2 be Riemann integrable functions on $[-\pi, \pi]$ such that, $\exists \, \gamma > 0$ for which*

$$\gamma x_1 \leq \sigma_i(\phi_r)(\omega) \leq \gamma x_2 \, \forall \, \omega, \quad i = 1, 2, \ldots, n \tag{5.5}$$

$$\textit{Let} \quad X_k(\omega) = \int_{-\pi}^{\omega} x_k(\tau)d\tau, \; k = 1, 2. \tag{5.6}$$

Then, for $b(P_i, C)$ as defined in (3.4), α_c and β_c as defined in (5.2)-(5.4) and the semi-norm $\|\cdot\|_\Phi$ as defined in (4.2),

1.

$$\frac{\alpha_c\, \delta_x(P_1, P_2, X_1)}{\sqrt{n}\, \|X_2\|_\Phi} \le \frac{\|(T(P_1, C_1, C_2) - T(P_2, C_1, C_2))\, r\|_{\mathcal{S}}}{\|r\|_{\mathcal{S}}} \le \frac{\beta_c\, \delta_x(P_1, P_2, X_2)}{\sqrt{n}\, \|X_1\|_\Phi\, b(P_1, C)\, b(P_2, C)}. \tag{5.7}$$

2. Further, if $C_1 = \tilde{V}^{-1}(\tilde{K}G_1)$ and $C_2 = C_1^{-1}C$, then

$$\frac{\delta_x(P_1, P_2, X_1)}{\sqrt{n}\, \|X_2\|_\Phi} \le \frac{\|(T(P_1, C_1, C_2) - T(P_2, C_1, C_2))\, r\|_{\mathcal{S}}}{\|r\|_{\mathcal{S}}} \le \frac{\delta_x(P_1, P_2, X_2)}{\sqrt{n}\, \|X_1\|_\Phi\, b(P_2, C)}. \tag{5.8}$$

Proof: See Appendix.

Several remarks on this result are in order.

- To guarantee that the filtered signal $T(P_i, C_1, C_2)\, r$ to be in $\mathcal{S}^n$, the impulse response of $T(P_i, C_1, C_2)$ should be in l_1 (*i.e.* should be absolutely summable) [1]. One simple way to ensure this is to impose exponential stability condition. From a practical point of view, of course, this is a perfectly sensible requirement.
- The equality (4.3) explains the presence of $\sqrt{n\|X_k\|_\Phi}$ in (5.7). To explore this further, let $\hat{r} \in \mathcal{S}^1$ be such that $\phi_{\hat{r}}$ is continuous. Let $X_1,\, X_2 \in \Phi$ be such that $\frac{dX_1}{d\omega} = \phi_{\hat{r}} = \frac{dX_2}{d\omega}$. Then $\|\hat{r}\|_{\mathcal{S}}^2 = \|X_1\|_\Phi = \|X_2\|_\Phi$. Define *spectral distribution function* [1]

  $$F_{\hat{r}}(\omega) = \int_{-\pi}^{\omega} \phi_{\hat{r}}(\tau)d\tau.$$

  Then $F_{\hat{r}}$ is continuous over $[-\pi, \pi]$ and $\frac{dF_{\hat{r}}}{d\omega} = \phi_{\hat{r}}$. Using this definition in (5.7) yields a simpler expression

  $$\alpha_c\, \delta_x(P_1, P_2, F_{\hat{r}}) \le \|(T(P_1, C_1, C_2) - T(P_2, C_1, C_2))\, \hat{r}\|_{\mathcal{S}} \le \frac{\beta_c\, \delta_x(P_1, P_2, F_{\hat{r}})}{b(P_1, C)\, b(P_2, C)}.$$

  To re-emphasise the main motivation of this work, note that $\delta_x(P_1, P_2, F_{\hat{r}})$ may be small for $F_{\hat{r}}$ of interest even if $\delta_\nu(P_1, P_2)$ is large.
- In proving theorem 1, we will use the equality

  $$\text{trace}\, R_r(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{trace}\, \phi_r(\omega)\, d\omega.$$

  This is certainly true when ϕ_r is continuous, but may not be true in general. For single input systems, ϕ_r may be allowed to have the form

  $$\phi_r(\omega) = \phi_{\hat{r}}(\omega) + \sum_{i=1}^{l} \delta(\omega - \omega_i)$$

  where $\phi_{\hat{r}}(\omega)$ is continuous, $\omega_i \in [-\pi, \pi]$ and $\delta(\cdot)$ is Dirac delta function. Impulses in the spectral density of r represent periodic excitation. It is reasonable to expect that the frequencies of periodic reference (or disturbance) are known. The bounds in (5.7)-(5.8) will still make sense if we allow jump discontinuities in X_1, X_2 at the frequencies of periodic excitation. For multi-input systems, it is more difficult to account for delta functions due to necessity of bounding singular values of spectrum.
- Note that the parameter γ in (5.5) doesn't appear in (5.7) or (5.8). This is important, since it implies that the bounds in (5.7)-(5.8) are *scale invariant*; in the sense that one only needs to know the bounds on the *shape* of singular values of spectral density ϕ_r. The actual magnitude of spectral density is irrelevant.

- For a multi-input system, these bounds make sense for comparing performance under simultaneous excitation of all inputs. To compare behaviour when $\sigma_n(\phi_r)(\omega) = 0 \,\forall\, \omega$ (*i.e.* to compare the response when some but not all inputs are excited), x_1 should be zero; but that makes the lower bound in (5.7) zero and the upper bound becomes unbounded. Even in that case, the following bound still holds:

$$\|(T(P_1, C_1, C_2) - T(P_2, C_1, C_2))\, r\|_{\mathcal{S}} \leq \frac{\beta_c\, \gamma\, \delta_x(P_1, P_2, X_2)}{b(P_1, C)\, b(P_2, C)}.$$

This may be easily shown following the steps of proof of theorem 1.

Suppose, P_2 is a model for a 'true' plant P_1. For a 'reasonable' controller C (*i.e.* which stabilises both the true plant and the model with adequate stability margins), theorem 1 shows that the difference in the designed and the achieved closed loop response to a persistent excitation $r \in \mathcal{S}^n$ is small if $\delta_x(P_1, P_2, \phi_r)$ is small. The results are relevant in comparing the designed and the achieved tracking performance (when r is a reference) and in comparing noise rejection of two closed loops (when r is a disturbance). These error bounds may also prove useful in assessing the suitability of a reduced order model to design a controller for a high order plant.

If $r_{\circ} \in \mathcal{S}^n$ is such that the $\phi_{r_0}(\omega) = I^{n \times n}$, then

$$\|T(P_1, C_1, C_2) - T(P_2, C_1, C_2) r_{\circ}\|_{\mathcal{S}} = \|T(P_1, C_1, C_2) - T(P_2, C_1, C_2)\|_2.$$

This observation leads to the following 2− norm inequality:

Corollary 1. *Suppose P_1, $P_2 \in \mathcal{P}^{m \times n}$ and a controller C stabilises both P_1 and P_2. Let $C = \tilde{V}^{-1}\tilde{U} = C_1 C_2$ where $\begin{bmatrix} -\tilde{U} & \tilde{V} \end{bmatrix}$ is normalised left graph symbol of C and C_1 is a square matrix function such that $\tilde{V} C_1 \in \mathcal{H}_\infty$, $\inf_\omega \underline{\sigma}(\tilde{V} C_1) > 0$. Further, suppose $T(P_1, C_1, C_2)$ as defined in (5.1) be exponentially stable. Then,*

$$\alpha_c \,\| \tilde{G}_2 G_1 \|_2 \leq \|T(P_1, C_1, C_2) - T(P_2, C_1, C_2)\|_2 \leq \frac{\beta_c \,\| \tilde{G}_2 G_1 \|_2}{b(P_1, C)\, b(P_2, C)}. \tag{5.9}$$

where α_c, β_c are as defined in (5.2)- (5.4).

Proof: follows from (5.7), with $\phi_r = I^{n \times n}$, $\frac{dX_1}{d\omega} = \frac{dX_2}{d\omega} = 1$. ∎

Given an unknown true plant P_0, a model P_θ and a spectral distribution F_r, it is not possible to measure $\delta_x(P_0, P_\theta, F_r)$ directly. The next section introduces bounds on $\delta_x(P_0, P_\theta, F_r)$ for a given single input 'true' plant P_0 and a model P_θ in terms of data from a time domain identification experiment.

6. Bounds on $\delta_x(P_0, P_\theta, X)$. The main result in this section is restricted to single input systems. A partial generalisation to the multiple input case is possible and is discussed at the end of the section.

Consider a plant $P_0 \in \mathcal{P}^{m \times 1}$. For a closed loop system as shown in figure 5.1 (with $P_i = P_0$), the *a posteriori data* is given by

$$z := \begin{bmatrix} y \\ u \end{bmatrix} = G_0(\tilde{K} G_0)^{-1}\, \tilde{V} C_1\, r + H(C, P_0) w \tag{6.1}$$

where $H(C, P_0)$ is a closed loop transfer function as defined in section 3, $H(C, P_0) = K(\tilde{G}_0 K)^{-1}\tilde{G}_0$. y, u (and possibly, r) are measured signals and $w = \begin{bmatrix} -w_1 \\ w_2 \end{bmatrix}$ is unmeasured noise or disturbance. Suppose, $r, w_1, \in \mathcal{S}^1$, $w_2 \in \mathcal{S}^m$ are such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} (r(t) w_i^T(t - \tau)) = 0 \,\forall\, \tau,\; i = 1, 2. \tag{6.2}$$

This deterministic assumption corresponds to the idea that the reference r and the noise signals w_i are uncorrelated. Also, suppose that the transfer function from $\begin{bmatrix} r \\ w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} y \\ u \end{bmatrix}$ is exponentially stable. Assume that ϕ_r is continuous and define the spectral distribution function of r as before,

$$F_r(\omega) = \int_{-\pi}^{\omega} \phi_r(\tau) d\tau.$$

The data z from (6.1) can be used for identifying a model for P_0. One common method for parametric, time domain identification is the prediction error method [3]. This method solves

$$\min_{\theta} \|\tilde{X}_\theta z\|_{\mathcal{S}} \quad \text{subject to} \quad \tilde{X}_\theta(\infty) = \begin{bmatrix} I & 0 \end{bmatrix}. \tag{6.3}$$

Here, $\tilde{X}_\theta$ is a left graph symbol of model parameterised by a (real) parameter vector θ and z is output-input data as in (6.1). This is sometimes referred to as the 'direct' prediction error in literature [12], as opposed to methods that use a measurable reference.

Prediction error identification approach has elegant statistical properties and is widely studied in literature; [3] offers a very comprehensive treatment. For a model P_θ obtained by the prediction error method, it is desired to obtain an estimate of $\delta_\nu(P_0, P_\theta, F_r)$. Let $\tilde{G}_\theta$ be the normalised left graph symbol of P_θ. The following result gives bounds on $\delta_\nu(P_0, P_\theta, F_r)$ in terms of time domain data:

THEOREM 2. *Let P_θ be a candidate model for P_0 with both $P_0, P_\theta \in \mathcal{P}^{m\times 1}$. Suppose, $r, w_1, \in \mathcal{S}^1$, $w_2 \in \mathcal{S}^m$ are such that (6.2) holds. Let $C = C_1 C_2$ be such that the transfer function from $\begin{bmatrix} r \\ w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} y \\ u \end{bmatrix}$ in figure 5.1 is exponentially stable. Let z be as defined in (6.1) and let α_c and β_c be as defined in (5.2)-(5.4). Then*

$$\frac{b^2(P_0, C)\, \|\tilde{G}_\theta\, z\|_{\mathcal{S}}^2 - \|\tilde{G}_0\, w\|_{\mathcal{S}}^2}{\beta_c^2} \leq \delta_\nu^2(P_0, P_\theta, F_r) \leq \frac{\|\tilde{G}_\theta\, z\|_{\mathcal{S}}^2 - b^2(P_\theta, C)\, \|\tilde{G}_0\, w\|_{\mathcal{S}}^2}{\alpha_c^2}. \tag{6.4}$$

Proof: See Appendix.

From (6.4), it follows that

$$\delta_\nu(P_0, P_\theta, F_r) \leq \frac{\|\tilde{G}_\theta\, z\|_{\mathcal{S}}}{\alpha_c}. \tag{6.5}$$

For a given model P_θ, measured data z and a given α_c the right hand side can be explicitly evaluated. Combining (5.7) and (6.5) gives

$$\frac{\|(T(P_0, C_1, C_2) - T(P_\theta, C_1, C_2))\, r\|_{\mathcal{S}}}{\|r\|_{\mathcal{S}}} \leq \frac{\|\tilde{G}_\theta\, z\|_{\mathcal{S}}}{\|r\|_{\mathcal{S}}} \frac{\beta_c}{\alpha_c\, b(P_0, C)\, b(P_\theta, C)}.$$

If $\frac{\|\tilde{G}_\theta\, z\|_{\mathcal{S}}}{\|r\|_{\mathcal{S}}}$ is small, any 'good' controller C (*i.e.* with sufficiently large robust stability margins $b(P_0, C)$ and $b(P_\theta, C)$) should yield a small difference in closed loop behaviour for excitation r. However, the upper bound above can not be evaluated from data due to the presence of an unknown term $b(P_0, C)$.

If $\|\tilde{G}_0 w\|_{\mathcal{S}} \approx 0$ and if $C_1 = \tilde{V}^{-1}$, $C_2 = \tilde{U}$, (6.4) yields an aesthetically pleasing expression

$$b(P_0, C)\, \|\tilde{G}_\theta\, z\|_{\mathcal{S}} \leq \delta_\nu(P_0, P_\theta, F_r) \leq \|\tilde{G}_\theta\, z\|_{\mathcal{S}}.$$

Note that the model P_θ need not be obtained from the same experimental data z used in the bounds; the results hold for *any* candidate model P_θ so far as the data z is generated according to (6.1) and (6.2) holds. In particular, note that (6.5) does not use any *a priori* assumptions about the order or the relative stability of the plant P_0. The exponential stability of the *closed loop* is the only assumption used here.

The result above is valid for single input systems. A generalisation to multivariable plants is possible in the case when the spectrum of r is a scalar function times identity.

COROLLARY 2. *Let P_θ be a candidate model for P_0 with both $P_0, P_\theta \in \mathcal{P}^{m\times n}$. Suppose, $r, w_1, \in \mathcal{S}^n$, $w_2 \in \mathcal{S}^m$ are such that (6.2) holds. Further, suppose $r \in \mathcal{S}^n$ is such that $\phi_r = x\, I^{n\times n}$ where x is a scalar, continuous spectrum. Let $X(\omega) = \int_{-\pi}^{\omega} x(\tau) d\tau$. Let $C = C_1 C_2$ be such that the transfer function from*

$\begin{bmatrix} r \\ w_1 \\ w_2 \end{bmatrix}$ *to* $\begin{bmatrix} y \\ u \end{bmatrix}$ *in figure 5.1 is exponentially stable. Let* z *be as defined in (6.1) and let* α_c *and* β_c *be as defined in (5.2)-(5.4). Then*

$$\frac{b^2(P_0, C)\, \|\tilde{G}_\theta\, z\|_S^2 - \|\tilde{G}_0\, w\|_S^2}{\beta_c^2} \leq \delta_x^2(P_0, P_\theta, X) \leq \frac{\|\tilde{G}_\theta\, z\|_S^2 - b^2(P_\theta, C)\, \|\tilde{G}_0\, w\|_S^2}{\alpha_c^2}. \tag{6.6}$$

Proof: This may be shown following the steps of the proof of theorem 2. Details are omitted.

REMARK 1. *Under the assumption (6.2), an upper bound similar to (6.5) also holds in the case of prediction error cost in (6.3). If* $\theta_\star$ *is an argument which minimises the cost in (6.3), it can be shown that*

$$\|\tilde{X}_{\theta_\star} X_0\, r\|_S \leq \|\tilde{X}_{\theta_\star} z\|_S.$$

where $X_0 = G_0(\tilde{K}G_0)^{-1}\tilde{V}C_1$. *This may be easily proved from the proof of theorem 2. The quantity* $\|\tilde{X}_{\theta_\star} X_0\, r\|_S$, *unlike* $\delta_x(P_0, P_{\theta_\star}, F_r)$, *depends on the choice and the configuration of controller* C.

REMARK 2. *The bounds presented above are 'distribution-free' and use only a non-probabilistic assumption (6.2) about noise. Developing results equivalent to theorem 2 in a probabilistic setting is an interesting and challenging area of future research.*

7. Examples. Here we consider some examples to see how this new measure can be useful in comparing closed loop response of systems to persistent excitation. Consider a pair of plants

$$P_1(z) = \frac{2(z+1)}{z-0.6},\ P_2(z) = \frac{(z+1)^2}{(z^2-0.6z+1.2)}.$$

The $\nu-$gap error between P_1 and P_2 is significantly large ($=0.64$). Suppose, the spectrum of interest is a low frequency spectrum which satisfies

$$\gamma\, x_1(\omega) \leq \phi_{r0} \leq \gamma\, x_2(\omega).$$

$$\text{for some } \gamma > 0, \text{ where } \quad x_1(\omega) = f_1(e^{j\omega})f_1(e^{-j\omega}),\ f_1(z) = \frac{0.01z}{z-0.99}$$
$$\text{and} \quad x_2(\omega) = f_2(e^{j\omega})f_2(e^{-j\omega}),\ f_2(z) = \frac{0.02z}{z-0.98}.$$

Note that the exact value of γ (and hence the energy in the signal) is immaterial. Let $F_i(\omega) = \int_{-\pi}^{\omega} x_i(\tau)d\tau$, $i = 1, 2$. Then

$$\frac{\delta_x(P_1, P_2, F_1)}{\|F_2\|_\Phi} = 0.0294 \qquad \text{and} \qquad \frac{\delta_x(P_1, P_2, F_2)}{\|F_1\|_\Phi} = 0.0724.$$

(These may be computed as 2$-$ norm of $\tilde{G}_2 G_1 f_i$, $i = 1, 2$). Thus, given a controller C which stabilises both P_1 and P_2 with a 'good' stability margins, the difference in closed loop gains $T(P_1, C_1, C_2) - T(P_2, C_1, C_2)$ over this spectrum is guaranteed to be small. In this particular case, a simple integral controller $C_1 = \frac{0.025(z+1)}{(z-1)}$, $C_2 = -1$ yields stability margins $b(P_1, C) = 0.408$, $b(P_2, C) = 0.401$.

For the same plants P_1, P_2 as above, if we consider $f_1(z) = f_2(z) = 1 - 0.9z^{-1}$, then

$$\frac{\delta_x(P_1, P_2, F_1)}{\|F_2\|_\Phi} = \frac{\delta_x(P_1, P_2, F_2)}{\|F_1\|_\Phi} = 0.4469$$

so that no controller can make the difference in closed loop gains $T(P_1, C_1, C_2) - T(P_2, C_1, C_2)$ smaller than $0.4469\,\alpha_c$ over this spectrum, with α_c as defined in (5.2).

Next, consider another pair of plants

$$P_3(z) = \frac{z-1}{z-0.99},\ P_4(z) = \frac{z-1}{z-1.01}.$$

The $\nu-$gap between P_3 and P_4 is 1. Suppose, P_3 and P_4 are to be compared from the perspective of white noise rejection. Then $\|\tilde{G}_4 G_3\|_2 = 0.055$, which means any 'reasonable' controller will yield a similar closed loop white noise rejection for both P_3 and P_4 (provided such a controller exists). The controller $C_1 = 1$, $C_2 = -1$ in this case yields $b(P_3, C) = b(P_4, C) = 0.7071$.

8. Conclusion. A new measure $\delta_x(\cdot,\cdot,X)$ is introduced for measuring distance between linear shift invariant systems. It is shown that this measure can be used to characterise the difference in closed loop behaviour of two feedback systems under a given persistent excitation. For a plant P_0 and a model P_θ, bounds on this measure with respect to a given reference spectrum are obtained in terms of data from a time domain identification experiment.

Appendix A.

First, some technical results necessary for proofs of lemma 1 and theorems 1-2 are given.

FACT 1. *For any complex matrix $Q \in \mathbb{C}^{m\times n}$, let Frobenius norm be defined as usual, $\|Q\|_F^2 =$ trace $(Q^*Q) =$ trace $(QQ^*) = \sum_{i=1}^m \sigma_i^2(Q)$, where $m = \min(m,n)$. For a pair of complex matrices Q, R of compatible dimensions, with R square and invertible, the following inequalities hold*

$$\underline{\sigma}(R)\|Q\|_F \le \|QR\|_F \le \overline{\sigma}(R)\,\|Q\|_F. \tag{A.1}$$

Proof: The upper bound is well known [2]. The lower bound follows from

$$\|Q\|_F = \|QRR^{-1}\|_F \le \overline{\sigma}(R^{-1})\|QR\|_F. \tag{A.2}$$

∎

The next result proves the the formula for $\inf_\omega \underline{\sigma}(\tilde{V}C_1)(e^{j\omega})$ stated in (5.3).

LEMMA 2. *Let $\begin{bmatrix} -\tilde{U} & \tilde{V}\end{bmatrix}$ be the normalised left inverse graph symbol for $C = \tilde{V}^{-1}\tilde{U}$. Then*

$$\inf_\omega \underline{\sigma}(\tilde{V})(e^{j\omega}) = \frac{1}{\sqrt{1+\|C\|_\infty^2}}. \tag{A.3}$$

Proof: From ([13], section 2.3),

$$\overline{\sigma}^2(C) = \frac{\overline{\sigma}^2(\tilde{U})}{1-\overline{\sigma}^2(\tilde{U})} \text{ and } \underline{\sigma}^2(\tilde{V}) = 1-\overline{\sigma}^2(\tilde{U}) \tag{A.4}$$

A rearrangement of (A.4) and taking the infimum of both sides leads to (A.3). ∎

The proofs that follow also use some identities from ([13], section 3.2) for manipulation of normalised graph symbols:

$$(\tilde{G}_1G_2)^*(\tilde{G}_1G_2) + (G_2^*G_1)(G_2^*G_1)^* = I, \tag{A.5}$$
$$(\tilde{G}_2G_1)^*(\tilde{G}_2G_1) + (G_2^*G_1)^*(G_2^*G_1) = I, \tag{A.6}$$
$$\tilde{G}_2^*\tilde{G}_2 + G_2G_2^* = I, \tag{A.7}$$
$$K(\tilde{G}_2K)^{-1}\tilde{G}_2 + G_2(\tilde{K}G_2)^{-1}\tilde{K} = I. \tag{A.8}$$

In the rest of the proofs, the argument $e^{j\omega}$ will be omitted for brevity wherever it is obvious from the context.

Proof of Lemma 1:

Property (4.5) is obvious. To prove (4.6), note that

$$\begin{aligned}\sigma_i^2(\tilde{G}_2G_1)(e^{j\omega}) &= 1-\sigma_{n-i+1}^2(G_2^*G_1)(e^{j\omega}) && \text{(from (A.5))}\\ &= \sigma_i^2(\tilde{G}_1G_2)(e^{j\omega}) && \text{(from (A.6))}.\end{aligned}$$

Next, using (A.7),

$$\delta_x^2(P_1,P_2,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{i=1}^n \sigma_i^2\left(\tilde{G}_2G_1\right)dX_0(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{i=1}^n \sigma_i^2\left(\tilde{G}_2\left(G_3G_3^* + \tilde{G}_3^*\tilde{G}_3\right)G_1\right)dX_0(\omega)$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{i=1}^n \sigma_i^2\left(\tilde{G}_2\,G_3\right)dX_0(\omega) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{i=1}^n \sigma_i^2\left(\tilde{G}_3\,G_1\right)dX_0(\omega) \tag{A.9}$$

$$\le \left(\delta_x^2(P_2,P_3,X_0) + \delta_x^2(P_3,P_1,X_0)\right) \tag{A.10}$$

from which (4.7) follows.

To prove (4.8), let $x(\omega) = \frac{dX_0}{d\omega}$ and note that

$$\int_{-\pi}^{\pi} \left(\sum_{i=1}^{n} \sigma_i^2 (\tilde{G}_2 G_1)(e^{j\omega}) \right) dX(\omega) \geq \min_{\omega \in [\omega_0, \omega_p]} x(\omega) \int_{\omega_0}^{\omega_p} \text{trace } \left((\tilde{G}_2 G_1)^* (\tilde{G}_2 G_1) \right) (e^{j\omega}) d\omega$$

On the other hand, if $X_0(\omega) = \sum_{i=1}^{\infty} a_i h(\omega - \omega_i)$, then from ([4],theorem 6.16),

$$\delta_x^2(P_1, P_2, X_0) = \frac{1}{2\pi} \sum_{i=1}^{\infty} a_i \text{ trace } \left((\tilde{G}_2 G_1)^* (\tilde{G}_2 G_1) \right) (e^{j\omega_i})$$

from which (4.9) follows. ∎

Proof of Theorem 1:
Note that

$$\delta_x^2(P_1, P_2, X_i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{trace } \left((\tilde{G}_2 G_1)^* (\tilde{G}_2 G_1) \right) x_i \, d\omega, \; i = 1, 2.$$

This follows from the definition (5.6) of $X_i(\omega)$ and from ([4], theorem 6.20).
Upper Bound: We have

$$\begin{aligned} \|(T(P_1, C_1, C_2) - T(P_2, C_1, C_2))\, r\|_{\mathcal{S}}^2 &= \left\| \left(G_1 (\tilde{K} G_1)^{-1} - G_2 (\tilde{K} G_2)^{-1} \right) (\tilde{V} C_1)\, r \right\|_{\mathcal{S}}^2 \\ &= \left\| \left(K (\tilde{G}_2 K)^{-1} \tilde{G}_2 G_1 (\tilde{K} G_1)^{-1} \right) (\tilde{V} C_1)\, r \right\|_{\mathcal{S}}^2 \end{aligned} \quad \text{(A.11)}$$

(pre-multiplying by the left hand side of equality in (A.7) and using the fact $\tilde{G}_2 G_2 = 0$)

$$\leq \gamma \left(\sup_{\omega} \overline{\sigma}^2 \, (\tilde{G}_2 K)^{-1} \right) \left(\sup_{\omega} \overline{\sigma}^2 \, (\tilde{K} G_1)^{-1} \right) \left(\sup_{\omega} \overline{\sigma}^2 \, (\tilde{V} C_1) \right) \delta_x^2(P_1, P_2, X_2). \quad \text{(A.12)}$$

where the last step uses

$$\overline{\sigma}\,(\phi_r) \sum_{i=1}^{n} \sigma_i^2 \, (\tilde{G}_2 G_1) \;\leq \gamma\, x_2 \sum_{i=1}^{n} \sigma_i^2 \, (\tilde{G}_2 G_1).$$

Also, note that

$$\frac{1}{n\,\gamma\,\|X_2\|_{\Phi}} \leq \frac{1}{\|r\|_{\mathcal{S}}^2} \leq \frac{1}{n\,\gamma\,\|X_1\|_{\Phi}}. \quad \text{(A.13)}$$

The result then follows using (A.11)-(A.13) and using the definitions of β_c and of $b(P_i, C)$ in (5.4) and (3.4) respectively.
Lower Bound: Put $L = (\tilde{K} G_1)^{-1}$, $Q = (\tilde{K} G_2)^{-1} (\tilde{K} G_1)$. Then

$$\begin{aligned} \left\| \left(G_1 (\tilde{K} G_1)^{-1} - G_2 (\tilde{K} G_2)^{-1} \right) (\tilde{V} C_1)\, r \right\|_{\mathcal{S}}^2 &= \left\| (G_1 - G_2 Q)\, L (\tilde{V} C_1)\, r \right\|_{\mathcal{S}}^2 \\ &\geq \alpha_c^2 \, \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{i=1}^{n} \sigma_i^2 \left((G_1 - G_2 Q) \phi_r^{\frac{1}{2}} \right) d\omega \quad \text{(using } \underline{\sigma}(L) \geq 1) \\ &\geq \frac{\gamma\, \alpha_c^2}{2\pi} \int_{-\pi}^{\pi} x_1 \sum_{i=1}^{n} \sigma_i^2 (G_1 - G_2 Q)\, d\omega \end{aligned}$$

$$\begin{aligned} \text{and} \quad x_1 \sum_{i=1}^{n} \sigma_i^2 \, ((G_1 - G_2 Q)) &= x_1 \sum_{i=1}^{n} \sigma_i^2 \left(\begin{bmatrix} \tilde{G}_2 \\ G_2^* \end{bmatrix} (G_1 - G_2 Q) \right) \quad \text{(using (A.6))} \\ &\geq x_1 \sum_{i=1}^{n} \sigma_i^2 \left(\tilde{G}_2 G_1 \right). \end{aligned} \quad \text{(A.14)}$$

Hence

$$\left\| \left(G_1(\tilde{K}G_1)^{-1} - G_2(\tilde{K}G_2)^{-1} \right) (\tilde{V}C_1)\, r \right\|_{\mathcal{S}}^2 \geq \frac{\gamma\, \alpha_c^2}{2\pi} \int_{-\pi}^{\pi} x_1 \sum_{i=1}^{n} \sigma_i^2 \left(\tilde{G}_2 G_1 \right) d\omega. \tag{A.15}$$

The result follows from (A.15) and (A.13).

The upper and lower bounds in the second part may be proven similarly substituting $C_1 = \tilde{V}^{-1}(\tilde{K}G_1)$ in (A.11) and (A.14). ■

Proof of Theorem 2:
From (6.1),

$$z = \begin{bmatrix} y \\ u \end{bmatrix} = G_0(\tilde{K}G_0)^{-1}\tilde{V}C_1 r + K(\tilde{G}_0 K)^{-1}\tilde{G}_0 w \tag{A.16}$$

so that

$$\|\tilde{G}_\theta z\|_{\mathcal{S}}^2 = \|(\tilde{G}_\theta G_0)(\tilde{K}G_0)^{-1}(\tilde{V}C_1)\, r\|_{\mathcal{S}}^2 + \|(\tilde{G}_\theta K)(\tilde{G}_0 K)^{-1}\, \tilde{G}_0\, w\|_{\mathcal{S}}^2 \tag{A.17}$$

(since $\phi_{r\omega} = 0$)

$$\leq \left(\sup_\omega \overline{\sigma}^2(\tilde{K}G_0)^{-1} \right) \left(\sup_\omega \overline{\sigma}^2(\tilde{V}C_1) \right) \delta_x^2(P_0, P_\theta, F_r) + \left(\sup_\omega \overline{\sigma}^2(\tilde{G}_0 K)^{-1} \right) \|\, \tilde{G}_0\, w\|_{\mathcal{S}}^2. \tag{A.18}$$

The lower bound follows from (A.1) using the definitions of β_c and $b(P_0, C)$. To derive the upper bound, note that

$$\begin{aligned} \|\tilde{G}_\theta z\|_{\mathcal{S}}^2 \geq & \left(\inf_\omega \underline{\sigma}^2\, (\tilde{K}G_0)^{-1} \right) \left(\inf_\omega \underline{\sigma}^2\, (\tilde{V}C_1) \right) \delta_x^2(P_0, P_\theta, F_r) \\ & + \left(\inf_\omega \underline{\sigma}^2\, (\tilde{G}_\theta K) \right) \left(\inf_\omega \underline{\sigma}^2\, (\tilde{G}_0 K)^{-1} \right) \|\, \tilde{G}_0\, w\|_{\mathcal{S}}^2. \end{aligned}$$

The result then follows using

$$\begin{aligned} & \inf_\omega \underline{\sigma}\, (\tilde{K}G_0)^{-1}(e^{j\omega}) \geq 1, \ \inf_\omega \underline{\sigma}\, (\tilde{G}_0 K)^{-1}(e^{j\omega}) \geq 1 \\ & \text{and } \inf_\omega \underline{\sigma}\, (\tilde{G}_\theta K)(e^{j\omega}) = b(P_\theta, C). \end{aligned}$$ ■

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