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Loop equations for multi-cut matrix models

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Abstract

The loop equation for the complex one-matrix model with a multi-cut structure is derived and solved in the planar limit. An iterative scheme for higher genus contributions to the free energy and the multi-loop correlators is presented for the two-cut model, where explicit results are given up to and including genus two. The double-scaling limit is analyzed and the relation to the one-cut solution of the hermitian and complex one-matrix model is discussed.

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1 Introduction

Within the topic of matrix models multi-cut solutions are of considerable interest. They are intimately related to the existence of multi-critical points. In order to reach, say, an mth critical point, $m-1$ coupling constants have to be introduced in the matrix potential and adjusted in the right way. This immediately leads to the possibility of multi-cut solutions because there can be as many cuts as minima of the potential.

Unfortunately there is not much known about higher-order contributions in the topological $1/N$ expansion for multi-cut solutions. The saddle-point approximation provides only the planar solution, whereas the full non-perturbative treatment with orthogonal polynomials has been successful only for one or at most two cuts except for special cases like degenerate minima of the potential [1]. The reason is that the appropriate ansatz for solving the string equation is not known in general. The assumption for the recursion coefficients yielding several continuous functions in the large-N limit does not match with the semi-classical analysis for higher-order potentials [2]. Numerical studies [3, 4, 5, 6, 7] have shown the existence of instabilities in the solution of the string equation for a variety of different potentials. This phenomenon has been subsumed under the catchword of 'chaos in matrix models'. There have been attempts to explain the origin of these oscillations [5] but a full understanding is still lacking.

Within the framework of the third method of solving matrix models, the technique of loop equations [8], there has been significant progress during the last years. Ambjørn *et al.* [9, 10, 11] have proposed a very effective scheme to calculate higher-genus contributions in the perturbative expansion. Making use of a redefinition from coupling constants to moments it allows one to determine all multi-loop correlators order by order in the genus expansion.

The aim of this paper is to demonstrate how this method can be applied to multi-cut solutions, where the complex matrix model [12] has been considered for simplicity. The loop equation [1] and the starting point, the planar solution of the one-loop correlator, can be obtained for any number of cuts. However, for more than two cuts technical difficulties enter the game via a new type of equation determining the edges of the cuts. So the complete iterative solution of the two-cut complex one-matrix model presented here may be seen as a first step towards a possible investigation of the 'chaotic phenomena' with the method of loop equations.

The paper is organized as follows. In section 2 the basic definitions are given. Section 3 sets up the loop equation for multi-cut correlators and its planar solution. In section 4 the iterative solution for two cuts is explained in detail, and explicit results for genus one and two are obtained. The last section before concluding is devoted to the double-scaling limit.

[1] The approach of [10] adopted here considerably differs from [13].

2 Basic Definitions

The complex one-matrix model is defined by the partition function

$$Z\,[N,\{g_i\}] \;=\; e^{N^2 F} \;=\; \int d\phi^\dagger d\phi \,\exp(-N\;\mathrm{Tr}V(\phi^\dagger\phi)) \tag{2.1}$$

with

$$V(\phi^\dagger\phi) \;=\; \sum_{j=1}^{\infty}\frac{g_j}{j}(\phi^\dagger\phi)^j\;, \tag{2.2}$$

where the integration is over complex $N\times N$ matrices. The generating functional or one-loop average is given by

$$W(p) \;=\; \frac{1}{N}\sum_{k=0}^{\infty}\frac{\langle\mathrm{Tr}(\phi^\dagger\phi)^k\rangle}{p^{2k+1}} \;=\; \frac{1}{N}\left\langle \mathrm{Tr}\frac{p}{p^2-\phi^\dagger\phi}\right\rangle\;. \tag{2.3}$$

Introducing the loop insertion operator

$$\frac{d}{dV}(p) \;\equiv\; -\sum_{j=1}^{\infty}\frac{j}{p^{2j+1}}\frac{d}{dg_j} \tag{2.4}$$

the generating functional can be obtained from the free energy

$$W(p) \;=\; \frac{d}{dV}(p)F+\frac{1}{p}\;. \tag{2.5}$$

More generally one gets the n-loop correlator by iterative application of the loop insertion operator to F,

$$W(p_1,\ldots,p_n) \;=\; \frac{d}{dV}(p_n)\frac{d}{dV}(p_{n-1})\cdots\frac{d}{dV}(p_1)F\;,\;\; n\geq 2\;, \tag{2.6}$$

where

$$W(p_1,\ldots,p_n) \;\equiv\; \sum_{k_1,\ldots,k_n=1}^{\infty}\frac{\langle\mathrm{Tr}(\phi^\dagger\phi)^{k_1}\cdots\mathrm{Tr}(\phi^\dagger\phi)^{k_n}\rangle_{conn}}{p_1^{2k_1+1}\cdots p_n^{2k_n+1}} \tag{2.7}$$

and *conn* refers to the connected part. As the multi-loop correlators and the free energy have the same $1/N$ expansion

$$W(p_1,\ldots,p_n) \;=\; \sum_{g=0}^{\infty}\frac{1}{N^{2g}}W_g(p_1,\ldots,p_n)\;, \tag{2.8}$$

$$F \;=\; \sum_{g=0}^{\infty}\frac{1}{N^{2g}}F_g\;, \tag{2.9}$$

relation (2.6) holds for each genus separately.

3 The Loop Equation

In this section the loop equation for the complex matrix model [14] and its planar solution will be given. The explicit formulas are restricted to the two-cut case for simplicity and for consistency with the following sections. The multi-cut case can be found in the appendix. The origin of technical difficulties for more than two cuts will be also explained in this section.

The form of the loop equation depends explicitly on the number of cuts of the one-loop correlator only via the contour $\mathcal{C}$ of the complex integral (see appendix A). One has

$$\oint_{\mathcal{C}} \frac{d\omega}{4\pi i} \frac{\omega V'(\omega)}{p^2 - \omega^2} W(\omega) \;=\; (W(p))^2 + \frac{1}{N^2} \frac{d}{dV}(p) W(p) \quad , \; p \notin \sigma \; . \tag{3.1}$$

Here, $V(\omega) = \sum_{j=1}^{\infty} \frac{g_j}{j} \omega^{2j}$, and the support of the eigenvalue density in the two-cut case is $\sigma = [-x, -y] \cup [y, x]$. The contour $\mathcal{C}$ depicted in Fig.1 encloses *all* eigenvalues in such a way that p can also take values on the open real interval between the cuts of $W(p)$. The generalization to s cuts is obvious.

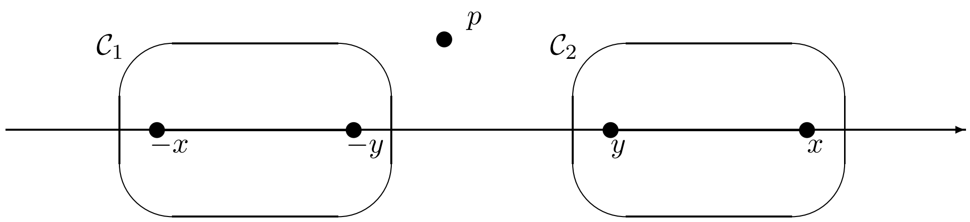

Figure 1: the integration contour $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$

Now the iterative solution of (3.1) works as follows [9, 10, 11]. First the planar solution $W_0(p)$ is determined by taking the limit $N \to \infty$, omitting the last term on the r.h.s.. This solution will then be used as a starting point for the iteration, which calculates $W_g(p)$ step by step from terms of lower genera. For the case of two cuts, $W_0(p)$ is given by

$$W_0(p) \;=\; \frac{1}{2} \oint_{\mathcal{C}} \frac{d\omega}{4\pi i} \frac{pV'(\omega)}{p^2 - \omega^2} \frac{\phi^{(0)}(\omega)}{\phi^{(0)}(p)} \tag{3.2}$$

with

$$\phi^{(0)}(\omega) \;\equiv\; \frac{1}{\sqrt{(\omega^2 - x^2)(\omega^2 - y^2)}} \; . \tag{3.3}$$

The extension to s cuts can be found in appendix B. Equation (3.2) differs from the one-cut solution not only by the second square root but also by a factor of p instead of ω in the numerator. This stems from the fact that depending on the number of cuts s being even or odd the complex function $\sqrt{(\omega^2-x_1^2)\cdots(\omega^2-x_s^2)}$ is to be defined as an even or odd function of ω respectively (see also appendix B).

From the unit normalization of the eigenvalue density it follows that

$$\lim_{p\to\infty} W(p) \;=\; \frac{1}{p}\;. \tag{3.4}$$

The leading asymptotic term must be accounted for already by the planar solution $W_0(p)$ as $1/p$ does not depend on N. Imposing this condition on eq. (3.2) one finds

$$\delta_{k,2} \;=\; \frac{1}{2}\oint_{\mathcal{C}} \frac{d\omega}{4\pi i}\omega^k V'(\omega)\phi^{(0)}(\omega)\;,\;\; k=0 \text{ and } 2\;, \tag{3.5}$$

which implicitly determines x and y as functions of the coupling constants g_i.

At this point it should be mentioned that for more than two cuts the condition (3.4) does not supply any more enough equations to determine all endpoints of the cuts. In the complex matrix model with an s-cut solution there are s such parameters x_i to be determined. In eq. (3.5) k then runs over $s, s-2, s-4,\ldots$ down to 0 or 1 depending on whether s is even or odd. So this yields only $(s+2)/2$ or $(s+1)/2$ equations for s even or odd. The missing equations can be derived from the requirement that the chemical potentials between the cuts are equal [15], namely

$$\int_{\bar{\sigma}_i} d\lambda\; \rho(\lambda) \;=\; 0\;,\;\; i=1,\ldots,s-1\;. \tag{3.6}$$

The eigenvalue density $\rho(\lambda)$ is given in the next section and the $\bar{\sigma}_i$ are the bounded connected components of the real complement of the support σ of $\rho(\lambda)$. Because of symmetry only the intervals $\bar{\sigma}_i$ on the positive real line need to be considered, which provides the remaining $(s-2)/2$ or $(s-1)/2$ equations. They lead to a more complicated dependence of the x_i containing elliptic integrals, except in the case of $s=1$ or 2 where they are trivially fulfilled. To see this the loop insertion operator is applied to eq. (3.6), which yields

$$0 \;=\; M_1^{(i)}\frac{dx_i^2}{dV}(p)\int_{\bar{\sigma}_j} d\lambda \frac{\sqrt{(\lambda^2-x_1^2)\cdot\ldots\cdot(\lambda^2-x_s^2)}}{(\lambda^2-x_i^2)}\cdot\begin{cases}\lambda & s \text{ even}\\ 1 & s \text{ odd}\;.\end{cases} \tag{3.7}$$

The moments $M_1^{(i)}$ are to be defined in the next section. Together with $\frac{d}{dV}(p)$ of eq. (3.5) the relations (3.6) determine the derivatives $\frac{dx_i^2}{dV}(p)$ for $i=1,\ldots,s$, which are needed in the iterative process. However, their complicated structure makes it hard to see whether a scheme for calculating higher genera can still be established.

Coming back to the iterative solution of the loop equation it turns out that after the insertion of the genus expansion eq. (2.8), $W_g(p)$ is determined by the following equation

$$(\hat{K}-2W_0(p))W_g(p) \;=\; \sum_{g'=1}^{g-1} W_{g'}(p)W_{g-g'}(p)+\frac{d}{dV}(p)W_{g-1}(p)\;,\;\; g\geq 1\;. \tag{3.8}$$

Here, the linear operator $\hat{K}$ is defined by

$$\hat{K}f(p) \;\equiv\; \oint_{\mathcal{C}} \frac{d\omega}{4\pi i} \frac{\omega V'(\omega)}{p^2-\omega^2} f(\omega)\ . \tag{3.9}$$

$W_g(p)$ is now expressed completely in terms of $W_{g\prime}(p)$ with $g\prime < g$ on the r.h.s. of eq. (3.8). Hence the next step is the inversion of the operator $(\hat{K} - 2W_0(p))$ acting on it. In contrast to the one-cut case this operation will involve zero modes contributing to $W_g(p)$ which have to be fixed.

4 The iterative solution

4.1 Change of variables

In analogy to [11] it is convenient to change variables from the coupling constants g_i to the moments M_k and J_k in the following way

$$\begin{aligned} M_k &\equiv \oint_{\mathcal{C}} \frac{d\omega}{4\pi i} \frac{V'(\omega)\phi^{(0)}(\omega)}{(\omega^2-x^2)^k}\ ,\ k \geq 1\ , \\ J_k &\equiv \oint_{\mathcal{C}} \frac{d\omega}{4\pi i} \frac{V'(\omega)\phi^{(0)}(\omega)}{(\omega^2-y^2)^k}\ ,\ k \geq 1\ . \end{aligned} \tag{4.1}$$

The advantage of these new variables is that, for given genus, F_g and the multi-loop correlators $W_g(p_1,\ldots,p_n)$ depend only on a finite number of moments instead of the infinite set of couplings. Moreover, the mth multi-critical point can be characterized by the vanishing of the first $m-1$ moments M_k or J_k. This happens whenever extra zeros of the eigenvalue density

$$\rho(\lambda) \;=\; \frac{1}{\pi}\,|\mathcal{I}\mathrm{m}W_0(\lambda)| \;=\; \frac{1}{4\pi}\,|M(\lambda)|\,\sqrt{(x^2-\lambda^2)(\lambda^2-y^2)}\ ,\ \lambda\in\sigma\ . \tag{4.2}$$

occur at either x^2 or y^2. The analytic part $M(\lambda)$, given in appendix B, must develop these extra zeros. Using eqs. (B.9) and (4.1) it can be seen that the moments defined above provide an expansion of $M(\lambda)$, i.e.

$$M(\lambda) \;=\; \sum_{k=1}^{\infty} \Big(M_k\lambda(\lambda^2-x^2)^{k-1} + J_k\lambda(\lambda^2-y^2)^{k-1} \Big)\ . \tag{4.3}$$

Calculating the moments in terms of the coupling constants yields

$$\begin{aligned} M_k &= g_{k+1} + \left((k+\frac{1}{2})x^2 + \frac{1}{2}y^2 \right) g_{k+2} + \ldots\ , \\ J_k &= g_{k+1} + \left((k+\frac{1}{2})y^2 + \frac{1}{2}x^2 \right) g_{k+2} + \ldots\ . \end{aligned} \tag{4.4}$$

4.2 Inversion of $(\hat{K} - 2W_0(p))$

In order to proceed in solving equation (3.8) it is necessary to find a set of basis functions for the operator $(\hat{K} - 2W_0(p))$ acting on $W_g(p)$ as follows

$$\begin{aligned} (\hat{K} - 2W_0(p))\, \chi^{(n)}(p) &= \frac{1}{(p^2-x^2)^n} \ , \ n \geq 1 \ , \\ (\hat{K} - 2W_0(p))\, \psi^{(n)}(p) &= \frac{1}{(p^2-y^2)^n} \ , \ n \geq 1 \ . \end{aligned} \tag{4.5}$$

As the main result, $W_g(p)$ will then be expressed in terms of these functions,

$$W_g(p) = \sum_{n=1}^{3g-1} A_g^{(n)} \chi^{(n)}(p) \ + B_g^{(n)} \psi^{(n)}(p) \ , \tag{4.6}$$

where the coefficients $A_g^{(n)}$ and $B_g^{(n)}$ only depend on the moments M_k and J_k and on x^2 and y^2.

Note that eq. (4.5) does not yet completely determine $\chi^{(n)}(p)$ and $\psi^{(n)}(p)$ due to a non-trivial kernel of $(\hat{K} - 2W_0(p))$. Because of eq. (3.4) only terms asymptotically of order $p^{-k}, k \geq 2$, can contribute to $W_g(p)$ for $g \geq 1$. In the one-cut case this made the definition of the basis unique in a simple way. The argument excluded that the zero mode $1/\sqrt{p^2-x^2}$ of $(\hat{K} - 2W_0(p))_{one-cut}$, which is asymptotically of order $\frac{1}{p}$, could be added to $W_g(p)$. However, here this is no longer the case as

$$\mathrm{Ker}(\hat{K} - 2W_0(p)) = \mathrm{Span}\left\{ p\phi^{(0)}(p),\ \frac{1}{p}\phi^{(0)}(p) \right\} \tag{4.7}$$

and

$$\frac{\phi^{(0)}(p)}{p} \sim \frac{1}{p^3} \ . \tag{4.8}$$

Hence such a term can be added to $W_g(p)$ in every step of the iteration. How shall $\chi^{(n)}(p)$ and $\psi^{(n)}(p)$ be fixed such that $W_g(p)$ is uniquely determined? By definition (see eq. (2.5)) $W_g(p)$ can be written as a total derivative

$$W_g(p) = \frac{d}{dV}(p)\ F_g \ , \ g \geq 1 \ . \tag{4.9}$$

In order to satisfy this equation, the p-dependence of $\chi^{(n)}(p)$ and $\psi^{(n)}(p)$ must be completely absorbed in terms of the type $\frac{dx^2}{dV}(p)$, $\frac{dy^2}{dV}(p)$, $\frac{dM_k}{dV}(p)$ and $\frac{dJ_k}{dV}(p)$. Consequently the basis functions $\chi^{(n)}(p)$ and $\psi^{(n)}(p)$ must be linear combinations of these, the coefficients of course again depending on moments, x^2 and y^2. In this manner the zero mode contributions to the basis, which are indeed necessary, become uniquely fixed. The final result, proven in appendix C, takes the following form

$$\begin{aligned} \chi^{(n)}(p) &\equiv \frac{1}{M_1}\left(\frac{1}{x^2}\Big(\phi_x^{(n)}(p) - \sum_{k=1}^{n-1} \chi^{(k)}(p) M_{n-k} \Big) - \sum_{k=1}^{n-1} \chi^{(k)}(p) M_{n-k+1} \right) \ , \\ \psi^{(n)}(p) &\equiv \frac{1}{J_1}\left(\frac{1}{y^2}\Big(\phi_y^{(n)}(p) - \sum_{k=1}^{n-1} \psi^{(k)}(p) J_{n-k} \Big) - \sum_{k=1}^{n-1} \psi^{(k)}(p) J_{n-k+1} \right) , \end{aligned} \tag{4.10}$$

where

$$\phi_x^{(n)}(p) \equiv \frac{p\phi^{(0)}(p)}{(p^2-x^2)^n} \ , \ n \geq 1 \ ,$$
$$\phi_y^{(n)}(p) \equiv \frac{p\phi^{(0)}(p)}{(p^2-y^2)^n} \ , \ n \geq 1 \ . \qquad (4.11)$$

4.3 The iterative procedure determining $W_g(p)$

In order to solve eq. (3.8), one first has to compute $\frac{d}{dV}(p)W_0(p) = W_0(p,p)$. To do so it is convenient to rewrite the loop insertion operator

$$\frac{d}{dV}(p) = \frac{\partial}{\partial V}(p) + \frac{dx^2}{dV}(p)\frac{\partial}{\partial x^2} + \frac{dy^2}{dV}(p)\frac{\partial}{\partial y^2} \ , \qquad (4.12)$$

where

$$\frac{\partial}{\partial V}(p) \equiv -\sum_{j=1}^{\infty} \frac{j}{p^{2j+1}}\frac{\partial}{\partial g_j} \ . \qquad (4.13)$$

For evaluating $\frac{dx^2}{dV}(p)$ and $\frac{dx^2}{dV}(p)$, eq. (4.12) is applied to eq. (3.5) which yields

$$\frac{dx^2}{dV}(p) = \frac{1}{M_1}\phi_x^{(1)}(p) \ , \ \ \frac{dy^2}{dV}(p) = \frac{1}{J_1}\ \phi_y^{(1)}(p) \ .^2 \qquad (4.14)$$

Now with eq. (4.12) $\frac{d}{dV}(p)W_0(p)$ can be calculated. Using the relation

$$\frac{\partial}{\partial V}(p)V'(\omega) = -\frac{2\omega p}{(p^2-\omega^2)^2} \ , \qquad (4.15)$$

the definitions (4.1) and deforming the contour of the remaining integral to infinity, one gets

$$W_0(p,p) = \frac{p^2 d^2}{16(p^2-x^2)^2(p^2-y^2)^2} \ , \qquad (4.16)$$

where

$$d \equiv x^2 - y^2. \qquad (4.17)$$

Before one can make use of the basis functions eq. (4.10) for $(\hat{K} - 2W_0(p))$ one has to decompose the r.h.s. of eq. (3.8) into fractions of the form $(p^2-x^2)^{-n}$ and $(p^2-y^2)^{-n}$. Doing so the coefficients $A_g^{(n)}$ and $B_g^{(n)}$ in (4.6) can now be identified for genus 1.

$$A_1^{(1)} = \frac{1}{16} - \frac{1}{8}\frac{x^2}{d} \ , \ \ A_2^{(1)} = \frac{x^2}{16}$$
$$B_1^{(1)} = \frac{1}{16} + \frac{1}{8}\frac{y^2}{d} \ , \ \ B_2^{(1)} = \frac{y^2}{16} \qquad (4.18)$$

[2] This simple relationship with the basis functions is spoiled for more than two cuts.

It is clear how to carry on the iteration process. However, in order to calculate $\frac{d}{dV}(p) W_g(p)$ it is convenient to rewrite the loop insertion operator once again,

$$\frac{d}{dV}(p) = \sum_{n=1}^{\infty} \frac{dM_n}{dV}(p) \frac{\partial}{\partial M_n} + \sum_{n=1}^{\infty} \frac{dJ_n}{dV}(p) \frac{\partial}{\partial J_n} + \frac{dx^2}{dV}(p) \frac{\partial}{\partial x^2} + \frac{dy^2}{dV}(p) \frac{\partial}{\partial y^2} \, , \tag{4.19}$$

$$\begin{aligned}\frac{dM_n}{dV}(p) &= -(n+\frac{1}{2})\phi_x^{(n+1)}(p) - \frac{1}{2}\sum_{k=1}^{n} \frac{(-1)^{k+n}}{d^{n-k+1}} \left(\phi_x^{(k)}(p) - M_k \frac{dy^2}{dV}(p) \right) \\ &\quad +(n+\frac{1}{2}) M_{n+1} \frac{dx^2}{dV}(p) \, .\end{aligned} \tag{4.20}$$

With the definition (4.1) of the moments and the form of $\frac{d}{dV}(p)$ in eq. (4.12) the relation is easily verified. The corresponding result for $\frac{dJ_n}{dV}(p)$ can be obtained by interchanging $x^2 \leftrightarrow y^2$ and $M_k \leftrightarrow J_k$. Looking back at the original loop equation (3.1) it is obvious that the free energy and the multi-loop correlators for all genera should be invariant under such an interchangement .

Using the loop insertion operator in the form of eq. (4.19) a lengthy calculation yields the following result for $g = 2$:

$$\begin{aligned}
A_2^{(1)} &= -\frac{5}{32}\frac{y^2 M_3}{d^3 (M_1)^3} - \frac{5}{32}\frac{x^2 J_3}{d^3 (J_1)^3} - \frac{1}{64}\frac{x^2 M_2 J_2}{d^3 (M_1)^2 (J_1)^2} + \frac{1}{128}\frac{M_2 J_2}{d^2 (M_1)^2 (J_1)^2} \\
&\quad + \frac{49}{256}\frac{y^2 (M_2)^2}{d^3 (M_1)^4} + \frac{49}{256}\frac{x^2 (J_2)^2}{d^3 (J_1)^4} + \frac{11}{128}\frac{x^2 M_2}{d^4 (M_1)^2 J_1} - \frac{11}{128}\frac{y^2 J_2}{d^4 M_1 (J_1)^2} \\
&\quad + \frac{67}{128}\frac{y^2 M_2}{d^4 (M_1)^3} - \frac{67}{128}\frac{x^2 J_2}{d^4 (J_1)^3} - \frac{3}{128}\frac{M_2}{d^3 (M_1)^2 J_1} - \frac{3}{128}\frac{J_2}{d^3 M_1 (J_1)^2} \\
&\quad + \frac{201}{256}\frac{y^2}{d^5 (M_1)^2} + \frac{201}{256}\frac{x^2}{d^5 (J_1)^2} + \frac{57}{64}\frac{x^2}{d^5 M_1 J_1} - \frac{57}{128}\frac{1}{d^4 M_1 J_1} \, , \\
A_2^{(2)} &= +\frac{5}{32}\frac{y^2 M_3}{d^2 (M_1)^3} + \frac{1}{128}\frac{x^2 M_2 J_2}{d^2 (M_1)^2 (J_1)^2} - \frac{49}{256}\frac{y^2 (M_2)^2}{d^2 (M_1)^4} \\
&\quad - \frac{3}{128}\frac{x^2 M_2}{d^3 (M_1)^2 J_1} + \frac{1}{16}\frac{x^2 J_2}{d^3 M_1 (J_1)^2} - \frac{3}{128}\frac{J_2}{d^2 M_1 (J_1)^2} - \frac{67}{128}\frac{y^2 M_2}{d^3 (M_1)^3} \\
&\quad - \frac{57}{128}\frac{x^2}{d^4 M_1 J_1} - \frac{201}{256}\frac{y^2}{d^4 (M_1)^2} + \frac{15}{128}\frac{1}{d^3 M_1 J_1} \, , \\
A_2^{(3)} &= -\frac{5}{32}\frac{x^2 M_3}{d (M_1)^3} + \frac{49}{256}\frac{x^2 (M_2)^2}{d (M_1)^4} - \frac{5}{128}\frac{x^2 J_2}{d^2 M_1 (J_1)^2} + \frac{67}{128}\frac{x^2 M_2}{d^2 (M_1)^3} \\
&\quad - \frac{49}{128}\frac{M_2}{d (M_1)^3} + \frac{15}{128}\frac{x^2}{d^3 M_1 J_1} + \frac{201}{256}\frac{x^2}{d^3 (M_1)^2} - \frac{189}{256}\frac{1}{d^2 (M_1)^2} \, , \\
A_2^{(4)} &= -\frac{49}{128}\frac{x^2 M_2}{d (M_1)^3} - \frac{189}{256}\frac{x^2}{d^2 (M_1)^2} + \frac{105}{256}\frac{1}{d (M_1)^2} \, , \\
A_2^{(5)} &= \frac{105}{256}\frac{x^2}{d (M_1)^2} \, , \\
B_2^{(i)} &= A_2^{(i)}(M_k \leftrightarrow J_k, x^2 \leftrightarrow y^2) \, , \; i = 1, \ldots, 5 \, .
\end{aligned} \tag{4.21}$$

$W_2(p)$ is then obtained by inserting eqs. (4.10) and (4.21) into eq. (4.6).

4.4 The iterative procedure for F_g

As it was already pointed out in section 4.2 and proven in appendix C the basis functions $\chi^{(n)}(p)$ and $\psi^{(n)}(p)$ can be written as linear combinations of derivatives with respect to $V(p)$. This requirement was imposed in order to fix the basis in a unique way. For genus 1 the explicit result reads

$$\begin{aligned}
\chi^{(1)}(p) &= \frac{1}{x^2}\frac{dx^2}{dV}(p) \;, \\
\chi^{(2)}(p) &= -\frac{2}{3}\frac{1}{x^2 M_1}\frac{dM_1}{dV}(p) - \frac{1}{x^4}\frac{dx^2}{dV}(p) - \frac{1}{3}\frac{1}{x^2}\frac{d}{dV}(p)\ln(d) \;, \\
\psi^{(1)}(p) &= \frac{1}{y^2}\frac{dy^2}{dV}(p) \;, \\
\psi^{(2)}(p) &= -\frac{2}{3}\frac{1}{y^2 J_1}\frac{dJ_1}{dV}(p) - \frac{1}{y^4}\frac{dy^2}{dV}(p) - \frac{1}{3}\frac{1}{y^2}\frac{d}{dV}(p)\ln(d) \;, \qquad (4.22)
\end{aligned}$$

which can easily be verified from the definitions. The relation (4.9) then allows to calculate F_g for any given $W_g(p)$. So from eq. (4.22) in combination with eqs. (4.18) and (4.6) F_1 can now be read off up to a constant

$$F_1 = -\frac{1}{24}\ln(M_1) - \frac{1}{24}\ln(J_1) - \frac{1}{6}\ln(d) \;. \qquad (4.23)$$

Continuing in the same manner and rewriting the basis as being sketched in appendix C the genus two result is obtained using eq. (4.21) after some tedious work,

$$\begin{aligned}
F_2 = & -\frac{35}{384}\frac{M_4}{d(M_1)^3} + \frac{35}{384}\frac{J_4}{d(J_1)^3} + \frac{43}{192}\frac{M_3}{d^2(M_1)^3} + \frac{43}{192}\frac{J_3}{d^2(J_1)^3} \\
& + \frac{29}{128}\frac{M_3 M_2}{d(M_1)^4} - \frac{29}{128}\frac{J_3 J_2}{d(J_1)^4} + \frac{1}{64}\frac{M_2 J_2}{d^2(M_1)^2(J_1)^2} \\
& - \frac{21}{160}\frac{(M_2)^3}{d(M_1)^5} + \frac{21}{160}\frac{(J_2)^3}{d(J_1)^5} - \frac{11}{40}\frac{(M_2)^2}{d^2(M_1)^4} - \frac{11}{40}\frac{(J_2)^2}{d^2(J_1)^4} \\
& - \frac{181}{480}\frac{M_2}{d^3(M_1)^3} + \frac{181}{480}\frac{J_2}{d^3(J_1)^3} - \frac{3}{64}\frac{M_2}{d^3(M_1)^2 J_1} + \frac{3}{64}\frac{J_2}{d^3 M_1 (J_1)^2} \\
& - \frac{181}{480}\frac{1}{d^4(M_1)^2} - \frac{181}{480}\frac{1}{d^4(J_1)^2} - \frac{5}{16}\frac{1}{d^4 M_1 J_1} \;. \qquad (4.24)
\end{aligned}$$

From eq. (4.6) and the procedure described just above it should have become obvious that F_g depends on at most $2(3g-2)$ moments.

Another remarkable fact is that the calculated F_1 and F_2 exactly coincide with those of the one-cut hermitian matrix model described in [11] when the identification of the moments M_k and J_k and the difference d is made (using the same notation). This coincidence away from the double-scaling limit cannot be merely pure coincidence. However, since the loop insertion operators are distinct the loop correlators will differ as it can already be seen for $W_1(p)$ and

$W_2(p)$. Relating the two models it has been mentioned in [16] that the complex matrix model corresponds to a hermitian matrix model with a general potential where the eigenvalues are restricted to be positive [3]. It seems that the repulsion at the origin ($\lambda = 0$) is reflected only in the multi-loop correlators but not seen in the free energy.

Finally a comparison to the one-cut solution of the complex matrix model presented in [10] can be made by letting $y \to 0$. The moments defined here and there then translate into each other.

$$\begin{aligned} M_k(g_{k+1} \to g_k) &\rightarrow \mathcal{I}_k \\ J_k &\rightarrow \mathcal{M}_k \end{aligned} \tag{4.25}$$

As one might have expected, the results for the free energy and the multi-loop correlators do not match except for the universal 2-loop correlator $W_0(p,p)$. For a given set of couplings the phase boundary between the one- and two-cut phase can be formulated. The free energy or the correlators can be compared on the boundary, inspecting the order of the phase transition and critical exponents.

4.5 The general structure of F_g and $W_g(p)$

The main result for F_g from the iterative solution of the loop equation can be written in the following way

$$F_g = \sum_{\alpha_i > 1, \beta_j > 1} \langle \alpha_1 \ldots \alpha_k; \beta_1 \ldots \beta_l | \alpha, \beta, \gamma \rangle_g \, \frac{M_{\alpha_1} \ldots M_{\alpha_k} J_{\beta_1} \ldots J_{\beta_l}}{d^\gamma (M_1)^\alpha (J_1)^\beta}, \quad g \geq 2 \,. \tag{4.26}$$

Here the brackets denote rational numbers and α, β and γ are non-negative integers. The summation-indices α_i and β_j take values in the interval $[2, 3g-2]$. For every genus g, F_g contains only finitely many terms with a finite number of moments. In particular F_g depends on at most $2(3g-2)$ different moments. This structure can either be proven along the same lines like in [10] or becomes clear when performing the first steps of the iteration. In perfect analogy to the one-cut case of the hermitian matrix model [11] several relations between the indices and powers in eq. (4.26) can be derived.

First of all because of the symmetry $x^2 \leftrightarrow y^2, M_k \leftrightarrow J_k$ the following holds

$$\langle \alpha_1 \ldots \alpha_k; \beta_1 \ldots \beta_l | \alpha, \beta, \gamma \rangle_g = (-1)^\gamma \langle \beta_1 \ldots \beta_l; \alpha_1 \ldots \alpha_k | \beta, \alpha, \gamma \rangle_g \,. \tag{4.27}$$

Defining

$$N_M = k - \alpha \,, \quad N_J = l - \beta \,, \tag{4.28}$$

it is true that

$$N_M \leq 0 \,, \quad N_J \leq 0 \,. \tag{4.29}$$

[3] This can be seen after diagonalisation when the eigenvalues of $\phi^\dagger \phi$ are considered to be $\lambda \geq 0$ instead of λ^2 with λ real here.

The invariance of the partition function $Z=\exp(\sum_g N^{2-2g}F_g)$ under the rescaling $N\to kN$ and $\rho(\lambda)\to\frac{1}{k}\rho(\lambda)$ for each genus yields

$$N_M+N_J \;=\; 2-2g\;. \tag{4.30}$$

The rescaling $N\to k^2N$ and $g_j\to k^{j-2}g_j$ implies

$$\begin{array}{rclcrcl} M_j & \to & k^{j-1}M_j\;, & & J_j & \to & k^{j-1}J_j\;, \\ x^2 & \to & k^{-1}x^2 & , & y^2 & \to & k^{-1}y^2 \end{array} \tag{4.31}$$

because of eq. (4.4). For F_g this reads

$$\sum_{i=1}^{k}(\alpha_i-1)+\sum_{j=1}^{l}(\beta_j-1)+\gamma=4g-4\;. \tag{4.32}$$

In the double-scaling limit in the next chapter further relations of this type will be derived allowing to decide which terms in eq. (4.26) will survive.

Turning to $W_g(p)$ the coefficients $A_g^{(n)}$ and $B_g^{(n)}$ have a similar structure like in eq. (4.26) since $W_g(p)$ follows from F_g by applying the loop insertion operator. In particular,

$$A_g^{(n)} \;=\; \sum_{\alpha_i>1,\beta_j>1}\langle\alpha_1\ldots\alpha_k;\beta_1\ldots\beta_l|\alpha,\beta,\gamma\rangle_g^{(n)}\;\frac{M_{\alpha_1}\ldots M_{\alpha_k}J_{\beta_1}\ldots J_{\beta_l}}{(M_1)^\alpha(J_1)^\beta}f(x^2,y^2) \tag{4.33}$$

for $g\geq 1$ and analogously for $B_g^{(n)}$. The only difference is that the indices α_i and β_j lie in the interval $[2,3g-n]$. Because of the same genus expansion eq. (4.28) is also valid here. So with the definition (4.10) of the basis functions $W_g(p)$ depends on at most $2(3g-1)$ moments.

5 The double-scaling limit

In the conventional double-scaling limit all matrix models belonging to the same universality class should be equivalent. Consequently all differences originating from the multi-cut structure should vanish in this limit. Having explicit results at hand for the one-cut hermitian and the one- and two-cut complex matrix model ([10],[11]), this can be checked as an example.

In the following the double-scaling limit will be performed for x^2 only; the one for y^2 is then easily obtained. For the mth multi-critical model the couplings are adjusted such that

$$\begin{array}{rcl} x^2 & = & x_c^2-a\Lambda^{\frac{1}{m}}\;, \\ p^2 & = & x_c^2+a\pi \end{array} \tag{5.1}$$

and y^2 does not scale. Here a is the scaling parameter, which becomes zero at the critical point. As it had been already mentioned in section 4.1 the eigenvalue density then develops $m-1$ extra zeros at x^2 and hence

$$M_k\;\sim a^{m-k}\;,\;k=1,\ldots,m-1\;, \tag{5.2}$$

whereas the J_k's do not scale. The resulting contribution to the free energy has the well known behavior

$$F_g \;\sim\; a^{(2-2g)(m+\frac{1}{2})} \;,\; g \geq 1. \tag{5.3}$$

Making use of eq. (4.26) for F_g leads to

$$\sum_{i=1}^{k}(m-\alpha_i) - \alpha(m-1) \geq m(2-2g) - g + 1 \;. \tag{5.4}$$

Since in the scaling limit the free energy should look the same for all multi-critical models it follows that

$$\begin{aligned} N_M &\;\geq\; 2-2g \;, \\ \sum_{i=1}^{k}(\alpha_i - 1) &\;\leq\; 3g-3 \;. \end{aligned} \tag{5.5}$$

From eqs. (4.30) and (4.29) the equality sign holds in the first equation. Only terms for which this is true also in the second line will contribute in the double-scaling limit. For example the genus two contribution to F will then be

$$F_2^{(d.s.l.)} \;=\; -\frac{35}{384}\frac{M_4}{d(M_1)^3} + \frac{29}{128}\frac{M_3 M_2}{d(M_1)^4} - \frac{21}{160}\frac{(M_2)^3}{d(M_1)^5} \;. \tag{5.6}$$

Switching to the one-loop correlator $W_g(p)$ the behavior of the basis functions in eq. (4.10) also has to be analyzed. Equations (5.1) and (5.2) lead to

$$\begin{aligned} \chi^{(n)}(\pi,\Lambda) &\;\sim\; a^{-m-n+\frac{1}{2}} \;, \\ \psi^{(n)}(\pi,\Lambda) &\;\sim\; a^{-\frac{1}{2}} \;. \end{aligned} \tag{5.7}$$

The known genus g contribution

$$W_g(\pi,\Lambda) \;\sim\; a^{(1-2g)(m+\frac{1}{2})-1} \tag{5.8}$$

together with eq. (4.33) requires for the $A_g^{(n)}$ terms that

$$\sum_{i=1}^{k}(m-\alpha_i) - \alpha(m-1) - m - n + \frac{1}{2} \geq m(1-2g) - g - \frac{1}{2}. \tag{5.9}$$

The same argument as above then yields

$$\begin{aligned} N_M &\;\geq\; 2-2g \;, \\ \sum_{i=1}^{k}(\alpha_i - 1) &\;\leq\; 3g-n-1 \;, \end{aligned} \tag{5.10}$$

where again only terms obeying equality appear in the scaling limit. A similar computation for $B_g^{(n)}$ reveals $N_M \geq 1-2g$, which clearly cannot be fulfilled as an equation together with eq. (4.30) because $N_J \leq 0$. So all the $B_g^{(n)}$ terms will disappear in the double-scaling limit. Finally the non-vanishing coefficients of $W_2(p)$ are given as an example.

$$
\begin{aligned}
A_2^{(3)} &= \frac{49}{256}\frac{x_c^2(M_2)^2}{d_c(M_1)^4} - \frac{5}{32}\frac{x_c^2 M_3}{d_c(M_1)^3} \\
A_2^{(4)} &= -\frac{49}{128}\frac{x_c^2 M_2}{d_c(M_1)^3} \\
A_2^{(5)} &= \frac{105}{256}\frac{x_c^2}{d_c(M_1)^2}
\end{aligned} \tag{5.11}
$$

In $W_2(p)$ the x_c^2-dependence cancels out because in the basis the second sum is suppressed in the scaling limit. This reproduces exactly the result for the one-cut hermitian matrix model in [11], where the equivalence to the one-cut complex matrix model had already been proven.

6 Conclusions

It has been shown how the powerful method of iteratively solving the loop equation by Ambjørn *et al.* [9, 10, 11] generalizes to the complex matrix model with more than one cut present. The loop equation for an arbitrary number of cuts was derived and solved for the one-loop correlator in the planar limit. In principle the procedure to find the genus g contribution is clear also for the multi-cut type. Nevertheless, for more than two cuts a new kind of equation determining the edges of the cuts enters and renders the computation technically much more involved.

The iterative scheme was then explicitly presented for the two-cut model, and results for genus one and two were obtained away from the double-scaling limit. Relations to the one-cut solution of the hermitian and complex matrix model were discussed, in particular in the case of the double-scaling limit and when the two cuts merge.

In order to attack the problem of instabilities in multi-cut solutions termed 'chaos in matrix models' a more complicated cut structure has to be examined, e.g. the hermitian model with two cuts for an arbitrary potential or simply just with three or more cuts. Up to now the instabilities have only been found within the picture of orthogonal polynomials. The hope is that they can be found also in the framework of loop equations and that a deeper understanding especially concerning correlation functions can be obtained in this way. These open problems are left subject to further investigations.

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A Derivation of the loop equation for the s-cut solution

The matrix integral eq. (2.1) is invariant under the following transformation

$$\phi \to \phi \left(1+\epsilon \frac{p}{p^2-\phi^\dagger\phi}\right) \ , \ \phi^\dagger \to \left(1+\epsilon \frac{p}{p^2-\phi^\dagger\phi}\right)\phi^\dagger \ . \tag{A.1}$$

The functional determinant and the change of the action is then given by

$$d\phi d\phi^\dagger \ \to \ d\phi d\phi^\dagger \left(1+2\epsilon p (\mathrm{Tr}\frac{p}{p^2-\phi^\dagger\phi})^2\right) \ , \tag{A.2}$$

$$V(\phi^\dagger\phi) \ \to \ V(\phi^\dagger\phi)+2\epsilon\frac{p\phi^\dagger\phi}{p^2-\phi^\dagger\phi}V'(\phi^\dagger\phi) \ , \tag{A.3}$$

where

$$V'(\phi^\dagger\phi) \ \equiv \ \sum_{n=1}^{\infty} g_n(\phi^\dagger\phi)^{n-1} \ . \tag{A.4}$$

The invariance of Z then reads

$$\left\langle (\mathrm{Tr}\frac{p}{p^2-\phi^\dagger\phi})^2\right\rangle - N\left\langle \mathrm{Tr}\frac{\phi^\dagger\phi}{p^2-\phi^\dagger\phi}V'(\phi^\dagger\phi)\right\rangle \ = \ 0 \ , \tag{A.5}$$

which is already almost the loop equation using eqs. (2.6) and (2.7)

$$\frac{1}{N}\left\langle \mathrm{Tr}\frac{\phi^\dagger\phi}{p^2-\phi^\dagger\phi}V'(\phi^\dagger\phi)\right\rangle \ = \ (W(p))^2+\frac{1}{N^2}\frac{d}{dV}(p)W(p) \ . \tag{A.6}$$

Now the density of the eigenvalues can be formally introduced as

$$\rho_N(\lambda) \equiv \frac{1}{N}\left\langle \sum_{i=1}^{N}\delta(\lambda-\lambda_i)\right\rangle \ . \tag{A.7}$$

With $\sigma = \cup_{i=1}^{s}\sigma_i$ being the support of our s-cut solution the explicit dependence on the number of cuts s enters the l.h.s. of eq. (A.6).

$$\begin{aligned}
\frac{1}{N}\left\langle \mathrm{Tr}\frac{\phi^\dagger\phi}{p^2-\phi^\dagger\phi}V'(\phi^\dagger\phi)\right\rangle \ &= \ \sum_{i=1}^{s}\int_{\sigma_i} d\lambda \rho_N(\lambda)\frac{\lambda^2 V'(\lambda^2)}{p^2-\lambda^2} \\
&= \ \sum_{i=1}^{s}\int_{\sigma_i} d\lambda \rho_N(\lambda)\oint_{\mathcal{C}}\frac{d\omega}{4\pi i}\frac{2\omega}{\omega^2-\lambda^2}\frac{\omega^2 V'(\omega^2)}{p^2-\omega^2} \\
&= \ \oint_{\mathcal{C}}\frac{d\omega}{4\pi i}W(\omega)\frac{2\omega^2 V'(\omega^2)}{p^2-\omega^2} \ .
\end{aligned} \tag{A.8}$$

Here $\mathcal{C}$ encloses all cuts without containing $\pm p$, which generalizes the two-cut case depicted in Figure 1 of section 3. With the following change of notation eq. (3.1) is finally obtained.

$$2\omega V'(\omega^2) = 2\sum_{n=1}^{\infty} g_n \omega^{2n-1} \equiv V'(\omega) \ . \tag{A.9}$$

B The planar solution of the loop equation

In the limit of $N \to \infty$ the loop equation (3.1) becomes

$$\oint_{\mathcal{C}} \frac{d\omega}{4\pi i} \frac{\omega V'(\omega)}{p^2 - \omega^2} W_0(\omega) \;=\; (W_0(p))^2 \; . \tag{B.1}$$

Deforming the contour $\mathcal{C}$ to infinity and using the fact that $W(p)$ and $V'(p)$ are odd functions by definition one gets the following contributions from the poles at $\pm p$ and ∞

$$(W_0(p))^2 \;=\; \frac{1}{2} V'(p) W_0(p) + \oint_{\mathcal{C}_\infty} \frac{d\omega}{4\pi i} \frac{\omega V'(\omega)}{p^2 - \omega^2} W_0(\omega) \; . \tag{B.2}$$

The solution of this quadratic equation for $W_0(p)$ of course reads

$$W_0(p) \;=\; \frac{1}{4} V'(p) \pm \sqrt{\frac{1}{16} (V'(p))^2 + Q(p)} \; , \tag{B.3}$$

with

$$Q(p) \;=\; \oint_{\mathcal{C}_\infty} \frac{d\omega}{4\pi i} \frac{\omega V'(\omega)}{p^2 - \omega^2} W_0(\omega) \tag{B.4}$$

to be calculated for any given potential with finitely many couplings. Making an ansatz for a solution with s cuts $W_0(p)$ looks the following

$$W_0(p) \;=\; \frac{1}{4} \left(V'(p) - M(p) \sqrt{(p^2 - x_1^2) \ldots (p^2 - x_s^2)} \right) \; , \tag{B.5}$$

where $M(p)$ is an analytic function. So

$$M(p) \;=\; \phi^{(0)}(p) (V'(p) - 4 W_0(p)) \; , \tag{B.6}$$

remembering

$$\phi^{(0)}(p) = \frac{1}{\sqrt{(p^2 - x_1^2) \ldots (p^2 - x_s^2)}} \;\equiv\; p^{-s} \left(1 - \frac{x_1^2}{p^2}\right)^{-\frac{1}{2}} \ldots \left(1 - \frac{x_s^2}{p^2}\right)^{-\frac{1}{2}} . \tag{B.7}$$

Now for s even (odd) $\phi^{(0)}(p)$ is defined as an even (odd) complex function of p and consequently

$$\begin{aligned} M(p) \;&=\; \frac{1}{2} \left(M(p) + (-1)^{s+1} M(-p) \right) \\ &=\; \frac{1}{2} \oint_{\mathcal{C}_\infty} \frac{d\omega}{2\pi i} M(\omega) \left(\frac{1}{\omega - p} + \frac{(-1)^{s+1}}{\omega + p} \right) \; . \end{aligned} \tag{B.8}$$

Reinserting eq. (B.6) into the integral in eq. (B.8) the term proportional to $W_0(p)$ drops out because of its asymptotic eq. (3.4) and hence

$$M(p) \;=\; \oint_{\mathcal{C}_\infty} \frac{d\omega}{4\pi i} \frac{2V'(\omega)}{\omega^2 - p^2} \phi^{(0)}(\omega) \cdot \begin{cases} p & s \text{ even} \\ \omega & s \text{ odd} \; . \end{cases} \tag{B.9}$$

Plugging this into eq. (B.5) again after a similar calculation $W_0(p)$ can then be expressed as

$$W_0(p) \;=\; \frac{1}{2} \oint_{\mathcal{C}} \frac{d\omega}{4\pi i} \frac{V'(\omega)}{p^2 - \omega^2} \frac{\phi^{(0)}(\omega)}{\phi^{(0)}(p)} \cdot \begin{cases} p & s \text{ even} \\ \omega & s \text{ odd} \; . \end{cases} \tag{B.10}$$

C Determination of the basis

As being described in section 4.2 our aim is to find a basis which can be expressed completely in terms of moments, x^2 and y^2 as well as total derivatives $\frac{d}{dV}(p)$ of them. The results for the latter were already given in eqs. (3) and (4.20). It will be shown by induction that the basis defined in eq. (4.10) can be expressed in the conjectured way. The starting point was made in eq. (4.22). Now assume that this holds for all $\chi^{(k)}(p)$, $k = 1, \dots, n-1$. Rearranging eq. (4.10) shows that the same is true then for the $\phi_x^{(k)}(p)$

$$\phi_x^{(k)}(p) \;=\; x^2 \sum_{l=1}^{k} \chi^{(l)}(p) M_{k-l+1} + \sum_{l=1}^{k-1} \chi^{(l)}(p) M_{k-l} \;,\; k = 1, \dots, n-1. \qquad \text{(C.1)}$$

It follows with eq. (4.20) that the remaining term in $\chi^{(n)}(p)$ can also be rewritten in the desired way.

$$\begin{aligned} \phi_x^{(n)}(p) \;=\; & \frac{1}{n-\frac{1}{2}} \left(-\frac{dM_{n-1}}{dV}(p) - \frac{1}{2} \sum_{k=1}^{n-1} \frac{(-1)^{n+k-1}}{d^{n-k}} \left(\phi_x^{(k)}(p) - M_k \frac{dy^2}{dV}(p) \right) \right) \\ & + \frac{dx^2}{dV}(p) M_n \end{aligned} \qquad \text{(C.2)}$$

The proof for $\psi^{(n)}(p)$ is going exactly along the same lines. The fixing of the basis is unique because the zero mode alone cannot be written as a derivative with respect to $V(p)$.

What remains to show is that $\chi^{(n)}(p)$ and $\psi^{(n)}(p)$ really form a basis like in eq. (4.5). First define in analogy with the one-cut case

$$\tilde{\chi}^{(n)}(p) \;\equiv\; \frac{1}{M_1} \left(\frac{1}{p^2} \phi_x^{(n)}(p) - \sum_{k=1}^{n-1} \tilde{\chi}^{(k)}(p) M_{n-k+1} \right) \;, \qquad \text{(C.3)}$$

which does not contain the zero modes yet. It is easily proven by induction that

$$(\hat{K} - 2W_0(p)) \tilde{\chi}^{(n)}(p) \;=\; \frac{1}{(p^2 - x^2)^n} \;,\; n \geq 1 \;, \qquad \text{(C.4)}$$

holds. Now the zero modes are added to $\tilde{\chi}^{(n)}(p)$,

$$\chi^{(n)}(p) \;\equiv\; \frac{1}{M_1} \left(\frac{1}{p^2} \phi_x^{(n)}(p) + \frac{(-1)^{n+1}}{x^{2n}} \frac{\phi^{(0)}(p)}{p} - \sum_{k=1}^{n-1} \chi^{(k)}(p) M_{n-k+1} \right) \;, \qquad \text{(C.5)}$$

so eq. (C.4) without tilde is still valid. Finally the equivalence of eq. (C.5) to the form in eq. (4.10) is again shown by induction. Proceeding in the same way for $\psi^{(n)}(p)$ completes the proof of this section.

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