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Universality of random matrices in the microscopic limit and the Dirac operator spectrum

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Abstract

We prove the universality of correlation functions of chiral unitary and unitary ensembles of random matrices in the microscopic limit. The essence of the proof consists in reducing the three-term recursion relation for the relevant orthogonal polynomials into a Bessel equation governing the local asymptotics around the origin. The possible physical interpretation as the universality of soft spectrum of the Dirac operator is briefly discussed.

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1 Introduction

Consider a four-dimensional gauge theory coupled to fermions in given representations of the gauge group, and with given global (and possibly spontaneously broken) flavor symmetries. In the chiral limit, the associated Dirac operator $\mathrm{i}\not{D}$ anticommutes with γ_5,

$$\{\mathrm{i}\not{D}, \gamma_5\} \;=\; 0 \; . \tag{1.1}$$

This means that the associated eigenvalues of $\mathrm{i}\not{D}$, defined by $\mathrm{i}\not{D}\phi_n = \lambda_n\phi_n$ appear symmetrically around zero: $\pm\lambda_n$. The corresponding eigenfunctions are, for $\lambda_n \neq 0$, ϕ_n and $\gamma_5\phi_n$. The accumulation of eigenvalues near $\lambda = 0$ determines whether or not chiral symmetry is spontaneously broken through the formation of a chiral condensate [1]. The study of the Dirac operator spectrum near $\lambda = 0$ is thus of fundamental importance for our understanding of chiral symmetry breaking in gauge theories. For example, for QCD with N_f flavors it is expected to lead to the conventional $\mathrm{SU}(N_f)_{\mathrm{L}} \times \mathrm{SU}(N_f)_{\mathrm{R}} \rightarrow \mathrm{SU}(N_f)_{\mathrm{V}}$ scenario, with all its implications in terms of low-energy effective Lagrangians.

In a series of papers [2, 3, 4], Verbaarschot and collaborators have added substantial new insight to this issue. Their central assertion is that the spectral density of the Dirac operator very close to the origin $\lambda = 0$ should be *universal*, depending only on the symmetries in question. One startling consequence of this conjecture is that the spectral density of the Dirac operator near the origin need not be computed in the gauge theory at all, but can be extracted from much simpler random matrix theories. The pertinent random matrix ensemble is determined by symmetry arguments alone. To find universal features of the Dirac spectrum near the origin, it is essential to first consider the problem in a finite volume V_4, corresponding in matrix-model terms to finite-size matrices, and then magnify the spectrum near the origin at a scale of order $1/V_4$. The usual spectral density $\rho(\lambda)$ is defined by the following average over gauge field configurations:

$$\rho(\lambda) = \sum_n \langle \delta(\lambda - \lambda_n) \rangle \; . \tag{1.2}$$

By the Banks-Casher relation* [1] (taking here the limit $V_4 \to \infty$),

$$\langle \bar{\psi}\psi \rangle = \frac{1}{V_4} \lim_{m\to 0} 2m \int_0^\infty \mathrm{d}\lambda \frac{\rho(\lambda)}{\lambda^2 + m^2} = \frac{\pi\rho(0)}{V_4} \; , \tag{1.3}$$

this spectral density, evaluated at the origin, is directly related to the appearance of a chiral condensate. It follows that the average spacing between eigenvalues becomes roughly constant near $\lambda = 0$ [5],

$$\Delta\lambda \sim \frac{\pi}{V_4 \langle \bar{\psi}\psi \rangle} \; . \tag{1.4}$$

Magnifying $\rho(\lambda)$ near $\lambda = 0$ according to the prescription given above thus entails the introduction of an associated *microscopic* spectral density $\rho_{\mathrm{S}}(\lambda)$, which is defined by [2]

$$\rho_{\mathrm{S}}(\lambda) \;=\; \lim_{V_4 \to \infty} \frac{1}{V_4} \rho\left(\frac{\lambda}{V_4}\right) \; . \tag{1.5}$$

* Applied naïvely here, ignoring the subtlety of regularization.

The precise statement is that this microscopic spectral density should be a universal function. The most compelling evidence comes from the fact that the microscopic spectral density defined as above, and evaluated in a particular random matrix theory, exactly reproduces the Leutwyler-Smilga spectral sum rules [5] and appropriate generalizations [6].

To understand the significance of this universality conjecture, let us for convenience first focus on one example, that of QCD with N_f massless fermions in the fundamental representation. As explained in refs. [2], the relevant matrix model partition function is given by a chiral unitary ensemble of the form

$$Z_{\chi\mathrm{UE}} = \int \mathrm{d}W \, \det{}^{N_f} \begin{pmatrix} 0 & W^\dagger \\ W & 0 \end{pmatrix} \mathrm{e}^{-N \mathrm{tr}\, V(W^\dagger W)} \ . \tag{1.6}$$

Here only symmetry arguments alone have selected the integration to be over complex $N \times N$ matrices, and the Dirac operator structure is encoded in the determinant. But such symmetry arguments alone place no restrictions on the form of $V(\lambda)$, which appears in the exponent. If indeed symmetry arguments alone should determine the microscopic spectral density of QCD from this matrix ensemble, then any reasonable choice of $V(\lambda)$ should give the same result for $\rho_{\mathrm{S}}(\lambda)$. This universality argument, if correct, allows us to evaluate the integral with a function $V(\lambda)$ of our own choice. Some hints supporting this universality had been found in refs. [7]. Clearly, the most convenient choice in to make the matrix integral Gaussian except for the determinant factor in front. This is the choice made in refs. [2].

It is important to separate the question of universality of the spectral density of the Dirac operator close to the origin into two parts. The first concerns the crucial jump from the d-dimensional gauge field integrations to the framework of zero-dimensional matrix models with similar symmetries. The second step concerns the conjecture about the universality of the microscopic spectral density *within the framework of matrix models.* The purpose of this paper is to prove the latter of these universality conjectures. The question of whether the universality extends all the way from zero-dimensional large-N matrix models to full-fledged quantum field theories such as QCD will of course remain unproven. But we shall show, within the context of large-N matrix models, under what conditions the microscopic spectral density and higher correlation functions are universal. This is a major step towards understanding the original issue, which concerns the possible universality of also the Dirac operator spectrum near $\lambda = 0$.

This paper is organized as follows. In section 2 we prove a theorem for the asymptotic behavior of generic orthogonal polynomials over a semi-infinite range. By making use of this theorem, we compute the associated universal form of the microscopic spectral density as well as all higher order correlation functions in the same limit. This universality class is of relevance for four-dimensional $\mathrm{SU}(N_c \geq 3)$ gauge theories with fermions in the fundamental representation. In section 3 we repeat the analysis for a class of orthogonal polynomials over an infinite range. This case is considered to be relevant for three-dimensional $\mathrm{SU}(N_c \geq 3)$ gauge theories with an even number of fermions in the fundamental representation. We conclude in section 4 with a discussion about the possible relation between the chiral-flavor (or parity-flavor) symmetry breaking in gauge theories and the now established universality of random matrices in the microscopic limit.

2 Chiral unitary ensembles

In this section we consider the chiral unitary ensemble:

$$Z = \int \mathrm{d}M \det{}^{N_f} M \, \exp\left\{-\frac{N}{2}\mathrm{tr}\, V(M^2)\right\}, \quad V(M^2) = \sum_{k\geq 1} \frac{g_k}{k} M^{2k}, \quad N_f = 0, 1, \cdots \tag{2.1}$$

where M stands for an $(N+N') \times (N+N')$ block hermitian matrix whose non-zero components are $N \times N'$ complex matrices on the off-diagonals,

$$M = \begin{pmatrix} 0 & W^\dagger \\ W & 0 \end{pmatrix} \tag{2.2}$$

and $\mathrm{d}M$ the Haar measure of W. We take $N \leq N'$ without loss of generality. An ensemble is called chiral unitary because of the invariance under the transformation

$$W \mapsto V^\dagger \, W \, U \ , \quad U \in \mathrm{U}(N) \ , \quad V \in \mathrm{U}(N'). \tag{2.3}$$

This model has the same global symmetries as a Euclidean four-dimensional $\mathrm{SU}(N_c \geq 3)$ gauge theory coupled to N_f massless fermions in the fundamental representation. The topological charge ν of the vacuum is identified with $|\nu| = N' - N$ (which is kept fixed) and the volume of space-time is $V_4 = N + N'$ (which is sent to infinity afterwards). Specifically, due to the "$\mathrm{U}(1)_\mathrm{A}$ symmetry"

$$\{M, \gamma_5\} = 0, \quad \gamma_5 = \begin{pmatrix} 1_N & 0 \\ 0 & -1_{N'} \end{pmatrix}, \tag{2.4}$$

all $(2N)$ non-zero eigenvalues M occurs in pairs with opposite signs.

The partition function (2.1) is expressible in terms of the component matrices as well as of the eigenvalues after integration over the angular coordinates $(U, V) \in \mathrm{U}(N) \times \mathrm{U}(N')/\mathrm{U}(1)^N$ [8],

$$\begin{aligned} Z &= \int \mathrm{d}W \det{}^{N_f} \left(W^\dagger W\right) \mathrm{e}^{-N \mathrm{tr} V(W^\dagger W)} \\ &\propto \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left(\mathrm{d}z_i^2 \, z_i^{2\alpha} \, \mathrm{e}^{-NV(z_i^2)}\right) \left|\det_{ij} z_j^{2(i-1)}\right|^2 \\ &= \int_0^{\infty} \prod_{i=1}^{N} \left(\mathrm{d}\lambda_i \, \lambda_i^\alpha \, \mathrm{e}^{-NV(\lambda_i)}\right) \left|\det_{ij} \lambda_j^{i-1}\right|^2 \ , \end{aligned} \tag{2.5}$$

where we have set $\alpha = N_f + |\nu|$ and suppressed an irrelevant constant of the angular integration. The above expression can be interpreted as a positive definite $N \times N$ hermitian matrix model in $H = W^\dagger W$ whose eigenvalues are $\lambda_1, \cdots, \lambda_N \geq 0$. Therefore the problem is reduced to finding a set of orthogonal polynomials $P_n^{(\alpha)}(\lambda)$ over the semi-infinite interval $[0,\infty)$ with the shown weight function,

$$\int_0^\infty \mathrm{d}\lambda\, \lambda^\alpha\, \mathrm{e}^{-NV(\lambda)} P_n^{(\alpha)}(\lambda)\, P_m^{(\alpha)}(\lambda) = h_n^{(\alpha)}\, \delta_{nm}. \tag{2.6}$$

The definition of the orthogonality relation (2.6) shows that the problem is well-defined for arbitrary real $\alpha > -1$. Here we shall only deal with α to be a nonnegative integer, which is motivated physically from the above.

In the orthogonal polynomial method, the polynomials $P_n^{(\alpha)}(\lambda)$ are usually normalized to be monic,

$$P_n^{(\alpha)}(\lambda) = \lambda^n + \cdots\ , \tag{2.7}$$

so that the Vandermonde determinant $\det_{ij} \lambda_j^{i-1}$ can be substituted by $\det_{ij} P_{i-1}^{(\alpha)}(\lambda_j)$ in the integrand. However, for the present purposes we will employ a different normalization by demanding $P_n^{(\alpha)}(0) = 1$:

$$P_n^{(\alpha)}(\lambda) = 1 + \cdots + p_n^{(\alpha)} \lambda^n. \tag{2.8}$$

This is because we shall seek a smooth limiting behavior near $\lambda = 0$ as n gets as large as N (which is taken to infinity), and $N^2\lambda$ is kept fixed at the same time (the microscopic limit). We thus start by the assumption that $P_n^{(\alpha)}(0) \neq 0$, and normalize the polynomials accordingly.[†] As we shall see shortly, this assumption is equivalent to having the origin included in the support of the spectral density.

2.1 Asymptotics of orthogonal polynomials

We shall prove the following theorem on the asymptotic behavior of these orthogonal polynomials:

Theorem 1: *Let $\{P_n^{(\alpha)}(\lambda)\}_{n=0,1,\cdots}$ be the set of polynomials orthogonal with respect to the measure*

$$\mathrm{d}\lambda\, \lambda^\alpha\, \mathrm{e}^{-N\,V(\lambda)}, \quad V(\lambda) = \sum_{k\geq 1} \frac{g_k}{k} \lambda^k, \quad \alpha = 0, 1, \cdots$$

over the range $[0,\infty)$, whose moments are all finite. If the polynomials can be normalized according to $P_n^{(\alpha)}(0) = 1$, then, for fixed $x = N^2\lambda$ and $t = n/N$, the following limiting relation holds:

$$\lim_{N\to\infty} P_n^{(\alpha)}\Big(\frac{x}{N^2}\Big)\Big|_{n=Nt} = \alpha!\, \frac{J_\alpha\,(u(t)\sqrt{x})}{(u(t)\sqrt{x}/2)^\alpha}, \tag{2.9}$$

where $u(t)$ is determined by

$$u(t) = \int_0^t \frac{\mathrm{d}t'}{\sqrt{r(t')}}, \qquad t = \sum_k \frac{g_k}{2} \binom{2k}{k} r(t)^k.$$

[†] We avoid a proliferation of new symbols by employing the same $P_n^{(\alpha)}(\lambda)$ and $h_n^{(\alpha)}$ in both conventions.

The proof of this theorem for $\alpha = 0$ was recently given by one of the authors [9], and is reproduced here for completeness. The central idea of how to prove it for arbitrary $\alpha = 1, 2, \ldots$ is to use induction, starting with the case $\alpha = 0$. Tracing it back, this means that the Bessel-type behavior of the polynomials is a consequence of the three-term recursion relation for the orthogonal polynomials. As $n, N \to \infty$ with $t = n/N$ fixed, this yields a certain "continuum limit" reminiscent of the derivation of string equations in the double-scaling limit. The recursion relations thus become the defining second order differential equations for Bessel (and Neumann) functions. It is the boundary conditions that finally uniquely specify the Bessel solutions. However, for simplicity we shall here show the proof by induction.

We start with the case $\alpha = 0$. The recursion relation for $P_n(\lambda) \equiv P_n^{(0)}(\lambda)$ takes the form (with $h_n \equiv h_n^{(0)}$ and $p_n \equiv p_n^{(0)}$):

$$
\begin{aligned}
\lambda\, P_n(\lambda) &= -r_n \left\{ P_{n+1}(\lambda) - P_n(\lambda) - \frac{h_n}{r_n}\frac{r_{n-1}}{h_{n-1}}\left(P_n(\lambda) - P_{n-1}(\lambda)\right) \right\} \qquad \left(r_n \equiv -\frac{p_n}{p_{n+1}} \right) \\
&\equiv \sum_m \hat{\lambda}_{nm}\, P_m(\lambda). \qquad (2.10)
\end{aligned}
$$

This is the well-known three-term recursion relation rewritten in our nonmonic normalization. The structure of the coefficients is dictated by a comparison of $O(\lambda^0)$ and $O(\lambda^{n+1})$ terms on both sides. The sets of unknowns $\{h_n\}$, $\{r_n\}$ can be determined iteratively by

$$
1 = -\int_0^\infty d\lambda \frac{d}{d\lambda}\left\{ e^{-N\,V(\lambda)} P_n(\lambda)\, P_n(\lambda) \right\} = N\, V'(\hat{\lambda})_{nn}\, h_n, \qquad (2.11)
$$

$$
0 = -\int_0^\infty d\lambda \frac{d}{d\lambda}\left\{ e^{-N\,V(\lambda)} \lambda\, P_n(\lambda)\, P_n(\lambda) \right\} = \left(N(\hat{\lambda}V'(\hat{\lambda}))_{nn} - 2n - 1 \right) h_n. \qquad (2.12)
$$

In the following we need to know the asymptotic behavior of r_n and h_n for

$$
n,\ N \to \infty \qquad \text{while} \qquad \frac{n}{N} = t \quad \text{is kept fixed.} \qquad (2.13)
$$

Eqs. (2.10), (2.11) and (2.12) tell us that they should behave as

$$
r_n = r\left(\frac{n}{N}\right) + \text{ higher orders in } \frac{1}{N}\,, \qquad h_n = \frac{1}{N}\, h\left(\frac{n}{N}\right) + \text{ higher orders in } \frac{1}{N}. \qquad (2.14)
$$

Then the leading behavior of the matrix $\hat{\lambda}$ and its powers is

$$
\begin{aligned}
\hat{\lambda}_{nm} &= r(\frac{n}{N})\,(-\delta_{n\,m-1} + 2\delta_{nm} - \delta_{n\,m+1})\,, \\
(\hat{\lambda}^k)_{nm} &= r(\frac{n}{N})^k \sum_{\ell=-k}^{k} (-)^\ell \binom{2k}{k+\ell} \delta_{n\,m+\ell} \qquad (2.15)
\end{aligned}
$$

so that eqs. (2.11) and (2.12) read

$$\sum_k g_k \binom{2k-2}{k-1} r(t)^{k-1} = \frac{1}{h(t)}, \tag{2.16}$$

$$\frac{1}{2}\sum_k g_k \binom{2k}{k} r(t)^k = t. \tag{2.17}$$

Note that eqs. (2.16) and (2.17) imply a universal relationship among total derivatives,

$$\mathrm{d}t = 2r\,\mathrm{d}\left(\frac{1}{h}\right) + \frac{1}{h}\mathrm{d}r = \sqrt{r}\,\mathrm{d}\left(\frac{2\sqrt{r}}{h}\right). \tag{2.18}$$

Next we expand the right hand side of the recursion relation (2.10) in terms of $1/N$ in the limit (2.13) to get

$$N^2\lambda\, P(n,N,\lambda) = -r(t)\left\{\frac{\mathrm{d}^2 P}{\mathrm{d}t^2} + \frac{h(t)}{r(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{r(t)}{h(t)}\right)\frac{\mathrm{d}P}{\mathrm{d}t}\right\} = -h(t)\frac{\mathrm{d}}{\mathrm{d}t}\frac{r(t)}{h(t)}\frac{\mathrm{d}}{\mathrm{d}t}P(n,N,\lambda), \tag{2.19}$$

where the argument N in $P(n,N,\lambda) \equiv P_n(\lambda)$ is to indicate explicitly the dependency via the coefficient in front of the potential. Only the leading terms up to $O(1/N)$ have been kept here. Eq. (2.19) tells us that that the arguments of the polynomial appear only in the combinations $t = n/N$ and $x = N^2\lambda$ in the large-N limit. The rescaled eigenvalue coordinate x is to be fixed finite hereafter, and is regarded as a parameter in the ordinary differential equation in t. Performing the change of variable

$$t \mapsto u(t) \equiv \frac{2\sqrt{r(t)}}{h(t)} = \int_0^t \frac{\mathrm{d}t'}{\sqrt{r(t')}}, \tag{2.20}$$

using the relationship (2.18), and neglecting higher order terms in $1/N$, eq. (2.19) reduces to Bessel equation of zeroth order:

$$\left(\frac{1}{u}\frac{\mathrm{d}}{\mathrm{d}u}u\frac{\mathrm{d}}{\mathrm{d}u} + x\right)P(u,x) = 0. \tag{2.21}$$

The unique solution to this equation which satisfies the following boundary condition at $t = n/N = 0$ ($u(0) = 0$),

$$P(0,x) = P_0(\lambda) = 1, \tag{2.22}$$

is

$$P(u,x) = J_0\left(u\sqrt{x}\right). \tag{2.23}$$

We now proceed for a generic integer α by induction. Given a set of polynomials $\{P_n^{(\alpha)}(\lambda)\}$ which are orthogonal with respect to the measure $\mathrm{d}\lambda\,\lambda^\alpha\,\mathrm{e}^{-NV(\lambda)}$ on $[0,\infty)$, and are normalized by $P_n^{(\alpha)}(0) = 1$ as in eq. (2.8), we note that

$$\tilde{P}_n^{(\alpha+1)}(\lambda) \equiv \frac{P_{n+1}^{(\alpha)}(\lambda) - P_n^{(\alpha)}(\lambda)}{\lambda} \tag{2.24}$$

are polynomials of order n. They are orthogonal to λ^k, $k = 0, \cdots, n-1$ with respect to the measure $\mathrm{d}\lambda\, \lambda^{\alpha+1}\, \mathrm{e}^{-NV(\lambda)}$ [10]:

$$
\begin{aligned}
& \int_0^\infty \mathrm{d}\lambda\, \lambda^{\alpha+1}\, \mathrm{e}^{-NV(\lambda)}\, \tilde{P}_n^{(\alpha+1)}(\lambda)\, \lambda^k \\
= & \int_0^\infty \mathrm{d}\lambda\, \lambda^{\alpha}\, \mathrm{e}^{-NV(\lambda)} \left(P_{n+1}^{(\alpha)}(\lambda) - P_n^{(\alpha)}(\lambda)\right) \lambda^k = 0.
\end{aligned}
\tag{2.25}
$$

Under the assumption that $\tilde{P}_n^{(\alpha+1)}(0) \neq 0$, the orthogonal polynomial normalized to $P_n^{(\alpha+1)}(0) = 1$ is thus given by

$$
P_n^{(\alpha+1)}(\lambda) = \frac{\tilde{P}_n^{(\alpha+1)}(\lambda)}{\tilde{P}_n^{(\alpha+1)}(0)}. \tag{2.26}
$$

The microscopic limit of the contiguous relation (2.24) reads

$$
\tilde{P}_{Nt}^{(\alpha+1)}\left(\frac{x}{N^2}\right) = \frac{N}{x}\frac{\partial}{\partial t} P_{Nt}^{(\alpha)}\left(\frac{x}{N^2}\right) + \text{ higher orders in } \frac{1}{N}. \tag{2.27}
$$

If we substitute eq. (2.9) for a given α into the above, we confirm that

$$
\tilde{P}_{Nt}^{(\alpha+1)}\left(\frac{x}{N^2}\right) = -N \frac{\alpha!}{\sqrt{r(t)}} \frac{J_{\alpha+1}\left(u(t)\sqrt{x}\right)}{\left(u(t)/2\right)^\alpha (\sqrt{x})^{\alpha+1}}, \tag{2.28}
$$

$$
\tilde{P}_{Nt}^{(\alpha+1)}(0) = -N \frac{u(t)/2}{(\alpha+1)\sqrt{r(t)}}. \tag{2.29}
$$

Thus eq. (2.9) holds also for $\alpha+1$ after normalizing according to (2.26). Together with the result (2.23) for $\alpha = 0$, this completes the proof.

We have worked out an alternative proof of the same theorem, which does not rely on induction. Rather, it is possible to convert the recursion relation for $P_n^{(\alpha)}(\lambda)$ directly into Bessel equation of order α, $\left(\frac{1}{u^{2\alpha+1}}\frac{\mathrm{d}}{\mathrm{d}u} u^{2\alpha+1} \frac{\mathrm{d}}{\mathrm{d}u} + x\right) P^{(\alpha)}(u,x) = 0$. Since it is lengthy, we do not reproduce it here.

2.2 Universal correlations

Theorem 1 can be used to establish a remarkable universal form of spectral correlators of the model (2.5). We shall consider the spectral density $\rho(\lambda)$, and higher order correlators, in the same microscopic limit as above.

To begin, we recall the expression for the integration kernel $K_N(\lambda, \mu)$ associated with the eigenvalue problem for the positive-definite hermitian matrix $H = W^\dagger W$,

$$
\begin{aligned}
K_N(\lambda,\mu) &= (\lambda\,\mu)^{\alpha/2}\, \mathrm{e}^{-\frac{N}{2}(V(\lambda)+V(\mu))}\, \frac{1}{N} \sum_{i=0}^{N-1} \frac{P_i^{(\alpha)}(\lambda) P_i^{(\alpha)}(\mu)}{h_i^{(\alpha)}} \\
&= (\lambda\,\mu)^{\alpha/2}\, \mathrm{e}^{-\frac{N}{2}(V(\lambda)+V(\mu))}\, \frac{1}{N} \frac{-r_{N-1}^{(\alpha)}}{h_{N-1}^{(\alpha)}} \frac{P_N^{(\alpha)}(\lambda) P_{N-1}^{(\alpha)}(\mu) - P_{N-1}^{(\alpha)}(\lambda) P_N^{(\alpha)}(\mu)}{\lambda - \mu}.
\end{aligned}
\tag{2.30}
$$

Here use has been made of the Christoffel-Darboux identity. In the large-N limit we may drop the superscript α in $r_n^{(\alpha)} = -p_n^{(\alpha)}/p_{n+1}^{(\alpha)}$. This is because the general recursion relation (the counterpart of eq. (2.12) for general α)

$$0 = N(\hat{\lambda}V'(\hat{\lambda}))_{nn} - 2n - \alpha - 1 \tag{2.31}$$

which determines $r^{(\alpha)}(t)$ leads, in the large-N limit (2.13), to the same (2.17) for any finite α.

On the other hand, the norms of the polynomials can be evaluated iteratively:

$$\begin{aligned}
\tilde{h}_n^{(\alpha+1)} &= \int_0^\infty \mathrm{d}\lambda\, \lambda^{\alpha+1}\, \mathrm{e}^{-NV(\lambda)}\, \tilde{P}_n^{(\alpha+1)}(\lambda)\tilde{P}_n^{(\alpha+1)}(\lambda) \\
&= \int_0^\infty \mathrm{d}\lambda\, \lambda^{\alpha}\, \mathrm{e}^{-NV(\lambda)} \left(P_{n+1}^{(\alpha)}(\lambda) - P_n^{(\alpha)}(\lambda)\right) \tilde{P}_n^{(\alpha+1)}(\lambda) \\
&= \int_0^\infty \mathrm{d}\lambda\, \lambda^{\alpha}\, \mathrm{e}^{-NV(\lambda)} \left(-P_n^{(\alpha)}(\lambda)\right) \frac{p_{n+1}^{(\alpha)}}{p_n^{(\alpha)}} P_n^{(\alpha)}(\lambda) = \frac{h_n^{(\alpha)}}{r_n} \;, && (2.32) \\
h_n^{(\alpha+1)} &= \frac{\tilde{h}_n^{(\alpha+1)}}{\tilde{P}_n^{(\alpha+1)}(0)^2} = h_n^{(\alpha)} \left(\frac{\alpha+1}{N\, u(t)/2}\right)^2 . && (2.33)
\end{aligned}$$

We therefore obtain

$$h_n^{(\alpha)} = \frac{1}{N^{2\alpha+1}} (\alpha!)^2 \left(\frac{2}{u(t)}\right)^{2\alpha} h(t) + \text{ higher orders in } \frac{1}{N} \quad \left(h(t) \equiv h^{(0)}(t)\right) . \tag{2.34}$$

Using Theorem 1 and inserting eq. (2.34) into (2.30), we obtain a universal form of the kernel (the Bessel kernel of order α) in the microscopic limit

$$\lim_{N\to\infty} \frac{1}{N} K_N(\frac{x}{N^2}, \frac{y}{N^2}) = \frac{u(1)}{2} \frac{\sqrt{x} J_{\alpha+1}(u(1)\sqrt{x}) J_\alpha(u(1)\sqrt{y}) - J_\alpha(u(1)\sqrt{x})\sqrt{y} J_{\alpha+1}(u(1)\sqrt{y})}{x-y} . \tag{2.35}$$

The only way this asymptotic kernel depends on the potential $V(\lambda)$ is through the parameter $u(1)$. As we shall show now, even this dependence is of a highly universal form. Let us compare eqs. (2.16) and (2.17) at $t = 1$ with the explicit expression for the large-N spectral density $\rho(z)$ having support on a single interval $[-a, a]$ [11].

$$\rho(z) = \frac{\sqrt{a^2 - z^2}}{2\pi} \sum_k g_k \sum_{n=0}^{k-1} \binom{2n}{n} \left(\frac{a^2}{4}\right)^n z^{2k-2n-2}, \tag{2.36}$$

$$\frac{1}{2} \sum_k g_k \binom{2k}{k} \left(\frac{a^2}{4}\right)^k = 1 . \tag{2.37}$$

This enables us to relate the parameters $r(1)$ and $u(1)$ to a and $\rho(0)$, respectively,

$$a \;=\; 2\sqrt{r(1)} \quad , \qquad \rho(z{=}0) \;=\; \frac{u(1)}{2\pi}. \tag{2.38}$$

The kernel (2.35) therefore only depends on the potential $V(\lambda)$ in the indirect way of setting the scale $\rho(0)$. It has clearly been the assumption throughout that $\rho(\lambda) \neq 0$. This was the starting point of the physical motivation from field theory (spontaneously broken chiral symmetries, which through the Banks-Casher relation (1.3) entails a non-vanishing spectral density at the origin). Moreover, it is known that the critical condition $\rho(0) = 0$ for a transition where the intervals of support move away from the origin is equivalent to $P_N(0) = 0$ for the monically normalized polynomials [8]. If we go back to our proof of Theorem 1, this was precisely the condition we had to impose in order to be able to find a smooth continuum limit of the orthogonal polynomials in the large-N limit. Thus, under the assumption that the normalization (2.8) is possible, the constants $u(1)$ and $r(1)$ are determined to be positive from the identification (2.38).

It finally remains us to relate the s-point correlation functions of σ_N (ρ_N) of eigenvalues of $H = W^\dagger W$ (or M) to the kernel K_N in eq. (2.35). The former is defined as

$$\sigma_N(\lambda_1, \cdots, \lambda_s) = \left\langle \prod_{a=1}^{s} \frac{1}{N} \mathrm{tr}\, \delta(\lambda_a - H) \right\rangle = \det_{1\le a,b\le s} K_N(\lambda_a, \lambda_b), \tag{2.39}$$

$$\rho_N(z_1, \cdots, z_s) = \left\langle \prod_{a=1}^{s} \frac{1}{2N} \mathrm{tr}\, \delta(z_a - M) \right\rangle = |z_1| \cdots |z_s|\, \sigma_N(z_1^2, \cdots, z_s^2), \tag{2.40}$$

respectively. Therefore all the formulae for their microscopic limits ($x_a = N^2\lambda_a$, $\zeta_a = N z_a$ fixed)

$$\sigma_{\mathrm{S}}(x_1, \cdots, x_s) \equiv \lim_{N\to\infty} \frac{1}{N^s} \sigma_N\Big(\frac{x_1}{N^2}, \cdots, \frac{x_s}{N^2}\Big), \tag{2.41}$$

$$\rho_{\mathrm{S}}(\zeta_1, \cdots, \zeta_s) \equiv \lim_{N\to\infty} \rho_N\Big(\frac{\zeta_1}{N}, \cdots, \frac{\zeta_s}{N}\Big) = |\zeta_1| \cdots |\zeta_s|\, \sigma_{\mathrm{S}}(\zeta_1^2, \cdots, \zeta_s^2), \tag{2.42}$$

that were previously calculated for the Laguerre (in the H-picture) or chiral Gaussian (in the M-picture) unitary ensemble [2, 12], hold universally.

In particular, the spectral density of the chiral unitary ensemble

$$\rho_N(z) = \left\langle \frac{1}{2N} \mathrm{tr}\, \delta(z - M) \right\rangle = |z| K_N(z^2, z^2) \tag{2.43}$$

takes the universal form

$$\rho_{\mathrm{S}}(\zeta) = (\pi\rho(0))^2\, |\zeta| \Big(J_\alpha^2(2\pi\rho(0)\zeta) - J_{\alpha+1}(2\pi\rho(0)\zeta) J_{\alpha-1}(2\pi\rho(0)\zeta) \Big) \tag{2.44}$$

in the microscopic limit. As expected from the Gaussian case, a matching condition between the microscopic and macroscopic (ordinary large-N) spectral densities is satisfied:

$$\lim_{\zeta\to\infty} \rho_{\mathrm{S}}(\zeta) = \rho(0). \tag{2.45}$$

3 Unitary ensembles

In this section we consider the unitary ensemble:

$$Z = \int \mathrm{d}M \, \det{}^{2\alpha} M \, \mathrm{e}^{-N\mathrm{tr}V(M^2)}, \quad V(M^2) = \sum_{k\geq 1} \frac{g_k}{2k} M^{2k}, \quad \alpha = 0, 1, \cdots \tag{3.1}$$

where M stands for an $N \times N$ hermitian matrix and $\mathrm{d}M$ its Haar measure. This model shares the same global symmetries with a Euclidean three-dimensional $\mathrm{SU}(N_c \geq 3)$ gauge theory coupled to $N_f = 2\alpha$ massless fermions in the fundamental representation [13]. The partition function (3.1) is expressible in terms of the eigenvalues after integration over the angular $\mathrm{U}(N)$ coordinates

$$Z = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left(\mathrm{d}\lambda_i \; \lambda_i^{2\alpha} \, \mathrm{e}^{-NV(\lambda_i^2)} \right) \left| \det_{i,j} \lambda_i^{j-1} \right|^2 . \tag{3.2}$$

Therefore the problem is reduced to finding a set of orthogonal polynomials $P_n^{(\alpha)}(\lambda)$ over the infinite interval $(-\infty, \infty)$ with the shown weight function,

$$\int_{-\infty}^{\infty} \mathrm{d}\lambda \, \lambda^{2\alpha} \, \mathrm{e}^{-NV(\lambda^2)} P_n^{(\alpha)}(\lambda) \, P_m^{(\alpha)}(\lambda) = h_n^{(\alpha)} \, \delta_{nm}. \tag{3.3}$$

3.1 Asymptotics of orthogonal polynomials

We shall prove the following theorem‡ on the asymptotic behavior of these orthogonal polynomials:

> **Theorem 2**: *Let $\{P_n^{(\alpha)}(\lambda)\}_{n=0,1,\cdots}$ be the set of polynomials orthogonal with respect to the measure*
>
> $$\mathrm{d}\lambda \, \lambda^{2\alpha} \, \mathrm{e}^{-NV(\lambda^2)}, \quad V(\lambda^2) = \sum_{k\geq 1} \frac{g_k}{2k} \, \lambda^{2k}, \quad \alpha = 0, 1, \cdots$$
>
> *over the range $(-\infty, \infty)$, whose moments are all finite. If the polynomials can be normalized according to $P_{2n}^{(\alpha)}(0) = P_{2n+1}^{(\alpha)}{}'(0) = 1$, then, for fixed $x = N\lambda$ and $t = 2n/N$, the following limiting relations hold:*
>
> $$\lim_{N\to\infty} P_{2n}^{(\alpha)}\left(\frac{x}{N}\right)\Big|_{n=Nt/2} = \Gamma(\alpha + \frac{1}{2}) \, \frac{J_{\alpha-\frac{1}{2}}(u(t)x)}{(u(t)x/2)^{\alpha-\frac{1}{2}}}, \tag{3.4a}$$
>
> $$\lim_{N\to\infty} N \, P_{2n+1}^{(\alpha)}\left(\frac{x}{N}\right)\Big|_{n=Nt/2} = x \, \Gamma(\alpha + \frac{3}{2}) \, \frac{J_{\alpha+\frac{1}{2}}(u(t)x)}{(u(t)x/2)^{\alpha+\frac{1}{2}}}, \tag{3.4b}$$
>
> *where $u(t)$ is determined by*
>
> $$u(t) = \int_0^t \frac{\mathrm{d}t'}{2\sqrt{r(t')}}, \qquad t = \sum_k \frac{g_k}{2} \binom{2k}{k} r(t)^k.$$

‡ Theorem 2 can be regarded as an extension of Theorem 1 for half-integer α.

The proof of this theorem for $\alpha = 0$ was sketched in ref. [14], and below we shall elaborate it in a rigorous form. The recursion relation for monically normalized polynomial

$$P_n(\lambda) \equiv P_n^{(0)}(\lambda) = \lambda^n + \cdots \tag{3.5}$$

takes the form

$$\lambda\, P_n(\lambda) = P_{n+1}(\lambda) + r_n\, P_{n-1}(\lambda), \tag{3.6}$$

with

$$r_n = \frac{h_n}{h_{n-1}}\ . \tag{3.7}$$

Since the monic normalization (3.5) does not distinguish the parity of n, neither does r_n. Eq. (3.6) immediately implies

$$0 = P_{2n+2}(0) + r_{2n+1}\, P_{2n}(0). \tag{3.8}$$

Due to the $\mathbf{Z}_2$ symmetry of the weight function, the polynomials have definite parities. Therefore we are lead to change the normalization of polynomials as follows:

$$\frac{P_{2n}(\lambda)}{P_{2n}(0)} \rightarrow P_{2n}(\lambda) = 1 + \cdots, \quad \frac{P_{2n+1}(\lambda)}{P_{2n+1}{}'(0)} \rightarrow P_{2n+1}(\lambda) = \lambda + \cdots. \tag{3.9}$$

Then the recursion relation (3.6) in this normalization, applied twice to $P_{2n}(\lambda)$, takes the form

$$\begin{aligned} \lambda^2\, P_{2n}(\lambda) &= -r_{2n+1}\,(P_{2n+2}(\lambda) - P_{2n}(\lambda)) + r_{2n}\,(P_{2n}(\lambda) - P_{2n-2}(\lambda)) \\ &\equiv \sum_m (\hat{\lambda}^2)_{2n,2m}\, P_{2m}(\lambda). \end{aligned} \tag{3.10}$$

Here use is made of eq. (3.8). A similar equation for $P_{2n+1}(\lambda)$ expresses the matrix elements $(\hat{\lambda}^2)_{2n+1,2m+1}$ in terms of the r_n's. The coefficients $\{r_n\}$ are iteratively determined by

$$0 = -\frac{1}{h_n}\int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{ \mathrm{e}^{-N\,V(\lambda^2)}\lambda\, P_n(\lambda)\, P_n(\lambda)\right\} = 2N(\hat{\lambda}^2 V'(\hat{\lambda}^2))_{nn} - 2n - 1. \tag{3.11}$$

The asymptotic behavior of r_n is determined in the large-N limit,

$$n,\ N \rightarrow \infty \qquad \text{while} \quad \frac{2n}{N} = t \quad \text{is kept fixed,} \tag{3.12}$$

to be

$$r_n = r\left(\frac{n}{N}\right) + \text{higher orders in } \frac{1}{N}. \tag{3.13}$$

Here $r(t)$ is given by

$$\frac{1}{2}\sum_k g_k \binom{2k}{k} r(t)^k = t. \tag{3.14}$$

Next we expand the right hand side of the recursion relation (3.10) in terms of $1/N$ to get $(P(2n, N, \lambda) \equiv P_{2n}(\lambda))$

$$N^2\lambda^2\, P(2n,N,\lambda) = -\left(4r(t)\frac{\mathrm{d}^2 P}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r(t)}{\mathrm{d}t}\frac{\mathrm{d}P}{\mathrm{d}t}\right) = -\left(2\sqrt{r(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 P(2n,N,\lambda). \tag{3.15}$$

It tells us that that the arguments of P appear only in the combinations $t = 2n/N$ and $x = N\lambda$ in the large-N limit. After the change of variables

$$t \mapsto u(t) \equiv \int_0^t \frac{\mathrm{d}t'}{2\sqrt{r(t')}}, \tag{3.16}$$

we equivalently have

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} + x^2\right) P(u, x) = 0. \tag{3.17}$$

The unique solution to this trigonometric differential equation which satisfies the boundary and parity conditions

$$P(0, x) = 1, \quad P(u, -x) = P(u, x) \tag{3.18}$$

is

$$P(u, x) = \lim_{N\to\infty} P_{2n}(\frac{x}{N})\Big|_{n=Nt/2} = \cos ux = \Gamma(\frac{1}{2})\frac{J_{-\frac{1}{2}}(ux)}{(ux/2)^{-\frac{1}{2}}}. \tag{3.19}$$

The odd-order polynomials are constructed out of normalized even-order ones as follows.

$$\tilde{P}_{2n+1}(\lambda) \equiv \frac{P_{2n+2}(\lambda) - P_{2n}(\lambda)}{\lambda} \tag{3.20}$$

are odd polynomials of order $2n+1$, and are orthogonal to λ^{2k+1}, $k = 0, \cdots, n-1$:

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{-NV(\lambda)} \tilde{P}_{2n+1}(\lambda)\, \lambda^{2k+1} \\ = &\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{-NV(\lambda)} \left(P_{2n+2}(\lambda) - P_{2n}(\lambda)\right) \lambda^{2k} = 0. \end{aligned} \tag{3.21}$$

The microscopic limit of (3.20) reads

$$\tilde{P}_{2n+1}(\frac{x}{N})\Big|_{n=Nt/2} = \frac{2}{x}\frac{\partial}{\partial t} P_{2n}(\frac{x}{N})\Big|_{n=Nt/2}. \tag{3.22}$$

Under the assumption that $\tilde{P}_{2n+1}{}'(0) \neq 0$, change of normalization to $P_{2n+1}{}'(0) = 1$ is implemented by

$$P_{2n+1}(\lambda) = \frac{\tilde{P}_{2n+1}(\lambda)}{\tilde{P}_{2n+1}{}'(0)}. \tag{3.23}$$

By inserting the asymptotic form of even-order polynomials (3.19) into (3.22) and normalizing according to (3.23), we conclude

$$\lim_{N\to\infty} N\, P_{2n+1}(\frac{x}{N})\Big|_{n=Nt/2} = \frac{\sin ux}{u} = x\,\Gamma(\frac{3}{2})\frac{J_{\frac{1}{2}}(ux)}{(ux/2)^{\frac{1}{2}}}. \tag{3.24}$$

Now we proceed for a generic α by induction. Given a set of polynomials $\{P_n^{(\alpha)}(\lambda)\}$ which are orthogonal with respect to the measure $\mathrm{d}\lambda\,\lambda^{2\alpha}\,\mathrm{e}^{-NV(\lambda^2)}$ on $[-\infty,\infty]$, and are normalized by $P_{2n}^{(\alpha)}(0)=P_{2n+1}^{(\alpha)}{}'(0)=1$, we note that

$$\tilde{P}_n^{(\alpha+1)}(\lambda)\equiv\frac{P_{n+2}^{(\alpha)}(\lambda)-P_n^{(\alpha)}(\lambda)}{\lambda^2} \tag{3.25}$$

are polynomials of order n. They are orthogonal to each other with respect to the measure $\mathrm{d}\lambda\,\lambda^{2(\alpha+1)}\,\mathrm{e}^{-NV(\lambda^2)}$. Under the assumption that $\tilde{P}_{2n}^{(\alpha+1)}(0),\tilde{P}_{2n+1}^{(\alpha+1)}{}'(0)\neq 0$, the correctly normalized orthogonal polynomials are given by

$$P_{2n}^{(\alpha+1)}(\lambda)=\frac{\tilde{P}_{2n}^{(\alpha+1)}(\lambda)}{\tilde{P}_{2n}^{(\alpha+1)}(0)},\quad P_{2n+1}^{(\alpha+1)}(\lambda)=\frac{\tilde{P}_{2n+1}^{(\alpha+1)}(\lambda)}{\tilde{P}_{2n+1}^{(\alpha+1)}{}'(0)}. \tag{3.26}$$

If we substitute eqs. (3.4) for a given α into the microscopic limit of eq. (3.25), we confirm that

$$\begin{aligned}
\tilde{P}_{2n}^{(\alpha+1)}(\frac{x}{N})\Big|_{n=Nt/2} &= -N\,\frac{\Gamma(\alpha+\frac{1}{2})}{\sqrt{r(t)}}\frac{J_{\alpha+\frac{1}{2}}\,(u(t)x)}{(u(t)/2)^{\alpha-\frac{1}{2}}\,x^{\alpha+\frac{1}{2}}}, && \text{(3.27a)}\\
\tilde{P}_{2n+1}^{(\alpha+1)}(\frac{x}{N})\Big|_{n=Nt/2} &= -\frac{\Gamma(\alpha+\frac{3}{2})}{\sqrt{r(t)}}\frac{J_{\alpha+\frac{3}{2}}\,(u(t)x)}{(u(t)x/2)^{\alpha+\frac{1}{2}}}, && \text{(3.27b)}\\
\tilde{P}_{2n}^{(\alpha+1)}(0)\Big|_{n=Nt/2} &= -N\,\frac{u(t)}{(2\alpha+1)\sqrt{r(t)}}, && \text{(3.28a)}\\
\tilde{P}_{2n+1}^{(\alpha+1)}{}'(0)\Big|_{n=Nt/2} &= -N\,\frac{u(t)}{(2\alpha+3)\sqrt{r(t)}}. && \text{(3.28b)}
\end{aligned}$$

Thus eqs. (3.4) hold also for $\alpha+1$ after normalizing according to (3.26). Together with the results (3.19) and (3.24) for $\alpha=0$, this completes the proof of Theorem 2.

3.2 Universal correlations

The integration kernel associated with the eigenvalue problem of M is (for brevity we set $N=\text{odd}=2\mathcal{N}+1$)

$$\begin{aligned}
K_N(\lambda,\mu)&=(\lambda\,\mu)^\alpha\,\mathrm{e}^{-\frac{N}{2}(V(\lambda^2)+V(\mu^2))}\,\frac{1}{N}\sum_{n=0}^{N-1}\frac{P_n^{(\alpha)}(\lambda)P_n^{(\alpha)}(\mu)}{h_n^{(\alpha)}}\\
&=(\lambda\,\mu)^\alpha\,\mathrm{e}^{-\frac{N}{2}(V(\lambda^2)+V(\mu^2))}\,\frac{1}{N}\left(\sum_{n=0}^{\mathcal{N}}\frac{1}{h_{2n}^{(\alpha)}}\right)\frac{P_{2\mathcal{N}+1}^{(\alpha)}(\lambda)P_{2\mathcal{N}}^{(\alpha)}(\mu)-P_{2\mathcal{N}}^{(\alpha)}(\lambda)P_{2\mathcal{N}+1}^{(\alpha)}(\mu)}{\lambda-\mu}
\end{aligned} \tag{3.29}$$

Here use is made of the Christoffel-Darboux identity, and the proportionality constant in the second line is determined by matching $O(\lambda^0\mu^0)$ terms. Noting that the orthogonality relation (3.3) in the microscopic limit

$$\frac{2}{N^{2\alpha+1}}\int_0^\infty \mathrm{d}x\,x^{2\alpha}\,P_{2n}^{(\alpha)}(\frac{x}{N})\Big|_{n=Nt/2}\,P_{2m}^{(\alpha)}(\frac{x}{N})\Big|_{m=Nt'/2}=h_{2n}^{(\alpha)}\,\frac{2}{N}\delta(t-t') \tag{3.30}$$

is identical to the Bessel closure relation (inversion of Hankel transform)

$$\int_0^\infty \mathrm{d}x\, x\, J_{\alpha-\frac{1}{2}}(ux) J_{\alpha-\frac{1}{2}}(u'x) = \frac{1}{u}\delta(u-u'), \tag{3.31}$$

we can determine the asymptotic form of the norm as

$$h_{2n}^{(\alpha)} = \frac{1}{N^{2\alpha}}\Gamma(\alpha+\frac{1}{2})^2 \left(\frac{2}{u(t)}\right)^{2\alpha} \sqrt{r(t)} + \text{higher orders in } \frac{1}{N}. \tag{3.32}$$

Accordingly we obtain

$$\sum_{n=0}^{\mathcal{N}} \frac{1}{h_{2n}^{(\alpha)}} = \frac{N^{2\alpha}}{\Gamma(\alpha+\frac{1}{2})^2\, 2^{2\alpha}} \cdot \frac{N}{2}\int_0^1 \mathrm{d}t \frac{u(t)^{2\alpha}}{\sqrt{r(t)}} = \frac{N^{2\alpha+1}}{\Gamma(\alpha+\frac{1}{2})^2\, 2^{2\alpha}} \frac{u(1)^{2\alpha+1}}{2\alpha+1}. \tag{3.33}$$

Using Theorem 2 and inserting eq. (3.33) into (3.29), we obtain the universal form of the kernel (the generalized sine kernel of order α) in the microscopic limit

$$\lim_{N\to\infty} \frac{1}{N} K_N(\frac{x}{N}, \frac{y}{N}) = \frac{u(1)}{2}\sqrt{xy} \frac{J_{\alpha+\frac{1}{2}}(u(1)x) J_{\alpha-\frac{1}{2}}(u(1)y) - J_{\alpha-\frac{1}{2}}(u(1)x) J_{\alpha+\frac{1}{2}}(u(1)y)}{x-y}. \tag{3.34}$$

Therefore all formulae for the correlation functions

$$\rho_N(\lambda_1, \cdots, \lambda_s) = \left\langle \prod_{a=1}^{s} \frac{1}{N} \mathrm{tr}\, \delta(\lambda_a - M) \right\rangle = \det_{1\le a,b\le s} K_N(\lambda_a, \lambda_b), \tag{3.35}$$

previously calculated for the Gaussian unitary ensemble in the microscopic limit [15, 12]

$$\rho_{\mathrm{S}}(x_1, \cdots, x_s) = \lim_{N\to\infty} \frac{1}{N^s} \rho_N(\frac{x_1}{N}, \cdots, \frac{x_s}{N}), \tag{3.36}$$

hold universally. Specifically, the spectral density $\rho_N(\lambda) = \langle 1/N\, \mathrm{tr}\, \delta(\lambda - M)\rangle = K_N(\lambda, \lambda)$ universally takes the form (now $\rho(0) = u(1)/\pi$)

$$\rho_{\mathrm{S}}(x) = \left(\frac{\pi\rho(0)}{2}\right)^2 x \left(J_{\alpha+\frac{1}{2}}^2 + J_{\alpha-\frac{1}{2}}^2 - J_{\alpha+\frac{1}{2}} J_{\alpha-\frac{3}{2}} - J_{\alpha-\frac{1}{2}} J_{\alpha+\frac{3}{2}}\right)(\pi\rho(0)x)\,. \tag{3.37}$$

It enjoys the matching condition

$$\lim_{x\to\infty} \rho_{\mathrm{S}}(x) = \rho(0). \tag{3.38}$$

4 Conclusion and speculation

In the present work we have proven two theorems concerning the asymptotic behavior of orthogonal polynomials for a broad class of measures. They show a remarkable asymptotic universality near the origin, a relation previously only known for the classical orthogonal polynomials of generalized Laguerre and Jacobi type [10]. Furthermore, we have used this highly universal behavior to prove universality conjectures regarding the spectral densities and their s-point correlators in the microscopic limit.

All of these proofs refer to the zero-dimensional matrix model language alone. And, as we have emphasized in the Introduction, the assumption that this universality can be applied to full d-dimensional quantum gauge theories with fermionic fields is *a priori* very far from obvious. Of course, our matrix model universality theorems are interesting in their own right. But from a particle physics perspective the real focus should be on the application of these theorems to the Dirac operator spectrum [2].

How could we imagine the link between random matrix models and gauge theories established in this connection? Consider a Euclidean four-dimensional $\mathrm{SU}(N_c)$ gauge theory coupled to N_f light fermions in, say, the fundamental representation. Assume that N_f is small enough to allow for the conventional scenario of spontaneous chiral symmetry breaking. Let us simply call such a theory "QCD". Its partition function can be written formally as

$$Z_{\mathrm{QCD}} = \int \frac{\mathcal{D}A_\mu}{(\mathrm{Gauge})} \prod_{f=1}^{N_f} \mathcal{D}\bar{\psi}_f \mathcal{D}\psi_f \, \mathrm{e}^{-S[A]-\int \bar{\psi}_f(\mathrm{i}\not{D}+m_f)\psi_f} \ , \tag{4.1}$$

where $S[A]$ is the gauge field part of the action. Consider the theory in a finite volume V_4. Let us now seek a low-energy effective description of this theory. One way is the chiral Lagrangian (χL) approach, which in this case introduces considerable simplification due to the suppression of all derivative terms (since one is interested only in the very soft modes) [5]. Here the chiral condensate at zero mass, $\langle\bar{\psi}\psi\rangle \equiv \Sigma$ is by construction assumed to be non-vanishing. Below is a schematic picture of how one could imagine random matrix theory to fit into this framework:

$$\begin{array}{ccc}
Z_{\mathrm{QCD}} = \int \frac{\mathcal{D}A_\mu}{(\mathrm{Gauge})}\, \mathrm{e}^{-S[A]} \prod_f \det(\mathrm{i}\not{D}+ m_f) & \xrightarrow{(\mathrm{b})} & Z_{\chi\mathrm{UE}} = \int \mathrm{d}M \, \mathrm{e}^{-NV(M)} \prod_f \det(M + m_f) \\
\downarrow (\mathrm{a}) & & (\mathrm{c}) \downarrow \\
Z_{\chi\mathrm{L}} = \int_{\mathrm{SU}(N_f)} \mathrm{d}U \, \mathrm{e}^{N\Sigma \, \mathrm{tr}\, mU \, \mathrm{e}^{\mathrm{i}\frac{\theta}{N_f}} + c.c.} & \xleftarrow{(\mathrm{d})} & Z_{\chi\mathrm{GUE}} = \int \mathrm{d}M \, \mathrm{e}^{-N\Sigma^2 \, \mathrm{tr}\, M^2} \prod_f \det(M + m_f)
\end{array}$$

This link (a) is, however, based on the global symmetry breaking pattern $\mathrm{SU}(N_f)_{\mathrm{L}} \times \mathrm{SU}(N_f)_{\mathrm{R}} \to \mathrm{SU}(N_f)_{\mathrm{V}}$ alone and thus by no means refers to the microscopic theory.

Let us now consider a different route (b)-(c)-(d): suppose we start with QCD in a finite volume, and with a sharp ultraviolet cut-off. (One would like to think of conventional lattice regularizations, were it not for the known difficulties of defining massless fermionic degrees of freedom in that case.) Consider the formal change of integration variables from the gauge field $A_\mu(x)$ to the Dirac operator $\mathrm{i}\not{D} = \not{A}(x) + \mathrm{i}\not{\partial} \equiv M$ itself (b). Since V_4 is the size of the matrix M, we write $N = V_4$. Although we can hardly imagine computing the Jacobian J of this transformation the resulting partition function would formally look like the random matrix model above, with $V(M) = S[A(M)] - \log J(M)$. Once phrased in a matrix model language, we can apply our Theorem 1 to substitute the uncomputable $V(M)$ by another simple measure, say that of the Gaussian unitary ensemble (GUE) with $V(M) = \Sigma^2 \operatorname{tr} M^2$. This GUE can be shown to be expressible as the zero-dimensional chiral Lagrangian in the microscopic limit (d) [2]. If the relation (b) between a properly regularized version of QCD and a particular random matrix model were not still very formal and weakly established, the diagram would otherwise close itself full circle. The diagram above at least serves as an intuitive picture outlining the logical routes between the different formulations, and the rôle played by our universality theorems proposed in this connection.

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