Microscopic Spectra of Dirac Operators and Finite-Volume Partition Functions

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Abstract

Exact results from random matrix theory are used to systematically analyse the relationship between microscopic Dirac spectra and finite-volume partition functions. Results are presented for the unitary ensemble, and the chiral analogs of the three classical matrix ensembles: unitary, orthogonal and symplectic, all of which describe universality classes of $SU(N_c)$ gauge theories with N_f fermions in different representations. Random matrix theory universality is reconsidered in this new light.

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1 Introduction

One puzzle of the random matrix theory approach [1, 2, 3] to the computation of Dirac operator spectra has been the case of one fermion species. While the (massless) microscopic spectral densities associated with the three chiral analogs of classical matrix ensembles have been found to be different [1], the corresponding finite-volume partition functions are all equal [4, 5][1]. This means that all three microscopic spectral densities, despite being very different, should lead to the same spectral sum rules of, for instance, the kind (ν indicates the (positive) topological charge):

$$\left\langle \sum_n{}' \frac{1}{\lambda_n^2} \right\rangle_\nu = \frac{1}{4(\nu+1)} \tag{1}$$

which indeed they do [1]. This holds also in the case of double-microscopic spectral densities, where fermion masses are kept finite (and scaled with volume at the same rate as the eigenvalues) [6].

As we shall show in this paper, there are in fact simultaneously two reasons for why three different universality classes of gauge theories can share the same finite-volume partition functions, have in common an infinite set of spectral sum rules, and yet have very different microscopic spectral densities. Let us first focus on the chiral unitary ensemble, which corresponds to $SU(N_c)$ gauge theories with $N_c \geq 3$ and N_f fermions in the fundamental representation of the gauge group. As was explained in ref. [7], to compute the (double-) microscopic spectral density $\rho_S^{(N_f)}(\zeta;\mu_1,\ldots,\mu_{N_f})$ corresponding to this random matrix ensemble, one needs, in addition to the finite-volume partition $\mathcal{Z}^{(N_f)}(\mu_1,\ldots,\mu_{N_f})$ of the theory with N_f flavors, also that of a theory with two additional fermion species (of imaginary mass), $\mathcal{Z}^{(N_f+2)}(\mu_1,\ldots,\mu_{N_f},i\zeta,i\zeta)$. So for $N_f=1$ we need also the partition function for $N_f=3$ (and the finite-volume partition functions corresponding to the two other random matrix ensembles for $N_f=3$ are *different*). In addition, and this is one of the main points of this paper, also the precise relationship between the (double-) microscopic spectral densities and the corresponding finite-volume partition functions turn out to be different.

One case which stands on a rather special footing is that of the (non-chiral) unitary ensemble. It has been conjectured to be relevant for QCD-like theories in (2+1) dimensions with an even number of fermions species. In section 2 of this paper we begin by deriving the relations which allow us to directly compute all double-microscopic spectral correlators from the finite-volume partition functions of QCD_3 alone. We also show how the universal limits of the orthogonal polynomials of the matrix model formulation can be computed directly from a QCD_3 partition function. In section 3 we turn to the chiral unitary ensemble described above, where we do the extension of the analysis of ref. [7] to the case of arbitrary topological charge ν, and also here demonstrate how the universal microscopic limit of the associated orthogonal polynomials are given in terms of finite-volume partition functions. In section 4 we describe the extent to which we have been able to derive analogous relations for the two remaining categories of (chiral) random matrices: the orthogonal and symplectic ensembles. We end in section 5 with our conclusions and a discussion of the notion of matrix-model universality [3, 8, 9], as seen in this new light.

[1]When considered in the appropriate "mesoscopic" scaling region. Finite-volume partition functions are always taken to be in this regime in what follows.

2 The Unitary Ensemble

The unitary ensemble has been conjectured to describe, in the microscopic limit, the Dirac operator spectrum of (2+1)-dimensional $SU(N_c)$ gauge theories with $N_c \geq 3$ and an *even* number of fermions N_f [2]. The relevant partition function of that ensemble is, for massive fermions,

$$\tilde{\mathcal{Z}}^{(N_f)}(m_1,\ldots,m_{N_f}) \;=\; \int dM \prod_{f=1}^{N_f} \det\left(M + im_f\right)\ \mathrm{e}^{-N\mathrm{tr}\,V(M^2)}\ , \tag{2}$$

where the integration is over the Haar measure of hermitian $N\times N$ matrices M. The even potential $V(M^2)$ has been left unspecified, and we write it in general as

$$V(M^2) = \sum_{k\geq 1} \frac{g_k}{2k} M^{2k}\ . \tag{3}$$

It is important to note that in order for this to possibly describe the microscopic limit of the above class of gauge theories, the even number of fermions have been regrouped into two sets of which one half is assigned the same values as the other half, except for a minus sign, *i.e.*, the mass matrix can be taken to be of the form

$$\mathrm{diag}(m_1, m_2, \ldots, m_{N_f/2}, -m_1, -m_2, \ldots, -m_{N_f/2})\ .$$

With this assignment, the fermions can effectively be regrouped into 4-spinors, the dynamics of which is reminiscent of the corresponding (3+1)-dimensional gauge theories. The associated "chiral symmetry", which really is a flavor symmetry, has been conjectured to break spontaneously according to $U(N_f)\rightarrow U(N_f/2)\times U(N_f/2)$ (recall that N_f is even) [10, 2]. An order parameter for this symmetry breaking pattern is the absolute value of the chiral condensate $\Sigma = \sum_i |\langle\bar{\psi}_i\psi_i\rangle|/N_f$. In matrix model terminology a non-zero condensate translates into a non-zero spectral density at the origin, $\rho(0)\neq 0$. For the underlying field theory, this is simply a (2+1)-dimensional generalization of the Banks-Casher relation between the chiral condensate and the spectral density of the Dirac operator, evaluated at the origin.

In terms of the eigenvalues λ_i of the hermitian matrix M we have, ignoring all irrelevant overall factors,

$$\tilde{\mathcal{Z}}^{(N_f)}(m_1,\ldots,m_{N_f}) \;=\; \int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(d\lambda_i \prod_{f=1}^{N_f/2}(\lambda_i^2+m_f^2)\ \mathrm{e}^{-NV(\lambda_i^2)}\right)\left|\det_{ij}\lambda_j^{i-1}\right|^2\ . \tag{4}$$

The imaginary part arising from the term im_f in eq. (2) has disappeared due to the pairwise grouping of masses discussed above, and the final expression is, except for the "fermion determinant", a conventional eigenvalue integration in the unitary ensemble with arbitrary even potential. The massive spectral correlators of this ensemble have recently been derived in the double-microscopic limit where masses and eigenvalues are considered on the same scale of magnification around the origin [8], and have been proved to be universal in the random matrix theory context [8].

We are now in a position to make use of a nice result from ref. [11] (see also ref. [12]), where the spectral two-point function, the kernel $K_N(x,y)$, was expressed in terms of a slightly modified random matrix integral. By definition,

$$K_N^{(N_f,\nu)}(x,x';m_1,\ldots,m_{N_f}) \;=\; e^{-\frac{N}{2}(V(x^2)+V(x'^2))}\prod_f^{N_f/2}\sqrt{(x^2+m_f^2)(x'^2+m_f^2)}\sum_{i=0}^{N-1}P_i(x)P_i(x')\ , \tag{5}$$

where $P_n(x)$ is an nth order polynomial orthogonal with respect to the even weight function

$$w(x) \;=\; \prod_{f=1}^{N_f/2} (x^2+m_f^2)\; \mathrm{e}^{-NV(x^2)} \;, \tag{6}$$

and this polynomial is hence also a function of all masses m_i. For convenience we consider here the normalization where the polynomials are chosen orthonormal on the real line. By making use of the orthonomality one can write the kernel in the form of a matrix integral:

$$\begin{aligned} K_N^{(N_f)}(x,x';m_1,\ldots,m_{N_f}) \;&=\; \frac{e^{-\frac{N}{2}(V(x^2)+V(x'^2))}\prod_f^{N_f/2}\sqrt{(x^2+m_f^2)(x'^2+m_f^2)}}{\tilde{\mathcal{Z}}^{(N_f)}(m_1,\ldots,m_{N_f})} \times \\ &\int_{-\infty}^{\infty}\prod_{i=1}^{N-1}\left(d\lambda_i(\lambda_i-x)(\lambda_i-x')\prod_{f=1}^{N_f/2}(\lambda_i^2+m_f^2)\mathrm{e}^{-NV(\lambda_i^2)}\right)\left|\det_{ij}\lambda_j^{i-1}\right|^2 . \end{aligned} \tag{7}$$

Although the last matrix integral is over $(N-1)$ eigenvalues only, this distinction becomes irrelevant in the large-N limit, which we will consider below. This gives us a very convenient expression for the spectral two-point function in the large-N limit:

$$\begin{aligned} K_N^{(N_f)}(x,x';m_1,\ldots,m_{N_f}) \;&=\; e^{-\frac{N}{2}(V(x^2)+V(x'^2))}\prod_f^{N_f/2}\sqrt{(x^2+m_f^2)(x'^2+m_f^2)} \\ &\times\;\frac{\tilde{\mathcal{Z}}^{(N_f+2)}(m_1,\ldots,m_{N_f},ix,ix')}{\tilde{\mathcal{Z}}^{(N_f)}(m_1,\ldots,m_{N_f})} \;, \end{aligned} \tag{8}$$

The partition functions are of course symmetric in the mass entries, and the position of the additional fermion masses in the argument list is therefore immaterial.

In writing the kernel in the form (8) we have made use of the observation that the insertion of the additional factor of $(\lambda_i-x)(\lambda_i-x')$ in the matrix integral can be viewed as considering a theory with two additional fermion species, of imaginary masses (cf. eq. (2)). Except for the shown prefactors, we have thus expressed the two-point spectral correlation function in terms of the ratio of two unitary-ensemble matrix model partition functions. All higher-order correlation functions are then also manifestly expressed in terms of these partition function through the usual factorization relation of the large-N limit. Finally, also the conventional (macroscopic) spectral density is expressed in such a manner, now in terms of two additional fermion species degenerate in (imaginary) masses:

$$\rho^{(N_f)}(x;m_1,\ldots,m_{N_f}) \;=\; \lim_{N\to\infty} K_N^{(N_f)}(x,x;m_1,\ldots,m_{N_f}) \;. \tag{9}$$

The above results hold in all generality in the planar limit. As in ref. [7], we now consider specifically the double-microscopic limit in which

$$\zeta \equiv \pi\rho(0)Nx \qquad \text{and} \qquad \mu_i \equiv \pi\rho(0)Nm_i \tag{10}$$

are kept fixed in the limit $N\to\infty$. If we identify $\Sigma=\pi\rho(0)$, this is precisely the mesoscopic scaling region of the finite volume partition functions of the underlying gauge theories. In ref. [2] these partition functions were argued to be describable in terms of a very simple chiral Lagrangian, in analogy with the Leutwyler-Smilga analysis in (3+1) dimensions:

$$\mathcal{Z}^{(N_f)}(\mu_1,\ldots,\mu_{N_f}) \;=\; \int dU \exp[Tr(\mathcal{M}U\Gamma_5U^\dagger)] \;. \tag{11}$$

Here $\mathcal{M}$ is the mass matrix discussed above, rescaled by $N\Sigma$:

$$\mathcal{M} \;=\; \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_{N_f/2}, -\mu_1, -\mu_2, \ldots, -\mu_{N_f/2}) \;, \tag{12}$$

and $\Gamma_5 \equiv (\mathbf{1}_{N_f/2}, -\mathbf{1}_{N_f/2})$ where $\mathbf{1}_{N_f/2}$ is an $(N_f/2)\times(N_f/2)$ unit matrix. The above partition function can be explicitly evaluated by making use of the Harish-Chandra–Itsykson-Zuber integral [13]. The result is [8]:

$$\mathcal{Z}^{(N_f)}(\mathcal{M}) \;=\; \frac{\det\begin{pmatrix} \mathbf{A}(\{\mu_i\}) & \mathbf{A}(\{-\mu_i\}) \\ \mathbf{A}(\{-\mu_i\}) & \mathbf{A}(\{\mu_i\}) \end{pmatrix}}{\Delta(\mathcal{M})} \;, \tag{13}$$

where the $(N_f/2)\times(N_f/2)$ matrix $\mathbf{A}(\{\mu_i\})$ is defined by

$$\mathrm{A}_{ij} \;\equiv\; (\mu_i)^{j-1} e^{\mu_i} \;. \tag{14}$$

The denominator is given by the Vandermonde determinant of rescaled masses:

$$\Delta(\mathcal{M}) \;=\; \prod_{i<j}^{N_f} (\mu_i - \mu_j) \;. \tag{15}$$

In the mesoscopic scaling limit the matrix model partition functions $\tilde{\mathcal{Z}}^{(N_f)}(\mu_1, \ldots, \mu_{N_f})$ should equal the field theory partition function $\mathcal{Z}^{(N_f)}(\mu_1, \ldots, \mu_{N_f})$ up to an irrelevant (mass-independent) normalization factor. Furthermore, in this scaling regime the prefactor of $\exp[-(N/2)(V(x^2)+V(x'^2))]$ in the expression for the kernel (8) becomes replaced by unity. For the kernel this leads us to the following master formula:

$$K_S^{(N_f)}(\zeta, \zeta'; \mu_1, \ldots, \mu_{N_f}) \;=\; C \prod_f^{N_f/2} \sqrt{(\zeta^2 + \mu_f^2)(\zeta'^2 + \mu_f^2)}\; \frac{\mathcal{Z}^{(N_f+2)}(\mu_1, \ldots, \mu_{N_f}, i\zeta, i\zeta')}{\mathcal{Z}^{(N_f)}(\mu_1, \ldots, \mu_{N_f})} \;. \tag{16}$$

Similarly, the double-microscopic spectral density becomes

$$\rho_S^{(N_f)}(\zeta; \mu_1, \ldots, \mu_{N_f}) \;=\; C \prod_f^{N_f/2} (\zeta^2 + \mu_f^2)\; \frac{\mathcal{Z}^{(N_f+2)}(\mu_1, \ldots, \mu_{N_f}, i\zeta, i\zeta)}{\mathcal{Z}^{(N_f)}(\mu_1, \ldots, \mu_{N_f})} \;, \tag{17}$$

and all other microscopic correlators are given by

$$\rho_S^{(N_f)}(\zeta_1, \ldots, \zeta_n; \mu_1, \ldots, \mu_{N_f}) \;=\; \det_{a,b} K_S^{(N_f)}(\zeta_a, \zeta_b; \mu_1, \ldots, \mu_{N_f}) \;. \tag{18}$$

In these expressions there is still one overall normalization factor, C, which remains to be fixed. As discussed in ref. [7], the simplest way to fix it is to make use of the matching condition between the microscopic spectral density and macroscopic spectral density,

$$\lim_{\zeta\to\infty} \rho_S^{(N_f)}(\zeta; \mu_1, \ldots, \mu_{N_f}) \;=\; \rho(0) \;=\; \frac{1}{\pi} \;, \tag{19}$$

where we have inserted the conventional normalization [1] (this conveniently makes $\mu_i = Nm_i$ and $\zeta_i = Nz_i$).

To illustrate the power of the master formula (16), consider the simplest case of quenched fermions, which formally corresponds to $N_f = 0$. In this case the origin $\lambda = 0$ is not singled out, and the kernel $K_S(\zeta,\zeta')$ should reduce to the famous sine-kernel of the unitary ensemble. Indeed, the relevant partition function for two flavors in this case reads, from eq. (13),

$$\mathcal{Z}(\mu_1,\mu_2) \;=\; \frac{2\sinh(\mu_1-\mu_2)}{\mu_1-\mu_2}\;, \tag{20}$$

while the finite-volume partition function for $N_f=0$ is a trivial constant, which we set to unity. The kernel is therefore given by

$$K_S(\zeta,\zeta') \;=\; C\,\frac{2\sin(\zeta-\zeta')}{\zeta-\zeta'}\;, \tag{21}$$

and the microscopic spectral density, as expected, becomes just a constant:

$$\rho_S^{(0)}(\zeta) \;=\; 2C\;. \tag{22}$$

The matching condition (19) hence gives $C = 1/(2\pi)$, and thus

$$K_S(\zeta,\zeta') \;=\; \frac{1}{\pi}\frac{\sin(\zeta-\zeta')}{\zeta-\zeta'}\;, \tag{23}$$

- a novel derivation of the sine-kernel.

Because we know the analytical form of the finite-volume partition function for any number N_f of (massive) fermion species, we can immediately write down the general expressions for all double-microscopic spectral correlators. From eq. (13) it follows that

$$K_S^{(N_f)}(\zeta_1,\zeta_2;\mu_1,\ldots,\mu_{N_f}) = \frac{-iC}{(\zeta_1-\zeta_2)\prod_f^{N_f/2}\sqrt{(\zeta_1^2+\mu_f^2)(\zeta_2^2+\mu_f^2)}}\,\frac{\det\begin{pmatrix}\mathbf{B}(\{\mu_i,\zeta_1\}) & \mathbf{B}(\{-\mu_i,-\zeta_1\})\\ \mathbf{B}(\{-\mu_i,\zeta_2\}) & \mathbf{B}(\{\mu_i,-\zeta_2\})\end{pmatrix}}{\det\begin{pmatrix}\mathbf{A}(\{\mu_i\}) & \mathbf{A}(\{-\mu_i\})\\ \mathbf{A}(\{-\mu_i\}) & \mathbf{A}(\{\mu_i\})\end{pmatrix}} \tag{24}$$

where $\mathbf{A}(\{\mu_i\})$ is the $(N_f/2)\times(N_f/2)$ matrix defined previously in eq. (14), and $\mathbf{B}(\{\mu_i,\zeta_i\})$ is an $(N_f/2{+}1)\times(N_f/2{+}1)$ matrix defined by

$$\begin{aligned} B_{kl} &= A_{kl} && \text{for}\;\; 1\le k\le \frac{N_f}{2}\;\;;\;\; 1\le l\le \frac{N_f}{2}+1\\ B_{kl} &= (i\zeta_{1,2})^{l-1}e^{i\zeta_{1,2}} && \text{for}\;\; k=\frac{N_f}{2}+1\;\;;\;\; 1\le l\le\frac{N_f}{2}+1\;, \end{aligned} \tag{25}$$

where in the last line the entry is either ζ_1 or ζ_2, as indicated explicitly in eq. (24).

It similarly follows from eq. (17) that the double-microscopic spectral density is

$$\rho_S^{(N_f)}(\zeta;\mu_1,\ldots,\mu_{N_f}) \;=\; \frac{-C}{\prod_f^{N_f/2}(\zeta_1^2+\mu_f^2)}\,\frac{\det\begin{pmatrix}\mathbf{B}(\{\mu_i,\zeta\}) & \mathbf{B}(\{-\mu_i,-\zeta\})\\ \mathbf{C}(\{-\mu_i,\zeta\}) & \tilde{\mathbf{C}}(\{\mu_i,-\zeta\})\end{pmatrix}}{\det\begin{pmatrix}\mathbf{A}(\{\mu_i\}) & \mathbf{A}(\{-\mu_i\})\\ \mathbf{A}(\{-\mu_i\}) & \mathbf{A}(\{\mu_i\})\end{pmatrix}}\;, \tag{26}$$

where $\mathbf{C}(\{\mu_i,\zeta\})$ and $\tilde{\mathbf{C}}(\{\mu_i,\zeta\})$ are $(N_f/2{+}1)\times(N_f/2{+}1)$ matrices defined by

$$C_{kl} \;=\; \tilde{C}_{kl} \;=\; A_{kl} \qquad \text{for}\;\; 1\le k\le\frac{N_f}{2}\;\;;\;\; 1\le l\le\frac{N_f}{2}+1\;,$$

$$C_{kl} \;=\; -\tilde{C}_{kl} \;=\; (i\zeta)^{l-2}e^{i\zeta}(i\zeta+l-1) \qquad \text{for} \;\; k=\frac{N_f}{2}+1 \;\;;\;\; 1\leq l\leq \frac{N_f}{2}+1 \;. \tag{27}$$

All other double-microscopic spectral correlators are then also known explicitly, using the relation (18).

It remains to fix the overall constant C in this general case. We again do it by the matching condition (19). This gives

$$C = \frac{1}{2\pi} \;.$$

All double-microscopic spectral correlators, for any even value of N_f, are then completely determined. Recently these double-microscopic spectral correlators were evaluated by means of random matrix theory, and the universality of the result in that framework was established [8]. The above expressions for the same quantities are more compact and convenient. We have explicitly checked in some special cases that the results of ref. [8] agree with those presented here.

It is interesting to note that not only can we compute the microscopic spectral correlators directly from the corresponding finite-volume partition functions, we can also derive the universal double-microscopic limits of the orthogonal polynomials from these partition functions. This despite of the fact that these orthogonal polynomials seem to have no clear interpretation in field theory language.

To derive expressions for the orthogonal polynomials, we make use of a convenient representation of these polynomials in terms of matrix integrals (see, *e.g.*, ref. [14]):

$$P_{2n}^{(N_f)}(\lambda;m_1,\ldots,m_{N_f}) \;=\; \frac{1}{\tilde{\mathcal{Z}}_{2n}^{(N_f)}(m_1,\ldots,m_{N_f})}\int_{-\infty}^{\infty}\prod_{i=1}^{2n}[d\lambda_i w(\lambda_i)]\left|\det{}_{ij}\lambda_j^{i-1}\right|^2\prod_{i=1}^{2n}(\lambda-\lambda_i) \tag{28}$$

Here,

$$\tilde{\mathcal{Z}}_{2n}^{(N_f)}(m_1,\ldots,m_{N_f}) \;=\; \int_{-\infty}^{\infty}\prod_{i}^{2n}[d\lambda_i w(\lambda_i)]\left|\det{}_{ij}\lambda_j^{i-1}\right|^2 \;, \tag{29}$$

and the measure factor $w(\lambda_i)$ is that of eq. (6). The relation (28) is readily verified by noting that it yields the required orthogonality relation. It also follows from (28) that the normalization corresponds to monic polynomials, *i.e.*, $P_n(\lambda)=\lambda^n+\ldots$.

One sees that for $n,N\to\infty$, with $t\equiv 2n/N$ fixed, eq. (28) becomes a relation between the orthogonal polynomials and (matrix model) partition functions, now involving an *odd* number of fermion species N_f+1 (the additional fermion having mass $i\lambda$). Let us now consider the double-microscopic scaling limit, and take $t=1$. The relation (28) then gives

$$P_{2n}^{(N_f)}(\zeta;\mu_1,\ldots,\mu_{N_f}) \;=\; C_1\frac{\mathcal{Z}^{(N_f+1)}(\mu_1,\ldots,\mu_{N_f},i\zeta)}{\mathcal{Z}^{(N_f)}(\mu_1,\ldots,\mu_{N_f})} \;, \tag{30}$$

where we again have replaced the matrix model partition functions by those of the finite-volume field theories (at the cost of introducing one overall proportionality constant C_1).

We have indicated the relation for the even polynomials only. This is actually all that is required, since we can construct the odd polynomials from the following relation [3]:

$$P_{2n+1}^{(N_f)}(\lambda;m_1,\ldots,m_{N_f}) \;=\; \frac{\tilde{P}_{2n+2}^{(N_f)}(\lambda;m_1,\ldots,m_{N_f})-\tilde{P}_{2n}^{(N_f)}(\lambda;m_1,\ldots,m_{N_f})}{\lambda} \;, \tag{31}$$

where the even polynomials $\tilde{P}_{2n}^{(N_f)}(\lambda; m_1, \ldots, m_{N_f})$ are those of eq. (28), but in a different normalization: $\tilde{P}_{2n}^{(N_f)}(0; m_1, \ldots, m_{N_f}) = 1$.

The distinction between odd and even polynomials in the manner shown above actually has a consistent interpretation in terms of the finite-volume partition functions from field theory. Verbaarschot and Zahed [2] have given these finite-volume partition functions in the form of integrals:

$$\mathcal{Z}^{(N_f+1)}(\mathcal{M}) = \int dU \ \cosh[Tr(\mathcal{M}U\Gamma U^\dagger)] \tag{32}$$

for N even, and

$$\mathcal{Z}^{(N_f+1)}(\mathcal{M}) = \int dU \ \sinh[Tr(\mathcal{M}U\Gamma U^\dagger)] \tag{33}$$

for N odd (where N denotes the three-volume). Here $\mathcal{M}$ is the mass matrix already rescaled by $N\Sigma$, whose diagonal form is

$$\mathcal{M} = \mathrm{diag}(\mu, \mu_1, \mu_2, \ldots, \mu_{N_f/2}, -\mu_1, -\mu_2, \ldots, -\mu_{N_f/2}) \ , \tag{34}$$

and $\Gamma \equiv \mathrm{diag}(\mathbf{1}_{N_f/2+1}, -\mathbf{1}_{N_f/2})$. We can choose to use just the even-N partition functions (32) to get the even polynomials (30), and then derive the odd polynomials from the relation (31). Alternatively, we can use the odd-N partition function directly (the analogue of the formula (28) for odd polynomials). The result will be the same, as follows by expanding the partition function (32) for $N+2$ in terms of the partition function for N and then comparing with eq. (31). In fact, just from the relations (31) and (30) it follows that if the even-N partition function is given by (32), then the odd-N partition function must necessarily be given by (33).

Both of the above partition functions are given in terms of integrals of the $SU(N_f+1)$-invariant Haar measure dU. These integrals can actually be evaluated rather easily by again making use of the Harish-Chandra–Itsykson-Zuber integration formula [13]. The case of one flavor of course stands on a special footing, and there one has, trivially,

$$\mathcal{Z}^{(1)}(\mu) = \cosh(\mu) \tag{35}$$

for N even, and

$$\mathcal{Z}^{(1)}(\mu) = \sinh(\mu) \tag{36}$$

for N odd. Here μ is the rescaled mass: $\mu = m\Sigma N$, and we have dropped all irrelevant (μ-independent) overall constants. For higher values of $N_f + 1$ we find

$$\mathcal{Z}^{(N_f+1)}(\{\mu; \mu_i\}) = \frac{1}{\Delta(\mathcal{M})}\left[\det \mathbf{D}(\{\mu;\mu_i\}) + (-1)^{N+N_f/2} \det \mathbf{D}(\{-\mu; -\mu_i\})\right] \ , \tag{37}$$

where $\Delta(\mathcal{M})$ is the the Vandermonde determinant of $\mathcal{M}$ eq. (34) and the additional sign $(-1)^{N_f/2}$ originates from $\Delta(-\mathcal{M})$. The matrix $\mathbf{D}$ is of size $(N_f+1)\times(N_f+1)$ and is given by

$$\begin{array}{lcll}
D_{1j} & = & \mu^{j-1} e^{\mu} & \text{for } \ 1 \le j \le \frac{N_f}{2} + 1 \\
D_{1j} & = & (-\mu)^{j-\frac{N_f}{2}-2} e^{-\mu} & \text{for } \ \frac{N_f}{2} + 2 \le j \le N_f + 1 \\
D_{ij} & = & \mu_{i-1}^{j-1} e^{\mu_{i-1}} & \text{for } \ 2 \le i \le \frac{N_f}{2} + 1 \ ; \ 1 \le j \le \frac{N_f}{2} + 1 \\
D_{ij} & = & (-\mu_{i-1})^{j-\frac{N_f}{2}-2} e^{-\mu_{i-1}} & \text{for } \ 2 \le i \le \frac{N_f}{2} + 1 \ ; \ \frac{N_f}{2} + 2 \le j \le N_f + 1
\end{array}$$

$$D_{ij} = \mu_{i-1}^{j-1} e^{\mu_{i-1}} \quad \text{for} \quad \frac{N_f}{2}+2 \le i \le N_f+1 \;\; ; \;\; 1 \le j \le \frac{N_f}{2}+1$$

$$D_{ij} = (-\mu_{i-1})^{j-\frac{N_f}{2}-2} e^{-\mu_{i-1}} \quad \text{for} \quad \frac{N_f}{2}+2 \le i \le N_f+1 \;\; ; \;\; \frac{N_f}{2}+2 \le j \le N_f+1 \; . \tag{38}$$

We are now ready to state the results. For the polynomials we get

$$P_N^{(N_f)}(\zeta;\mu_1,\ldots,\mu_{N_f}) = \frac{C_1(-1)^{N_f/2}}{\prod_f^{N_f/2}(\zeta^2+\mu_f^2)} \frac{\det \mathbf{D}(\{i\zeta;\mu_i\}) + (-1)^{N+N_f/2}\det \mathbf{D}(\{-i\zeta;-\mu_i\})}{\det \left(\begin{array}{cc} \mathbf{A}(\{\mu_i\}) & \mathbf{A}(\{-\mu_i\}) \\ \mathbf{A}(\{-\mu_i\}) & \mathbf{A}(\{\mu_i\}) \end{array}\right)} \tag{39}$$

where in matrix $\mathbf{D}$ eq. (38) μ has been replaced by $i\zeta$ and $\mathbf{A}$ is the matrix defined in eq. (14). The normalization constant C_1 remains undetermined, but can of course in any case be chosen at will. In [8] universal expressions for the same polynomials have been derived from random matrix theory. In contrast to the situation for the correlation functions eqs. (24) and (26) the expression for the polynomials obtained here are less compact that in [8]. We have checked explicitly in some special cases that both results agree.

To illustrate the simplicity of the derivation presented here, let us look at the special case of quenched fermions $N_f = 0$. From eqs. (35) and (36) we can immediately read off the well-known asymptotic behavior of the orthogonal polynomials in the microscopic limit:

$$P_{2n}^{(0)}(\zeta) = C_1 \cos(\zeta) \; , \qquad P_{2n+1}^{(0)}(\zeta) = C_1 \sin(\zeta) \; . \tag{40}$$

Although we needed to evaluate the partition function for an odd number of fermions N_f+1 to obtain the polynomials for an even number N_f, we have otherwise refrained from providing all the corresponding odd-N_f results for the double-microscopic spectral correlators, as well as the formulas for the orthogonal polynomials with an odd number of fermions. In fact, the physical interpretation of the odd case is not as straightforward as that of the even case. For example, in the massless limit the spectral density as normally defined is only positive definite for even N [2]. Moreover, the orthogonal polynomial technique does not directly apply, due to the measure being odd under parity (in eigenvalue space) for odd N. Indeed, even the formula (28) for the orthogonal polynomials will not be valid in that case, due to the non-existence of normalizable orthogonal polynomials. However, both partition functions (32) and (33), are completely well-defined, and can of course be used for convenience to compute the orthogonal polynomials for *even* N_f, as we have just shown.

3 The Chiral Unitary Ensemble

For the Dirac operator spectrum, this case corresponds to the gauge group $SU(N_c), N_c \ge 3$ with N_f fermions in the fundamental representation. The matrix model partition function reads, in the sector of topological charge ν (for convenience we shall consider $\nu \ge 0$ throughout), [1]

$$\tilde{\mathcal{Z}}_\nu^{(N_f)}(m_1,\ldots,m_{N_f}) = \int dW \prod_{f=1}^{N_f} \det(iM+m_f) \; \exp\left[-\frac{N}{2}\mathrm{tr}\, V(M^2)\right] \; , \tag{41}$$

where

$$M = \left(\begin{array}{cc} 0 & W^\dagger \\ W & 0 \end{array}\right) \; , \tag{42}$$

where W is a rectangular complex matrix of size $N \times (N+\nu)$, which is integrated over with the Haar measure. The space-time volume V of the gauge theory is, in the large-N (and large-V) limit identified with $2N$.

Introducing the eigenvalues λ_i of the hermitian matrix $W^\dagger W$, $\tilde{\mathcal{Z}}_\nu$ can be written

$$\tilde{\mathcal{Z}}_\nu^{(N_f)}(m_1,\ldots,m_{N_f}) \;=\; \prod_{f=1}^{N_f}(m_f^\nu)\int_0^\infty \prod_{i=1}^{N}\left(d\lambda_i\;\lambda_i^\nu\;\prod_{f=1}^{N_f}(\lambda_i+m_f^2)\;\mathrm{e}^{-NV(\lambda_i)}\right)\left|\det{}_{ij}\lambda_j^{i-1}\right|^2 \;. \tag{43}$$

We have ignored all unimportant factors that arise from the angular integrations. Since the partition function for N_f fermions in the sector of topological charge ν is related in a simple way to the partition function of the same N_f fermions plus ν additional fermions of zero mass (in the sector of zero topological charge), one can in principle restrict attention to the $\nu=0$ sector. It is nevertheless worthwhile to point out that the whole analysis which leads to a relation between the double-microscopic spectral correlators and the finite-volume partition functions carries over to the case of $\nu\neq 0$. This will lead us to very compact expressions for these double-microscopic spectral correlators.

The necessary generalization of the previous analysis to the present case of a *chiral* unitary ensemble with measure (43) is straightforward. We are here interested in the spectral correlations of M-eigenvalues z_i rather than those of W-eigenvalues $\lambda_i=z_i^2$. Because the whole procedure is identical to that of the previous section, and because the case $\nu=0$ already has been worked out in detail [7], we shall be brief. The two-point correlator, the kernel, is

$$\begin{aligned}&K_N^{(N_f,\nu)}(z,z';m_1,\ldots,m_{N_f}) \;=\\ &e^{-\frac{N}{2}(V(z^2)+V(z'^2))}(zz')^{\nu+\frac{1}{2}}\prod_f^{N_f}\sqrt{(z^2+m_f^2)(z'^2+m_f^2)}\sum_{i=0}^{N-1}P_i(z^2)P_i(z'^2)\;,\end{aligned} \tag{44}$$

where $P_i(z^2)$ are the orthonormal polynomials associated with the above matrix model. The kernel can now be expressed as a normalized random matrix integral:

$$\begin{aligned}K_N^{(N_f,\nu)}(z,z';m_1,\ldots,m_{N_f}) &= \frac{e^{-\frac{N}{2}(V(z^2)+V(z'^2))}(zz')^{\nu+\frac{1}{2}}\prod_f^{N_f}\sqrt{(z^2+m_f^2)(z'^2+m_f^2)}}{\tilde{\mathcal{Z}}_\nu^{(N_f)}(m_1,\ldots,m_{N_f})}\\ \times\prod_f^{N_f}(m_f^\nu)\int_0^\infty\prod_{i=1}^{N-1}&\left(d\lambda_i\lambda_i^\nu(\lambda_i-z^2)(\lambda_i-z'^2)\prod_f^{N_f}(\lambda_i+m_f^2)\mathrm{e}^{-NV(\lambda_i)}\right)\left|\det{}_{ij}\lambda_j^{i-1}\right|^2\;.\end{aligned} \tag{45}$$

Thus, in the large-N limit we have

$$\begin{aligned}K_N^{(N_f,\nu)}(z,z';m_1,\ldots,m_{N_f}) \;&=\; e^{-\frac{N}{2}(V(z^2)+V(z'^2))}(-1)^\nu\sqrt{zz'}\prod_f^{N_f}\sqrt{(z^2+m_f^2)(z'^2+m_f^2)}\\ &\quad\times\frac{\tilde{\mathcal{Z}}_\nu^{(N_f+2)}(m_1,\ldots,m_{N_f},iz,iz')}{\tilde{\mathcal{Z}}_\nu^{(N_f)}(m_1,\ldots,m_{N_f})}\;,\end{aligned} \tag{46}$$

By means of the usual factorization property, all higher n-point spectral correlation functions are then also explicitly expressed in terms of the two matrix model partition functions $\tilde{\mathcal{Z}}_\nu^{(N_f)}$ and $\tilde{\mathcal{Z}}_\nu^{(N_f+2)}$. The spectral density corresponds to the two additional (imaginary) masses being equal, as in eq. (9).

We now turn to the double-microscopic limit in which $\zeta \equiv zN2\pi\rho(0)$ and $\mu_i \equiv m_iN2\pi\rho(0)$ are kept fixed as $N\to\infty$. The prefactor $\exp[-(N/2)(V(z^2)+V(z'^2))]$ again becomes replaced by unity, and by identifying $\Sigma = 2\pi\rho(0)$, we can now compare with the field theory finite-volume partition functions. This gives us the master formula

$$K_S^{(N_f,\nu)}(\zeta,\zeta';\mu_1,\ldots,\mu_{N_f}) \;=\; C_2\sqrt{\zeta\zeta'}\prod_f^{N_f}\sqrt{(\zeta^2+\mu_f^2)(\zeta'^2+\mu_f^2)}\;\frac{\mathcal{Z}_\nu^{(N_f+2)}(\mu_1,\ldots,\mu_{N_f},i\zeta,i\zeta')}{\mathcal{Z}_\nu^{(N_f)}(\mu_1,\ldots,\mu_{N_f})}\;. \quad (47)$$

where the partition functions are those of the finite-volume field theories. Similarly, for the double-microscopic spectral density,

$$\rho_S^{(N_f,\nu)}(\zeta;\mu_1,\ldots,\mu_{N_f}) \;=\; C_2|\zeta|\prod_f^{N_f}(\zeta^2+\mu_f^2)\;\frac{\mathcal{Z}_\nu^{(N_f+2)}(\mu_1,\ldots,\mu_{N_f},i\zeta,i\zeta)}{\mathcal{Z}_\nu^{(N_f)}(\mu_1,\ldots,\mu_{N_f})}\;. \quad (48)$$

All double-microscopic n-point correlation functions are again given by the factorization formula (18).

A simple example which illustrates how powerful the above relations can be is that of the quenched case $N_f=0$. All we need is the finite-volume QCD partition function for two massive fermions of degenerate masses $i\mu/(N\Sigma)$. This was evaluated analytically already by Leutwyler and Smilga [4] and found to be, in their normalization,

$$\mathcal{Z}_\nu^{(2)}(i\mu,i\mu) \;=\; I_\nu(i\mu)^2 - I_{\nu+1}(i\mu)I_{\nu-1}(i\mu)\;, \quad (49)$$

where $I_n(x)$ is the nth modified Bessel function. The corresponding denominator in eq. (47) is again an irrelevant constant which we can set to unity. This gives

$$\rho_S^{(0,\nu)}(\zeta) = C_2(-1)^\nu|\zeta|\left[J_\nu(\zeta)^2 - J_{\nu+1}(\zeta)J_{\nu-1}(\zeta)\right]\;. \quad (50)$$

The matching condition (19) yields

$$C_2 = \frac{1}{2}(-1)^\nu\;,$$

and hence

$$\rho_S^{(0,\nu)}(\zeta) = \frac{1}{2}|\zeta|\left[J_\nu(\zeta)^2 - J_{\nu+1}(\zeta)J_{\nu-1}(\zeta)\right]\;, \quad (51)$$

which is the known result [1]. Furthermore, by the previous considerations (cf. eq. (43)) we also know that the general case of N_f massless fermions simply is equivalent to a shift $\nu\to\nu+N_f$. We thus recover the general massless result [1] without any effort:

$$\rho_S^{(N_f,\nu)}(\zeta) \;=\; \frac{1}{2}|\zeta|\left[J_{N_f+\nu}(\zeta)^2 - J_{N_f+\nu+1}(\zeta)J_{N_f+\nu-1}(\zeta)\right]\;. \quad (52)$$

We now need the general analytical expression for the finite-volume partition function for this case, with N_f fermions of arbitrary masses [15] (see also ref. [16]). It can conveniently be written [7]

$$\mathcal{Z}_\nu^{(N_f)}(\mu_1,\ldots,\mu_{N_f}) \;=\; \frac{\det\mathcal{A}(\{\mu_i\})}{\Delta(\mu^2)} \quad (53)$$

where the $N_f\times N_f$ matrix $\mathcal{A}(\{\mu_i\})$ is

$$\mathcal{A}_{ij} \;=\; \mu_i^{j-1}I_{\nu+j-1}(\mu_i)\;, \quad (54)$$

and $\Delta(\mu^2)$ again indicates the Vandermonde determinant of the μ_i^2.

For the numerator of eq. (47) we need the $(N_f+2)\times(N_f+2)$ matrix $\mathcal{A}$ with two of the entries being imaginary. This means that

$$\mathcal{A}_{ij} \;=\; i^{\nu}(-\zeta_i)^{j-1}J_{\nu+j-1}(\zeta_i) \qquad \text{for} \;\; i=1,2 \;, \tag{55}$$

and otherwise (for $i\geq 3$) as in eq. (54). For convenience we pull out a factor of (-1) from every second column of the matrix $\mathcal{A}$, and also the factor of i^{ν} from the first two rows. This yields an overall factor of $(-1)^{\nu+[N_f/2]}$ where $[x]$ denotes the integer part of x. Thus,

$$K_S^{(N_f,\nu)}(\zeta_1,\zeta_2;\mu_1,\ldots,\mu_{N_f}) = C_2\;\frac{(-1)^{\nu+[N_f/2]+1}\sqrt{\zeta_1\zeta_2}}{(\zeta_1^2-\zeta_2^2)\prod_f\sqrt{(\zeta_1^2+\mu_f^2)(\zeta_2^2+\mu_f^2)}}\;\frac{\det\mathcal{B}}{\det\mathcal{A}}\;, \tag{56}$$

where the $(N_f+2)\times(N_f+2)$ matrix $\mathcal{B}$ is defined by

$$\begin{aligned}\mathcal{B}_{ij} \;&=\; (\zeta_i)^{j-1}J_{\nu+j-1}(\zeta_i) && \text{for} \;\; i=1,2\\ \mathcal{B}_{ij} \;&=\; (-\mu_{i-2})^{j-1}I_{\nu+j-1}(\mu_{i-2}) && \text{for} \;\; 3\leq i\leq N_f+2\;,\end{aligned} \tag{57}$$

and the $N_f\times N_f$ matrix $\mathcal{A}$ is as in (54). Using the Bessel relation

$$\frac{d}{dx}\left[x^nJ_{n+m}(x)\right] \;=\; x^nJ_{n+m-1}(x) - mx^{n-1}J_{n+m}(x) \tag{58}$$

we find the corresponding double-microscopic spectral density:

$$\rho_S^{(N_f,\nu)}(\zeta;\mu_1,\ldots,\mu_{N_f}) \;=\; C_2\;\frac{(-1)^{\nu+[N_f/2]+1}|\zeta|}{2\prod_f(\zeta^2+\mu_f^2)}\;\frac{\det\tilde{\mathcal{B}}}{\det\mathcal{A}}\;, \tag{59}$$

where the $(N_f+2)\times(N_f+2)$ matrix $\tilde{\mathcal{B}}$ is defined by

$$\tilde{\mathcal{B}}_{1j} \;=\; \zeta^{j-2}J_{\nu+j-2}(\zeta) \tag{60}$$

and $\tilde{\mathcal{B}}_{ij}=\mathcal{B}_{ij}$ for $i\neq 1$. The general n-point correlators follow from eqs. (18) and (56). Using the matching condition (19) gives

$$C_2 \;=\; (-1)^{\nu+[N_f/2]}\;,$$

and everything is now determined.

For $\nu=0$ the results (56) and (59) agree with what has recently been obtained by a direct computation in random matrix theory [9, 17]. While no explicit expressions were given for the case $\nu\neq 0$ in ref. [9], it could in principle be extracted from the general formulae for $\nu=0$ by setting ν fermion masses equal to zero in a theory of $N_f+\nu$ fermions. The result done in that way should of course agree with our explicit formula given above. Indeed, we have managed to prove by induction that the compact formulas (56) and (59) also follow from the $\nu=0$ results given in ref. [9].

As in the previous section for the unitary ensemble, we now show that also in the chiral case the universal limit of the orthogonal polynomials can be obtained directly from the finite-volume partition functions alone. The extension of the formula (30) to this chiral case is straightforward. There is now no parity "quantum number" for the polynomials, and the universal double-microscopic limit is therefore unique, independent of whether the polynomials are of odd or even order. The explicit formula reads

$$P_n^{(N_f,\nu)}(\lambda; m_1,\ldots,m_{N_f}) \;=\; \frac{\prod_f^{N_f}(m_f^\nu)}{\tilde{\mathcal{Z}}_{\nu,n}^{N_f}(m_1,\ldots,m_{N_f})}\int_0^\infty \prod_{i=1}^n [d\lambda_i w(\lambda_i)] \left|\det{}_{ij}\lambda_j^{i-1}\right|^2 \prod_{i=1}^n(\lambda-\lambda_i)\ , \tag{61}$$

where now the weight function $w(\lambda)$ is given by

$$w(\lambda) \;=\; \lambda^\nu \prod_{f=1}^{N_f}\left[(\lambda+m_f^2)\right]\ \mathrm{e}^{-NV(\lambda)}\ , \tag{62}$$

and

$$\tilde{\mathcal{Z}}_{\nu,n}^{(N_f)} \;=\; \prod_f^{N_f}(m_f^\nu)\int_0^\infty \prod_i^n [d\lambda_i w(\lambda_i)] \left|\det{}_{ij}\lambda_j^{i-1}\right|^2\ , \tag{63}$$

Also here one easily sees that the normalization is such that the polynomials $P_n^{(N_f,\nu)}(\lambda; m_1,\ldots,m_{N_f})$ are monic.

In the limit $n\to\infty$ and $N\to\infty$ with $t = n/N$ fixed, the relation (61) determines the orthogonal polynomials in terms of the matrix model partition functions. By going to the double-microscopic scaling regime with $t=1$ where we can make the identification with the field theory partition functions, this gives us the relation

$$P_N^{(N_f,\nu)}(\zeta^2;\mu_1,\ldots,\mu_{N_f}) \;=\; C_3(-1)^N(i\zeta)^{-\nu}\,\frac{\mathcal{Z}_\nu^{(N_f+1)}(\mu_1,\ldots,\mu_{N_f},i\zeta)}{\mathcal{Z}_\nu^{(N_f)}(\mu_1,\ldots,\mu_{N_f})}\ , \tag{64}$$

where the normalization constant C_3 is still undetermined and we have passed to scaled M-eigenvalues. This relation is just as in the ordinary unitary case. For the numerator we need the $(N_f+1)\times(N_f+1)$ matrix $\mathcal{A}$ eq. (54) with one imaginary entry,

$$\mathcal{A}_{1j} \;=\; i^\nu(-\zeta)^{j-1}J_{\nu+j-1}(\zeta) \tag{65}$$

and otherwise as in eq. (54). In order to fix the constant C_3 and to compare with [9] we choose the normalization $P_N^{(N_f,\nu)}(0;\mu_1,\ldots,\mu_{N_f})=1$. We then obtain

$$P_N^{(N_f,\nu)}(\zeta^2;\mu_1,\ldots,\mu_{N_f}) \;=\; \zeta^{-\nu}\prod_f^{N_f}\frac{\mu_f^2}{\zeta^2+\mu_f^2}\frac{\det\mathcal{D}}{\det\tilde{\mathcal{A}}} \tag{66}$$

where

$$\begin{aligned}\mathcal{D}_{1j} \;&=\; (-\zeta)^{j-1}J_{\nu+j-1}(\zeta) && \text{for}\ \ 1\le j\le N_f+1\ ,\\ \mathcal{D}_{ij} \;&=\; \mu_{i-1}^{j-1}I_{\nu+j-1}(\mu_{i-1}) && \text{for}\ \ 2\le i\le N_f+1\ ;\quad 1\le j\le N_f+1\ ,\end{aligned} \tag{67}$$

and

$$\tilde{\mathcal{A}}_{ij} \;=\; \mu_i^j I_{\nu+j}(\mu_i) \qquad \text{for}\ \ 1\le i,j\le N_f\ . \tag{68}$$

For $\nu=0$ the above expression matches to the result for the polynomials in [9], which was derived from random matrix theory. Here, eq. (66) directly gives the results for ν zero modes, which is equivalent to set ν fermion masses to 0 in [9]. We have proven by induction that the results of ref. [9] lead to precisely the same formula for arbitrary ν as shown above.

4 The Symplectic and Orthogonal Ensembles

As follows from the general classification of universality classes [1], gauge group $SU(2)$ and N_f fermions in the fundamental representation correspond in matrix model language to the orthogonal ensemble, while gauge group $SU(N_c)$ with N_f fermions in the adjoint representation of the gauge group correspond to the symplectic ensemble.

The symplectic and orthogonal matrix ensembles are somewhat more complicated from an analytical point of view due to the non-existence of simple orthogonal-polynomial methods for those cases. The closest one apparently can get is based on the so-called quaternion method, which can be phrased in terms of skew-orthogonal (as opposed to truly orthogonal) polynomials [12]. In this chapter we shall consider what may be the closest symplectic and orthogonal ensemble analogues of the relations derived above for the unitary and chiral unitary ensembles.

We begin with the chiral symplectic matrix ensemble, as this is the case for which we most easily can derive useful relations that connect finite-volume partition functions to the associated Dirac spectra. In the language of random matrix theory, the partition function for the chiral symplectic ensemble is as in eq. (41), except that now the integration is over matrices W whose elements are quaternion real [1]. In terms of the eigenvalues λ_i of the hermitian matrix $W^\dagger W$, $\tilde{\mathcal{Z}}_\nu$ can now be written (we follow the conventional normalization, where the symplectic matrix model potential is rescaled by a factor of 2 compared with the chiral unitary case):

$$\tilde{\mathcal{Z}}_\nu^{(N_f)}(m_1,\ldots,m_{N_f}) \;=\; \prod_{f=1}^{N_f}(m_f^\nu)\int_0^\infty \prod_{i=1}^{N}\left(d\lambda_i\;\lambda_i^{2\nu+1}\;\prod_{f=1}^{N_f}(\lambda_i+m_f^2)\;\mathrm{e}^{-2NV(\lambda_i)}\right)\left|\det{}_{ij}\lambda_j^{i-1}\right|^4 \;. \quad (69)$$

Our goal is now to find the closest analogues of the master formulas (16) and (47). We shall make good use of some general relations that have been derived by Mahoux and Mehta [12]. Throughout this section we will use their notation here.

The problem we encounter is that the quantity that most closely corresponds to the kernel of the now skew-orthogonal polynomials now is a *quaternion* $f_4(\lambda_i,\lambda_j)$, which can be represented by a 2×2 matrix. The correlation functions of eigenvalues are then given by quaternion determinants $\det[f_4(\lambda_i,\lambda_j)]_m$ of the kernel $f_4(\lambda_i,\lambda_j)$. We have not been able to express this kernel itself in terms of matrix model (and thus also finite-volume field theory) partition functions, but only the determinants of this kernel, which are real valued functions. This will directly give us the expressions for the correlators, where we display the eigenvalue density and the density-density correlator as examples. Going back to eigenvalues z_i of the Dirac operator rather than $\lambda_i = z_i^2$ the spectral density can be obtained in the following way:

$$\begin{aligned}
\rho_4^{(N_f,\nu)}(z;m_1,\ldots,m_{N_f}) \;&=\; \frac{1}{N}\det[f_4(z,z)]_1\\
&=\; z^{4\nu+3}\prod_{f=1}^{N_f}(z^2+m_f^2)\;\mathrm{e}^{-2NV(z^2)}\,\frac{\prod_{f=1}^{N_f}(m_f^\nu)}{\tilde{\mathcal{Z}}_\nu^{(N_f)}(m_1,\ldots,m_{N_f})}\\
&\quad\times\int_0^\infty\prod_{i=1}^{N-1}\left(dz_i\;z_i^{4\nu+3}\prod_{f=1}^{N_f}(z_i^2+m_f^2)\;\mathrm{e}^{-2NV(z_i^2)}|z^2-z_i^2|^4\right)\left|\det{}_{ij}z_j^{2i-2}\right|^4\\
&=\; i^{-4\nu}z^3\prod_{f=1}^{N_f}(z^2+m_f^2)\;\mathrm{e}^{-2NV(z^2)}\,\frac{\tilde{\mathcal{Z}}_\nu^{(N_f+4)}(m_1,\ldots,m_{N_f},\{iz\})}{\tilde{\mathcal{Z}}_\nu^{(N_f)}(m_1,\ldots,m_{N_f})}\;. \qquad (70)
\end{aligned}$$

In the first step we have made use of Theorem 1.2 of ref. [12] in the form

$$\int dz_{p+1}\det[f_4(z_i,z_j)]_{p+1} \;=\; (N-p)\det[f_4(z_i,z_j)]_p \quad , \tag{71}$$

using their explicit expression for the partition function. We have slightly generalized the measure of ref. [12] here to include the massive fermions and zeromodes. In the second step we have replaced the integral over $N-1$ eigenvalues by the matrix model partition function with 4 equal massive flavors of imaginary mass iz, ignoring their difference in the large-N limit.

When replacing the matrix model partition function in the scaling limit by its mesoscopic field theory counterpart, as was done in the previous sections, we obtain the following relation:

$$\rho_S^{(N_f,\nu)}(\zeta;\mu_1,\ldots,\mu_{N_f}) \;=\; C_4\;\zeta^3\prod_{f=1}^{N_f}(\zeta^2+\mu_f^2)\;\frac{\mathcal{Z}_\nu^{(N_f+4)}(\mu_1,\ldots,\mu_{N_f},\{i\zeta\})}{\mathcal{Z}_\nu^{(N_f)}(\mu_1,\ldots,\mu_{N_f})}\;. \tag{72}$$

Unfortunately the needed finite-volume partition function of 4 or more fermions in the adjoint representation is not known at present and thus the eigenvalue density cannot be further evaluated yet. But since from matrix model calculations the eigenvalue density is known to be expressible in terms of integrals of Bessel functions [18] (see also [19]) a relatively simple expression for the field theory partitions functions should exist. (Simple analytical formulas are indeed known at present up to $N_f=2$ [4, 6]).

As a second example we derive in a similar way an expression for the density-density correlator:

$$\begin{aligned}
\rho_4^{(N_f,\nu)}(z,z';\{m_f\}) \;&=\; \frac{1}{N(N-1)}\det[f_4(z,z')]_2\\
&=\; |z^2-z'^2|^4(zz')^{4\nu+3}\prod_{f=1}^{N_f}(z^2+m_f^2)(z'^2+m_f^2)\;\mathrm{e}^{-2N(V(z^2)+V(z'^2))}\,\frac{\prod_{f=1}^{N_f}(m_f^\nu)}{\tilde{\mathcal{Z}}_\nu^{(N_f)}(\{m_f\})}\\
&\times\;\int_0^\infty\prod_{i=1}^{N-2}\left(dz_i z_i^{4\nu+3}\prod_{f=1}^{N_f}(z_i^2+m_f^2)\;\mathrm{e}^{-2NV(z_i^2)}|z^2-z_i^2|^4|z'^2-z_i^2|^4\right)\left|\det_{ij}z_j^{2i-2}\right|^4\\
&=\; (-1)^{4\nu}|z^2-z'^2|^4(zz')^3\prod_{f=1}^{N_f}(z^2+m_f^2)(z'^2+m_f^2)\;\mathrm{e}^{-2N(V(z^2)+V(z'^2))}\\
&\times\;\frac{\tilde{\mathcal{Z}}_\nu^{(N_f+4+4)}(m_1,\ldots,m_{N_f},\{iz\},\{iz'\})}{\tilde{\mathcal{Z}}_\nu^{(N_f)}(m_1,\ldots,m_{N_f})}\quad .
\end{aligned} \tag{73}$$

and thus in the double-microscopic limit,

$$\rho_S^{(N_f,\nu)}(\zeta,\zeta';\mu_1,\ldots,\mu_{N_f}) = \tilde{C}_4|\zeta^2-\zeta'^2|^4(\zeta\zeta')^3\prod_{f=1}^{N_f}(\zeta^2+\mu_f^2)(\zeta'^2+\mu_f^2)\frac{\mathcal{Z}_\nu^{(N_f+8)}(\mu_1,\ldots,\mu_{N_f},\{i\zeta\},\{i\zeta'\})}{\mathcal{Z}_\nu^{(N_f)}(\mu_1,\ldots,\mu_{N_f})} \tag{74}$$

In the unitary ensembles the knowledge of this two-point correlator is equivalent to knowing the kernel as well, since its connected part is the square of the kernel $\rho_{con}(z,z')=-K(z,z')^2$. However, in the symplectic case the kernel $f_4(z_i,z_j)$ is not a symmetric function and does not factorize. All higher correlation functions can actually be expressed by partition functions in an analogous way as above.

Finally we turn to the case of the orthogonal ensemble, for which the relevant partition function, when expressed in terms of eigenvalue integrals, reads

$$\tilde{\mathcal{Z}}_{\nu}^{(N_f)}(m_1,\ldots,m_{N_f}) \;=\; \prod_{f=1}^{N_f}(m_f^{\nu})\int_0^{\infty}\prod_{i=1}^{N}\left(d\lambda_i\ \lambda_i^{\frac{\nu}{2}-\frac{1}{2}}\ \prod_{f=1}^{N_f}(\lambda_i+m_f^2)\ \mathrm{e}^{-\frac{N}{2}V(\lambda_i)}\right)\left|\det{}_{ij}\lambda_j^{i-1}\right| \ . \qquad (75)$$

Since the orthogonal case can only be treated on the same footing as the symplectic case using quaternions [12], we will follow the same procedure as above:

$$\begin{aligned}\rho_1^{(N_f,\nu)}(z;\{m_i\}) &= \frac{1}{N}\det[f_1(z,z)]_1\\ &= z^{\nu}\prod_{f=1}^{N_f}(z^2+m_f^2)\ \mathrm{e}^{-\frac{N}{2}V(z^2)}\frac{\prod_{f=1}^{N_f}(m_f^{\nu})}{\tilde{\mathcal{Z}}_{\nu}^{(N_f)}(m_1,\ldots,m_{N_f})}\\ &\times\int_0^{\infty}\prod_{i=1}^{N-1}\left(dz_i\ z_i^{\nu}\prod_{f=1}^{N_f}(z_i^2+m_f^2)\ \mathrm{e}^{-\frac{N}{2}V(z_i^2)}|z^2-z_i^2|\right)\left|\det{}_{ij}z_j^{2i-2}\right| \qquad (76)\end{aligned}$$

and

$$\begin{aligned}\rho_1^{(N_f,\nu)}(z,z';\{m_i\}) &= \frac{1}{N(N-1)}\det[f_1(z,z')]_2\\ &= |z^2-z'^2|(zz')^{\nu}\prod_{f=1}^{N_f}(z^2+m_f^2)(z'^2+m_f^2)\mathrm{e}^{-\frac{N}{2}(V(z^2)+V(z'^2))}\frac{\prod_{f=1}^{N_f}(m_f^{\nu})}{\tilde{\mathcal{Z}}_{\nu}^{(N_f)}(m_1,\ldots,m_{N_f})}\\ &\times\int_0^{\infty}\prod_{i=1}^{N-2}\left(dz_i\ z_i^{\nu}\prod_{f=1}^{N_f}(z_i^2+m_f^2)\ \mathrm{e}^{-\frac{N}{2}V(z_i^2)}|z^2-z_i^2||z'^2-z_i^2|\right)\left|\det{}_{ij}z_j^{2i-2}\right|. \qquad (77)\end{aligned}$$

It is straightforward to go to the double-microscopic limit of these expressions. However, in this case the absolute value inside the eigenvalue integrals eqs. (76) and (77) prevents us from immediately identifying them with the matrix model partition function with additional masses in any simple way. But since analogously to the symplectic ensemble an expression for the microscopic density in terms of integrals of Bessel functions is known [1], similar relations to finite-volume partition functions should exist.

In this context it is particularly interesting to consider those relations between the kernels of the chiral symplectic/orthogonal ensembles and the chiral unitary ensemble which very recently have been derived by Şener and Verbaarschot [20]. According to our present viewpoint, these relations not only extend the universality proof of ref. [3] to these ensembles, but also provide completely surprising and non-trivial identities among finite-volume partition functions for different effective field theories. An understanding of these new relations between effective partition functions would be very desirable at this point.

5 Conclusions

In this paper we have systematically explored the relationship between spectral correlators of the Dirac operator in the double-microscopic scaling region, and the corresponding finite-volume partition functions. Based on relations that can be proven in random matrix theory, we have shown how to extract the universal properties from these finite-volume partition functions alone. The most powerful results are the two master formulas for the unitary and chiral unitary ensembles. These allow for a complete determination of all double-microscopic spectral correlators for those cases in terms of finite-volume partition functions. One of the surprising conclusions is that one can also derive the universal limits of the orthogonal polynomials of random matrix theory from the associated field theory partition functions.

The resulting relations are therefore not just of interest from the field theory point of view (where they indicate that the formulation in terms of large-N random matrix theory to some extent can be avoided), but also from the viewpoint of random matrix theory. Indeed, in section 2 we have illustrated this in another way by showing how the famous (bulk) sine kernel of the unitary ensemble can be derived neatly from a simple $SU(2)$ "chiral lagrangian". Many other examples can surely be found.

The cases most exhaustively solved are those of the unitary and chiral unitary matrix ensembles. Here all pertinent information for the orthogonal polynomials, their kernel and thus all correlation functions are derivable from the corresponding finite-volume partitions, suitably extended to include more fermionic species of imaginary masses. For the chiral symplectic random matrix ensemble we have chosen a different way. We have directly related the correlation functions to the corresponding field theory partition functions with 4-fold degenerate additional fermion species. These relations may turn out to provide the most easy analytical derivation of the involved quantities. For the chiral orthogonal case there is a highly suggestive relation, which, however, relates the spectral density to the partition function of a theory where the absolute value of the determinant of the Dirac operator enters. More work is required here, to either relate this partition function to the conventional one, or to construct a chiral lagrangian that would correspond to taking the absolute value of the Dirac determinant.

It is also worthwhile to reconsider the notion of random matrix universality in this new light. Of course, there is no substitute for the complete mathematical proof [3]. But we can gain substantial insight into the mechanism of universality by tracing the disappearance of the matrix model potential $V(\lambda)$ in the relevant expressions. If we, for example, return to eq. (8) of random matrix theory and the master formula (16), we see that in the double-microscopic scaling limit the prefactor of $\exp[-(N/2)(V(x^2)+V(x'^2))]$ simply becomes replaced by unity. This is the only place where the random matrix theory potential enters *explicitly* (and where it disappears in the scaling limit, leaving a universal result). Corresponding factors of the potentials disappear in the other analogous relations. One should not be misled by these simple observations to conclude that universality of the matrix model results can be understood in such simple terms alone. To some extent the notion of universality is simply built into the crucial identification between matrix model and field theory partition functions in the mesoscopic, or double-microscopic, scaling regime. Indeed, the disappearance of factors such as $\exp[-(N/2)(V(x^2)+V(x'^2))]$ in the appropriate scaling limit is not the sole mechanism behind the proven universality of random matrix theory results. An obvious counterexample is provided by the recent study of microscopic limits of random matrix theories for which the macroscopic spectral density $\rho(0)$ at the origin precisely is vanishing [21]. Here exponential prefactors such as those discussed above do *not* approach unity in the microscopic scaling limits, and still universality can be proven [21]. It would be interesting to find the chiral lagrangian analogues of these multicritical cases (for which $\rho(0)=0$[2]), and explore the relations discussed here in that more general context.

[2]There are hence no Goldstone modes, and no obvious group manifold on which to base the effective Lagrangian of the lightest hadronic excitations. A new principle seems needed to derive the corresponding effective Lagrangian.

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