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Multicritical Matrix Models and the Chiral Phase Transition

G. Akemann*

Centre de Physique Théorique CNRS,
Case 907 Campus de Luminy, F-13288 Marseille Cedex 9, France

Universality of multicritical unitary matrix models is shown and a new scaling behavior is found in the microscopic region of the spectrum, which may be relevant for the low energy spectrum of the Dirac operator at the chiral phase transition.

1. Introduction

In the last five years analytical results from random matrix theory have been successfully applied in the investigation of Dirac operator eigenvalues provided by QCD lattice data (see [1] for a recent review). A crucial point for their applicability is the question of matrix model universality when replacing the Euclidean finite-volume QCD-partition function by its matrix model counter part:

$$\mathcal{Z} \;=\; \int dW dW^{\dagger} \det{}^{N_f} \begin{pmatrix} 0 & W^{\dagger} \\ W & 0 \end{pmatrix} \; \mathrm{e}^{-NTrV(WW^{\dagger})} \;\; , \; V(\lambda^2) = \sum_{k=1}^{p} \frac{g_{2k}}{2k} \lambda^{2k} \;\; . \tag{1}$$

Here the chiral unitary ensemble with W $N \times (N+\nu)$ complex has been chosen as an example, which corresponds to QCD_4 with N_f massless fermions, ν zeromodes and gauge group $SU(N_c \geq 3)$ in the fundamental representation. It is not fixed apriori which matrix model average $\exp(-NV)$ is to be taken since it cannot be derived from the effective Yang-Mills action, that averages the determinant of the Dirac operator in QCD. The matrix model calculation, if applicable, should therefore be independent of the details of the potential V and thus be universal. This question has been entirely answered for the unitary ensembles in the phase where the chiral symmetry is broken [2,3]. The link between the matrix model eigenvalue correlations and the chiral condensate is given by the Banks-Casher formula $\rho(0) = | < \bar{q}q > |/\pi$ [4]. However, the matrix model potential V can be also tuned in such a way that the eigenvalue density at the origin $\rho(0)$ vanishes. These multicritical points [5] may therefore serve as a class of models for the chiral phase transition, where the results presented here mainly summarize [6]. In contrast to [7,8] the transition is not driven by an external parameter introduced to model the effect of temperature T or finite baryon density μ in the matrix model.

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2. The chiral phase transition at multicriticality

There are different large-N limits in which the matrix model eq.(1) can be investigated. In the *macroscopic limit* $N(\lambda_i - \lambda_j) \gg 1$, where the λ_i are the eigenvalues of W, the eigenvalue density becomes a smooth function with finite support $[-a, a]$

$$\rho(\lambda) \;=\; \sum_{k=0}^{p-1} C_k(g_i)\lambda^{2k}\sqrt{a^2-\lambda^2} \quad . \tag{2}$$

It is nonuniversal since the coefficients $C_k(g_i)$ depend explicitly on the couplings of the unknown potential $V(\lambda^2)$ in the measure (see e.g. [9]). Consequently the macroscopic density cannot lead to any quantitative prediction on the Dirac operator spectrum[2]. This has to be compared to the macroscopic density of the Dirac operator eigenvalues [10]

$$\rho_{Dirac}(\lambda) \;=\; -\frac{1}{\pi} < \bar{q}q > \;+\; \frac{(N_f^4-4) < \bar{q}q >^2}{32\pi^2 N_f F_\pi^4}|\lambda| \;+\; o(\lambda) \quad , \tag{3}$$

which is clearly a non universal, model dependent quantity as well as it contains the pion decay constant F_π as a physical parameter.

Still, the macroscopic matrix model density carries qualitative information. If it is identified with the Dirac operator eigenvalue density [11], the chiral condensate enters the matrix model through the Banks-Casher relation [4]. In the microscopic limit it will be the only physical parameter for all matrix model correlation functions which constitutes their predictive power [1]. The fact that the first coefficient of the macroscopic density eq.(2) $C_0(g_i) \sim < \bar{q}q >$ is vanishing or not distinguishes between the matrix model being in the symmetric or broken phase. If the coupling constants g_i are adjusted such that the first $m-1$ coefficients vanish, $C_0 = ... = C_{m-1} = 0$ (m-th multicriticality), a class of possible models for the chiral phase transition is obtained, where the density vanishes as an even power $\rho(\lambda) \sim \lambda^{2m}$ at the origin. A minimal potential describing such a multicritical behavior is given by [12]

$$V_m'(\lambda) \;=\; k(m)\lambda^{2m+1}\left(1-\frac{1}{\lambda^2}\right)^{1/2}\Big|_+ \quad , \quad k(m) = 2^{2m+1}\frac{(m+1)!(m-1)!}{(2m-1)!} \; , \tag{4}$$

where the index + indicates to take only positive powers when expanding in $1/\lambda^2$ and $a=1$ has been chosen. The corresponding critical density reads

$$\rho_m(\lambda) \;=\; \frac{1}{2\pi}k(m)\lambda^{2m}\sqrt{1-\lambda^2} \; . \tag{5}$$

The parameter replacing $< \bar{q}q >$ is given by the first non vanishing term $C_m(g_i) \sim \rho^{(2m)}(0)$ in eq.(2). The question of universality can then be addressed by perturbing the minimal potentials (4) by higher order terms while maintaining the same critical behavior. The phase transitions corresponding to these multicritical points are all of third order as their free energy $F = 1/N^2 \ln \mathcal{Z}$ behaves like [5]

$$F \;\sim\; (g-g_*)^{2+\frac{1}{m}} \quad , \; m \in \mathrm{N} \; , \tag{6}$$

[2]The connected density-density correlator (and all higher correlators) *is* universal in the macroscopic limit [9] as it only depends on V through the support a. However, the support of the Dirac operator eigenvalues is a cut-off dependent quantity, such that macroscopic universality has no physical implications here.

when approaching the critical point. Investigations using matrix model [8] or renormalization group techniques [13] (see proceedings) have shown that the $(T-\mu)$-phase diagram for $<\bar{q}q>$ consists of a first order (small T) and a second order line (small μ). In the tricritical point where the two lines meet the class of transitions provided by the above multicritical matrix models may become relevant.

3. A new microscopic scaling limit

In the *microscopic limit* correlations of eigenvalues are considered which are in a distance of the order $1/N$. Since the spectrum of the Dirac operator close to the origin is of interest a scaling variable $x=N\lambda$ is defined and kept finite as $N\to\infty$. The rescaled microscopic eigenvalue density reads

$$\rho_S(x) \equiv \lim_{N\to\infty} \frac{1}{N}\rho(\lambda=\frac{x}{N}) \quad . \tag{7}$$

It is a universal function since it depends on the potential V only through the macroscopic density at zero, $\rho(0)$, and thus it is parameterized by chiral condensate $<\bar{q}q>$ only. This has been shown for the unitary ensembles for an arbitrary polynomial potential at any given N_f and ν for the massless [2] and massive case [3].

In order to extend universality to the multicritical points of the unitary ensembles the way to take the microscopic limit has to be modified. At the phase transition $<\bar{q}q>\rightarrowtail 0$ the appropriate scaling behavior for the m-th multicritical point is found to be [6]

$$N^{\frac{1}{2m+1}}\lambda = x \quad , \; m\in \mathrm{N} \quad , \tag{8}$$

and the microscopic density in terms of the new scaling variable reads

$$\rho_S(x) \equiv \lim_{N\to\infty} N^{-\frac{1}{2m+1}}\rho(N^{-\frac{1}{2m+1}}x) \quad . \tag{9}$$

The same phenomenon may be expected for the Dirac operator eigenvalues on the lattice at the transition. The new universality classes eq.(9) will be parameterized by $\rho^{(2m)}(0)$ in analogy to the broken phase [6]. Recently the multicritical behavior of a different matrix model with random and deterministic degrees of freedom has been studied [14]. The authors find a modification of the microscopic scaling at multicriticality as well, with exponents $N^{\frac{2k+1}{2k+2}}\lambda=x$, $k\in$N. For $k=1$ such a model has been applied to the chiral phase transition [15] (see proceedings).

4. Universality at multicritical points

In this section a brief summary of the analytical and numerical results of [6] is given. First the results for the unitary ensemble (QCD_3)

$$\mathcal{Z} = \int dM \det{}^{2N_f}(M)\; e^{-NTrV(M)} \sim \int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(d\lambda_i|\lambda_i|^{2N_f}e^{-NV(\lambda_i)}\right)\left|\det{}_{ij}\lambda_j^{i-1}\right|^2 \quad , \tag{10}$$

with potential $V(M)=\sum\frac{g_{2k}}{2k}M^{2k}$ are given and then extended to the chiral unitary ensemble. In order to determine correlation functions of eigenvalues a differential equation for the wavefunctions

$$\psi_m(\lambda) \equiv |\lambda|^{N_f}e^{-\frac{N}{2}V(\lambda)}P_m(\lambda) \quad , \int_{-\infty}^{\infty}d\lambda\;\psi_m(\lambda)\psi_n(\lambda) = \delta_{mn} \; , \tag{11}$$

is derived following [16] and then solved in the microscopic limit eq.(8). The $P_n(\lambda)$ are polynomials orthonormal to the measure absorbed in the wavefunctions $\psi_n(\lambda)$. The spectral kernel

$$K_N(\lambda,\mu) \;=\; c_N \frac{\psi_N(\mu)\psi_{N-1}(\lambda)-\psi_N(\lambda)\psi_{N-1}(\mu)}{\mu-\lambda} \tag{12}$$

then determines all n-point correlation functions from taking its determinant. The set of polynomials $P_n(\lambda)$ obeys the following properties

$$\begin{aligned} \lambda P_n(\lambda) &= c_{n+1}P_{n+1}(\lambda) + c_n P_{n-1}(\lambda)\;, \\ P_n{}'(\lambda) &\equiv A_n(\lambda)P_{n-1}(\lambda) - B_n(\lambda)P_n(\lambda)\;, \end{aligned} \tag{13}$$

where the boundary condition determining the recursion coefficients c_n and the functions $A_n(\lambda)$ and $B_n(\lambda)$ can be found in [16]. From eqs.(13) the authors derive that the wavefunctions satisfy the following differential equation for finite N

$$\psi_n''(\lambda) - F_N(\lambda)\psi_n'(\lambda) + G_N(\lambda)\psi_n(\lambda) \;=\; 0\;, \tag{14}$$

where the $F_N(\lambda)$ and $G_N(\lambda)$ are given functions of A_N, B_N and the potential V as well as derivatives of them [16]. Taking the scaling limit in the broken phase together with the relation $\rho(\lambda)=\lim_{N\to\infty}\frac{A_N(\lambda)}{N\pi}\sqrt{1-(\lambda/a)^2}$ ([16]) leads to a universal differential equation for $\psi_N(x)$ of Bessel type first derived in [2]. When going to the m-th multicritical point in the scaling limit eq.(8) the resulting differential equation will no longer be solvable analytically, the $F_N(\lambda)$ and $G_N(\lambda)$ being rational functions [6].

In the simplest example the $m=1$-critical potential $V(\lambda)=-4\lambda^2+\frac{1}{4}g\lambda^4$ at $g=16$ leads to the following coefficients in terms of the scaling variable $x=N^{1/3}\lambda$:

$$\begin{aligned} N^{-\frac{1}{3}}F_N(x) &= \frac{2gx}{u_+ + gx^2} \\ N^{-\frac{2}{3}}G_N(x) &= \frac{u_+u_-}{4} + ((-)^N N_f - \frac{1}{2})v + \frac{u_+v+(-)^N 2gN_f}{u_+ + gx^2} + \frac{g^2x^4}{4} + \frac{(-)^N N_f - N_f^2}{x^2}\;. \end{aligned} \tag{15}$$

The constants $u_\pm = g(2f^2(0)\pm f'(0))$ and $v=2gf(0)$ have to be determined numerically by solving an auxiliary Painlevé II equation for the recursion coefficients [12] at $z=0$

$$0 \;=\; gf(z)^3 - zf(z) - \frac{g}{8}f''(z) + \frac{N_f}{2}\;. \tag{16}$$

Still, the issue of universality of the two differential equations and thus of $\psi_N(x)$ can be addressed analytically. Perturbing the critical quartic potential by a sextic term $g_6\lambda^6/6$ and maintaining $\rho(\lambda)\sim\lambda^2$ all expressions eqs.(15) and (16) remain valid when replacing $g\to g_* = g+g_6/2$. Thus $g_*=\rho''(0)/\pi$ plays the role of a universal parameter in the solution $\psi_N(x)$ and via the kernel eq.(12) in all microscopic correlation functions. Below the numerical solution for the universal microscopic density eq.(9) for $m=1$ is given. The x^2-growth is due to the matching condition $\lim_{x\to\infty}\rho_S''(x)=\rho''(0)$ (dotted line).

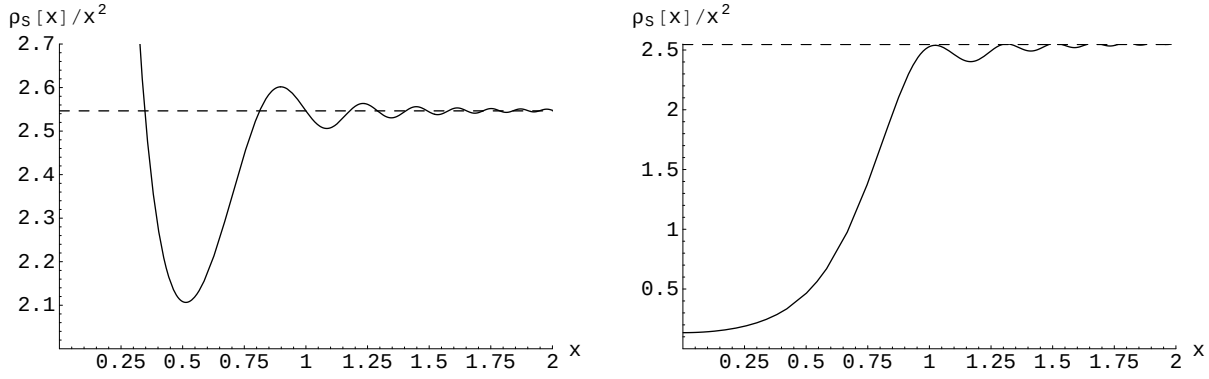

Figure 1. The $m=1$ multicritical density at $N_f=0$ (left) and $N_f=1$ (right).

For higher m the functions $F_N(x)$ and $G_N(x)$ become much more involved. However, when making an approximation to the exact differential equation (14) in the scaling limit (8) it is possible to obtain an equation which is analytically solvable for any m. Imposing

$$1 \ll k(m)\, x^{2m} \tag{17}$$

on x for the critical potentials (4) leads to the approximate equation

$$\psi_N''(x) - \frac{2m}{x}\psi_N'(x) + \left(\frac{\pi}{2(2m!)}\rho^{(2m)}(0)x^{4m} + \frac{(-1)^N N_f(2m+1) - N_f^2}{x^2}\right)\psi_N(x) \;=\; 0, \tag{18}$$

which is again of Bessel type. Here the constant $k(m) = 2\pi\rho^{(2m)}(0)/(2m)!$ has been replaced. The solution reads[3]

$$\psi_N(x) \;\sim\; \sqrt{X}\, J_{\frac{N_f}{2m+1}-\frac{(-)^N}{2}}(X) \;\;,\; X = \frac{\pi\rho^{(2m)}(0)}{(2m+1)!}\; x^{2m+1} \;\;. \tag{19}$$

The appearance of the parameter $\rho^{(2m)}(0)$ in the approximate solutions makes it highly suggestive to conjecture new universality classes for all multicritical points with $m \geq 2$.

The above results can be easily translated to the multicritical chiral unitary ensembles. The wavefunctions $\tilde{\psi}_n(\lambda)$ corresponding to the partition function eq.(1) can be simply related to those of the unitary ensemble above

$$\tilde{\psi}_m(\lambda^2;\; N_f+\nu) \;=\; \psi_{2m}(\lambda;\; N_f+\nu+\frac{1}{2}) \;\;, \tag{20}$$

when shifting N_f in eq.(10) by $\nu + 1/2$. The approximate analytic solution can thus be immediately read off from eq.(19). The final expression for the approximate microscopic density of the m-th multicritical chiral unitary ensemble reads

$$\rho_S(x) = \frac{\pi\rho^{(2m)}(0)x^{2m}}{2\,(2m)!}\left(X\left(J_{\beta+\frac{1}{2}}(X)^2 + J_{\beta-\frac{1}{2}}(X)^2\right) - \frac{2N_f}{2m+1}J_{\beta+\frac{1}{2}}(X)J_{\beta-\frac{1}{2}}(X)\right) \;\;, \tag{21}$$

with $\beta \equiv \frac{2(N_f+\nu)+1}{2(2m+1)}$. It matches to the exact noncritical density for $m=0$ [2].

[3]To fix a unique solution regularity and normalizability have been imposed as in the broken phase, although eq.(18) is no longer valid at $x=0$.

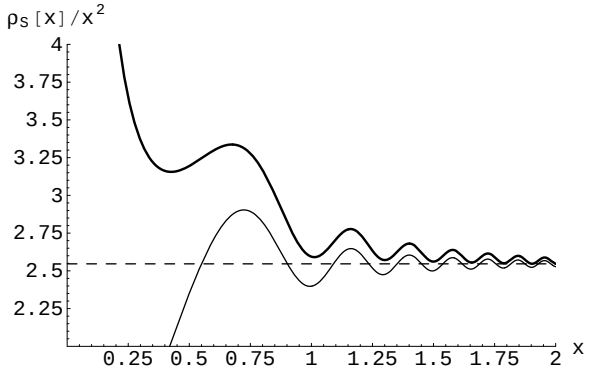

Figure 2. A comparison between the numerical solution for the $m = 1$ critical microscopic density for $N_f + \nu = 0$ (upper line) and the analytical approximation (lower line) which breaks down at small x (see eq.(17)).

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