Robust mixed H_2/H_∞ control for a class of nonlinear stochastic systems

F. Yang, Z. Wang and D.W.C. Ho

Abstract: The problem of mixed H_2/H_∞ control is considered for a class of uncertain discrete-time nonlinear stochastic systems. The nonlinearities are described by statistical means of the stochastic variables and the uncertainties are represented by deterministic norm-bounded parameter perturbations. The mixed H_2/H_∞ control problem is formulated in terms of the notion of exponentially mean-square quadratic stability and the characterisations of both the H_2 control performance and the H_∞ robustness performance. A new technique is developed to deal with the matrix trace terms arising from the stochastic nonlinearities and the well-known S-procedure is adopted to handle the deterministic uncertainities. A unified framework is established to solve the addressed mixed H_2/H_∞ control problem using a linear matrix inequality approach. Within such a framework, two additional optimisation problems are discussed, one is to optimise the H_∞ robustness performance, and the other is to optimise the H_2 control performance. An illustrative example is provided to demonstrate the effectiveness of the proposed method.

1 Introduction

In engineering practice, it is always welcome to design a controller that achieves multiple objectives. A typical example is the mixed H_2/H_∞ control scheme, which attempts to capture the benefits of both the H_2 control performance and the H_∞ robustness performance simultaneously. In general, a pure H_2 controller is designed for a good measure of transient performance [1], whereas a pure H_∞ control framework is developed for robustness with respect to disturbances and system uncertainities. Therefore the mixed H_2/H_∞ multiobjective design framework has a better and clearer physical interpretation and has received much attention from the control research community in the past few decades.

For linear deterministic systems, the mixed H_2/H_∞ control problems have been extensively studied. For example, algebraic approaches to mixed H_2/H_∞ control problems have been proposed in [2] and a time domain Nash game approach has been provided to solve the mixed H_2/H_∞ in [1, 3]. Moreover, some efficient numerical methods for mixed H_2/H_∞ control problems have been developed based on a convex optimisation approach in [4–6]. In particular, since the linear matrix inequality (LMI) approach has proven to be a very effective numerical optimisation algorithm [7], it has been employed to design both linear state feedback and output feedback controllers subject to H_2/H_∞ criterion, see, for example, [8]. It is noted that the mixed H_2/H_∞ control theories have already been applied to various engineering fields [9–11]. Parallel to the mixed H_2/H_∞ control problem, the mixed H_2/H_∞ filtering problem has also been well studied, see [12–14] and the references therein.

For nonlinear deterministic systems, the mixed H_2/H_∞ control problem has gained some research interests, see, for example, [15], where the solutions have been characterised in terms of the cross-coupled Hamilton–Jacobi–Issacs (HJI) partial differential equations. Since it is difficult to solve the cross-coupled HJI partial differential equations either analytically or numerically, Chen *et al.* [16] have used the Takagi and Sugeno (T–S) fuzzy linear model to approximate the nonlinear system, and solutions to the mixed H_2/H_∞ fuzzy output feedback control problem have been obtained via an LMI approach.

On the other hand, since stochastic modelling has been playing a more and more important role in engineering designs [17, 18], the stochastic H_∞ control problem has attracted growing research attention recently. Many research results have been available which, unfortunately, are mainly for linear stochastic systems. In [17], a stochastic-bounded real lemma has been developed to solve the H_∞ control problem for stochastic linear systems with state- and control-dependent noises. The results have been extended to the H_∞ control problem for discrete-time stochastic linear systems with the state- and control-dependent noises [19]. A robust stochastic H_∞ control problem has been addressed in [20] to deal with the systems in the presence of stochastic uncertainty. Very recently, a stochastic mixed H_2/H_∞ control problem has been considered for the system with the state-dependent noises in [21], where sufficient conditions have been provided in terms of the existence of the solutions of cross-coupled Riccati equations. However, there are very few results on the mixed H_2/H_∞ control problem for nonlinear stochastic systems. In [22], an elegant LMI approach has been developed to deal with the analysis problem for a class of systems with stochastic nonlinearities, where the nonlinearities characterised by statistical means were first introduced in [23]. Unfortunately, the robustness issue in the presence of parameter uncertainties has not been addressed.

In this paper, we aim to substantially extend part of the analysis results in [22] to the uncertain systems, derive

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the explicit expressions of the upper bounds for the robust H_2 and H_∞ performances and deal with the corresponding robust mixed H_2/H_∞ control problem using the LMI approach. Specifically, we are interested in designing a state feedback controller such that, for all admissible stochastic nonlinearities and deterministic uncertainities, the closed-loop system is exponentially mean-square quadratically stable, the H_2 control performance is achieved and the prescribed disturbance attenuation level is guaranteed in an H_∞ sense. The nonlinearities considered in this paper, which are characterised by statistical means of the stochastic variables, are shown to be more general than many well-studied nonlinearities in the literature concerning nonlinear stochastic systems. The parameter uncertainties are assumed to be norm-bounded and enter the system matrices. A new technique is developed to deal with the matrix trace terms arising from the stochastic nonlinearities, and the well-known S-procedure is adopted to handle the deterministic uncertainties. The solution to the mixed H_2/H_∞ control problem is enforced within a unified LMI framework. In order to demonstrate the flexibility of the proposed framework, we will examine two types of the optimisation problems that optimise either the H_2 control performance or the H_∞ robustness performance, and a numerical example is provided to illustrate the effectiveness of the proposed design method.

The remainder of this paper is organised as follows. In Section 2, a class of uncertain discrete-time nonlinear stochastic systems is described and the mixed H_2/H_∞ control problem for the systems is formulated. In Section 3, the system analysis problem is considered, where the existence conditions for the solution to the mixed H_2/H_∞ control problem are derived, by introducing the notion of exponentially mean-square quadratic stability and by characterising the H_2 control performance and the H_∞ robustness performance. An LMI algorithm is developed in Section 4 to design the mixed H_2/H_∞ controller for the systems with stochastic nonlinearities and deterministic norm-bounded parameter uncertainties. An illustrative example is presented in Section 5 to demonstrate the applicability of the method and some concluding remarks are provided in Section 6.

Notation: The notation used here is fairly standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices, and $\mathbb{I}^+$ stands for the set of non-negative integers. The notation $\boldsymbol{X} \geq \boldsymbol{Y}$ (respectively, $\boldsymbol{X} > \boldsymbol{Y}$), where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are symmetric matrices, means that $\boldsymbol{X} - \boldsymbol{Y}$ is positive semi-definite (respectively, positive definite). tr($\boldsymbol{A}$) represents the trace of matrix $\boldsymbol{A}$. $\mathbb{E}\{x\}$ stands for the expectation of stochastic variable x and $\mathbb{E}\{x|y\}$ for the expectation of x conditional on y. The superscript 'T' denotes the transpose. $\lambda_{\max}(\boldsymbol{M})$ stands for the maximum eigenvalue of matrix $\boldsymbol{M}$. diag$\{\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_n\}$ denotes a block diagonal matrix whose diagonal blocks are given by $\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_n$, and in symmetric block matrices, $*$ is used as an ellipsis for terms induced by symmetry.

2 Problem formulation

Consider the following class of discrete-time systems with stochastic nonlinearities and deterministic norm-bounded parameter uncertainties

$$x_{k+1} = (\boldsymbol{A} + \boldsymbol{H}_1\boldsymbol{F}\boldsymbol{E})x_k + f(x_k, u_k) + \boldsymbol{B}_1 w_k + \boldsymbol{B}_2 u_k \quad (1)$$

$$z_{\infty k} = \boldsymbol{L}_\infty x_k \quad (2)$$

$$z_{2k} = \boldsymbol{L}_2 x_k \quad (3)$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^r$ is the control input, $z_{\infty k} \in \mathbb{R}^{p1}$ is a combination of the states to be controlled (with respect to H_∞-norm constraints), $z_{2k} \in \mathbb{R}^{p2}$ is another combination of the states to be controlled (with respect to H_2-norm constraints), $w_k \in \mathbb{R}^m$ is the process noise, which is a zero mean Gaussian white noise sequences with covariance R and $\boldsymbol{A}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{L}_\infty$, $\boldsymbol{L}_2$, $\boldsymbol{H}_1$ and $\boldsymbol{E}$ are known real matrices with appropriate dimensions.

The matrix $\boldsymbol{F} \in \mathbb{R}^{i\times j}$, which may be time-varying, represents the deterministic parameter uncertainties, that is

$$\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} \leq I \quad (4)$$

The deterministic uncertain matrix $\boldsymbol{F}$ is said to be admissible if it satisfies the condition (4).

The function $f(x_k, u_k)$: $\mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ is a stochastic nonlinear function of the states and control inputs, which is assumed to have the following first moment for all x_k and u_k

$$\mathbb{E}\{f_k|x_k, u_k\} = 0 \quad (5)$$

with its covariance given by

$$\mathbb{E}\{f_k f_k^{\mathrm{T}}|x_k, u_k\} = \sum_{i=1}^{q} \boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}\left(x_k^{\mathrm{T}}\boldsymbol{\Gamma}_i x_k + u_k^{\mathrm{T}}\boldsymbol{\Pi}_i u_k\right) \quad (6)$$

where $\boldsymbol{\theta}_i$ $(i = 1, \ldots, q)$ is known column vector, $\boldsymbol{\Gamma}_i$ and $\boldsymbol{\Pi}_i$ $(i = 1, \ldots, q)$ are known positive-definite matrices with appropriate dimensions.

Remark 1: Note that the output matrices $\boldsymbol{L}_\infty$ and $\boldsymbol{L}_2$ can be chosen to be identical in practical design. Furthermore, the structure of the deterministic uncertainties in (4) has been used in many works concerning robust control and filtering problems, see, for example, [24, 25].

We are now in a position to discuss the generality of the nonlinear description in (5) and (6). As pointed out in [23], such a description encompasses many well-studied nonlinearities in stochastic systems, which enables the designer to deal with:

- Linear systems with state- and control-dependent multiplicative noises $D_1x_k\xi_{k1} + D_2u_k\xi_{k2}$, where ξ_{k1} and ξ_{k2} are zero mean, uncorrelated noise sequences.
- Nonlinear systems with random vectors dependent on the norms of states and control inputs, that is $\|x_k\|D_1\xi_{k1} + \|u_k\|D_2\xi_{k2}$, where ξ_{k1} and ξ_{k2} are zero mean, uncorrelated noise sequences.
- Nonlinear systems with a random sequence dependent on the sign of a nonlinear function of states and control inputs, that is, sign$[\phi(x_k, u_k)](D_1x_k\xi_{k1} + D_2u_k\xi_{k2})$, where ξ_{k1} and ξ_{k2} are zero mean, uncorrelated noise sequences.
- Other models that have been discussed in [23].

One can see that some of the most important uncertain nonlinear stochastic models can be special cases of the system given in (1)–(6).

We now consider the following state feedback controller for the system (1)

$$u_k = \boldsymbol{K}x_k \quad (7)$$

where $\boldsymbol{K}$ is the state feedback gain to be determined.

The closed-loop system is governed as follows by substituting (7) into (1)

$$x_{k+1} = \boldsymbol{A}_K x_k + f(x_k, \boldsymbol{K}x_k) + \boldsymbol{B}_1 w_k \quad (8)$$

where

$$\boldsymbol{A}_K = \boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{K} + \boldsymbol{H}_1\boldsymbol{F}\boldsymbol{E} \quad (9)$$

Before giving our design goal, we introduce the following notion of exponentially quadratic stability in the mean-square sense for the closed-loop system (8).

Definition 1: The system (8) is said to be exponentially mean-square quadratically stable if, with $w_k = 0$, there exist constants $\alpha \geq 1$ and $\tau \in (0, 1)$ such that

$$\mathbb{E}\{\|x_k\|^2\} \leq \alpha\tau^k \mathbb{E}\{\|x_0\|^2\}, \quad \forall x_0 \in \mathbb{R}^n, \quad k \in \mathbb{I}^+ \tag{10}$$

for all admissible uncertainties satisfying (4).

The purpose of this paper is to seek a state feedback controller of the form (7), for the system (1), such that for all stochastic nonlinearities and all admissible deterministic uncertainties, the closed-loop system is exponentially mean-square quadratically stable, and additional H_2 control performance constraint and H_∞ robustness performance constraint are also satisfied. In other words, we aim to design a controller such that the closed-loop system satisfies the following requirements (Q1) and (Q2), simultaneously:

(Q1) For a given constant $\beta > 0$, the system (8) is exponentially mean-square quadratically stable and the following constraint is satisfied

$$J_2 = \lim_{k\to\infty} \mathbb{E}\{\|z_{2k}\|^2\} < \beta \tag{11}$$

(Q2) For a given $\gamma > \gamma_0 > 0$, the system (8) is exponentially mean-square quadratically stable and the following constraint is achieved

$$\sum_{k=0}^{\infty} \mathbb{E}\{\|z_{\infty k}\|^2\} < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{\|w_k\|^2\} \tag{12}$$

for all non-zero w_k under zero initial condition, where γ_0 is the minimum attenuation level.

The design problem stated above will be referred to as the robust-mixed H_2/H_∞ control problem for the nonlinear stochastic system (1)–(6).

3 Robust mixed H_2/H_∞ analysis problem

To facilitate our discussion on the H_2 control problem (Q1) and the H_∞ control problem (Q2), we need the following technical results.

Lemma 1 [22, 26]: Given the feedback gain matrix $\boldsymbol{K}$. The system (8) is exponentially mean-square quadratically stable if, for all admissible uncertainties, there exists a positive definite matrix $\boldsymbol{P}$ satisfying

$$\boldsymbol{A}_K^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_K - \boldsymbol{P} + \sum_{i=1}^{q}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})\,\mathrm{tr}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}\boldsymbol{P}) < 0 \tag{13}$$

Lemma 2 [22]: If the system (8) is exponentially mean-square quadratically stable, then

$$\rho\left\{\boldsymbol{A}_K \otimes \boldsymbol{A}_K + \sum_{i=1}^{q} \mathrm{st}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}})\,\mathrm{st}^{\mathrm{T}}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})\right\} < 1 \tag{14}$$

or equivalently

$$\rho\left\{\boldsymbol{A}_K^{\mathrm{T}} \otimes \boldsymbol{A}_K^{\mathrm{T}} + \sum_{i=1}^{q} \mathrm{st}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})\,\mathrm{st}^{\mathrm{T}}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}})\right\} < 1 \tag{15}$$

where $\otimes$ is the Kronecker product of matrices, ρ is the spectral radius of a matrix and st stands for the stack of a matrix that forms a vector out of the columns of the matrix.

Lemma 3: Consider the system

$$\xi_{k+1} = \boldsymbol{M}\xi_k + f(\xi_k) \tag{16}$$

where $\mathbb{E}\{f_k|\xi_k\} = 0$, and $\mathbb{E}\{f_k f_k^{\mathrm{T}}|\xi_k\} = \sum_{i=1}^{q} \boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}(\xi_k^{\mathrm{T}}\Xi_i\xi_k)$, $\boldsymbol{\theta}_i$ $(i = 1, \ldots, q)$ are known column vectors, Ξ_i $(i = 1, \ldots, q)$ are known positive-definite matrices with appropriate dimensions. If the system (16) is exponentially mean-square stable, and there exists a symmetric matrix $\boldsymbol{Y}$ satisfying

$$\boldsymbol{M}^{\mathrm{T}}\boldsymbol{Y}\boldsymbol{M} - \boldsymbol{Y} + \sum_{i=1}^{q} \Xi_i\,\mathrm{tr}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}Y) < 0 \tag{17}$$

then $\boldsymbol{Y} \geq 0$.

Lemma 3 can be easily proved by using the Lyapunov method, hence the proof is omitted.

3.1 H_2 control problem

Define the state covariance by

$$\begin{aligned}\boldsymbol{Q}_k &:= \mathbb{E}\{x_k x_k^{\mathrm{T}}\}\\ &= \mathbb{E}\{[x_{1,k} \quad x_{2,k} \quad \cdots \quad x_{n,k}][x_{1,k} \quad x_{2,k} \quad \cdots \quad x_{n,k}]^{\mathrm{T}}\}\end{aligned} \tag{18}$$

and then the Lyapunov-type equation that governs the evolution of the state covariance matrix $\boldsymbol{Q}_k$ can be derived from the system (8) and the relation (7) as follows

$$\boldsymbol{Q}_{k+1} = \boldsymbol{A}_K\boldsymbol{Q}_k\boldsymbol{A}_K^{\mathrm{T}} + \sum_{i=1}^{q} \boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}\,\mathrm{tr}[\boldsymbol{Q}_k(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})] + \boldsymbol{B}_1\boldsymbol{R}\boldsymbol{B}_1^{\mathrm{T}} \tag{19}$$

We rewrite (19) in the form of the stack matrix by

$$\mathrm{st}(\boldsymbol{Q}_{k+1}) = \Psi \cdot \mathrm{st}(\boldsymbol{Q}_k) + \mathrm{st}\,(\boldsymbol{B}_1\boldsymbol{R}\boldsymbol{B}_1^{\mathrm{T}}) \tag{20}$$

where

$$\Psi := \boldsymbol{A}_K \otimes \boldsymbol{A}_K + \sum_{i=1}^{q} \mathrm{st}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}})\,\mathrm{st}^{\mathrm{T}}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})$$

If the system (8) is exponentially mean-square quadratically stable, it follows from Lemma 2 that $\rho(\Psi) < 1$ and $\boldsymbol{Q}_k$ in (20) converges to a constant matrix $\boldsymbol{Q}$ when $k \to \infty$, that is

$$\boldsymbol{Q} = \lim_{k\to\infty} \boldsymbol{Q}_k \tag{21}$$

Therefore H_2 performance can be written by

$$J_2 = \lim_{k\to\infty} \mathbb{E}\{\|z_{2k}\|^2\} = \lim_{k\to\infty} \mathrm{tr}[\boldsymbol{L}_2\boldsymbol{Q}_k\boldsymbol{L}_2^{\mathrm{T}}] = \mathrm{tr}[\boldsymbol{L}_2\boldsymbol{Q}\boldsymbol{L}_2^{\mathrm{T}}] \tag{22}$$

In order to make sure that the H_2 performance and H_∞ performance can be tackled within the same framework by using a unified LMI approach, we will need to derive an alternative expression of the H_2 performance (22). Suppose now that there exists a matrix $\hat{\boldsymbol{P}}_k > 0$ such that the following backward recursion is satisfied

$$\begin{aligned}\hat{\boldsymbol{P}}_k = {} & \boldsymbol{A}_K^{\mathrm{T}}\hat{\boldsymbol{P}}_{k+1}\boldsymbol{A}_K + \sum_{i=1}^{q}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})\,\mathrm{tr}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}\hat{\boldsymbol{P}}_{k+1})\\ & + \boldsymbol{L}_2^{\mathrm{T}}\boldsymbol{L}_2\end{aligned} \tag{23}$$

which can be rearranged in terms of the stack operator as follows

$$\mathrm{st}(\hat{\boldsymbol{P}}_k) = \Phi \cdot \mathrm{st}(\hat{\boldsymbol{P}}_{k+1}) + \mathrm{st}(\boldsymbol{L}_2^{\mathrm{T}}\boldsymbol{L}_2) \tag{24}$$

where

$$\Phi := \boldsymbol{A}_K^{\mathrm{T}} \otimes \boldsymbol{A}_K^{\mathrm{T}} + \sum_{i=1}^{q} \mathrm{st}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})\,\mathrm{st}^{\mathrm{T}}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}})$$

If the system (8) is exponentially mean-square quadratically stable, then it follows from Lemma 2 that $\rho(\Phi) < 1$ and $\hat{\boldsymbol{P}}_k$ in (24) converges to $\hat{\boldsymbol{P}}$ when $k \to \infty$, that is

$$\hat{\boldsymbol{P}} = \lim_{k\to\infty} \hat{\boldsymbol{P}}_k \tag{25}$$

Hence, in the steady state, (23) becomes

$$\hat{\boldsymbol{P}} = \boldsymbol{A}_K^{\mathrm{T}}\hat{\boldsymbol{P}}\boldsymbol{A}_K + \sum_{i=1}^{q}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})\,\mathrm{tr}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}\hat{\boldsymbol{P}}) + \boldsymbol{L}_2^{\mathrm{T}}\boldsymbol{L}_2 \tag{26}$$

Summing up (23)–(26), we obtain the following result that gives an alternative to the H_2 performance and facilitates our later consideration on the H_∞ performance constraint.

Theorem 1: If the system (8) is exponentially mean-square quadratically stable, H_2 performance can be expressed in terms of $\hat{\boldsymbol{P}}$ as follows

$$J_2 = \mathrm{tr}[\boldsymbol{R}\boldsymbol{B}_1^{\mathrm{T}}\hat{\boldsymbol{P}}\boldsymbol{B}_1] \tag{27}$$

where $\hat{\boldsymbol{P}} > 0$ is the solution to (26).

Proof: Noting that

$$\begin{aligned}
&\lim_{k\to\infty} \mathrm{tr}\{\boldsymbol{Q}_{k+1}\hat{\boldsymbol{P}}_{k+1} - \boldsymbol{Q}_k\hat{\boldsymbol{P}}_k\}\\
&= \lim_{k\to\infty} \mathrm{tr}\Bigg\{\Bigg[\boldsymbol{A}_K\boldsymbol{Q}_k\boldsymbol{A}_K^{\mathrm{T}} + \sum_{i=1}^{q}\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}\mathrm{tr}[\boldsymbol{Q}_k(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})]\\
&\quad + \boldsymbol{B}_1\boldsymbol{R}\boldsymbol{B}_1^{\mathrm{T}}\Bigg]\hat{\boldsymbol{P}}_{k+1} - \boldsymbol{Q}_k\Bigg[\boldsymbol{A}_K^{\mathrm{T}}\hat{\boldsymbol{P}}_{k+1}\boldsymbol{A}_K + \sum_{i=1}^{q}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})\\
&\quad \times \mathrm{tr}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}\hat{\boldsymbol{P}}_{k+1}) + \boldsymbol{L}_2^{\mathrm{T}}\boldsymbol{L}_2\Bigg]\Bigg\}\\
&= 0
\end{aligned} \tag{28}$$

Therefore we have

$$\mathrm{tr}[\boldsymbol{L}_2\boldsymbol{Q}\boldsymbol{L}_2^{\mathrm{T}}] = \mathrm{tr}[\boldsymbol{R}\boldsymbol{B}_1^{\mathrm{T}}\hat{\boldsymbol{P}}\boldsymbol{B}_1] \tag{29}$$

and the proof follows from (22) immediately. □

Remark 2: We use (27) to compute the H_2 performance instead of (22). The reason is that the H_2 control performance and H_∞ robustness performance need to be characterised as a similar structure so that the solution to the mixed H_2/H_∞ control problem can be obtained by using a unified LMI approach. We will see in the next subsection that the structure of (27) is similar to that for the H_∞ robustness performance.

Notice that the system model in (1)–(3) involves parameter uncertainties, and hence the exact H_2 performance (27) cannot be obtained by simply solving (26). One way to deal with this problem is to provide an upper bound for the H_2 performance. Suppose that there exists a positive definite matrix $\boldsymbol{P}$ such that the following matrix inequality is satisfied

$$\boldsymbol{A}_K^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_K - \boldsymbol{P} + \sum_{i=1}^{q}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})\,\mathrm{tr}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}\boldsymbol{P}) + \boldsymbol{L}_2^{\mathrm{T}}\boldsymbol{L}_2 < 0 \tag{30}$$

Now we are ready to give the upper bound for $\hat{\boldsymbol{P}}$. Comparing (26) to (30), we obtain the following main result in this subsection.

Theorem 2: If there exists a positive definite matrix $\boldsymbol{P}$ satisfying (30), then the system (8) is exponentially mean-square quadratically stable

$$\hat{\boldsymbol{P}} \le \boldsymbol{P} \tag{31}$$

and

$$\mathrm{tr}[\boldsymbol{R}\boldsymbol{B}_1^{\mathrm{T}}\hat{\boldsymbol{P}}\boldsymbol{B}_1] \le \mathrm{tr}[\boldsymbol{R}\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_1] \tag{32}$$

where $\hat{\boldsymbol{P}} > 0$ satisfies (26).

Proof: It is obvious that (30) implies (13), and then it follows directly from Lemma 1 that the system (8) is exponentially mean-square quadratically stable. Hence, the solution $\hat{\boldsymbol{P}} > 0$ to (26) exists. Subtracting (30) from (26) yields

$$\begin{aligned}
&\boldsymbol{A}_K^{\mathrm{T}}(\boldsymbol{P} - \hat{\boldsymbol{P}})\boldsymbol{A}_K - (\boldsymbol{P} - \hat{\boldsymbol{P}})\\
&\quad + \sum_{i=1}^{q}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})\,\mathrm{tr}[\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}(\boldsymbol{P} - \hat{\boldsymbol{P}})] < 0
\end{aligned} \tag{33}$$

which indicates from Lemma 3 that $\boldsymbol{P} - \hat{\boldsymbol{P}} \ge 0$. Furthermore, (31) implies (32), and this completes the proof. □

The corollary given below follows immediately from Theorem 2 and (11).

Corollary 1: If there exists a positive definite matrix $\boldsymbol{P}$ satisfying (30) and $\mathrm{tr}[\boldsymbol{R}\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_1] < \beta$, where $\beta > 0$ is a given scalar, then the system (8) is exponentially mean-square quadratically stable, and (11) is satisfied for $\beta > 0$.

3.2 H_∞ control problem

Contrary to the standard H_∞ performance formulation, we shall use the expression (12) to describe the H_∞ performance of the stochastic system, where the expectation operator is utilised on both the controlled output and the disturbance input, see [18] for more details.

The following lemma can be proved along a similar line in [21].

Lemma 4: Given a scalar $\gamma > 0$ and a feedback gain matrix $\boldsymbol{K}$. The system (8) is exponentially mean-square quadratically stable and the H_∞-norm constraint (12) is achieved for all non-zero w_k, if there exists a positive-definite matrix $\boldsymbol{P}$ satisfying

$$\begin{bmatrix} \boldsymbol{A}_K^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_K - \boldsymbol{P} + \sum_{i=1}^{q}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})\,\mathrm{tr}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}\boldsymbol{P}) + \boldsymbol{L}_\infty^{\mathrm{T}}\boldsymbol{L}_\infty & \boldsymbol{A}_K^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_1 \\ \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_K & \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_1 - \gamma^2 I \end{bmatrix} < 0 \tag{34}$$

for all admissible uncertainties.

Up to now, the H_2 control problem and the H_∞ control problem have been considered separately. Before proceeding to the next Section, we will need to discuss the mixed H_2/H_∞ analysis problem.

3.3 Robust mixed H_2/H_∞ analysis problem

In order to realise our design goals (Q1) and (Q2) simultaneously, it can be easily seen that the robust mixed H_2/H_∞ control problem addressed in Section 2 can be restated as follows.

Problem A: Design a controller (7) such that there exists a positive definite matrix $\boldsymbol{P}$ satisfying the following inequalities

$$\mathrm{tr}[\boldsymbol{R}\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_1] < \beta \tag{35}$$

$$\boldsymbol{A}_K^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_K - \boldsymbol{P} + \sum_{i=1}^{q}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})\,\mathrm{tr}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}\boldsymbol{P}) + \boldsymbol{L}_2^{\mathrm{T}}\boldsymbol{L}_2 < 0 \tag{36}$$

$$\begin{bmatrix} \boldsymbol{A}_K^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_K - \boldsymbol{P} + \sum_{i=1}^{q}(\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}}\boldsymbol{\Pi}_i\boldsymbol{K})\,\mathrm{tr}(\boldsymbol{\theta}_i\boldsymbol{\theta}_i^{\mathrm{T}}\boldsymbol{P}) + \boldsymbol{L}_\infty^{\mathrm{T}}\boldsymbol{L}_\infty & \boldsymbol{A}_K^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_1 \\ \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_K & \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_1 - \gamma^2 I \end{bmatrix} < 0 \tag{37}$$

The purpose of Problem A is to find a controller (7) so as to ensure that (35)–(37) are satisfied for all admissible uncertainties, and subsequently the stability, the H_2 and H_∞ constraints are all achieved. Note that at this stage, such a problem is still complicated since the matrix trace terms and the uncertainty $\boldsymbol{F}$ are involved in (35)–(37). Our goal in the next Section is therefore to develop an LMI approach to designing the desired controller based on (35)–(37).

4 Robust mixed H_2/H_∞ controller design

In this Section, we will present the solution to the robust H_2/H_∞ state feedback controller design problem for the discrete-time systems with stochastic nonlinearities and deterministic norm-bounded parameter uncertainty. In other words, we aim to design the controller that satisfies the performance requirements (Q1) and (Q2) simultaneously. In order to develop a unified LMI framework, the main task at this stage is to deal with the matrix trace terms (nonlinear term) and handle the uncertainties in the matrix inequalities (35)–(37), such that Problem A can be converted into a convex optimisation problem that is easy to be solved.

Before giving our main result, we recall the following useful lemmas.

Lemma 5 (Schur complement) [7]: Given constant matrices $\boldsymbol{L}_1, \boldsymbol{L}_2, \boldsymbol{L}_3$ where $\boldsymbol{L}_1 = \boldsymbol{L}_1^{\mathrm{T}}$ and $0 < \boldsymbol{L}_2 = \boldsymbol{L}_2^{\mathrm{T}}$, then

$$\boldsymbol{L}_1 + \boldsymbol{L}_3^{\mathrm{T}}\boldsymbol{L}_2^{-1}\boldsymbol{L}_3 < 0$$

if and only if

$$\begin{bmatrix} \boldsymbol{L}_1 & \boldsymbol{L}_3^{\mathrm{T}} \\ \boldsymbol{L}_3 & -\boldsymbol{L}_2 \end{bmatrix} < 0$$

or equivalently

$$\begin{bmatrix} -\boldsymbol{L}_2 & \boldsymbol{L}_3 \\ \boldsymbol{L}_3^{\mathrm{T}} & \boldsymbol{L}_1 \end{bmatrix} < 0$$

Lemma 6 (S-procedure) [7, 14]: Let $\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}}$, $\boldsymbol{H}$ and $\boldsymbol{E}$ be real matrices of appropriate dimensions, with $\boldsymbol{F}$ satisfying (4), then

$$\boldsymbol{M} + \boldsymbol{H}\boldsymbol{F}\boldsymbol{E} + \boldsymbol{E}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}} < 0 \tag{38}$$

if and only if, there exists a positive scalar $\varepsilon > 0$ such that

$$\boldsymbol{M} + \varepsilon\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}} + \frac{1}{\varepsilon}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E} < 0 \tag{39}$$

or equivalently

$$\begin{bmatrix} \boldsymbol{M} & \varepsilon\boldsymbol{H} & \boldsymbol{E}^{\mathrm{T}} \\ \varepsilon\boldsymbol{H}^{\mathrm{T}} & -\varepsilon I & 0 \\ \boldsymbol{E} & 0 & -\varepsilon I \end{bmatrix} < 0 \tag{40}$$

In order to recast Problem A into a convex optimisation problem, we first tackle the matrix trace terms in (35)–(37) by introducing new variables, which is actually one of the technical contributions in this paper. The following theorem presents sufficient conditions for solving Problem A.

Theorem 3: Given constants $\gamma > 0$, $\beta > 0$ and the feedback gain matrix $\boldsymbol{K}$. If there exists positive-definite matrix $\boldsymbol{P} > 0$ and $\boldsymbol{\Theta} > 0$, and positive scalars $\alpha_i > 0$ $(i = 1, \ldots, q)$ such that the following matrix inequalities

$$\mathrm{tr}(\boldsymbol{\Theta}) < \beta \tag{41}$$

$$\begin{bmatrix} -\boldsymbol{\Theta} & \boldsymbol{R}^{1/2}\boldsymbol{B}_1^{\mathrm{T}} \\ \boldsymbol{B}_1\boldsymbol{R}^{1/2} & -\boldsymbol{P}^{-1} \end{bmatrix} < 0 \tag{42}$$

$$\begin{bmatrix} -\alpha_i & \alpha_i\boldsymbol{\theta}_i^{\mathrm{T}} \\ \alpha_i\boldsymbol{\theta}_i & -\boldsymbol{P}^{-1} \end{bmatrix} < 0 \quad (i = 1, \ldots, q) \tag{43}$$

$$\begin{bmatrix} -\boldsymbol{P} & \boldsymbol{A}_K^{\mathrm{T}} & \boldsymbol{\Gamma}_1^{1/2} & \cdots & \boldsymbol{\Gamma}_q^{1/2} & \boldsymbol{K}^{\mathrm{T}} & \cdots & \boldsymbol{K}^{\mathrm{T}} & \boldsymbol{L}_2^{\mathrm{T}} \\ \boldsymbol{A}_K & -\boldsymbol{P}^{-1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \boldsymbol{\Gamma}_1^{1/2} & 0 & -\alpha_1 I & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{\Gamma}_q^{1/2} & 0 & 0 & \cdots & -\alpha_q I & 0 & \cdots & 0 & 0 \\ \boldsymbol{K} & 0 & 0 & \cdots & 0 & -\alpha_1\boldsymbol{\Pi}_1^{-1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{K} & 0 & 0 & \cdots & 0 & 0 & \cdots & -\alpha_q\boldsymbol{\Pi}_q^{-1} & 0 \\ \boldsymbol{L}_2 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -I \end{bmatrix} < 0 \tag{44}$$

$$
\begin{bmatrix}
-\boldsymbol{P} & 0 & \boldsymbol{A}_K^{\mathrm{T}} & \boldsymbol{\Gamma}_1^{1/2} & \cdots & \boldsymbol{\Gamma}_q^{1/2} & \boldsymbol{K}^{\mathrm{T}} & \cdots & \boldsymbol{K}^{\mathrm{T}} & \boldsymbol{L}_\infty^{\mathrm{T}} \\
0 & -\gamma^2 I & \boldsymbol{B}_1^{\mathrm{T}} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\boldsymbol{A}_K & \boldsymbol{B}_1 & -\boldsymbol{P}^{-1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\boldsymbol{\Gamma}_1^{1/2} & 0 & 0 & -\alpha_1 I & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\boldsymbol{\Gamma}_q^{1/2} & 0 & 0 & 0 & \cdots & -\alpha_q I & 0 & \cdots & 0 & 0 \\
\boldsymbol{K} & 0 & 0 & 0 & \cdots & 0 & -\alpha_1 \boldsymbol{\Pi}_1^{-1} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\boldsymbol{K} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\alpha_q \boldsymbol{\Pi}_q^{-1} & 0 \\
\boldsymbol{L}_\infty & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -I
\end{bmatrix} < 0 \qquad (45)
$$

hold, then (35)–(37) are satisfied.

Proof: Define new variables at $\alpha_i > 0$ $(i = 1, \ldots, q)$ satisfying

$$
\mathrm{tr}(\boldsymbol{\theta}_i \boldsymbol{\theta}_i^{\mathrm{T}} \boldsymbol{P}) < \alpha_i^{-1} \quad (i = 1, \ldots, q) \qquad (46)
$$

Using the property of matrix trace and Lemma 5 (Schur complement), we have

$$
\mathrm{tr}(\boldsymbol{\theta}_i \boldsymbol{\theta}_i^{\mathrm{T}} \boldsymbol{P}) = \boldsymbol{\theta}_i^{\mathrm{T}} \boldsymbol{P} \boldsymbol{\theta}_i < \alpha_i^{-1} \Longleftrightarrow \begin{bmatrix} -\alpha_i & \alpha_i \boldsymbol{\theta}_i^{\mathrm{T}} \\ \alpha_i \boldsymbol{\theta}_i & -\boldsymbol{P}^{-1} \end{bmatrix} < 0 \quad (i = 1, \ldots, q) \qquad (47)
$$

which implies (43).

Next, we prove that (44) is equivalent to

$$
\boldsymbol{A}_K^{\mathrm{T}} \boldsymbol{P} \boldsymbol{A}_K - \boldsymbol{P} + \sum_{i=1}^{q} \alpha_i^{-1} (\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}} \boldsymbol{\Pi}_i \boldsymbol{K}) + \boldsymbol{L}_2^{\mathrm{T}} \boldsymbol{L}_2 < 0 \qquad (48)
$$

By using Lemma 5 (Schur complement) to (48), we have

$$
\begin{bmatrix} -\boldsymbol{P} + \sum_{i=1}^{q} \alpha_i^{-1} (\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}} \boldsymbol{\Pi}_i \boldsymbol{K}) + \boldsymbol{L}_2^{\mathrm{T}} \boldsymbol{L}_2 & \boldsymbol{A}_K^{\mathrm{T}} \\ \boldsymbol{A}_K & -\boldsymbol{P}^{-1} \end{bmatrix} < 0 \qquad (49)
$$

$$
\Longleftrightarrow
$$

$$
\begin{bmatrix}
-\boldsymbol{P} + \boldsymbol{L}_2^{\mathrm{T}} \boldsymbol{L}_2 & \boldsymbol{A}_K^{\mathrm{T}} & \boldsymbol{\Gamma}_1^{1/2} & \cdots & \boldsymbol{\Gamma}_q^{1/2} & \boldsymbol{K}^{\mathrm{T}} & \cdots & \boldsymbol{K}^{\mathrm{T}} \\
\boldsymbol{A}_K & -\boldsymbol{P}^{-1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\boldsymbol{\Gamma}_1^{1/2} & 0 & -\alpha_1 I & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\boldsymbol{\Gamma}_q^{1/2} & 0 & 0 & \cdots & -\alpha_q I & 0 & \cdots & 0 \\
\boldsymbol{K} & 0 & 0 & \cdots & 0 & -\alpha_1 \boldsymbol{\Pi}_1^{-1} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\boldsymbol{K} & 0 & 0 & \cdots & 0 & 0 & \cdots & -\alpha_q \boldsymbol{\Pi}_q^{-1}
\end{bmatrix} < 0 \qquad (50)
$$

which is equivalent to (44). Moreover, it follows from (46) and (48) that

$$
\begin{aligned}
&\boldsymbol{A}_K^{\mathrm{T}} \boldsymbol{P} \boldsymbol{A}_K - \boldsymbol{P} + \sum_{i=1}^{q} (\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}} \boldsymbol{\Pi}_i \boldsymbol{K})\, \mathrm{tr}(\boldsymbol{\theta}_i \boldsymbol{\theta}_i^{\mathrm{T}} \boldsymbol{P}) + \boldsymbol{L}_2^{\mathrm{T}} \boldsymbol{L}_2 \\
&\quad < \boldsymbol{A}_K^{\mathrm{T}} \boldsymbol{P} \boldsymbol{A}_K - \boldsymbol{P} + \sum_{i=1}^{q} \alpha_i^{-1} (\boldsymbol{\Gamma}_i + \boldsymbol{K}^{\mathrm{T}} \boldsymbol{\Pi}_i \boldsymbol{K}) + \boldsymbol{L}_2^{\mathrm{T}} \boldsymbol{L}_2 < 0
\end{aligned} \qquad (51)
$$

which proves (36).

Similarly, by using Lemma 5 (Schur complement), one can see that (45) implies (37). It remains to show that (41) and (42) indicate (35). Since (42) is equivalent to

$$
\boldsymbol{R}^{1/2} \boldsymbol{B}_1^{\mathrm{T}} \boldsymbol{P} \boldsymbol{B}_1 \boldsymbol{R}^{1/2} < \boldsymbol{\Theta} \qquad (52)
$$

it follows from (41) and (52) that

$$
\mathrm{tr}(\boldsymbol{R} \boldsymbol{B}_1^{\mathrm{T}} \boldsymbol{P} \boldsymbol{B}_1) = \mathrm{tr}(\boldsymbol{R}^{1/2} \boldsymbol{B}_1^{\mathrm{T}} \boldsymbol{P} \boldsymbol{B}_1 \boldsymbol{R}^{1/2}) < \mathrm{tr}(\boldsymbol{\Theta}) < \beta \qquad (53)
$$

The proof is complete. □

Remark 3: In Theorem 3, we provide sufficient conditions for satisfying (35)–(37), where the nonlinear matrix trace terms are handled so as to form a convex optimisation problem. The possible conservatism caused by such a transformation can be reduced by making the values of $\mathrm{tr}(\boldsymbol{\theta}_i \boldsymbol{\theta}_i^{\mathrm{T}} \boldsymbol{P})$ as close as possible to the value of α_i^{-1} when solving the LMIs. This will be demonstrated later in Section 5.

In the following, we will continue to 'eliminate' the uncertainty $\boldsymbol{F}$ contained in (44) and (45) by using the well-known S-procedure technique, and then the desired robust mixed H_2/H_∞ controller could be obtained via an LMI approach by solving Problem A.

Theorem 4: Given constants $\gamma > 0$ and $\beta > 0$. If there exists positive-definite matrix $\boldsymbol{X} > 0$ and $\boldsymbol{\Theta} > 0$, a real matrix $\boldsymbol{G}$, positive scalars $\alpha_i > 0$ $(i = 1, \ldots, q)$ and $\varepsilon_i > 0$

$(i = 1, 2)$ such that the following linear matrix inequalities

$$
\begin{aligned}
&[1 \quad 0 \quad \cdots \quad 0]\boldsymbol{\Theta}[1 \quad 0 \quad \cdots \quad 0]^{\mathrm{T}} + \cdots \\
&\quad + [0 \quad \cdots \quad 0 \quad 1]\boldsymbol{\Theta}[0 \quad \cdots \quad 0 \quad 1]^{\mathrm{T}} < \beta
\end{aligned} \tag{54}
$$

$$
\begin{bmatrix} -\boldsymbol{\Theta} & \boldsymbol{R}^{1/2}\boldsymbol{B}_1^{\mathrm{T}} \\ \boldsymbol{B}_1\boldsymbol{R}^{1/2} & -\boldsymbol{X} \end{bmatrix} < 0 \tag{55}
$$

$$
\begin{bmatrix} -\alpha_i & \alpha_i\boldsymbol{\theta}_i^{\mathrm{T}} \\ \alpha_i\boldsymbol{\theta}_i & -\boldsymbol{X} \end{bmatrix} < 0 \quad (i = 1, \ldots, q) \tag{56}
$$

$$
\left[\begin{array}{ccccccccccc}
-\boldsymbol{X} & * & * & * & * & * & * & * & * & * & * \\
\boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}_2\boldsymbol{G} & -\boldsymbol{X} & * & * & * & * & * & * & * & * & * \\
\boldsymbol{\Gamma}_1^{1/2}\boldsymbol{X} & 0 & -\alpha_1 I & * & * & * & * & * & * & * & * \\
\cdots & \cdots & \cdots & \cdots & * & * & * & * & * & * & * \\
\boldsymbol{\Gamma}_q^{1/2}\boldsymbol{X} & 0 & 0 & \cdots & -\alpha_q I & * & * & * & * & * & * \\
\boldsymbol{G} & 0 & 0 & \cdots & 0 & -\alpha_1\boldsymbol{\Pi}_1^{-1} & * & * & * & * & * \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * & * & * & * \\
\boldsymbol{G} & 0 & 0 & \cdots & 0 & 0 & \cdots & -\alpha_q\boldsymbol{\Pi}_q^{-1} & * & * & * \\
\boldsymbol{L}_2\boldsymbol{X} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -I & * & * \\
0 & \varepsilon_1\boldsymbol{H}_1^{\mathrm{T}} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -\varepsilon_1 I & * \\
\boldsymbol{E}\boldsymbol{X} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & -\varepsilon_1 I
\end{array}\right] < 0 \tag{57}
$$

$$
\left[\begin{array}{cccccccccccc}
-\boldsymbol{X} & * & * & * & * & * & * & * & * & * & * & * \\
0 & -\gamma^2 I & * & * & * & * & * & * & * & * & * & * \\
\boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}_2\boldsymbol{G} & \boldsymbol{B}_1 & -\boldsymbol{X} & * & * & * & * & * & * & * & * & * \\
\boldsymbol{\Gamma}_1^{1/2}\boldsymbol{X} & 0 & 0 & -\alpha_1 I & * & * & * & * & * & * & * & * \\
\cdots & \cdots & \cdots & \cdots & \cdots & * & * & * & * & * & * & * \\
\boldsymbol{\Gamma}_q^{1/2}\boldsymbol{X} & 0 & 0 & 0 & \cdots & -\alpha_q I & * & * & * & * & * & * \\
\boldsymbol{G} & 0 & 0 & 0 & \cdots & 0 & -\alpha_1\boldsymbol{\Pi}_1^{-1} & * & * & * & * & * \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * & * & * & * \\
\boldsymbol{G} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\alpha_q\boldsymbol{\Pi}_q^{-1} & * & * & * \\
\boldsymbol{L}_\infty\boldsymbol{X} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -I & * & * \\
0 & 0 & \varepsilon_2\boldsymbol{H}_1^{\mathrm{T}} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -\varepsilon_2 I & * \\
\boldsymbol{E}\boldsymbol{X} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & -\varepsilon_2 I
\end{array}\right] < 0 \tag{58}
$$

are feasible, then there exists a state feedback controller of the form (7) such that the requirements (Q1) and (Q2) are satisfied for all stochastic nonlinearities and all admissible deterministic uncertainties. Moreover, the desired controller (7) can be determined by

$$
\boldsymbol{K} = \boldsymbol{G}\boldsymbol{X}^{-1} \tag{59}
$$

Proof: In view of Theorem 3, we just need to show that (44) holds if and only if there exists a positive scalar ε_1 such that (57) holds, and (45) holds if and only if there exists a positive scalar ε_2 such that (58) holds.

Rewrite the condition (44) in the form of (38) as follows

$$
\hat{\boldsymbol{M}} + \hat{\boldsymbol{H}}\boldsymbol{F}\hat{\boldsymbol{E}} + \hat{\boldsymbol{E}}^{\mathrm{T}}\hat{\boldsymbol{F}}^{\mathrm{T}}\hat{\boldsymbol{H}}^{\mathrm{T}} < 0 \tag{60}
$$

where

$$
\hat{\boldsymbol{M}} = \left[\begin{array}{ccccccccc}
-\boldsymbol{P} & (\boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{K})^{\mathrm{T}} & \boldsymbol{\Gamma}_1^{1/2} & \cdots & \boldsymbol{\Gamma}_q^{1/2} & \boldsymbol{K}^{\mathrm{T}} & \cdots & \boldsymbol{K}^{\mathrm{T}} & \boldsymbol{L}_2^{\mathrm{T}} \\
\boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{K} & -\boldsymbol{P}^{-1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\boldsymbol{\Gamma}_1^{1/2} & 0 & -\alpha_1 I & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\boldsymbol{\Gamma}_q^{1/2} & 0 & 0 & \cdots & -\alpha_q I & 0 & \cdots & 0 & 0 \\
\boldsymbol{K} & 0 & 0 & \cdots & 0 & -\alpha_1\boldsymbol{\Pi}_1^{-1} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\boldsymbol{K} & 0 & 0 & \cdots & 0 & 0 & \cdots & -\alpha_q\boldsymbol{\Pi}_q^{-1} & 0 \\
\boldsymbol{L}_2 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -I
\end{array}\right]
$$

$$
\hat{\boldsymbol{H}} = [0 \quad \boldsymbol{H}_1^{\mathrm{T}} \quad 0 \quad \cdots \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0]^{\mathrm{T}}
$$

$$
\hat{\boldsymbol{E}} = [\boldsymbol{E} \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0]
$$

Applying Lemma 6 to (60) to 'eliminate' the uncertainty $\boldsymbol{F}$, we know that (44) holds if and only if there exists a positive scalar parameter ε_1 such that the following LMI holds

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * & * & * & * & * & * & * & * \\ \boldsymbol{A}+\boldsymbol{B}_2\boldsymbol{K} & -\boldsymbol{P}^{-1} & * & * & * & * & * & * & * & * & * \\ \boldsymbol{\Gamma}_1^{1/2} & 0 & -\alpha_1 I & * & * & * & * & * & * & * & * \\ \cdots & \cdots & \cdots & \cdots & * & * & * & * & * & * & * \\ \boldsymbol{\Gamma}_q^{1/2} & 0 & 0 & \cdots & -\alpha_q I & * & * & * & * & * & * \\ \boldsymbol{K} & 0 & 0 & \cdots & 0 & -\alpha_1 \boldsymbol{\Pi}_1^{-1} & * & * & * & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * & * & * & * \\ \boldsymbol{K} & 0 & 0 & \cdots & 0 & 0 & \cdots & -\alpha_q \boldsymbol{\Pi}_q^{-1} & * & * & * \\ \boldsymbol{L}_2 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -I & * & * \\ 0 & \varepsilon_1 \boldsymbol{H}_1^{\mathrm{T}} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -\varepsilon_1 I & * \\ \boldsymbol{E} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & -\varepsilon_1 I \end{bmatrix} < 0 \quad (61)$$

Letting

$$\boldsymbol{X} = \boldsymbol{P}^{-1} \quad (62)$$

and

$$\boldsymbol{G} = \boldsymbol{K}\boldsymbol{X} \quad (63)$$

and performing the congruence transformation diag$\{\boldsymbol{P}^{-1}, I, I, \ldots, I, I, \ldots, I, I, I, I\}$ to (61), we obtain (57).

Similarly, applying Lemma 6 to (45) and performing the congruence transformation diag$\{\boldsymbol{P}^{-1}, I, I, I, \ldots, I, I, \ldots, I, I, I, I\}$, we get (58) from (62) and (63). Furthermore, (55) and (56) are obtained from (42) and (43) by using (62). Using the property of matrix trace, (54) is derived from (41). This completes the proof. □

Remark 4: The robust mixed H_2/H_∞ controller can be obtained by solving LMIs (54)–(58) in Theorem 4. The LMIs can be solved efficiently via interior point method [7]. Note that LMIs (54)–(58) are affine in the scalar positive parameters ε_1 and ε_2. Hence, they can be defined as LMI variables in order to increase the possibility of the solutions and decrease conservatism with respect to the uncertainty $\boldsymbol{F}$.

So far the controller has been designed which satisfies the requirements (Q1) and (Q2). Because of the advantages of LMI formulations, the results in Theorem 4 also suggest the following two optimisation problems that would be interesting to control engineers:

(P1) *The optimal H_∞ control problem with H_2 performance constraints for uncertain nonlinear stochastic systems*

$$\min_{Q>0, G, \alpha_1, \ldots, \alpha_q, \varepsilon_1>0, \varepsilon_2>0} \gamma \quad \text{subject to (54)–(58)}$$
$$\text{for some given } \beta \quad (64)$$

(P2) *The optimal H_2 control problem with H_∞ performance constraints for uncertain nonlinear stochastic systems*

$$\min_{Q>0, G, \alpha_1, \ldots, \alpha_q, \varepsilon_1>0, \varepsilon_2>0} \beta \quad \text{subject to (54)–(58)}$$
$$\text{for some given } \gamma \quad (65)$$

Remark 5: In many engineering applications, the performances constraints are often specified a priori. For example, in Theorem 4, the controller is designed after H_∞ performance and H_2 performance are prescribed. In fact, however, we can obtain an improved performance by optimisation method. The problem (P1) will help exploit the design freedom to meet the optimal H_∞ performance under a prescribed β. The problem (P2) will find an optimal solution among them to achieve the H_2 performance under a prescribed γ_2. These are certainly attractive because the addressed multiobjective problems can be solved while a local optimal performance can also be achieved, and the computation is efficient by using the Matlab LMI toolbox.

5 Illustrative example

Consider a discrete-time system described by (1)–(3) with stochastic nonlinearities and deterministic norm-bounded parameter uncertainties as follows

$$\boldsymbol{A} = \begin{bmatrix} -0.5 & 0 & -0.8 \\ 0 & -1.2 & 0 \\ 0.6 & 0 & 0.6 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0.3 \\ 0 \\ 0.2 \end{bmatrix}$$

$$\boldsymbol{B}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{L}_\infty = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{L}_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{H}_1 = \begin{bmatrix} 0.5 \\ 0.6 \\ 0 \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 0.8 & 0 & 0 \end{bmatrix}$$

where w_k is a zero mean Gaussian white noise sequence with covariance $R = 0.1$. The deterministic uncertainty $\boldsymbol{F}$ satisfies the condition (4), and the stochastic nonlinear function $f(x_k, u_k)$ satisfies

$$\mathbb{E}\{f_k | x_k, u_k\} = 0$$
$$\mathbb{E}\{f_k f_k^{\mathrm{T}} | x_k, u_k\}$$
$$= \begin{bmatrix} 0.1 \\ 0.2 \\ 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \\ 0 \end{bmatrix}^{\mathrm{T}} \left(x_k^{\mathrm{T}} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} x_k \right.$$
$$\left. + 0.6 u_k^{\mathrm{T}} u_k + x_k^{\mathrm{T}} \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} x_k + 0.8 u_k^{\mathrm{T}} u_k \right)$$

Now, let us examine the following three cases.

Case 1: $\gamma^2 = 0.5$, $\beta = 0.1$.

This case is exactly concerned with the addressed robust H_2/H_∞ control problem, hence can be tackled by using Theorem 4 with $q = 2$. In theory, the solution set is large, and we just provide one solution by employing the Matlab

LMI toolbox

$$X = \begin{bmatrix} 0.3628 & 0.1928 & -0.1127 \\ 0.1928 & 7.1889 & 5.4349 \\ -0.1127 & 5.4349 & 5.2622 \end{bmatrix}$$

$$G = [-0.0864 \quad -4.6579 \quad -4.0586]$$

$$\varepsilon_1 = 0.3826, \qquad \varepsilon_2 = 0.3813$$

$$\alpha_1 = 26.3259, \qquad \alpha_2 = 26.3549, \qquad \boldsymbol{\Theta} = 0.0716$$

$$\mathrm{tr}(\boldsymbol{\theta}_1 \boldsymbol{\theta}_1^{\mathrm{T}} X^{-1}) = 0.0377 < \alpha_1^{-1} = 0.0380$$

$$\mathrm{tr}(\boldsymbol{\theta}_2 \boldsymbol{\theta}_2^{\mathrm{T}} X^{-1}) = 0.0377 < \alpha_2^{-1} = 0.0379$$

$$K = [-0.2732 \quad -0.2422 \quad -0.5270]$$

Case 2: $\beta = 0.1$.

In this case, we wish to design the controller which minimises the H_∞ performance under the H_2 performance constraints. That is, we want to solve the problem (P1). Solving the optimisation problem (64) using LMI toolbox yields the minimum value $\gamma^2_{\min} = 0.3958$ and

$$X = \begin{bmatrix} 0.5657 & 0.3112 & -0.1701 \\ 0.3112 & 9.8502 & 7.5724 \\ -0.1701 & 7.5724 & 7.2331 \end{bmatrix}$$

$$G = [-0.1399 \quad -6.4779 \quad -5.5926]$$

$$\varepsilon_1 = 0.9033, \qquad \varepsilon_2 = 0.9008$$

$$\alpha_1 = 37.4043, \qquad \alpha_2 = 37.4862, \qquad \boldsymbol{\Theta} = 0.0661$$

$$\mathrm{tr}(\boldsymbol{\theta}_1 \boldsymbol{\theta}_1^{\mathrm{T}} X^{-1}) = 0.0263 < \alpha_1^{-1} = 0.0267$$

$$\mathrm{tr}(\boldsymbol{\theta}_2 \boldsymbol{\theta}_2^{\mathrm{T}} X^{-1}) = 0.0263 < \alpha_2^{-1} = 0.0267$$

$$K = [-0.2583 \quad -0.2583 \quad -0.5089]$$

Case 3: $\gamma^2 = 0.5$.

We now deal with the problem (P2). Solving the optimisation problem (65), we obtain the minimum H_2 performance $\beta_{\min} = 0.0.0319$, and

$$X = \begin{bmatrix} 0.5829 & 0.3100 & -0.1853 \\ 0.3100 & 10.4074 & 8.0742 \\ -0.1853 & 8.0742 & 7.6853 \end{bmatrix}$$

$$G = [-0.1368 \quad -6.8501 \quad -5.9134]$$

$$\varepsilon_1 = 0.9687, \qquad \varepsilon_2 = 0.9673$$

$$\alpha_1 = 38.4558, \qquad \alpha_2 = 38.4895, \qquad \boldsymbol{\Theta} = 0.0319$$

$$\mathrm{tr}(\boldsymbol{\theta}_1 \boldsymbol{\theta}_1^{\mathrm{T}} X^{-1}) = 0.0259 < \alpha_1^{-1} = 0.0260$$

$$\mathrm{tr}(\boldsymbol{\theta}_2 \boldsymbol{\theta}_2^{\mathrm{T}} X^{-1}) = 0.0259 < \alpha_2^{-1} = 0.0260$$

$$K = [-0.2516 \quad -0.2652 \quad -0.4969]$$

The results show that the designed system can satisfy H_2 control performance and H_∞ disturbance rejection performance simultaneously. In Case 2, in order to achieve a better disturbance rejection performance, the optimisation algorithm (P1) is employed to obtain the optimal solution. Similarly, to get a better H_2 control performance, the optimisation algorithm (P2) is applied to obtain the optimal solution in Case 3. Furthermore, we can see from the results that the values of $\mathrm{tr}(\boldsymbol{\theta}_i \boldsymbol{\theta}_i^{\mathrm{T}} X^{-1})$ $(i = 1, 2)$ is very close to α_i^{-1} $(i = 1, 2)$ in all three cases, hence the possible conservatism could be significantly reduced.

Remark 6: Within the LMI framework developed in this paper, we can show that there are some trade-off that can be used for satisfying specific performance requirements. For example, the H_∞ performance will be improved if the H_2 performance constraints become more relaxed (larger). Also, if the value of the H_∞ performance constraint is allowed to be increased, then the H_2 performance can be further reduced. Hence, the proposed approach allows much flexibility in making compromise between the H_2 performance and the H_∞ performance, while the essential multiple objectives can all be met simultaneously.

6 Conclusions

A robust mixed H_2/H_∞ controller has been designed in this paper for a class of uncertain discrete-time nonlinear stochastic systems. A key technique has been used to deal with the matrix trace terms arising from the stochastic nonlinearities, and the well-known S-procedure has been adopted to handle the deterministic uncertainties. A unified framework has been established to solve the addressed mixed H_2/H_∞ control problem, and sufficient conditions for the solvability of the mixed H_2/H_∞ control problem have been given in terms of a set of feasible LMIs. Two types of the optimisation problems have been proposed by optimising either the H_2 performance or the H_∞ performance. Our method can also be extended to output feedback case, and the results will appear in the near future.

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8 References

1 Chen, X., and Zhou, K.: 'Multiobjective H_2 and H_∞ control design', *SIAM J. Control Optim.*, 2001, **40**, (2), pp. 628–660

2 Bernstein, D.S., and Haddad, W.M.: 'LQG control with an H_∞ performance bound: Ariccati equation approach', *IEEE Trans. Autom. Control*, 1989, **34**, (3), pp. 293–305

3 Limebeer, D.J.N., Anderson, B.D.O., and Hendel, B.: 'A Nash game approach to mixed H_2/H_∞ control', *IEEE Trans. Autom. Control*, 1994, **39**, (1), pp. 69–82

4 Khargonekar, P.P., and Rotea, M.A.: 'Mixed H_2/H_∞ control: a convex optimization approach', *IEEE Trans. Autom. Control*, 1991, **36**, (7), pp. 824–837

5 Scherer, C.W.: 'Multiobjective H_2/H_∞ control', *IEEE Trans. Autom. Control*, 1995, **40**, (6), pp. 1054–1062

6 Zhang, L., Huang, B., and Lam, J.: 'LMI synthesis of H_2 and mixed H_2/H_∞ controllers for singular systems', *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, 2003, **50**, (9), pp. 615–626

7 Boyd, S., Ghaoui, L.E., Feron, E., and Balakrishnan, V.: 'Linear matrix inequalities in system and control theory' (SIAM Studies in Applied Mathematics, Philadelphia, 1994)

8 Scherer, C., Gahient, P., and Chilali, M.: 'Multiobjective output-feedback control via LMI optimization', *IEEE Trans. Autom. Control*, 1997, **42**, pp. 896–911

9 Khosrowjerdi, M.-J., Nikoukhah, R., and Safari-Shad, N.: 'Fault detection in a mixed H_2/H_∞ setting'. Proc. IEEE Conf. Decision and Control, Maui, Hawaii, 2003, pp. 1461–1466

10 Rotstein, H., Sznaier, M., and Idan, M.: 'Mixed H_2/H_∞ filtering theory and an aerospace application', *Int. J. Robust Nonlinear Control*, 1996, **6**, pp. 347–366

11 Yang, C.-D., and Sun, Y.P.: 'Mixed H_2/H_∞ state-feedback design for microsatellite attitude control', *Control Eng. Pract.*, 2002, **10**, pp. 951–970
12 Hung, Y.S., and Yang, F.: 'Robust H_∞ filtering with error variance constraints for uncertain discrete time-varying systems with uncertainty', *Automatica*, 2003, **39**, (7), pp. 1185–1194
13 Theodor, Y., and Shaked, U.: 'A dynamic game approach to mixed H_2/H_∞ estimation', *Int. J. Robust Nonlinear Control*, 1996, **6**, pp. 331–345
14 Yang, F., and Hung, Y.S.: 'Robust mixed H_2/H_∞ filtering with regional pole assignment for uncertain discrete-time systems', *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, 2002, **49**, (8), pp. 1236–1241
15 Lin, W.: 'Mixed H_2/H_∞ control of nonlinear systems'. Proc. IEEE Conf. Decision and Control, New Orleans, LA, December 1995, pp. 333–338
16 Chen, B.S., Tseng, C.S., and Uang, H.J.: 'Mixed H_2/H_∞ fuzzy output feedback control design for nonlinear dynamic systems: an LMI approach', *IEEE Trans. Fuzzy Syst.*, 2000, **8**, (6), pp. 249–265
17 Hinrichsen, D., and Pritchard, A.J.: 'Stochastic H_∞', *SIAM J. Control Optim.*, 1998, **36**, pp. 1504–1538
18 Wang, H.: 'Bounded dynamic stochastic systems: modelling and control' (Springer-Verlag, London, 2000)
19 Bouhtouri, A.E., Hinrichsen, D., and Pritchard, A.J.: 'H_∞-type control for discrete-time stochastic systems', *Int. J. Robust Nonlinear Control*, 1999, **9**, (13), pp. 923–948
20 Ugrinovskii, V.A.: 'Robust H_∞ control in the presence of stochastic uncertainty', *Int. J. Control*, 1998, **71**, pp. 219–237
21 Chen, B.S., and Zhang, W.: 'Stochastic H_2/H_∞ control with state-dependent noise', *IEEE Trans. Autom. Control*, 2004, **49**, (1), pp. 45–57
22 Yaz, Y.I., and Yaz, E.E.: 'On LMI formulations of some problems arising in nonlinear stochastic system analysis', *IEEE Trans. Autom. Control*, 1999, **44**, (4), pp. 813–816
23 Jacobson, D.H.: 'A general result in stochastic optimal control of nonlinear discrete-time systems with quadratic performance criteria', *J. Math. Anal. Appl.*, 1974, **47**, pp. 153–161
24 Wang, Z., and Qiao, H.: 'Robust filtering for bilinear uncertain stochastic discrete-time systems', *IEEE Trans. Signal Process.*, 2002, **50**, (3), pp. 560–567
25 Yang, F., Wang, Z., and Hung, Y.S.: 'Robust Kalman filtering for discrete time-varying uncertain systems with multiplicative noise', *IEEE Trans. Autom. Control*, 2002, **47**, (7), pp. 1179–1183
26 Subramanian, A., and Sayed, A.H.: 'Multiobjective filter design for uncertain stochastic time-delay systems', *IEEE Trans. Autom. Control*, 2004, **49**, (1), pp. 149–154