Spectra of massive QCD Dirac Operators from Random Matrix Theory: all three chiral symmetry breaking patterns*

G. Akemann[a†] and E. Kanzieper[b]

[a]Max-Planck-Institut für Kernphysik, Postfach 103980, D-69029 Heidelberg, Germany

[b] Hitachi Cambridge Laboratory, Madingley Road, Cambridge CB3 0HE, United Kingdom,
The Abdus Salam International Centre for Theoretical Physics P.O.B. 586, 34100 Trieste, Italy

The microscopic spectral eigenvalue correlations of QCD Dirac operators in the presence of dynamical fermions are calculated within the framework of Random Matrix Theory (RMT). Our approach treats the low–energy correlation functions of all three chiral symmetry breaking patterns (labeled by the Dyson index $\beta = 1, 2$ and 4) on the same footing, offering a unifying description of massive QCD Dirac spectra. RMT universality is explicitly proven for all three symmetry classes and the results are compared to the available lattice data for $\beta = 4$.

1. Introduction

Random Matrix Theory (RMT) has turned out to be a very fruitful tool in studying the phenomenon of chiral symmetry breaking in low–energy QCD [1]. First proposed as a purely phenomenological approach, it has recently been put onto firm field theoretic grounds after the analytic RMT predictions have been reproduced within the framework of finite–volume partition functions and partially quenched chiral perturbation theory using supersymmetry [2] and replica [3] techniques.

Similarly to previously studied sum rules [4], RMT solutions for spectral statistics serve as a more detailed test of QCD lattice data, in particular for a given sector of topological charge ν (which counts the number of zero modes of the QCD Dirac operator). This has become possible due to recent developments in lattice gauge theory. Namely, the Ginsparg-Wilson relation was shown to provide an exact chiral symmetry on the lattice together with a well defined topological charge ν [5] (and these proceedings). The classification of the three different χSB patterns according to gauge group and representation [6] has been confronted to lattice data and good agreement has been found for the spectral density and distribution of the smallest Dirac eigenvalue for *massless* lattice data of all χSB patterns and for different values of ν (see e.g. [7], Fig. 2).

It has to be mentioned that the direct field theoretic calculation of spectral correlators is much more cumbersome as compared to RMT. The former approach has not led to any explicit analytic results beyond the spectral density due to enormous increase of dimensionality of auxiliary (supermatrix) fields involved. So far, only spectral densities for massless flavors (in all three χSB patterns) and for one single massive flavor (in $SU(N_c \geq 3)$ in the fundamental representation) have been derived [2]. In contrast, the classic RMT technique is free of the above technical complications and allows computing the higher order correlation functions with the same ease (see Ref. [8] for comparison with lattice data). This advantage of a RMT description becomes even more significant for dynamical fermions. The correlation functions with an arbitrary number of massive flavors N_f have been calculated for fundamental $SU(N_c\geq 3)$ [9,10] ($\beta=2$) and recently for the two remaining χSB patterns [11,12] ($\beta=1,4$).

In the present communication, we report on our results [11] relevant for gauge groups $SU(2)$ in the fundamental ($\beta = 1$) and $SU(N_c)$ in the adjoint representation ($\beta = 4$). Our predictions are compared to $SU(2)$ lattice data [13] for dynamical staggered fermions with 4 degenerate flavors.

---
*presented by G.A. at Lattice 2000, Bangalore, India

†partial support by EU TMR grant ERBFMRXCT97-0122

2. RMT results for massive flavors

Let us briefly recall the connection between RMT and low–energy QCD. The Dirac operator spectrum at the origin is related to the chiral condensate Σ, the order parameter of χSB, through the Banks-Casher relation [14] $\Sigma = \lim_{V\to\infty} \pi\rho(0)/V$, where V is the Euclidean space-time volume. Here, the spectral density of the Dirac operator is given by the average $\rho(\lambda) = \langle\sum_k \delta(\lambda-\lambda_k)\rangle$ over all gauge field configurations and the λ_k are the Dirac operator eigenvalues. In the limit $\Lambda^{-1} \ll V \ll m_\pi^{-1}$ [4], where m_π is the pion mass and Λ is the scale of the lightest non-Goldstone particle, the QCD partition function is dominated by zero momentum modes of the Goldstone fields and hence collapses into a simple group integral [4]. As a result, the partition function only depends on the global symmetries of the QCD Dirac operator and contains just the rescaled quark masses $\mu_f = m_f V\Sigma$ as parameters. In a sector with fixed topological charge ν, it coincides with the corresponding RMT partition function once the space-time volume V is identified with the size n of the corresponding random matrix. Analogously, all correlation functions can also be computed from RMT provided the matrix eigenvalues are appropriately rescaled, $\xi_k = \lambda_k V\Sigma = \lambda_k n\pi\rho(0)$, where $V\to\infty$ or $n\to\infty$ is taken; $\rho(0)$ denotes the RMT spectral density. This provides us with a parameter free prediction.

The joint probability density function of chiral RMT associated with N_f massive quarks in the sector of topological charge ν is defined as

$$P_n^{(N_f,\nu,\beta)}(\lambda_1,\ldots,\lambda_n) = \frac{1}{Z_n^{(N_f,\nu,\beta)}(\{m\})} \qquad (1)$$
$$\times |\Delta_n(\{\lambda\})|^\beta \prod_{i=1}^{n}[w_{\beta,\nu}(\lambda_i)\prod_{f=1}^{N_f} m_f^\nu(\lambda_i+m_f^2)].$$

Here, $\beta = 1, 2$ and 4 labels the symmetry of the matrix ensemble to be orthogonal ($\beta = 1$), unitary ($\beta = 2$) or symplectic ($\beta = 4$) in correspondence with the three χSB patterns [6]. The partition function appearing in the normalization is obtained by integrating over all eigenvalues λ_k. The k-point correlation function is determined by integrating over $n-k$ eigenvalues only:

$$R_{n,k}^{(N_f,\nu,\beta)}(\lambda_1,\ldots,\lambda_k) = \frac{n!}{(n-k)!} \qquad (2)$$
$$\times \int_0^{+\infty} d\lambda_{k+1}\ldots d\lambda_n\; P_n^{(N_f,\nu,\beta)}(\lambda_1,\ldots,\lambda_n).$$

Here, $\Delta_n(\{\lambda\}) = \prod_{i<j}^{n}(\lambda_i-\lambda_j)$ is the Vandermonde determinant and the weight function is given by

$$w_{\beta,\nu}(\lambda) = \lambda^{\frac{\beta}{2}\nu+\frac{\beta}{2}-1}e^{-\beta V(\lambda)}, \qquad (3)$$

where $V(\lambda)$ is a finite-polynomial confinement potential whose form is not fixed apriori. Although the simplest choice $V(\lambda) = \Sigma^2\lambda$ defining the Gaussian ensemble leads to significant mathematical simplifications it cannot be derived from the QCD Lagrangian. It is therefore crucial to show that the RMT results for the rescaled k-point correlation functions, Eq. (6), are universal and do not depend[3] on this choice for $V(\lambda)$.

In the following we present a unified way to explicitly calculate, and prove, the RMT universality of massive spectral correlators for all three χSB patterns, $\beta = 1, 2$ and 4. Our strategy is to express the *massive* spectral correlators in terms of the known *massless* ones; the latter have already been shown to be universal [15]. To proceed, we assume that the massive fermions are β–fold degenerate. With the help of the identity

$$\frac{\Delta_{n+N_f}(\{\lambda\},\{-m^2\})}{\Delta_{N_f}(\{-m^2\})}$$
$$= \Delta_n(\{\lambda\})\prod_{i=1}^{n}\prod_{f=1}^{N_f}(\lambda_i+m_f^2)\ , \qquad (4)$$

the joint probability density $P_n^{(\beta N_f,\nu,\beta)}$ associated with the β–fold degenerate massive fermions of total amount βN_f can be rewritten through the massless joint probability density $P_{n+N_f}^{(0,\nu,\beta)}$ with n positive $\{\lambda_i\}$ and N_f negative $\{-m_f^2\}$ eigenvalues. This leads us to the remarkable identity [11] which holds in all generality for finite n:

$$R_{n,k}^{(\beta N_f,\nu,\beta)}(\lambda_1,\ldots,\lambda_k) \qquad (5)$$

[3]The only condition is that the macroscopic RMT spectral density has to obey $\rho(0)\neq 0$.

$$= \frac{R^{(0,\nu,\beta)}_{n+N_f,k+N_f}(\lambda_1,\ldots,\lambda_k,-m_1^2,\ldots,-m_{N_f}^2)}{R^{(0,\nu,\beta)}_{n+N_f,N_f}(-m_1^2,\ldots,-m_{N_f}^2)}$$

In order to compare with QCD we have to perform the microscopic large-n limit as mentioned above. It reduces to evaluating the rescaled (or unfolded) correlators

$$\rho_S^{(N_f,\nu,\beta)}(\xi_1,\ldots,\xi_k) = \lim_{n\to\infty}\prod_{i=1}^{k}\left(\frac{2|\xi_i|}{n\pi\rho(0)}\right) \times R^{(N_f,\nu,\beta)}_{n,k}\left(\frac{\xi_1^2}{(n\pi\rho(0))^2},\ldots,\frac{\xi_k^2}{(n\pi\rho(0))^2}\right) \quad (6)$$

with a similar rescaling of the masses. Here, we have switched from positive to real Dirac operator eigenvalues. It is easy to see that Eqs. (5) and (6) result in the following expression for the microscopic k-point correlation function with βN_f masses:

$$\rho_S^{(\beta N_f,\nu,\beta)}(\xi_1,\ldots,\xi_k) = \frac{\rho_S^{(0,\nu,\beta)}(\xi_1,\ldots,\xi_k,i\mu_1,\ldots,i\mu_{N_f})}{\rho_S^{(0,\nu,\beta)}(i\mu_1,\ldots,i\mu_{N_f})} \ . \quad (7)$$

Here, $\rho_S^{(0,\nu,\beta)}$ is the *massless* correlation function which is entirely known [16] in terms of determinants ($\beta = 2$) or quaternion determinants ($\beta = 1, 4$). Since the universality of massless correlation functions has already been firmly established, the universality of the massive ones automatically follows. Alternative representations of massive correlation functions were derived in Ref. [12] using Gaussian ensembles. There, the mass degeneracy for $\beta=4$ is partially lifted to be two-fold.

In the simplest situation of the spectral density with β degenerate massive fermions, Eq. (7) reduces to

$$\rho_S^{(\beta,\nu,\beta)}(\xi) = \rho_S^{(0,\nu,\beta)}(\xi) + \frac{\rho_S^{(0,\nu,\beta)}(\xi,i\mu)_{\rm conn}}{\rho_S^{(0,\nu,\beta)}(i\mu)} \ . \quad (8)$$

The full density with β dynamical flavors is thus given by the quenched density $\rho_S^{(0,\nu,\beta)}$ and the mass dependent correction term expressed through the connected part of the massless two-point correlation function $\rho_S^{(0,\nu,\beta)}(\xi,i\mu)_{\rm conn}$.

The application of Eq. (8) to the symmetry class $\beta = 1$ was discussed in Ref. [11] and we will not consider it in what follows. However, it is instructive to consider the simplest example, the symmetry class $\beta = 2$. The connected part of the two-point correlation function is proportional to the square of the unitary kernel

$$K_\alpha(\xi,\eta) = \frac{\xi J_{\alpha+1}(\xi)J_\alpha(\eta) - \eta J_{\alpha+1}(\eta)J_\alpha(\xi)}{2(\xi^2-\eta^2)}. \quad (9)$$

Combining Eqs. (8) and (9), we easily arrive at the result of Ref. [10] for the microscopic density with two degenerate flavors,

$$\rho_S^{(2,\nu,2)}(\xi) = \rho_S^{(0,\nu,2)}(\xi) - 2|\xi|\frac{K_\nu(\xi,i\mu)^2}{K_\nu(i\mu,i\mu)}. \quad (10)$$

Interestingly, the result of Ref. [10] for non-degenerate masses μ_1 and μ_2 can be put into the same form

$$\rho_S^{(2,\nu,2)}(\xi) = \rho_S^{(0,\nu,2)}(\xi) - 2|\xi|\frac{K_\nu(\xi,i\mu_1)K_\nu(\xi,i\mu_2)}{K_\nu(i\mu_1,i\mu_2)}$$

of the quenched density plus a mass-dependent correction.

At $\beta = 4$, the microscopic density for four degenerate massive fermions is given by Eq. (8) with the massless microscopic density [17]

$$\rho_S^{(0,\nu,4)}(\xi) = 2|\xi| \left[2K_{2\nu+1}(2\xi,2\xi) - \frac{J_{2\nu}(2\xi)}{4\xi}\int_0^{2\xi}dt J_{2\nu+2}(t)\right] \ , \quad (11)$$

and the connected part of the massless two-point correlation function [8]

$$\rho_S^{(0,\nu,4)}(\xi,\eta)_{\rm conn} = -f(\xi,\eta)\partial_\xi\partial_\eta f(\xi,\eta) + \partial_\xi f(\xi,\eta)\partial_\eta f(\xi,\eta) \ , \quad (12)$$

where

$$f(\xi,\eta) = \frac{\eta}{2}\int_0^{2\xi}dt K_{2\nu}(t,2\eta) - \frac{\xi}{2}\int_0^{2\eta}dt K_{2\nu}(2\xi,t).$$

Performing the analytic continuation from J-Bessel to I-Bessel functions completes the solution of Eq. (8).

In Fig. 1 the microscopic massive density $\rho_S^{(4,\nu,4)}$ described by Eqs. (8), (11) and (12) is plotted for $\nu = 0$ versus the lattice data of Ref.

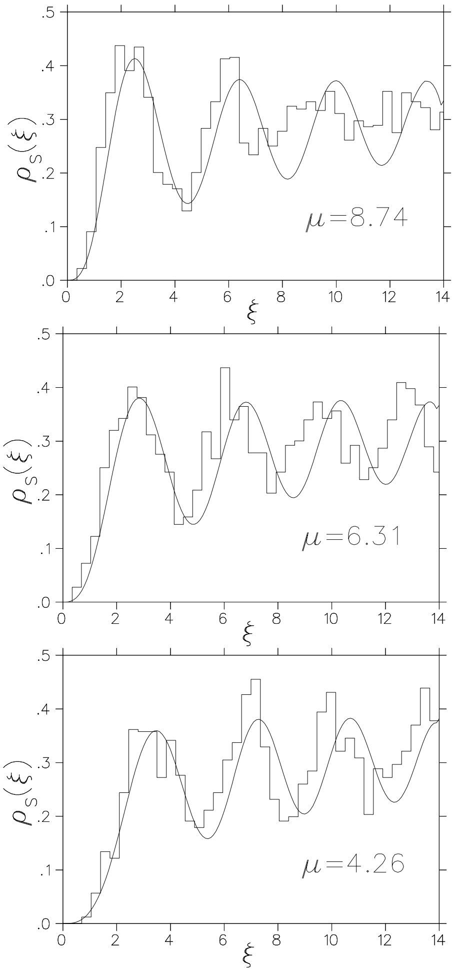

Figure 1. The microscopic density $\rho_S(\xi)$ plotted against lattice data for different values of μ.

[13] with gauge group $SU(2)$ in the fundamental representation. Because of using staggered fermions symmetry class $\beta = 4$ applies. A reasonable agreement between our parameter–free theoretical prediction and the lattice data is observed. The chiral condensate has been obtained from the Banks-Casher relation. A fit to the best value of Σ could improve the systematic shift for higher values of ξ due to finite-size effect and statistics as is discussed in Ref. [13].

REFERENCES

1. J.J.M. Verbaarschot and T. Wettig, Ann. Rev. Nucl. Part. Sci. (2000) to appear, `hep-ph/0003017`.
2. J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. **B540**, 317 (1999); P.H. Damgaard et al., ibid. **B547**, 305 (1999); D. Toublan and J.J.M. Verbaarschot, ibid. **B560**, 259 (1999).
3. P.H. Damgaard, Phys. Lett. **B476** (2000) 465; P.H. Damgaard and K. Splittorff, Phys. Rev. **D62** (2000) 054509; D. Dalmazi and J.J.M. Verbaarschot, `hep-th/0005229`.
4. H. Leutwyler and A. Smilga, Phys. Rev. **D46** (1992) 5607.
5. F. Niedermayer, Nucl. Phys. Proc. Suppl. **73**, (1999) 105 and references therein.
6. J.J.M. Verbaarschot, Phys. Rev. Lett. **72** (1994) 2531.
7. R.G. Edwards et al., Phys. Rev. Lett. **82**, 4188 (1999).
8. J.-Z. Ma, T. Guhr and T. Wettig, Eur. Phys. J. **A2** (1998) 87; Erratum-ibid. 425.
9. J. Jurkiewicz, M.A. Nowak and I. Zahed, Nucl. Phys. **B478** (1996) 605; Erratum: ibid. **B513** (1998) 759.
10. P.H. Damgaard and S.M. Nishigaki, Nucl. Phys. **B518** (1998) 495; T. Wilke, T. Guhr, and T. Wettig, ibid. **D57** (1998) 6486.
11. G. Akemann and E. Kanzieper, Phys. Rev. Lett. **85** (2000) 1174.
12. T. Nagao and S.M. Nishigaki, Phys. Rev. **D62** (2000) 065006; ibid 065007.
13. M.E. Berbenni-Bitsch, S. Meyer and T. Wettig, Phys. Rev. **D58** (1998) 71502.
14. T. Banks and A. Casher, Nucl. Phys. **B169** (1980) 103.
15. G. Akemann et al., Nucl. Phys. **B487** (1997) 721; E. Kanzieper and V. Freilikher, Philos. Magazine **B77** (1998) 1161; M.K. Şener and J.J.M. Verbaarschot, Phys. Rev. Lett. **81** (1998) 248; H. Widom, J. Stat. Phys. **94** (1999) 347; B. Klein and J.J.M. Verbaarschot, `hep-th/0004119`.
16. M.L. Mehta, *Random Matrices* (Academic, San Diego, 1991).
17. T. Nagao and P.J. Forrester, Nucl. Phys. **B435** (1995) 401.