THE GIERER-MEINHARDT SYSTEM ON A COMPACT TWO-DIMENSIONAL RIEMANNIAN MANIFOLD: INTERACTION OF GAUSSIAN CURVATURE AND GREEN'S FUNCTION

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Abstract. In this paper, we rigorously prove the existence and stability of single-peaked patterns for the singularly perturbed Gierer-Meinhardt system on a compact two-dimensional Riemannian manifold without boundary which are far from spatial homogeneity. Throughout the paper we assume that the activator diffusivity ϵ^2 is small enough.

We show that for the threshold ratio $D \sim \frac{1}{\epsilon^2}$ of the activator diffusivity ϵ^2 and the inhibitor diffusivity D, the Gaussian curvature and the Green's function interact.

A convex combination of the Gaussian curvature and the Green's function together with their derivatives are linked to the peak locations and the $o(1)$ eigenvalues. A nonlocal eigenvalue problem (NLEP) determines the $O(1)$ eigenvalues which all have negative part in this case.

Résumé. Dans ce papier, nous rigoureusement étudions le singulièrement préoccupé Système de Gierer-Meinhardt sur une compacte variété de Riemann deux dimensionnelle. Nous prouvez qu'il existe une solution stationnaire avec un pic d'activateur qui sont loin de homogénéité spatiale. Partout dans le papier nous supposons que le diffusivity d'activateur ϵ^2 est assez petit.

Nous le montrons pour le rapport de seuil $D \sim \frac{1}{\epsilon^2}$ pour le diffusivity de l'activateur, ϵ^2, et le diffusivity de l'inhibiteur, D, il y a une action réciproque de la courbure de Gauss et de la fonction de Green.

Une combinaison convexe de la courbure de Gauss et de la fonction de Green avec leurs dérivés est reliée aux position du maximum et le eigenvalues le $o(1)$. Un problème eigenvalue nonlocal (NLEP) détermine le eigenvalues le $O(1)$ que tous ayez la partie négative dans ce cas-là.

(Titre: Le système de Gierer-Meinhardt sur une compacte variété de Riemann deux dimensionnelle: l'Action réciproque de la courbure de Gaussian et de la fonction de Green)

1. Introduction

1.1. **The problem.** We look for nontrivial steady states to the Gierer-Meinhardt system defined on a compact two-dimensional Riemannian manifold $(\mathcal{S}, g)$ without boundary. The equation can be stated as follows ([14, 28]):

$$\begin{cases} A_t = d\triangle_g A - A + \frac{A^2}{H} & \text{in } \mathcal{S}, \\ \tau H_t = D\triangle_g H - H + A^2 & \text{in } \mathcal{S}, \end{cases} \tag{1.1}$$

where $A = A(p,t)$, $H = H(p,t) > 0$ represent the activator and inhibitor concentrations, respectively, at a point $p \in \mathcal{S}$ and at time $t > 0$; their corresponding diffusivities are denoted by d, $D > 0$; $\tau \geq$ is the time-relaxation constant of the inhibitor; $\triangle_g$ denotes the Laplace-Beltrami operator with respect to the metric tensor g.

For convenience, we define ϵ and β by $d = \epsilon^2$ and $D = \frac{1}{\beta^2}$, and we will work with these new parameters throughout the paper.

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We shall consider the weak coupling case (as in [50]), i.e. we consider pairs of parameters (ϵ, β) such that $\epsilon, \beta \to 0$ (hence, $d \to 0$ and $D \to \infty$). More specifically, we will always assume that

$$\epsilon \text{ is small enough.} \tag{1.2}$$

We further assume the asymptotic relation

$$\lim \frac{\beta^2}{\epsilon^2} = \kappa > 0. \tag{1.3}$$

We will see that the relation (1.3) for the diffusion constants is essential for the rest of the paper. In particular, under this assumption we will be able to introduce a function $F(p)$, $p \in \mathcal{S}$, which is a convex combination of the Gaussian curvature and the Green's function and will be crucial in deriving results on existence and stability. Here κ indicates the relative strength in the coupling of the Gaussian curvature and the Green's function.

1.2. **Motivation.** This Gierer-Meinhardt system (1.1) is used to model morphogenesis.

Morphogenesis is the development of an organism from a single cell. This complex process can be understood by dividing it into several elementary steps, such as the change of cell shapes, cell to cell interaction, growth, and cell movement. One of the most important of these steps is the formation of a spatial pattern of cell structure, starting from an almost homogeneous cell distribution.

Turing in his pioneering work in 1952 [40] proposed that a patterned distribution of two chemical substances, called the morphogens, could trigger the emergence of such a cell structure. He also gives the following explanation for the formation of the morphogenetic pattern: It is assumed that one of the morphogens, in this case the activator, diffuses slowly and the other, in this case the inhibitor, diffuses much faster. In the mathematical framework of a coupled system of reaction-diffusion equations with hugely different diffusion coefficients he shows by linear stability analysis that the homogeneous state may possess instabilities. In particular, a small perturbation of spatially homogeneous initial data may evolve to a stable spatially complex pattern of the morphogens.

Since the work of Turing, lots of models have been proposed and analyzed to explore this phenomenon, which is now called Turing instability, and its implications for the understanding of various patterns more fully. One of the most famous of these models is the Gierer-Meinhardt system ([14, 28]).

In domains with zero curvature (i.e. domains in $\mathbb{R}^n$, in particular for space dimensions $n = 1, 2$), there are various results for this system some of which are given at the end of this introduction. However, there are few results, if any, that deal with a curved manifold, and perhaps the biologically most interesting domain is the two-dimensional Riemannian manifold. This may correspond to any membrane structure, e.g. cell, in which the Gierer-Meinhardt system correctly models the biological phenomena observed.

In previous works on two-dimensional flat domains, various authors showed that as $\epsilon \to 0$ there are multi-peak patterns which exhibit a **"point condensation phenomenon"**. By this we mean that the peaks become narrower and narrower and eventually shrink to the set of points itself. In fact, their spatial extent is of order $O(\epsilon)$. We also say that the spike solutions "concentrate" at the set of points. Furthermore, we remark that the maximum values of activator and inhibitor both diverge to $+\infty$.

In this paper we consider a single-spike solution on a Riemannian manifold. We explicitly give a rigorous construction of single-peaked stationary states by using the powerful method of Lyapunov-Schmidt reduction. Locally, in a normal neighborhood of a point, this enables us to reduce the infinite-dimensional problem of finding an equilibrium state to (1.1) to the finite-dimensional problem of locating the point at which the spike concentrates.

We will give criteria for existence and stability explicitly in terms of a function on the manifold defined as a convex combination of the Gaussian curvature function and the Green's function. In [50], it was found that the Green's function plays such a role. However, in our case, the Green's function is replaced by the convex combination of the Gaussian curvature and the Green's function which indicates that they interact in an essential way.

We will rigorously answer the following questions: How can we construct these spiky solutions? Where is the peak located? When are these solutions stable?

We give a sufficient condition for the location of this point in terms of a non-degenerate critical point of the gradient of the convex combination of Gaussian curvature and Green's function.

Concerning stability we study the eigenvalues of the order $O(1)$ (called "large eigenvalues") and of the order $o(1)$ (called "small eigenvalues") separately. We show that the small eigenvalues are linked to the spike locations by the Hessian of this convex combination of Gaussian curvature and Green's function. If the real parts of its eigenvalues are both negative, the spiky steady state for the Gierer-Meinhardt system (1.1) is linearly stable.

1.3. **The geometric setting.** Before describing the main results of this paper in detail we introduce some notations. Let $\mathcal{S}$ be a compact two-dimensional Riemannian manifold without boundary. Let $T_p\mathcal{S}$ be the tangent plane to $\mathcal{S}$ at p, and given an orthonormal basis $\{e_1(p), e_2(p)\}$ of $T_p\mathcal{S}$, we can obtain, via the exponential map $\exp_p : T_p\mathcal{S} \to \mathcal{S}$, a natural correspondence $x_1e_1(p) + x_2e_2(p) \mapsto q = \exp_p(x_1e_1(p) + x_2e_2(p))$.

To give an explicit chart, let us denote by $E_p : \mathbb{R}^2 \to T_p\mathcal{S}$ the map $E_p(x_1, x_2) = x_1e_1(p) + x_2e_2(p)$. Then there is a maximal $\delta_p > 0$ such that

$$E_p^{-1} \circ \exp_p^{-1} : B_g(p, \delta_p) \to B(0, \delta_p) \subset \mathbb{R}^2$$

is a diffeomorphism. Moreover, since $\mathcal{S}$ is compact, we actually have an injectivity radius $i_g > 0$ so that

$$X_p := E_p^{-1} \circ \exp_p^{-1} \; : \; B_g(p, i_g) \to B(0, i_g) \tag{1.4}$$

is a diffeomorphism for every $p \in \mathcal{S}$. The values of this natural chart X_p are called (geodesic) normal coordinates about p.

We assume that the exponential map is smooth (C^∞). Moreover, since the tangent bundle $T\mathcal{S}$ has a natural differentiable structure, we may choose the basis $\{e_1(p), e_2(p)\}$ of $T_p\mathcal{S}$ to be smooth. Thus any smooth function f defined on $\mathcal{S}$ by means of the normal coordinates varies smoothly with p as well as the coordinates (x_1, x_2).

We define cut-off functions as follows: let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function which is equal to 1 for $|y| < 0.5$ and equal to 0 for $|y| > 0.75$. For $p \in \mathcal{S}$ we introduce

$$\chi_{\delta_0,p}(q) = \chi\left(\frac{d_g(p,q)}{\delta_0}\right), \qquad q \in \mathcal{S}, \tag{1.5}$$

and we choose $\delta_0 = i_g$. We set $\chi_{\delta_0}(x) = \chi(x/\delta_0)$ for $x \in R^2$.

We denote the geodesic gradient of f by $\nabla_g f$. Written in normal coordinates, the partial derivatives of f with respect to (x_1, x_2) are denoted by ∇f. We will frequently consider rescaled normal coordinates $y = x/\epsilon$.

We now introduce function spaces. We define

$$L^2(\mathcal{S}) = \{u \text{ measurable function defined on } \mathcal{S} \text{ s.t. } \int_{\mathcal{S}} u^2(p) dv_g(p) < \infty\},$$

where dv_g denotes the Riemannian measure with respect to the metric g. We further set

$$H^1(\mathcal{S}) = \{u \in L^2(\mathcal{S}) \,:\, \nabla_g u \in L^2(\mathcal{S})\}.$$

We use analogous definitions for other Sobolev spaces.

Let $H^1_\epsilon(\mathcal{S})$ be the Sobolev space $H^1(\mathcal{S})$ equipped with the inner product

$$< u, v >_{H^1_\epsilon(\mathcal{S})} = \frac{1}{\epsilon^2}\left(\epsilon^2 \int_{\mathcal{S}} \nabla_g u \cdot \nabla_g v \, dv_g + \int_{\mathcal{S}} uv \, dv_g\right).$$

This induces the norm

$$\|u\|^2_{H^1_\epsilon(\mathcal{S})} = \frac{1}{\epsilon^2}\left(\epsilon^2 \int_{\mathcal{S}} \nabla_g u \cdot \nabla_g v \, dv_g + \int_{\mathcal{S}} uv \, dv_g\right).$$

In the same way we define $L^2_\epsilon(\mathcal{S})$ and $H^2_\epsilon(\mathcal{S})$ and other Sobolev spaces.

Now we introduce a Green's function G_0 which we need to formulate our main results. We set $G_0 \,:\, \mathcal{S} \times \mathcal{S} \backslash \{(p, q) \in \mathcal{S} \times \mathcal{S} :\; p = q\} \to \mathbb{R}$ uniquely defined by

$$\begin{cases} \triangle_g G_0(p, q) - \frac{1}{|\mathcal{S}|} + \delta_p(q) = 0 \quad \text{in } \mathcal{S}, \\ \int_{\mathcal{S}} G_0(p, q) dv_g(q) = 0. \end{cases} \tag{1.6}$$

(For basic properties and a constructive proof of its existence, see [2]).

Next, we denote by

$$\frac{1}{2\pi} \log \frac{1}{d_g(p, q)} \chi_{\delta_0, p}(q) \quad \text{and} \quad R_0(p, q) := \frac{1}{2\pi} \log \frac{1}{d_g(p, q)} \chi_{\delta_0, p}(q) - G_0(p, q) \tag{1.7}$$

the singular and regular parts of G_0, respectively, where $d_g(p, q)$ is the geodesic distance between $p \in \mathcal{S}$ and $q \in \mathcal{S}$. We set

$$R(p) = R_0(p, p). \tag{1.8}$$

Note that $R_0 \in C^\infty(\mathcal{S} \times \mathcal{S})$ and $R \in C^\infty(\mathcal{S})$.

Now we proceed to define a function on the manifold that is essential for our existence and stability results. Let $F : \mathcal{S} \to \mathbb{R}$ be the function defined by

$$F(p) := c_1 K(p) + c_2 R(p), \tag{1.9}$$

where $K(p)$ denotes the Gauss curvature on $\mathcal{S}$, $R(p)$ denotes the diagonal of the regular part of the Green's function defined in (1.8),

$$c_1 = \frac{\pi}{4} \int_0^\infty (w')^2 r^3 \, dr, \quad c_2 = \frac{|\mathcal{S}|\pi}{2} \frac{\beta^2}{\epsilon^2} \int_0^\infty w^2 \, r \, dr, \quad w' = \frac{\partial w}{\partial r}$$

and w is the unique solution of the problem

$$\begin{cases} \Delta w - w + w^2 = 0, \quad w > 0 \quad \text{in } \mathbb{R}^2, \\ w(0) = \max_{y \in \mathbb{R}^2} w(y), \quad w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{1.10}$$

For existence and uniqueness of the solutions of (1.10) we refer to [15, 26]. We also recall that

$$w(y) \sim |y|^{-1/2} e^{-|y|} \quad \text{as } |y| \to \infty. \tag{1.11}$$

Note that $F(p) \in C^\infty(\mathcal{S})$.

Let us write

$$\mathcal{M}(p) = \left(\nabla^2 F(p)\right), \tag{1.12}$$

where $\nabla^2 F$ is the Hessian of the function F on $\mathcal{S}$ with respect to normal coordinates, so that $\mathcal{M}(p)$ is a 2×2 matrix with components $\frac{\partial^2 F}{\partial x_j \partial x_k}(p)$, $j, k = 1, 2$.

Likewise, the derivatives of the Green's function in normal coordinates are denoted by

$$\nabla_x R_0(p,q) \quad \text{derivative of the first component,}$$

$$\nabla_z R_0(p,q) \quad \text{derivative of the second component.}$$

Using the relation $R(p) = R_0(p,p)$, we have

$$\nabla R(p) = (\nabla_x + \nabla_z) R_0(p,p),$$

$$\nabla^2 R(p) = (\nabla_x^2 + 2\nabla_x \nabla_z + \nabla_z^2) R_0(p,p)$$

$$= 2(\nabla_x^2 + \nabla_x \nabla_z) R_0(p,p)$$

since $R_0(p,q)$ is symmetric in its arguments p, q.

Remark. $\mathcal{M}(p)$ will be evaluated using a normal coordinate system, but the eigenvalues of $\mathcal{M}(p)$ (and hence its negative-definiteness which we will assume) will be independent of the choice of coordinates. Moreover, the entries of $\mathcal{M}(p)$ vary differentiably with p because the basis of the tangent plane $T_p\mathcal{S}$, namely $\{e_1(p), e_2(p)\}$, is chosen to vary differentiably with p.

1.4. **The main results.** The stationary system for (1.1) is the following system of elliptic equations:

$$\begin{cases} \epsilon^2 \Delta_g A - A + \frac{A^2}{H} = 0, \quad A > 0 \quad \text{in } \mathcal{S}, \\ \frac{1}{\beta^2} \Delta H - H + A^2 = 0, \quad H > 0 \quad \text{in } \mathcal{S}. \end{cases} \tag{1.13}$$

Our first theorem concerns the existence of single-peaked solutions whose position is determined by an interaction of the local geometry and the Green's function.

Theorem 1.1. *Let $p^0 \in \mathcal{S}$ be a non-degenerate critical point of $F(p)$ (defined in (1.9)), i.e.*

$$\nabla F(p^0) = 0, \quad det(\nabla^2 F(p^0)) \neq 0. \tag{1.14}$$

Then, under the assumptions (1.2) and (1.3), problem (1.1) has a positive spiky steady state (A_ϵ, H_ϵ) with the following properties:

(1) $A_\epsilon(x) = \xi_\epsilon (w(\frac{x-p^\epsilon}{\epsilon}) + O(\epsilon^2))$ uniformly for $x \in \mathcal{S}$, where w is the unique solution of (1.10) and

$$\xi_\epsilon = \frac{|\mathcal{S}|}{\epsilon^2 \int_{\mathbb{R}^2} w^2(y)\,dy} + O\left(\log \frac{1}{\epsilon}\right). \tag{1.15}$$

Furthermore, $p^\epsilon \to p^0$ as $\epsilon \to 0$.

(2) $H_\epsilon(x) = \xi_\epsilon (1 + O(\epsilon^2))$ uniformly for $x \in \mathcal{S}$.

Next we study the stability and instability of the K-peaked solutions constructed in Theorem 1.1. To this end, we need to study the following eigenvalue problem

$$\mathcal{L}_\epsilon \begin{pmatrix} \phi_\epsilon \\ \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} \epsilon^2 \Delta_g \phi_\epsilon - \phi_\epsilon + 2\frac{A_\epsilon}{H_\epsilon}\phi_\epsilon - \frac{A_\epsilon^2}{H_\epsilon^2}\psi_\epsilon \\ \frac{1}{\tau}(\frac{1}{\beta^2}\Delta_g \psi_\epsilon - \psi_\epsilon + 2A_\epsilon \phi_\epsilon) \end{pmatrix} = \lambda_\epsilon \begin{pmatrix} \phi_\epsilon \\ \psi_\epsilon \end{pmatrix}, \qquad (1.16)$$

where (A_ϵ, H_ϵ) is the solution constructed Theorem 1.1 and $\lambda_\epsilon \in \mathbb{C}$, the set of complex numbers.

We say that (A_ϵ, H_ϵ) is **linearly stable** if the spectrum $\sigma(\mathcal{L}_\epsilon)$ of $\mathcal{L}_\epsilon$ lies in the left half plane $\{\lambda \in \mathbb{C}: \ \text{Re}\ (\lambda) < 0\}$. On the other hand, (A_ϵ, H_ϵ) is called **linearly unstable** if there exists an eigenvalue λ_ϵ of $\mathcal{L}_\epsilon$ with $\text{Re}(\lambda_\epsilon) > 0$. (From now on, we use the notations linearly stable and linearly unstable in this sense.)

Our second main result, which is on stability, is stated as follows.

Theorem 1.2. *Let p^0 is a non-degenerate local maximum point of $F(p)$ (defined by (1.9)), i.e.*

$$(*) \quad \nabla F(p^0) = 0, \quad \nabla^2 F(p^0) \quad \textit{is negative definite.} \qquad (1.17)$$

Under the assumptions (1.2) and (1.3), let (A_ϵ, H_ϵ) be the single-peaked solution constructed in Theorem 1.1 whose peak approaches p^0.

Then there exists a unique $\tau_1 > 0$ such that for $\tau < \tau_1$, (A_ϵ, H_ϵ) is linearly stable, while for $\tau > \tau_1$, (A_ϵ, H_ϵ) is linearly unstable.

Remark. The condition (*) on the locations p^0 arises in the study of small ($o(1)$) eigenvalues. For any compact two-dimensional Riemannian manifold without boundary, the functional $F(p)$, defined by (1.9), always admits a global maximum at some $p^0 \in \mathcal{S}$ since it is a continuous function defined on a compact set. We believe that for **generic** manifolds, this global maximum point p^0 is non-degenerate.

We believe that for other types of critical points of $F(p)$, such as saddle points, the solution constructed in Theorem 1.1 should be linearly unstable. We are not able to prove this at the moment, since the operator $\mathcal{L}_\epsilon$ is **not self-adjoint.** The difficulty is in controlling the small eigenvalues of the linearization.

We now comment on some related work.

Generally speaking, system (1.13) is difficult to solve since it does neither have a **variational structure** nor **a priori estimates**. One way to study (1.13) is to examine the so-called **shadow system**. Namely, we let $D \to +\infty$ first. It is known (see [21, 30, 37]) that the study of the shadow system amounts to the study of the following single equation for $p = 2$:

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0, \quad u > 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \qquad (1.18)$$

Equation (1.18) has a variational structure and has been studied by numerous authors. It is known that equation (1.18) has both boundary spike solutions and interior spike solutions. For existence of boundary spike solutions, see [16, 31, 32, 33, 46, 47] and the references therein. For existence of interior spike solutions, see [17, 35] and the references therein. For stability of spike solutions see [34, 44, 45].

Next we review some results for bumps, spikes and related patterns in the Gierer-Meinhardt system. Ground states on the real line are studied in [8, 10, 11, 54] and for the whole $\mathbb{R}^2$ in [9]. Multiple spikes for an interval are studied in [18, 19, 25, 39, 43] and for bounded two-dimensional domains in [23, 24, 33, 48, 49, 50, 51, 52]. Hopf bifurcation of spikes is investigated in [6, 41, 42]. For dynamics we refer to [4, 5, 12, 20, 38]. Steady states with spherical layers have been constructed in [25, 36]. Stripes have been studied in [22]. Nonlocal eigenvalue problems related to the one in this paper have been studied in [44, 45, 53].

The existence of spikes for single semilinear elliptic PDEs on manifolds has been investigated in [3, 7, 29].

The structure of the paper is as follows:

Section 2: Preliminaries
- 2.1 Two Eigenvalue Problems
- 2.2 Calculating the Height of the Peak

Section 3: Existence – Proof of Theorem 1.1

Section 4: Stability – Proof of Theorem 1.2
- 4.1 Study of Large Eigenvalues
- 4.2 Further Improvement of Solutions
- 4.3 Study of Small Eigenvalues

Appendix A: Expansion of the Laplace-Beltrami Operator

Appendix B: Some Technical Calculations

Throughout the paper $C > 0$ is a generic constant which is independent of ϵ and β and may change from line. We always assume that $p \in \overline{\Lambda_\delta}$, where

$$\Lambda_\delta = \mathcal{S} \cap B_g(p^0, \delta) \tag{1.19}$$

and $\delta = \epsilon^\alpha$ for some $0 < \alpha < 1$. To simplify our notation, we use $e.s.t.$ to denote exponentially small terms in the corresponding norms, more precisely, $e.s.t. = O(e^{-c/\epsilon})$ for some $c > 0$. The notation $A(\epsilon) \sim B(\epsilon)$ means that $\lim_{\epsilon \to 0} \frac{A(\epsilon)}{B(\epsilon)} = c_0 > 0$, for some positive number c_0.

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2. Preliminaries

2.1. **Two eigenvalue problems.** Let w be the unique solution of (1.10). In this subsection, we study two eigenvalue problems.

Let

$$L_0\phi = \Delta\phi - \phi + 2w\phi, \quad \phi \in H^2(\mathbb{R}^2). \tag{2.1}$$

We first recall the following well-known result:

Lemma 2.1. *The eigenvalue problem*

$$L_0\phi = \mu\phi, \quad \phi \in H^2(\mathbb{R}^2), \tag{2.2}$$

admits the following set of eigenvalues

$$\mu_1 > 0, \quad \mu_2 = \mu_3 = 0, \quad \mu_4 < 0, \dots . \tag{2.3}$$

The eigenfunction Φ_0 corresponding to μ_1 can be made positive and radially symmetric; the space of eigenfunctions corresponding to the eigenvalue 0 *is*

$$K_0 := span \left\{ \frac{\partial w}{\partial y_j}, j = 1, 2 \right\}. \tag{2.4}$$

Proof: This lemma follows from Theorem 2.1 of [27] and Lemma C of [32]. □

Next, we consider the following nonlocal eigenvalue problem

$$L\phi := \Delta\phi - \phi + 2w\phi - \gamma \frac{\int_{\mathbb{R}^2} w\phi}{\int_{\mathbb{R}^2} w^2} w^2 = \lambda_0 \phi, \;\; \phi \in H^2(\mathbb{R}^2), \tag{2.5}$$

where $\gamma = \dfrac{\mu}{1+\tau\lambda_0}$ and $\mu > 0$, $\tau \geq 0$.

Problem (2.5) plays the key role in the study of large eigenvalues (Subsection 4.1 below).

We have the following result:

Lemma 2.2. *Let* $\gamma = \frac{\mu}{1+\tau\lambda_0}$ *where* $\mu > 0, \tau \geq 0$ *and let* L *be defined by (2.5).*

(1) Suppose that $\mu > 1$. *Then there exists a unique* $\tau = \tau_1 > 0$ *such that for* $\tau > \tau_1$ *(2.5) admits an eigenvalue with* $Re(\lambda) > 0$. *Further, for* $\tau < \tau_1$, *all nonzero eigenvalues of problem (2.5) satisfy* $Re(\lambda) < 0$. *At* $\tau = \tau_1$, L *has a Hopf bifurcation.*

(2) Suppose that $\mu < 1$. *Then* L *admits an eigenvalue* λ_0 *with* $Re(\lambda_0) > 0$.

Proof: Lemma 2.2 has been proved as Theorem 2.2 in [50].

□

2.2. **Calculating the height of the peak.** In this subsection, we formally calculate the height of the peak as needed in the sections below. In particular, we introduce the scale $\xi_{\epsilon,p}$ given in (2.17). For the asymptotic regime $\epsilon \to 0$ and $\beta \to 0$, it is found that the height does not depend on the spike location in leading order, but only in higher order.

For $\beta > 0$, let $G_\beta(p,q)$ be the Green's function given by

$$\Delta_g G_\beta(p,q) - \beta^2 G_\beta(p,q) + \delta_q = 0 \quad \text{in } \mathcal{S}. \tag{2.6}$$

From (2.6) we get

$$\int_{\mathcal{S}} G_\beta(p,q)\, dv_g(p) = \beta^{-2}.$$

Set

$$G_\beta(p,q) = \frac{\beta^{-2}}{|\mathcal{S}|} + \overline{G}_\beta(p,q). \tag{2.7}$$

Then

$$\begin{cases} \Delta_g \bar{G}_\beta(p,q) - \beta^2 \bar{G}_\beta(p,q) - \frac{1}{|\mathcal{S}|} + \delta_q = 0 \quad \text{in } \mathcal{S}, \\ \displaystyle\int_{\mathcal{S}} \bar{G}_\beta(p,q)\, dv_g(p) = 0. \end{cases} \tag{2.8}$$

Let $G_0(p,q)$ be the Green's function given by (1.6). Let $G_{0,1}$ be defined by

$$\Delta_g G_{0,1}(p,q) - G_{0,1}(p,q) = 0, \quad \int_{\mathcal{S}} G_{0,1}(p,q)\, dv_g(p) = 0. \tag{2.9}$$

Note that

$$\begin{aligned} G_{0,1}(p,q) &= \int_{\mathcal{S}} G_0(p,r) G_0(r,q) dv_g(r) \\ &= \frac{1}{8\pi} d_g(p,q)^2 \log \frac{1}{d_g(p,q)} + O(d_g(p,q)^2). \end{aligned}$$

Next we rewrite the Green's functions in terms of geodesic normal coordinates. Let us define explicitly

$$G_{0,p}(x,z) := G_0(q,r), \qquad \text{where } x = X_p(q) \in B(0,\delta_0),\ z = X_p(r) \in B(0,\delta_0). \tag{2.10}$$

In the same way, we define $R_{0,p}$, $G_{0,1,p}$ and $G_{\beta,p}$.

The equations (1.6), (2.8) and (2.9) imply that

$$\begin{aligned}\bar{G}_{\beta,p}(x,z) &= G_{0,p}(x,z) + \beta^2 G_{0,1,p}(x,z) + O(\beta^4)\\ &= G_{0,p}(x,z) + O\left(\beta^2|x-z|^2 \log\frac{1}{|x-z|} + \beta^4\right)\end{aligned}$$

in the operator norm of $L^2(\mathcal{S}) \to H^2(\mathcal{S})$. (Note that the embedding of $H^2(\mathcal{S})$ into $L^\infty(\mathcal{S})$ is compact.) Hence

$$G_{\beta,p}(x,z) = \frac{\beta^{-2}}{|\mathcal{S}|} + G_{0,p}(x,z) + O\left(\beta^2|x-z|^2\log\frac{1}{|x-z|} + \beta^4\right) \tag{2.11}$$

in the operator norm of $L^2(\mathcal{S}) \to H^2(\mathcal{S})$.

Now we introduce $w_0 \in H^2(\mathbb{R}^2)$ to be the unique rotationally symmetric solution of the equation

$$\begin{aligned}&\Delta w_0 - w_0 - \frac{1}{3}K(p)\epsilon^2 r w_0'\\ &+\frac{w_0^2}{1+\frac{\epsilon^2\xi_{\epsilon,p}\beta^2}{2\pi}\int \log\frac{|z|}{|y-z|}w_0^2(z)\,dz} = 0, \quad y \in \mathbb{R}^2,\end{aligned} \tag{2.12}$$

where $K(p)$ is the Gaussian curvature at $p \in \mathcal{S}$.

Existence and uniqueness of w_0 can be derived as follows:

Note that the operator

$$L_0 : H_r^2(\mathbb{R}^2) \to L_r^2(\mathbb{R}^2), \quad L_0\phi := \Delta\phi - \phi + 2w\phi,$$

where $H_r^2(\mathbb{R}^2)$ and $L_r^2(\mathbb{R}^2)$ are the spaces of radially symmetric functions in $H^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$, respectively, is invertible with a bounded inverse. Therefore it follows by the implicit function theorem, applied at $\epsilon = 0$, that (2.12) at has a unique rotationally symmetric solution w_0 if ϵ is small enough. Further, the implicit function theorem implies that $\|w_0 - w\|_{H^2(\mathbb{R}^2)} = O(\epsilon^2)$.

Let us assume that a single spike solution (A_ϵ, H_ϵ) of (1.13) in leading order satisfies (this statement will be proved rigorously):

$$\begin{cases} A_{\epsilon,p}(q) \sim \xi_{\epsilon,p} w_0\left(X_p(q)/\epsilon\right)\chi_{\delta_0,p}(q),\\ H_{\epsilon,p}(p) = \xi_{\epsilon,p},\end{cases} \tag{2.13}$$

where w is the unique solution of (1.10), $\xi_{\epsilon,p}$ is the height of the peak and $p \in \overline{\Lambda_\delta}$ is the location of the peak, where the latter two are to be determined later.

Then from the equation for H_ϵ,

$$\Delta_g H_\epsilon - \beta^2 H_\epsilon + \beta^2 A_\epsilon^2 = 0,$$

we get, using (2.11) and (2.13),

$$\begin{aligned}\xi_{\epsilon,p} &= \int_{\mathcal{S}} G_\beta(p,q)\beta^2\xi_{\epsilon,p}^2\left(w_0\left(X_p(q)/\epsilon\right)\chi_{\delta_0,p}(q)\right)^2\,dv_g(q)\\ &= \int_{\mathcal{S}}\left(\frac{\beta^{-2}}{|\mathcal{S}|} + G_0(p,q) + O\left(\epsilon^2 + \beta^2\epsilon^4\log\frac{1}{\epsilon} + \beta^4\right)\right)\\ &\qquad \beta^2\xi_{\epsilon,p}^2\left(w_0\left(\frac{X_p(q)}{\epsilon}\right)\right)^2\,dv_g(q)\end{aligned}$$

$$= \int_{\mathbb{R}^2} \left(\frac{\epsilon^2}{|\mathcal{S}|} + \beta^2 \epsilon^2 G_{0,p}(0, \epsilon z) + O\left(\beta^2 \epsilon^4 + \beta^4 \epsilon^4 \log \frac{1}{\epsilon} + \beta^6 \epsilon^2 \right) \right)$$
$$\xi_{\epsilon,p}^2 \left(w_0(z)\right)^2 \sqrt{|g|(\epsilon z)}\, dz$$
$$= \int_{\mathbb{R}^2} \left(\frac{\epsilon^2}{|\mathcal{S}|} + \beta^2 \epsilon^2 G_{0,p}(0, \epsilon z) + O\left(\epsilon^6 \log \frac{1}{\epsilon} \right) \right)$$
$$\left(1 - \frac{1}{6} K(p) |z|^2 \epsilon^2 - \frac{1}{12} (\nabla K(p) \cdot z) |z|^2 \epsilon^3 - \frac{1}{40} (z^t \nabla^2 K(p) z) |z|^2 \epsilon^4 + \frac{1}{120} K(p)^2 |z|^4 \epsilon^4 + O(\epsilon^5) \right)$$
$$\xi_{\epsilon,p}^2 \left(w_0(z)\right)^2 \, dz.$$

Thus

$$\frac{1}{\epsilon^2 \xi_{\epsilon,p}} = \left(\frac{1}{|\mathcal{S}|} + \frac{\beta^2}{2\pi} \log \frac{1}{\epsilon} - \beta^2 R_0(p,p) \right) \left(\int_{\mathbb{R}^2} w_0^2(z)\, dz - \frac{\epsilon^2 K(p)}{6} \int_{\mathbb{R}^2} |z|^2 w_0^2(z)\, dz \right)$$
$$+ \frac{\beta^2}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|z|} w_0^2(z)\, dz + O(\epsilon^4). \tag{2.14}$$

From (2.14) we get an expansion of $\xi_{\epsilon,p}$, where $\xi_{\epsilon,p}$ depends on p not in leading order but only in higher order ϵ^2.

Define

$$\xi_{\epsilon,p} = \frac{\hat{\xi}_{\epsilon,p} |\mathcal{S}|}{\epsilon^2 \int_{\mathbb{R}^2} w_0^2 \, dy}. \tag{2.15}$$

Then from (2.14) we get

$$\hat{\xi}_{\epsilon,p} = 1 + O\left(\epsilon^2 \log \frac{1}{\epsilon} \right), \tag{2.16}$$

which is clearly equivalent to

$$\xi_{\epsilon,p} = \frac{|\mathcal{S}|}{\epsilon^2 \int_{\mathbb{R}^2} w^2(y)\, dy} \left(1 + O\left(\epsilon^2 \log \frac{1}{\epsilon} \right) \right). \tag{2.17}$$

In this subsection, we have calculated the height of the peak under the assumption that its shape is given. In the next section, we provide a rigorous proof for the existence of equilibrium states.

3. Existence

3.1. **Reduction to finite dimensions.** Let us start to prove Theorem 1.1.

In this subsection, we use the Liapunov-Schmidt process to reduce the PDE problem to a finite dimensional problem. In the next subsection, we will solve this reduced problem. Such a procedure has been used in the study of the Gierer-Meinhardt system for Neumann problems in bounded two-dimensional subdomains of $\mathbb{R}^2$ [48, 49, 50].

We rescale the amplitudes

$$a(p) = \frac{1}{\xi_{\epsilon,p}} A(p), \quad p \in \mathcal{S},$$
$$h(p) = \frac{1}{\xi_{\epsilon,p}} H(p), \quad p \in \mathcal{S},$$

where $\xi_{\epsilon,p}$ is given in (2.17).

Then an equilibrium solution (a, h) has to solve the following rescaled Gierer-Meinhardt system:

$$\begin{cases} \epsilon^2 \Delta_g a - a + \frac{a^2}{h} = 0, \quad a > 0 \text{ in } \mathcal{S}, \\ \Delta_g h - \beta^2 h + \beta^2 \xi_{\epsilon,p} a^2 = 0, \quad h > 0 \text{ in } \mathcal{S}. \end{cases} \tag{3.1}$$

(This rescaling is introduced to achieve $a = O(1)$, $h = O(1)$ for the amplitudes.)

For any function $u \in H^2(\mathcal{S})$, let $T_\beta[u]$ denote the unique solution to the second equation of (3.1):

$$\triangle_g h - \beta^2 h + \beta^2 \xi_\epsilon u = 0 \quad \text{in } \mathcal{S}.$$

Note that $T_\beta : L^2(\mathcal{S}) \to H^2(\mathcal{S})$ is a linear operator and using (2.6), we can write down the solution by the formula

$$T_\beta[u](q) = \beta^2 \xi_\epsilon \int_{\mathcal{S}} G_\beta(q,r) u(r) dv_g(r) \tag{3.2}$$

Therefore, to solve the rescaled system (3.1), it suffices to find a zero of the operator $S_\epsilon : H^2(\mathcal{S}) \to L^2(\mathcal{S})$ defined by

$$S_\epsilon[u] \ := \ \epsilon^2 \triangle_g u - u + \frac{u^2}{T_\beta[u^2]}. \tag{3.3}$$

Let us now define our approximate solution to (3.3) to be

$$a_{\epsilon,p}(q) := w_0(X_p(q)/\epsilon)\chi_{\delta_0,p}(q) \quad \text{for } q \in \mathcal{S} \tag{3.4}$$

and set $h_{\epsilon,p} = T_\beta[a_{\epsilon,p}^2]$. Recall that w_0 has been defined in (2.12).

We now derive some key estimates for the existence proof. By (2.13), we already know $h_{\epsilon,p}(p) = 1$, but we would also like to estimate $h_{\epsilon,p}(q)$ for $q \in B_g(p, \delta_0)$. To this end, we calculate via the Green's function G_β defined in (2.6) and its expansion up to $O(\beta^2)$ given in (2.11),

$$\begin{aligned}
h_{\epsilon,p}(q) &= h_{\epsilon,p}(p) + h_{\epsilon,p}(q) - h_{\epsilon,p}(p) \\
&= 1 + \beta^2 \xi_\epsilon \int_{\mathcal{S}} \left(G_\beta(q,r) - G_\beta(p,r)\right) a_{\epsilon,p}^2(r) dv_g(r) \\
&= 1 + \beta^2 \xi_\epsilon \int_{B_g(p,\delta_0)} \left(G_0(q,r) - G_0(p,r)\right) w_0^2(X_p(q)/\epsilon) dv_g(r) + O(\beta^4) \\
&= 1 + \epsilon^2 \beta^2 \xi_\epsilon \int_{B(0,\delta_0/\epsilon)} \left(G_{0,p}(\epsilon y, \epsilon z) - G_{0,p}(0, \epsilon z)\right) w_0^2(z) \sqrt{|g|(\epsilon z)} dz + O(\beta^4) \\
&= 1 + \epsilon^2 \beta^2 \xi_\epsilon \int_{B(0,\delta_0/\epsilon)} \left(\frac{1}{2\pi} \log \frac{|z|}{|y-z|} + R_{0,p}(\epsilon y, \epsilon z) - R_{0,p}(0, \epsilon z) \right) w_0^2(z) dz + O(\epsilon^4) \\
&= 1 + \beta^2 \frac{|\mathcal{S}|}{2\pi \int w_0^2} \int_{\mathbb{R}^2} \log \frac{|z|}{|y-z|} w_0^2(z) dz + \epsilon \beta^2 |\mathcal{S}| \, y \cdot \nabla_x R_0(p,p) + O(\epsilon^4) \\
&= 1 + \beta^2 h_0(y) + \epsilon \beta^2 |\mathcal{S}| \, y \cdot \nabla_x R_0(p,p) + O(\epsilon^4),
\end{aligned}$$

changing variables by $y = X_p(q)/\epsilon$, $z = X_p(r)/\epsilon$ and using the estimate of the volume element (5.2) to obtain the last expression, where

$$h_0(y) = \frac{|\mathcal{S}|}{2\pi \int w_0^2} \int_{\mathbb{R}^2} \log \left| \frac{z}{y-z} \right| w_0^2(z) \, dz. \tag{3.5}$$

Thus we have the following estimate:

Lemma 3.1. *Let p be fixed. Then for $q \in B_g(p, \delta_0)$, we have the expansion*

$$h_{\epsilon,p}(q) = 1 + \beta^2 h_0(X_p(q)/\epsilon) + \epsilon \beta^2 \frac{|\mathcal{S}|}{2} \, (X_p(q)/\epsilon) \cdot \nabla_x R_0(p,p) + O(\epsilon^4), \tag{3.6}$$

where h_0 has been defined in (3.5).

Next we estimate $S_\epsilon[a_{\epsilon,p}]$. Using the above expansion (3.6), the expansion of $\epsilon^2\triangle_g$ given in (5.3), the equation of w_0 (2.12) and Lemma 6.1,

$$\begin{aligned} S_\epsilon[a_{\epsilon,p}] &= \epsilon^2\triangle_g a_{\epsilon,p} - a_{\epsilon,p} + \frac{a_{\epsilon,p}^2}{h_{\epsilon,p}} \\ &= \triangle w_0 - w_0 + w_0^2 - \frac{1}{3}K(p) r w_0' \epsilon^2 \\ &\quad -\beta^2 h_0(y) w_0^2(y) - \epsilon\beta^2 \frac{|\mathcal{S}|}{2} \nabla R(p)\cdot y\, w_0^2(y) \\ &\quad -\frac{1}{6}(\nabla K(p)\cdot y) r w_0' \epsilon^3 + \frac{1}{6} R_1[w_0]\epsilon^3 + O(\epsilon^4|y|^4) \\ &= -\frac{\beta^2}{\epsilon^2}\frac{|\mathcal{S}|}{2}\nabla R(p)\cdot y\, w_0^2(y)\epsilon^3 - \frac{1}{6}\nabla K(p)\cdot y\, r w_0'(y)\epsilon^3 + \frac{1}{6}R_1[w_0](y)\epsilon^3 \\ &\quad + O(\epsilon^4|y|^4), \end{aligned}$$

since $w_0(y) = w_0(|y|)$.

Thus we have derived the following key estimate

Lemma 3.2. *For $q \in B_g(p,\delta_0)$, let $y = X_p(q)/\epsilon$. Then*

$$S_\epsilon[a_{\epsilon,p}](y) = -\frac{\beta^2}{\epsilon^2}\frac{|\mathcal{S}|}{2}\nabla R(p)\cdot y\, w_0^2(y)\epsilon^3 - \frac{1}{6}\nabla K(p)\cdot y\, r w_0'(y)\epsilon^3 + \frac{1}{6}R_1[w_0](y)\epsilon^3 + O(\epsilon^4|y|^4). \quad (3.7)$$

For $j = 1, 2$, define

$$Z_{\epsilon,p}^j(q) := \frac{\partial w}{\partial y_j}(X_p(q)/\epsilon)\chi_{\delta_0}(X_{p^\epsilon}(q)) \quad (3.8)$$

So $\left\langle Z_{\epsilon,p}^1, Z_{\epsilon,p}^2\right\rangle_{L_\epsilon^2(\mathcal{S})} = \int_{B(0,\delta_0/\epsilon)} \frac{\partial w}{\partial y_1}\frac{\partial w}{\partial y_2} dy + \text{e.s.t.} = \text{e.s.t.}.$

Further, we compute $\left\|Z_{\epsilon,p}^j\right\|_{L_\epsilon^2(\mathcal{S})} = \pi\int_0^\infty (w'(r))^2\, r dr + \text{e.s.t.}$

Next, we define our approximate kernel and cokernel as

$$K_{\epsilon,p} := \text{span}\left\{Z_{\epsilon,p}^1, Z_{\epsilon,p}^2\right\} \subset H_\epsilon^2(\mathcal{S}),$$

$$C_{\epsilon,p} := \text{span}\left\{Z_{\epsilon,p}^1, Z_{\epsilon,p}^2\right\} \subset L_\epsilon^2(S).$$

We then let $K_{\epsilon,p}^\perp$ and $C_{\epsilon,p}^\perp$ denote the orthogonal complement with respect to the scalar product $L_\epsilon^2(\mathcal{S})$ in $H_\epsilon^2(\mathcal{S})$ and $L_\epsilon^2(\mathcal{S})$, respectively.

Next we study several linear operators.

Let $\tilde{L}_{\epsilon,p} : H_\epsilon^2(\mathcal{S}) \to L_\epsilon^2(\mathcal{S})$ defined by

$$\begin{aligned} \tilde{L}_{\epsilon,p}\phi : &= S_\epsilon'[a_{\epsilon,p}]\,\phi \\ &= \epsilon^2\triangle_g\phi - \phi + \frac{2a_{\epsilon,p}}{h_{\epsilon,p}}\phi - \frac{a_{\epsilon,p}^2}{h_{\epsilon,p}^2}\psi, \end{aligned}$$

where $h_{\epsilon,p} = T_\beta[a_{\epsilon,p}^2]$, $\psi = T_\beta[2a_{\epsilon,p}\phi]$.

Let $\pi_{\epsilon,p}$ denote the projection in $L_\epsilon^2(\mathcal{S})$ onto $C_{\epsilon,p}^\perp$. We are going to show that the equation

$$\pi_{\epsilon,p}\circ S_\epsilon[a_{\epsilon,p}+\phi] = 0 \quad (3.9)$$

has the unique solution $\phi_{\epsilon,p} \in K_{\epsilon,p}^\perp$, provided ϵ is small enough.

Let

$$L_{\epsilon,p} \,:\, K_{\epsilon,p}^\perp \to C_{\epsilon,p}^\perp, \quad L_{\epsilon,p}\phi = \left(\pi_{\epsilon,p}\circ\tilde{L}_{\epsilon,p}\right)\phi \quad (3.10)$$

be the corresponding linearized operator.

As a preparation, we first give two propositions which show the invertibility of $L_{\epsilon,p}$.

Proposition 3.3. *There exist $\epsilon_0 > 0$ and $C > 0$ such that for any $p \in \mathcal{S}$ and $\epsilon \in (0, \epsilon_0)$,*

$$\|L_{\epsilon,p}\phi\|_{L^2_\epsilon(\mathcal{S})} \geq C \|\phi\|_{H^2_\epsilon(\mathcal{S})}$$

for any $\phi \in K^{\perp}_{\epsilon,p}$.

Proof: We proceed by proving a contradiction. Assume there are sequences $\epsilon_k \to 0$, $p_k \in \mathcal{S}$ such that $p_k \to p^0$, $\phi_k \in K^{\perp}_{\epsilon,p_k}$ with $\|\phi_k\|_{H^2_\epsilon(\mathcal{S})} = 1$, but

$$\|L_{\epsilon_k,p_k}\phi_k\|_{L^2_\epsilon(\mathcal{S})} \to 0. \tag{3.11}$$

Let us decompose $\phi_k = \phi_{k,1} + \phi_{k,2}$, where $\phi_{k,1} = (\chi_{\delta_0} \circ X_{p_k})\, \phi_k$.

At first (after rescaling) $\phi_{k,1}$ is only defined for $y \in B(0, \delta_0/\epsilon_k)$. Then by a standard procedure we extend $\phi_{k,1}$ to a function defined on $\mathbb{R}^2$ such that

$$\|\phi_{k,1}\|_{H^2(\mathbb{R}^2)} \leq C\|\phi_{k,1}\|_{H^2_{\epsilon_k}(\mathcal{S})}.$$

Since $\|\phi_k\|_{H^2_{\epsilon_k}(\mathcal{S})} = 1$, we have $\|\phi_{k,1}\|_{H^2(\mathbb{R}^2)} \leq C$.

Thus we may also assume that $\phi_{k,1}$ has a weak limit in $H^2_{loc}(\mathbb{R}^2)$ and therefore also a strong limit in $L^2_{loc}(\mathbb{R}^2)$ and $L^\infty_{loc}(\mathbb{R}^2)$. Call this limit ϕ_1.

Further, $\phi_{k,2} \to \phi_2$, where ϕ_2 satisfies

$$\Delta\phi_2 - \phi_2 = 0 \quad \text{in } \mathbb{R}^2.$$

Therefore, $\phi_2 = 0$ and $\|\phi_{k,2}\|_{H^2_{\epsilon_k}(\mathcal{S})} \to 0$ as $k \to \infty$.

Using the expansion of h_{ϵ_k} (3.6), we get $h_{\epsilon_k} \to 1$ in $H^2_\epsilon(\mathcal{S})$. Next we calculate

$$\begin{aligned}
\psi_k &= T_{\beta_k}[2a_{\epsilon_k,p_k}\phi_k] \\
&= \beta_k^2\xi_{\epsilon_k}\int_{\mathcal{S}} G_{\beta_k}(p,q)2a_{\epsilon_k,p_k}\phi_k dv_g(q) \\
&= 2\epsilon_k^2\beta_k^2\xi_{\epsilon_k}\int_{B(0,\delta_0/\epsilon_k)}\left(\frac{\beta_k^{-2}}{|\mathcal{S}|} + \frac{1}{2\pi}\log\frac{1}{\epsilon_k|y-z|} + R(\epsilon_k y, \epsilon_k z)\right) w(z)\phi_{k,1}(z)dz + o(1) \\
&= 2\frac{\int_{\mathbb{R}^2} w(z)\phi(z)dz}{\int_{\mathbb{R}^2} w^2(z)dz} + o(1).
\end{aligned}$$

Hence, with the knowledge of the expansion of $\epsilon_k^2\triangle_g$ in (5.3), and taking $k \to \infty$, we obtain from (3.11) the limiting problem

$$\triangle\phi_1 - \phi_1 + 2w\phi_1 - 2\frac{\int_{\mathbb{R}^2} w(z)\phi_1(z)dz}{\int_{\mathbb{R}^2} w^2(z)dz}w^2 = 0, \tag{3.12}$$

where $C_0 := \text{span}\left\{\frac{\partial w}{\partial y_j},\ j = 1, 2\right\}$, and $C_0^\perp$, $K_0^\perp$ denote the orthogonal complement with respect to the inner product of $L^2(\mathbb{R}^2)$ in the spaces $L^2(\mathbb{R}^2)$ and $H^2(\mathbb{R}^2)$, respectively.

Taking limits, ϕ_1 satisfies

$$\phi_1 \in \left\{\phi \in H^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} \phi\frac{\partial w}{\partial y_j}dy = 0,\ j = 1,\ 2\right\} = K_0^\perp.$$

Since for $L_0 := \triangle - 1 + 2w$, $L_0 w = w^2$, (3.12) can be rewritten as

$$L_0\left(\phi_1 - 2\frac{\int_{\mathbb{R}^2} w(z)\phi_1(z)dz}{\int_{\mathbb{R}^2} w^2(z)dz}w\right) = 0. \tag{3.13}$$

Now, by Lemma 2.1, we have that L_0 is invertible from $K_0^\perp$ to $C_0^\perp$, so

$$\phi_1 - 2\frac{\int_{\mathbb{R}^2} w(z)\phi_1(z)dz}{\int_{\mathbb{R}^2} w^2(z)dz} w = 0.$$

Multiplying by w and integrating, one sees that

$$\int_{\mathbb{R}^2} w(z)\phi_1(z)dz = 0$$

so that $\phi_1 = 0$ which is a contradiction since our assumption $\|\phi_k\|_{H^2_\epsilon(\mathcal{S})} = 1$ implies $\|\phi_1\|_{H^2(\mathbb{R}^2)} > 0$.

Proposition 3.4. *There exists $\epsilon_2 > 0$ such that for all $\epsilon \in (0, \epsilon_2)$, $L_{\epsilon,p}$ is surjective for any $p \in \mathcal{S}$.*

Proof: The argument is similar to the proof of Proposition 4.3 in [50] and of Proposition 3.3 above. It is therefore omitted. □

By the two previous propositions we have that $L_{\epsilon,p} : K_{\epsilon,p}^\perp \to C_{\epsilon,p}^\perp$ is invertible. Let us call the inverse $L_{\epsilon,p}^{-1}$. Now we are in a position to solve the equation (3.9) by a fixed point argument. Indeed, we apply $L_{\epsilon,p}^{-1}$ to (3.9), and regrouping we can write

$$\phi = -(L_{\epsilon,p}^{-1} \circ \pi_{\epsilon,p})(S_\epsilon[a_{\epsilon,p}]) - (L_{\epsilon,p}^{-1} \circ \pi_{\epsilon,p})(N_{\epsilon,p}(\phi)) \equiv M_{\epsilon,p}(\phi), \tag{3.14}$$

where

$$N_{\epsilon,p}(\phi) = S_\epsilon[a_{\epsilon,p} + \phi] - S_\epsilon[a_{\epsilon,p}] - S'_\epsilon[a_{\epsilon,p}]\phi$$

and the operator $M_{\epsilon,p}$ is defined by (3.14) for $\phi \in H^2_\epsilon(\mathcal{S})$. We are going to show that the operator $M_{\epsilon,p}$ is a contraction on

$$B_{\epsilon,\eta} \equiv \{\phi \in H^2_\epsilon(\mathcal{S}) \,:\, \|\phi\|_{H^2_\epsilon(\mathcal{S})} < \eta\} \tag{3.15}$$

if η and ϵ are small enough. We have by Lemma 3.2 and Propositions 3.3 and 3.4 that

$$\begin{aligned} \|M_{\epsilon,p}(\phi)\|_{H^2_\epsilon(\mathcal{S})} &\leq C(\|\pi_{\epsilon,p} \circ N_{\epsilon,p}(\phi)\|_{L^2_\epsilon(\mathcal{S})} + \|\pi_{\epsilon,p} \circ S_\epsilon[a_{\epsilon,p}]\|_{L^2_\epsilon(\mathcal{S})}) \\ &\leq C(\eta^2 + O(\epsilon^3)), \end{aligned}$$

where $C > 0$ is independent of $\eta > 0$ and $\epsilon > 0$. Similarly we can show

$$\|M_{\epsilon,p}(\phi) - M_{\epsilon,p}(\phi')\|_{H^2_\epsilon(\mathcal{S})} \leq C\eta\|\phi - \phi'\|_{H^2_\epsilon(\mathcal{S})},$$

where $C > 0$ is independent of $\eta > 0$ and $\epsilon > 0$. If we choose η and ϵ small enough (more precisely, if we choose (i) η small enough and (ii) $\epsilon^3 \sim \eta$), then $M_{\epsilon,p}$ is a contraction on $B_{\epsilon,\eta}$. The existence of a unique fixed point $\phi_{\epsilon,p} \in B_{\epsilon,\eta}$ now follows from the Contraction Mapping Principle. Since $\phi_{\epsilon,p}$ is a solution of (3.14), we have thus proved

Proposition 3.5. *There is $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, and for arbitrary $p \in \mathcal{S}$, there exists a unique $\phi_{\epsilon,p} \in K_{\epsilon,p}^\perp$ satisfying $S_\epsilon[a_{\epsilon,p} + \phi_{\epsilon,p}] \in C_{\epsilon,p}$ and*

$$\|\phi_{\epsilon,p}\|_{H^2_\epsilon(\mathcal{S})} \leq C\epsilon^3. \tag{3.16}$$

3.2. **The reduced problem.** By Proposition 3.5, for each $p \in \mathcal{S}$, we have

$$S_\epsilon[a_{\epsilon,p} + \phi_{\epsilon,p}] \in C_{\epsilon,p}$$

for ϵ small enough. Now, to solve the equation $S_\epsilon[a_{\epsilon,p} + \phi_{\epsilon,p}] = 0$ exactly, we have to further choose a p^ϵ such that

$$S_\epsilon[a_{\epsilon,p^\epsilon} + \phi_{\epsilon,p^\epsilon}] \in C^\perp_{\epsilon,p^\epsilon}.$$

This is a finite dimensional problem and we are looking for a point $p^\epsilon \in \mathcal{S}$ at which constructing a single spike is possible. We will show that it is possible to construct a spike close to any given non-degenerate critical point of $F = c_1 K + c_2 R$.

To this end, let us define a vector field $W_\epsilon : \mathcal{S} \to \mathbb{R}^2$ by

$$W_{\epsilon,j}(p) := \frac{1}{\epsilon} \int_{\mathcal{S}} S_\epsilon[a_{\epsilon,p} + \phi_{\epsilon,p}](q) Z^j_{\epsilon,p}(q) dv_g(q)$$

$$= \frac{1}{\epsilon^3} \int_{B(0,\delta_0/\epsilon)} S_\epsilon[a_{\epsilon,p} + \phi_{\epsilon,p}](X_p^{-1}(\epsilon y)) \frac{\partial w}{\partial y_j}(y) dy + O(\epsilon^2)$$

and $W_\epsilon(p) = (W_{\epsilon,1}(p), W_{\epsilon,2}(p))$ with our approximate kernel defined in (3.8). Note that W_ϵ is continuous on $\mathcal{S}$, and we would like to find a zero to W_ϵ.

We now calculate the asymptotic expansion of $W_{\epsilon,j}(p)$:

$$\begin{aligned} W_{\epsilon,j}(p) &= \frac{1}{\epsilon^3} \int_{B(0,\delta_0/\epsilon)} S_\epsilon[a_{\epsilon,p}](X_p^{-1}(\epsilon y)) \frac{\partial w}{\partial y_j}(y) dy \\ &\quad + \frac{1}{\epsilon^3} \int_{B(0,\delta_0/\epsilon)} \left(S'_\epsilon[a_{\epsilon,p}]\phi_{\epsilon,p}\right)(X_p^{-1}(\epsilon y)) \frac{\partial w}{\partial y_j}(y) dy + O(\epsilon^2) \\ &= I_1 + I_2 + O(\epsilon^2), \end{aligned}$$

where I_1 and I_2 are defined at the last equality in an obvious manner.

Using our key estimate (3.7), we calculate

$$\begin{aligned} I_1 &= -\frac{\beta^2}{\epsilon^2} \frac{|\mathcal{S}|\pi}{2} \int_{\mathbb{R}^2} \nabla R(p) \cdot y\, w_0^2(y) \frac{\partial w}{\partial y_i} dy \\ &\quad - \frac{1}{6} \int_0^\infty \left(\nabla K(p) \cdot y\ (Q - 2P)[w_0](y) + R_1[w_0](y)\right) \frac{\partial w}{\partial y_i} dr + O(\epsilon). \end{aligned}$$

Now

$$\begin{aligned} & \int_{\mathbb{R}^2} \nabla R(p) \cdot y\, w_0^2(y) \frac{\partial w}{\partial y_i} dy \\ =\ & \frac{\partial R}{\partial x_i}(p) \int_{\mathbb{R}^2} y_i\, w^2(y) \frac{\partial w}{\partial y_i} dy + O(\epsilon^2) \\ =\ & -\frac{1}{2} \frac{\partial R}{\partial x_i}(p) \int_{\mathbb{R}^2} w^2(y) dy + O(\epsilon^2), \end{aligned}$$

using Pohozaev identity which gives $\frac{1}{2}\int_{\mathbb{R}^2} w^2(y)dy = \frac{1}{3}\int_{\mathbb{R}^2} w^3(y)dy$. Next, by Lemma 6.2, we have

$$\begin{aligned} & \int_0^\infty \left(\nabla K(p) \cdot y\ (Q - 2P)[w_0](y) + R_1[w_0](y)\right) \frac{\partial w}{\partial y_i} dr \\ =\ & -\frac{3\pi}{2} \frac{\partial K}{\partial y_j}(p) \int_0^\infty r^3 \left(w'\right)^2 dr + O(\epsilon^2). \end{aligned}$$

Together we have

$$\begin{aligned} I_1 &= \frac{\beta^2}{\epsilon^2}\frac{|\mathcal{S}|\pi}{4}\frac{\partial R}{\partial x_i}(p)\int_{\mathbb{R}^2} w^2(y)dy \\ &\quad +\frac{\pi}{4}\frac{\partial K}{\partial x_j}(p)\int_0^\infty r^3\,(w')^2\,dr + O(\epsilon^2). \end{aligned}$$

This is our main term. Next we compute:

$$I_2 = \frac{1}{\epsilon^3}\int_{\mathbb{R}^2} S_\epsilon'[a_{\epsilon,p}]\phi_{\epsilon,p}\frac{\partial w}{\partial y_j}\,dy + O(\epsilon^2) = O(\epsilon^2)$$

since

$$\|\phi_{\epsilon,p}\|_{H^2_\epsilon(\mathcal{S})} = O(\epsilon^3)$$

and

$$\begin{aligned} S_\epsilon'[a_{\epsilon,p}]\frac{\partial w_0}{\partial y_j} &= \Delta_{g,y}\frac{\partial w_0}{\partial y_j} - \frac{\partial w_0}{\partial y_j} + 2\frac{a_{\epsilon,p}}{h_{\epsilon,p}}\frac{\partial w_0}{\partial y_j} - \frac{a^2_{\epsilon,p}}{h^2_{\epsilon,p}}\frac{\partial w_0}{\partial y_j} + O(\epsilon^2) \\ &= O(\epsilon^2) - \frac{a^2_{\epsilon,p}}{h^2_{\epsilon,p}}\frac{\partial w_0}{\partial y_j}, \end{aligned}$$

where

$$\int_{\mathbb{R}^2}\frac{a^2_{\epsilon,p}}{h^2_{\epsilon,p}}\frac{\partial w_0}{\partial y_j}\,dy = \int_{\mathbb{R}^2} w_0^2\frac{\partial w_0}{\partial y_j}\,dy + O(\epsilon^2) = O(\epsilon^2)$$

by our choice of approximate solution w_0 given in (2.12) and the expansions of Δ_g given in (5.3) and $h_{\epsilon,p}$ in (3.6).

In conclusion, we get

$$W_\epsilon = \nabla F(p) + o(1) \quad \text{for all } p \in \overline{\Lambda_\delta}, \tag{3.17}$$

where $o(1)$ is a continuous function of p which tends to 0 as $\epsilon \to 0$ uniformly in $\overline{\Lambda_\delta}$.

At p^0, we have $\nabla F(p^0) = 0$, $\det(\nabla^2 F(p^0) \neq 0$ by (1.14). (Recall that $\det(\nabla^2 F(p^0))$ is independent of the choice of tangent plane basis, and the entries of $\nabla F(p)$ in local coordinates vary differentiably with p.)

By (3.17), for ϵ small enough W_ϵ has exactly one zero in $\overline{\Lambda_\delta}$. We compute the mapping degree of W_ϵ for the set $\overline{\Lambda_\delta}$ and the value 0 as follows:

$$\deg(W_\epsilon, 0, B_g(p^0,\eta)) = \text{sign}\ \det(-\nabla^2 F(p^0)) = \text{sign}\ \det(-\mathcal{M}(p^0)) \neq 0.$$

Therefore, standard degree theory implies that for ϵ small enough, there exists a $p^\epsilon \in \overline{\Lambda_\delta}$ such that $W_\epsilon(p^\epsilon) = 0$ and, by (3.17), we have $p^\epsilon \to p^0$.

Thus we have proved the following proposition.

Proposition 3.6. *For ϵ sufficiently small there exist points $p^\epsilon \in \overline{\Lambda_\delta}$ with $p^\epsilon \to p^0$ such that $W_\epsilon(p^\epsilon) =$ 0.*

Finally, we prove Theorem 1.1.

Proof: By Proposition 3.6, for $\epsilon \to 0$ there exist points $p^\epsilon \to p^0$ such that $W_\epsilon(p^\epsilon) = 0$. In other words, $S_\epsilon[a_{\epsilon,p^\epsilon} + \phi_{\epsilon,p^\epsilon}] = 0$. We set $\xi_\epsilon = \xi_{\epsilon,p^\epsilon}$. Let $A_\epsilon = \xi_\epsilon(a_{\epsilon,p^\epsilon} + \phi_{\epsilon,p^\epsilon})$ and $H_\epsilon = \xi_\epsilon(h_{\epsilon,p^\epsilon} + \psi_{\epsilon,p^\epsilon})$. It is easy to see that $H_\epsilon = \xi_\epsilon T_\beta[a_{\epsilon,p^\epsilon} + \phi_{\epsilon,p^\epsilon}] > 0$. Hence $A_\epsilon \geq 0$. By applying the Maximum Principle on sets of the type $B_g(p, \delta_0/\epsilon)$ which are a covering of $\mathcal{S}$, we derive $A_\epsilon > 0$. Therefore (A_ϵ, H_ϵ) satisfies Theorem 1.1. □

4. Stability Analysis

4.1. **Study of Large Eigenvalues.** We consider the stability of the one-spike steady state (A_ϵ, H_ϵ) constructed in Theorem 1.1.

Linearizing the system (1.1) around the equilibrium states $(A_\epsilon + \phi_\epsilon e^{\lambda_\epsilon t}, H_\epsilon + \psi_\epsilon e^{\lambda_\epsilon t})$, we obtain the following eigenvalue problem

$$\begin{cases} \Delta_{g,y}\phi_\epsilon - \phi_\epsilon + 2\frac{A_\epsilon}{H_\epsilon}\phi_\epsilon - \frac{A_\epsilon^2}{H_\epsilon^2}\psi_\epsilon = \lambda_\epsilon\phi_\epsilon, \\ \frac{1}{\beta^2}\Delta_{g,x}\psi_\epsilon - \psi_\epsilon + 2A_\epsilon\phi_\epsilon = \tau\lambda_\epsilon\psi_\epsilon, \end{cases} \tag{4.1}$$

where λ_ϵ is some complex number and

$$\phi_\epsilon \in H_\epsilon^2(\mathcal{S}),\ \psi_\epsilon \in H^2(\mathcal{S}). \tag{4.2}$$

Let

$$a_\epsilon = \xi_\epsilon^{-1} A_\epsilon = a_{\epsilon,p^\epsilon}, \quad h_\epsilon = \xi_\epsilon^{-1} H_\epsilon = h_{\epsilon,p^\epsilon}, \tag{4.3}$$

where $\xi_\epsilon = \xi_{\epsilon,p^\epsilon}$.

Then (4.1) becomes

$$\begin{cases} \Delta_y\phi_\epsilon - \phi_\epsilon + 2\frac{a_\epsilon}{h_\epsilon}\phi_\epsilon - \frac{a_\epsilon^2}{h_\epsilon^2}\psi_\epsilon = \lambda_\epsilon\phi_\epsilon, \\ \frac{1}{\beta^2}\Delta\psi_\epsilon - \psi_\epsilon + 2\xi_\epsilon a_\epsilon\phi_\epsilon = \tau\lambda_\epsilon\psi_\epsilon. \end{cases} \tag{4.4}$$

In this subsection, we study the large eigenvalues, i.e., we assume that $|\lambda_\epsilon| \geq c > 0$ for ϵ small. Furthermore, we may assume that $(1+\tau)c < \frac{1}{2}$. If $\mathrm{Re}(\lambda_\epsilon) \leq -c$, we are done since then λ_ϵ is a stable large eigenvalue. Therefore we may assume that $\mathrm{Re}(\lambda_\epsilon) \geq -c$ and for a subsequence $\epsilon \to 0$, $\lambda_\epsilon \to \lambda_0 \neq 0$.

We shall derive the limiting eigenvalue problem which is a NLEP. Then we will apply the key reference is Lemma 2.2 to derive a stability result.

The second equation in (4.4) is equivalent to

$$\Delta\psi_\epsilon - \beta^2(1+\tau\lambda_\epsilon)\psi_\epsilon + 2\beta^2\xi_\epsilon a_\epsilon\phi_\epsilon = 0. \tag{4.5}$$

We introduce the complex constant

$$\beta_{\lambda_\epsilon} = \beta\sqrt{1+\tau\lambda_\epsilon}, \tag{4.6}$$

where in $\sqrt{1+\tau\lambda_\epsilon}$ we take the principal part of the square root. This means that the real part of $\sqrt{1+\tau\lambda_\epsilon}$ is positive, which is possible since $\mathrm{Re}(1+\tau\lambda_\epsilon) \geq 1 - \tau c \geq \frac{1}{2}$.

Let us assume that

$$\|\phi_\epsilon\|_{H_\epsilon^2(\mathcal{S})} = 1. \tag{4.7}$$

We cut off ϕ_ϵ as follows: Introduce

$$\phi_{\epsilon,1}(y) = \phi_\epsilon(y)\chi_{\delta_0,p^\epsilon}(\epsilon y), \tag{4.8}$$

where $\epsilon y = X_{p^\epsilon}(q)$ and $\chi_{\delta_0,p^\epsilon}$ was introduced in (1.5).

As in the proof of Proposition 3.3, we extend $\phi_{\epsilon,1}$ to a function defined on $\mathbb{R}^2$ such that

$$\|\phi_{\epsilon,1}\|_{H^2(\mathbb{R}^2)} \leq C\|\phi_{\epsilon,1}\|_{H_\epsilon^2(\mathcal{S})}.$$

Since $\|\phi_\epsilon\|_{H_\epsilon^2(\mathcal{S})} = 1$, we have $\|\phi_{\epsilon,1}\|_{H^2(\mathbb{R}^2)} \leq C$.

By taking a subsequence of ϵ, we may also assume that $\phi_{\epsilon,1}$ has a limit in $H^2_{loc}(\mathbb{R}^2)$ which we call ϕ_1.

We have by (4.5)

$$\psi_\epsilon(p) = 2\beta^2\xi_\epsilon \int_{\mathcal{S}} G_{\beta_{\lambda_\epsilon}}(p,q)a_\epsilon(\frac{q}{\epsilon})\phi_\epsilon(\frac{q}{\epsilon})\,dv_g(q). \tag{4.9}$$

For $p = p^\epsilon$, we calculate

$$\psi_\epsilon(p^\epsilon) = 2\beta^2 \int_{\mathcal{S}} G_{\beta_{\lambda_\epsilon}}(p^\epsilon,q)\xi_\epsilon w_0(X_{p^\epsilon}(q)/\epsilon)\chi_{\delta_0,p^\epsilon}(q)\phi_{\epsilon,1}(\frac{X_{p^\epsilon}(q)}{\epsilon})\,dv_g(q) + o(1)$$

$$= 2\beta^2 \int_{\mathcal{S}} \left(\frac{(\beta_{\lambda_\epsilon})^{-2}}{|\mathcal{S}|} + G_0(p^\epsilon,q) + O(|\beta_{\lambda_\epsilon}|^2)\right)\xi_\epsilon w(X_{p^\epsilon}(q)/\epsilon)\phi_{\epsilon,1}(X_{p^\epsilon}(q)/\epsilon)\,dv_g(q) + o(1)$$

$$= 2\epsilon^2 \int_{\mathbb{R}^2} \left(\frac{1}{|\mathcal{S}|(1+\tau\lambda_\epsilon)} + \beta^2 G_{0,p^\epsilon}(0,\epsilon z) + O(|\beta_{\lambda_\epsilon}|^4)\right)\xi_\epsilon w(z)\phi_{\epsilon,1}(z)\,dz + o(1)$$

$$= 2\frac{1}{|\mathcal{S}|(1+\tau\lambda_\epsilon)}\xi_\epsilon\epsilon^2 \int_{\mathbb{R}^2} w(z)\phi_{\epsilon,1}(z)\,dz + o(1). \tag{4.10}$$

Substituting (4.10) into the first equation (4.4), letting $\epsilon \to 0$ and using (2.17), we arrive at the following nonlocal eigenvalue problem (NLEP)

$$\Delta\phi_1 - \phi_1 + 2w\phi_1 - \frac{2}{1+\tau\lambda_0}\frac{\int_{\mathbb{R}^2} w\phi_1}{\int_{\mathbb{R}^2} w^2}w^2 = \lambda_0\phi_1. \tag{4.11}$$

By Lemma 2.2, problem (4.11) is stable if $\tau < \tau_1$, which implies that the large eigenvalues of (4.4) are stable.

If $\tau > \tau_1$, by Theorem 2.2, problem (4.11) has an eigenvalue λ_0 with Re $(\lambda_0) \geq c_0$ for some $c_0 > 0$.

By a compactness argument given in Section 2 of [6], it follows that problem (4.4) also admits an eigenvalue λ_ϵ with $\lambda_\epsilon = \lambda_0 + o(1)$. This implies that problem (4.4) is unstable.

This finishes the proof of Theorem 1.2 in the large eigenvalue case.

□

4.2. **Further improvement of solutions.** In this subsection, we further improve our expansion to the solutions derived in Section 3.

More precisely, we will show that

$$\begin{cases} A_\epsilon(q) = \xi_\epsilon\left[\left(w_0(\frac{x}{\epsilon}) + \epsilon^3 w_2^0(\frac{x}{\epsilon}) + \epsilon^4 w_1^0(\frac{x}{\epsilon}) + \epsilon^4 w_3^0(\frac{x}{\epsilon})\right)\chi_{\delta_0})(x) + O(\epsilon^5)\right], \\ H_\epsilon(p^\epsilon) = \xi_\epsilon(1+O(\epsilon^4)), \end{cases} \tag{4.12}$$

where $q = X_{p^\epsilon}^{-1}(x)$, the amplitude ξ_ϵ is given by $\xi_\epsilon = \xi_{\epsilon,p^\epsilon}$ and w_0, w_2^0, w_1^0, w_3^0 are suitably chosen functions; $w_0 = w + O(\epsilon^2)$ has been defined in (2.12) and in this subsection we will introduce w_2^0, w_1^0, w_3^0.

First we know from the existence proof that

$$\nabla(c_1K(p^\epsilon) + c_2R(p^\epsilon)) = O(\epsilon^2), \tag{4.13}$$

see (3.17).

By the non-degeneracy of the critical point p^0 for the function F we derive $p^\epsilon = p^0 + O(\epsilon^2)$ so that

$$\nabla K(p^\epsilon) = \nabla K(p^0) + O(\epsilon^2),$$

$$\nabla R(p^\epsilon) = \nabla R(p^0) + O(\epsilon^2).$$

We now expand the one-spike solution A_ϵ. First we define $w_2 = \epsilon^3 w_2^0$ as follows: Let $w_2^0 \in H^2(\mathbb{R}^2)$ be the unique solution of the problem

$$L_0 w_2^0 - \frac{2\int w_0 w_2^0}{\int w_0^2} w_0^2 + \frac{1}{6}(\nabla K(p^0)\cdot y)(Q-2P)[w_0] + \frac{1}{6}R_1[w_0]$$
$$+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{1}{2}(\nabla R(p^0)\cdot y)w_0^2 = 0,$$
$$w_2 \perp \frac{\partial w_0}{\partial y_j}, \quad j=1,2, \tag{4.14}$$

where

$$L_0\phi = \Delta\phi - \phi + 2w_0\phi.$$

We recall that w_0 has been defined in (2.12). Note that w_2 is an odd function. The solution w_2^0 exists and is unique because (4.13) implies that the following solvability condition holds:

$$\frac{1}{6}(\nabla K(p^0)\cdot y)(Q-2P)[w_0] + \frac{1}{6}R_1[w_0]$$
$$+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{1}{2}(\nabla R(p^0)\cdot y)w_0^2 \perp \frac{\partial w_0}{\partial y_j}, \quad j=1,2.$$

This follows by an argument as in the proof of Proposition 3.3, using the fact that by Lemma 2.1 we have that L_0 is invertible from $K_0^\perp$ to $C_0^\perp$.

Second we define $w_1 = \epsilon^4 w_1^0$, where $w_1^0 \in H^2(\mathbb{R}^2)$ is the unique solution of the problem

$$L_0 w_1^0 - 2\frac{\int w_0 w_1^0}{\int w_0^2} w_0^2 + \frac{1}{20}(y^t\nabla^2 K(p^0)y)(Q-2P)[w_0] + \frac{1}{10}R_2[w_0]$$
$$+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{1}{2}(y^t\nabla_x^2 R_0(p^0,p^0)y)w_0^2 = 0,$$
$$w_1^0 \perp \frac{\partial w_0}{\partial y_j}, \quad j=1,2. \tag{4.15}$$

The solution exists because the following solvability condition holds:

$$\frac{1}{20}(y^t\nabla^2 K(p^0)y)(Q-2P)[w_0] + \frac{1}{10}R_2[w_0]$$
$$+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{1}{2}(y^t\nabla_x^2 R_0(p^0,p^0)y)w_0^2 \perp \frac{\partial w_0}{\partial y_j}, \quad j=1,2,$$

since this expression is even in y.

Third we set $w_3 = \epsilon^4 w_3^0$, where $w_3^0 \in H^2(\mathbb{R}^2)$ is the unique solution of the problem

$$L_0 w_3^0 - 2\frac{\int w_0 w_3^0}{\int w_0^2} w_0^2 - \frac{2}{90}K^2(p^0)\epsilon^4 r^3 w_0' = 0,$$
$$w_3^0 \perp \frac{\partial w_0}{\partial y_j}, \quad j=1,2. \tag{4.16}$$

The solution exists because the following solvability condition holds:

$$\frac{2}{90}K^2(p^0)\epsilon^4 r w_0' \perp \frac{\partial w_0}{\partial y_j}, \quad j=1,2,$$

since this expression rotationally symmetric.

We remark that it does not matter if we use w_0 or w in the definitions of w_2, w_1, w_3 since the difference is $O(\epsilon^5)$. Neither does it matter if we use p^0 or p^ϵ since the error caused is $O(\epsilon^5)$, and for simplicity we use p^0.

Now it easy to see that $S_\epsilon[(w_0+\epsilon^3 w_2^0+\epsilon^4 w_1^0+\epsilon^4 w_3^0)\chi_{\delta_0}] = O(\epsilon^5)$ since by the definition of w_0 and w_i^0, $i=1,2,3$, all the terms up to order ϵ^4 cancel. Using Liapunov-Schmidt reduction as in Proposition 3.5, we finally have

$$a_\epsilon = (w_0+\epsilon^3 w_2^0+\epsilon^4 w_1^0+\epsilon^4 w_3^0)\chi_{\delta_0}+\phi_\epsilon^\perp, \tag{4.17}$$

where $\phi_\epsilon^\perp \in K_{\epsilon,p^\epsilon}^\perp$ and $\|\phi_\epsilon^\perp\|_{H^2(\mathbb{R}^2)} = O(\epsilon^5)$. Further, w_0, w_3^0 are radially symmetric, w_2^0 is odd, w_1^0 is even.

Let us derive from the defining equations for w_0 and w_i^0 identities to be used in the stability proof.

Applying $\frac{\partial}{\partial y_j}$ in (2.12) gives:

$$\Delta\frac{\partial w_0}{\partial y_j}-\frac{\partial w_0}{\partial y_j}-\frac{1}{3}K(p^\epsilon)\epsilon^2\frac{\partial}{\partial y_j}(rw_0')+2\frac{w_0}{1+\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int\log\frac{|z|}{|y-z|}w_0^2(z)\,dz}\frac{\partial w_0}{\partial w_j}$$

$$+\frac{w_0^2}{\left(1+\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int\log\frac{|z|}{|y-z|}w_0^2(z)\,dz\right)^2}\left(-\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int\frac{\partial}{\partial y_j}(\log\frac{|z|}{|y-z|})w_0^2(z)\,dz\right)=0, \quad y\in\mathbb{R}^2. \tag{4.18}$$

Taking $\frac{\partial}{\partial y_j}$ in (4.14), we get

$$L_0\frac{\partial w_2^0}{\partial y_j}+2w_2^0\frac{\partial w_0}{\partial y_j}-\frac{2\int w_0w_2^0}{\int w_0^2}2w_0\frac{\partial w_0}{\partial y_j}$$
$$+\frac{1}{6}\frac{\partial K}{\partial x_j}(p^0)(Q-2P)\,[w_0]+\frac{1}{6}(\nabla K(p^0)\cdot y)\frac{\partial}{\partial y_j}(Q-2P)\,[w_0]+\frac{1}{6}\frac{\partial}{\partial y_j}R_1\,[w_0]$$
$$+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{1}{2}(\frac{\partial R}{\partial x_j}(p^0))w_0^2+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{1}{2}(\nabla R(p^0)\cdot y)2w_0\frac{\partial w_0}{\partial y_j}=0. \tag{4.19}$$

Applying $\frac{\partial}{\partial y_j}$ in (4.15), we get

$$L_0\frac{\partial w_1^0}{\partial y_j}+2w_1^0\frac{\partial w_0}{\partial y_j}-2\frac{\int w_0w_1^0}{\int w_0^2}2w_0\frac{\partial w_0}{\partial y_j}+\frac{1}{10}(\frac{\partial}{\partial x_j}\nabla K(p^0)\cdot y)(Q-2P)[w_0]$$
$$+\frac{1}{20}(y^t\nabla^2K(P^\epsilon)y)\frac{\partial}{\partial y_j}(Q-2P)[w_0]+\frac{1}{10}\frac{\partial}{\partial y_j}R_2[w_0]$$
$$+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{1}{2}(y^t\nabla_x^2R_0(p^0,p^0)y)2w_0\frac{\partial w_0}{\partial y_j}+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}(\frac{\partial}{\partial x_j}\nabla_xR_0(p^0,p^0)\cdot y)w_0^2=0. \tag{4.20}$$

Taking $\frac{\partial}{\partial y_j}$ in (4.16), we get

$$L_0\frac{\partial w_3^0}{\partial y_j}+2w_3^0\frac{\partial w_0}{\partial y_j}-2\frac{\int w_0w_3^0}{\int w_0^2}2w_0\frac{\partial w_0}{\partial y_j}-\frac{1}{45}K^2(0)\epsilon^4\frac{\partial}{\partial y_j}(r^3w_0')=0. \tag{4.21}$$

These relations will be needed in the study of the small eigenvalues.

4.3. **Study of Small Eigenvalues.** We now study (4.4) for small eigenvalues. Namely, we assume that $\lambda_\epsilon\to0$ as $\epsilon\to0$. We will show that the small eigenvalues are related to the matrix

$$\mathcal{M}(p^0)=\nabla^2\left(c_1K(p^0)+c_2R(p^0)\right),$$

where

$$c_1=\frac{\pi}{4}\int_0^\infty(w')^2r^3\,dr, \quad c_2=\frac{\beta^2}{\epsilon^2}\frac{|\mathcal{S}|\pi}{2}\int_0^\infty w^2\,r\,dr,$$

which has been introduced in (1.12)

Let us assume that condition (*) holds true. That is, all eigenvalues of the matrix $\mathcal{M}(p^0)$ are negative. The main result which we derive in this subsection says that if $\lambda_\epsilon \to 0$, then

$$\lambda_\epsilon \sim \sigma_0 \frac{\epsilon^4}{\int (\frac{\partial w}{\partial y_1})^2\, dy}, \tag{4.22}$$

where σ_0 is an eigenvalue of $\mathcal{M}(p^0)$. From (4.22), we see that all small eigenvalues of $\mathcal{L}_\epsilon$ are stable, provided that condition (*) holds.

Again let (A_ϵ, H_ϵ) be the equilibrium state of (1.13) which has been rigorously constructed in Theorem 1.1 and (a_ϵ, h_ϵ) be the rescaled solution given by (4.3).

For the eigenfunction we set

$$\phi_\epsilon = \sum_{k=1}^{2} a_k^\epsilon \left(\frac{\partial w_0}{\partial y_k} + \epsilon^3 \frac{\partial w_2^0}{\partial y_k} + \epsilon^4 \frac{\partial w_1^0}{\partial y_k} + \epsilon^4 \frac{\partial w_3^0}{\partial y_k} \right) \chi_{\delta_0}(\epsilon y) + \phi^\perp \tag{4.23}$$

where a_k^ϵ are some constant complex coefficients and

$$\phi^\perp \perp \tilde{K}_\epsilon := \operatorname{span} \left\{ \frac{\partial w_0}{\partial y_k} \chi_{\delta_0} \ : \ k = 1, 2 \right\} \subset H_\epsilon^2(\mathcal{S}). \tag{4.24}$$

Our proof will consist of two steps. First we will show that $\|\phi^\perp - \epsilon^3 \phi_2^0\|_{H_\epsilon^2(\mathcal{S})} = O(\epsilon^5)$, where $\|\phi_2^0\|_{H_\epsilon^2(\mathcal{S})} = O(1)$ and ϕ_2^0 is radially symmetric. Second we will derive algebraic equations for the coefficients a_1^ϵ, a_2^ϵ.

As a preparation, we need to compute $L_g \left[\left(\frac{\partial w_0}{\partial y_j} + \epsilon^3 \frac{\partial w_2^0}{\partial y_j} + \epsilon^4 \frac{\partial w_1^0}{\partial y_j} + \epsilon^4 \frac{\partial w_3^0}{\partial y_j} \right) \chi_{\delta_0} \right]$, where

$$L_g \phi = \Delta_g \phi - \phi + \frac{2 a_\epsilon \phi}{T[a_\epsilon^2]} - \frac{a_\epsilon^2}{T[a_\epsilon^2]^2} T[2 a_\epsilon \phi]$$

for $\phi \in H_\epsilon^2(\mathcal{S})$ and w_0, w_1^0, w_2^0, w_3^0 have been defined in (2.12), (4.15), (4.14), (4.16), respectively. To this end, we make some preparations.

Using the expansion of Δ_g given in (5.3) and the relations

$$\frac{1}{T[a_\epsilon^2]} = \frac{1}{1 + \frac{\epsilon^2 \xi_\epsilon \beta^2}{2\pi} \int \log \frac{|z|}{|y-z|}}$$
$$-2\epsilon^3 T[w_0 w_2^0] + |\mathcal{S}| \frac{\beta^2}{\epsilon^2} \frac{\epsilon^3}{2} (\nabla R(p^0) \cdot y)$$
$$-2\epsilon^4 T[w_0 (w_1^0 + w_3^0)] + |\mathcal{S}| \frac{\beta^2}{\epsilon^2} \frac{\epsilon^4}{2} (y^t \nabla_x^2 R_0(p^0, p^0) y) + O(\epsilon^5),$$
$$T \left[2 a_\epsilon \left(\frac{\partial w_0}{\partial y_j} + \epsilon^3 \frac{\partial w_2^0}{\partial y_j} + \epsilon^4 \frac{\partial w_1^0}{\partial y_j} + \epsilon^4 \frac{\partial w_3^0}{\partial y_j} \right) \right]$$
$$= -|\mathcal{S}| \frac{\beta^2}{\epsilon^2} \frac{\epsilon^3}{2} w_0^2 \int (\nabla R(p^0) \cdot z) 2 w_0 \frac{\partial w_0}{\partial z_j}\, dz (\int w_0^2\, dz)^{-1}$$
$$-|\mathcal{S}| \frac{\beta^2}{\epsilon^2} \epsilon^4 w_0^2 \int (y^t \nabla_x \nabla_z R_0(p^0, p^0) z) 2 w_0 \frac{\partial w_0}{\partial z_j}\, dz (\int w_0^2\, dz)^{-1} + O(\epsilon^5),$$

we get (recall that $a_\epsilon = (w_0 + \epsilon^3 w_2^0 + \epsilon^4 w_1^0 + \epsilon^4 w_3^0) \chi_{\delta_0} + O(\epsilon^5)$):

$$L_g \left[\left(\frac{\partial w_0}{\partial y_j} + \epsilon^3 \frac{\partial w_2^0}{\partial y_j} + \epsilon^4 \frac{\partial w_1^0}{\partial y_j} + \epsilon^4 \frac{\partial w_3^0}{\partial y_j} \right) \chi_{\delta_0} \right]$$
$$= \Delta_g \left(\frac{\partial w_0}{\partial y_j} + \epsilon^3 \frac{\partial w_2^0}{\partial y_j} + \epsilon^4 \frac{\partial w_1^0}{\partial y_j} + \epsilon^4 \frac{\partial w_3^0}{\partial y_j} \right) - \left(\frac{\partial w_0}{\partial y_j} + \epsilon^3 \frac{\partial w_2^0}{\partial y_j} + \epsilon^4 \frac{\partial w_1^0}{\partial y_j} + \epsilon^4 \frac{\partial w_3^0}{\partial y_j} \right)$$
$$+ \frac{2 a_\epsilon}{T[a_\epsilon^2]} \left(\frac{\partial w_0}{\partial y_j} + \epsilon^3 \frac{\partial w_2^0}{\partial y_j} + \epsilon^4 \frac{\partial w_1^0}{\partial y_j} + \epsilon^4 \frac{\partial w_3^0}{\partial y_j} \right) - \frac{a_\epsilon^2}{(T[a_\epsilon^2])^2} T \left[2 a_\epsilon \left(\frac{\partial w_0}{\partial y_j} + \epsilon^3 \frac{\partial w_2^0}{\partial y_j} + \epsilon^4 \frac{\partial w_1^0}{\partial y_j} + \epsilon^4 \frac{\partial w_3^0}{\partial y_j} \right) \right] + O(\epsilon^5)$$

$$
\begin{aligned}
&= \Delta \frac{\partial w_0}{\partial y_j} - \frac{\partial w_0}{\partial y_j} + \frac{1}{3} K(p^0)\epsilon^2 \left(Q\left[\frac{\partial w_0}{\partial y_j}\right] - 2P\left[\frac{\partial w_0}{\partial y_j}\right]\right) \\
&+2\frac{w_0}{1+\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int \log\frac{|z|}{|y-z|} w_0^2(z)\,dz}\frac{\partial w_0}{\partial w_j} \\
&+\frac{w_0^2}{\left(1+\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int \log\frac{|z|}{|y-z|} w_0^2(z)\,dz\right)^2}\left(-\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int \log\frac{|z|}{|y-z|} 2w_0(z)\frac{\partial w_0}{\partial z_j}(z)\,dz\right) \\
&+\Delta\epsilon^3\frac{\partial w_2^0}{\partial y_j} - \epsilon^3\frac{\partial w_2^0}{\partial y_j} \\
&+\frac{1}{6}(\nabla K(p^0)\cdot y)\epsilon^3\left(Q\left[\frac{\partial w_0}{\partial y_j}\right] - 2P\left[\frac{\partial w_0}{\partial y_j}\right]\right) + \frac{1}{6}\epsilon^3 R_1\left[\frac{\partial w_0}{\partial y_j}\right] \\
&+2\epsilon^3 w_0\frac{\partial w_2^0}{\partial y_j} + 2\epsilon^3 w_2^0\frac{\partial w_0}{\partial y_j} - 2\epsilon^3 w_0\frac{\partial w_0}{\partial y_j}\frac{\int 2w_0 w_2^0}{\int w_0^2} \\
&+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{\epsilon^3}{2} 2w_0\frac{\partial w_0}{\partial y_j}(\nabla_x R_0(p^\epsilon,p^\epsilon)\cdot y) \\
&+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{\epsilon^3}{2} w_0^2\int(\nabla_x R_0(p^\epsilon,p^\epsilon)\cdot z)2w_0\frac{\partial w_0}{\partial z_j}\,dz(\int w_0^2\,dz)^{-1} \\
&+\Delta\epsilon^4\frac{\partial w_1^0}{\partial y_j} - \epsilon^4\frac{\partial w_1^0}{\partial y_j} \\
&+\frac{1}{20}(y^t\nabla^2 K(p^0)y)\epsilon^4\left(Q\left[\frac{\partial w_0}{\partial y_j}\right] - 2P\left[\frac{\partial w_0}{\partial y_j}\right]\right) + \frac{1}{10}\epsilon^4 R_2\left[\frac{\partial w_0}{\partial y_j}\right] \\
&+2\epsilon^4 w_0\frac{\partial w_1^0}{\partial y_j} + 2\epsilon^4 w_1^0\frac{\partial w_0}{\partial y_j} - 2\epsilon^4 w_0\frac{\partial w_0}{\partial y_j}\frac{\int 2w_0 w_1^0}{\int w_0^2} \\
&+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\epsilon^4 2w_0\frac{\partial w_0}{\partial y_j}\frac{1}{2}(y^t\nabla_x^2 R_0(p^\epsilon,p^\epsilon)y) \\
&+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\epsilon^4 w_0^2\int(y^t\nabla_x\nabla_z R_0(p^\epsilon,p^\epsilon)z)2w_0\frac{\partial w_0}{\partial z_j}\,dz(\int w_0^2\,dz)^{-1} \\
&+\epsilon^4\Delta\frac{\partial w_3^0}{\partial y_j} - \epsilon^4\frac{\partial w_3^0}{\partial y_j} \\
&+\frac{1}{45}K^2(0)|y|^2\epsilon^4\left(3Q\left[\frac{\partial w_0}{\partial y_j}\right] - 4P\left[\frac{\partial w_0}{\partial y_j}\right]\right) \\
&+2\epsilon^4 w_0\frac{\partial w_3^0}{\partial y_j} + 2\epsilon^4 w_3^0\frac{\partial w_0}{\partial y_j} - 2\epsilon^4 w_0\frac{\partial w_0}{\partial y_j}\frac{\int 2w_0 w_3^0}{\int w_0^2} + O(\epsilon^5).
\end{aligned} \tag{4.25}
$$

We now consider the contributions in (4.25) coming from w_0, w_2^0, w_1^0, w_3^0 separately.

Using (4.18), we get

$$
\begin{aligned}
&\Delta \frac{\partial w_0}{\partial y_j} - \frac{\partial w_0}{\partial y_j} + \frac{1}{3} K(p^\epsilon)\epsilon^2 \left(Q\left[\frac{\partial w_0}{\partial y_j}\right] - 2P\left[\frac{\partial w_0}{\partial y_j}\right]\right) \\
&+2\frac{w_0}{1+\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int \log\frac{|z|}{|y-z|} w_0^2(z)\,dz}\frac{\partial w_0}{\partial w_j} \\
&+\frac{w_0^2}{\left(1+\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int \log\frac{|z|}{|y-z|} w_0^2(z)\,dz\right)^2}\left(-\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int \log\frac{|z|}{|y-z|} 2w_0(z)\frac{\partial w_0}{\partial z_j}(z)\,dz\right) \\
&= \frac{\epsilon^2}{3}K(p^\epsilon)(Q-2P)\left[\frac{\partial w_0}{\partial y_j}\right] + \frac{\epsilon^2}{3}K(p^\epsilon)\frac{\partial}{\partial y_j}(rw_0')
\end{aligned}
$$

$$+\frac{w_0^2}{\left(1+\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int\log\frac{|z|}{|y-z|}w_0^2(z)\,dz\right)^2}\left(-\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int(\log\frac{|z|}{|y-z|})2w_0(z)\frac{\partial}{\partial z_j}w_0(z)\,dz\right)$$
$$-\frac{w_0^2}{\left(1+\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int\log\frac{|z|}{|y-z|}w_0^2(z)\,dz\right)^2}\left(-\frac{\epsilon^2\xi_\epsilon\beta^2}{2\pi}\int\frac{\partial}{\partial y_j}(\log\frac{|z|}{|y-z|})w_0^2(z)\,dz\right)+O(\epsilon^5). \tag{4.26}$$

We show that all terms in (4.26) vanish, except for the error terms of order $O(\epsilon^5)$, by the following identities: First we consider the coefficients of $\frac{1}{3}K(p^\epsilon)\epsilon^2$:

$$(Q-2P)\left[\frac{\partial w_0}{\partial y_j}\right]+\frac{\partial}{\partial y_j}(rw_0')$$
$$=-P\left[\frac{\partial w_0}{\partial y_j}\right]+\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial w_0}{\partial y_j}\right)+\frac{\partial}{\partial y_j}(rw_0')$$
$$=-rw_0''\cos\theta-w_0'\cos\theta+(rw_0')'\cos\theta=0. \tag{4.27}$$

Second we compute

$$\frac{\partial}{\partial y_j}[\int\log\frac{|z|}{|y-z|}w_0^2(z)\,dz]=-\int\frac{\partial}{\partial z_j}\log\frac{|z|}{|y-z|}w_0^2(z)\,dz$$
$$=\int\log\frac{|z|}{|y-z|}2w_0(z)\frac{\partial w_0}{\partial z_j}\,dz.$$

Using (4.19) we get

$$\Delta\epsilon^3\frac{\partial w_2^0}{\partial y_j}-\epsilon^3\frac{\partial w_2^0}{\partial y_j}$$
$$+\frac{1}{6}(\nabla K(p^0)\cdot y)\epsilon^3\left(Q\left[\frac{\partial w_0}{\partial y_j}\right]-2P\left[\frac{\partial w_0}{\partial y_j}\right]\right)+\frac{1}{6}\epsilon^3R_1\left[\frac{\partial w_0}{\partial y_j}\right]$$
$$+2\epsilon^3w_0\frac{\partial w_2^0}{\partial y_j}+2\epsilon^3w_2^0\frac{\partial w_0}{\partial y_j}-2\epsilon^3w_0\frac{\partial w_0}{\partial y_j}\frac{\int 2w_0w_2^0}{\int w_0^2}$$
$$+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\epsilon^3 2w_0\frac{\partial w_0}{\partial y_j}(\nabla_xR_0(p^\epsilon,p^\epsilon)\cdot y)$$
$$+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\epsilon^3w_0^2\int(\nabla_zR_0(p^0,p^0)\cdot z)2w_0\frac{\partial w_0}{\partial z_j}\,dz(\int w_0^2\,dz)^{-1}.$$
$$=\frac{\epsilon^3}{6}(\nabla K(p^0)\cdot y)\left[(Q-2P)\left[\frac{\partial w_0}{\partial y_j}\right]-\frac{\partial}{\partial y_j}(Q-2P)[w_0]\right]-\frac{\epsilon^3}{6}\frac{\partial K}{\partial x_j}(p^0)(Q-2P)[w_0]$$
$$+\frac{\epsilon^3}{6}\left[R_1\left[\frac{\partial w_0}{\partial y_j}\right]-\frac{\partial}{\partial y_j}R_1[w_0]\right]$$
$$-|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{\epsilon^3}{2}(\frac{\partial R}{\partial x_j}(p^0))w_0^2$$
$$+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\epsilon^3w_0^2\int(\nabla_zR_0(p^0,p^0)\cdot z)2w_0\frac{\partial w_0}{\partial z_j}\,dz(\int w_0^2\,dz)^{-1}+O(\epsilon^5). \tag{4.28}$$

We apply (4.27) and the identity

$$R_1\left[\frac{\partial w_0}{\partial y_1}\right]-\frac{\partial}{\partial y_1}R_1[w_0]$$
$$=\frac{\partial K}{\partial x_1}(p^0)\left(y_1\frac{\partial w}{\partial y_1}+y_2\frac{\partial w}{\partial y_2}\right)+\frac{\partial K}{\partial x_2}(p^0)\left(y_1\frac{\partial w}{\partial y_2}-y_2\frac{\partial w}{\partial y_1}\right)$$
$$=\frac{\partial K}{\partial x_1}(p^0)\,(rw')\,,$$

for $j=1$ (the case $j=2$ is handled with minor change), the term in (4.28) simplifies to

$$\frac{\epsilon^3}{3}\frac{\partial}{\partial x_j}K(p^0)(rw_0') - |\mathcal{S}|\frac{\beta^2}{\epsilon^2}\epsilon^3\frac{\partial R}{\partial x_j}(p^0)w_0^2 + O(\epsilon^5). \tag{4.29}$$

Using (4.20) we get

$$\begin{aligned}
&\Delta\epsilon^4\frac{\partial w_1^0}{\partial y_j} - \epsilon^4\frac{\partial w_1^0}{\partial y_j}\\
&+\frac{1}{20}(y^t\nabla^2K(p^0)y)\epsilon^4\left(Q\left[\frac{\partial w_0}{\partial y_j}\right] - 2P\left[\frac{\partial w_0}{\partial y_j}\right]\right) + \frac{1}{10}\epsilon^4 R_2\left[\frac{\partial w_0}{\partial y_j}\right]\\
&+2\epsilon^4 w_0\frac{\partial w_1^0}{\partial y_j} + 2\epsilon^4 w_1^0\frac{\partial w_0}{\partial y_j} - 2\epsilon^4 w_0\frac{\partial w_0}{\partial y_j}\frac{\int 2w_0w_1^0}{\int w_0^2}\\
&+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{1}{2}(y^t\nabla_x^2 R_0(p^0,p^0)y)2w_0\frac{\partial w_0}{\partial y_j}\\
&+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\epsilon^4 w_0^2\int(y^t\nabla_x\nabla_z R_0(p^0,p^0)z)2w_0\frac{\partial w_0}{\partial z_j}\,dz\left(\int w_0^2\,dz\right)^{-1}\\
&=\frac{\epsilon^4}{20}y^t\nabla^2K(p^0)y\left[(Q-2P)\left[\frac{\partial w_0}{\partial y_j}\right] - \frac{\partial}{\partial y_j}(Q-2P)[w_0]\right] - \frac{\epsilon^4}{10}\left(\frac{\partial}{\partial x_j}\nabla K(p^0)\cdot y\right)(Q-2P)[w_0]\\
&+\frac{\epsilon^4}{10}\left[R_2\left[\frac{\partial w_0}{\partial y_j}\right] - \frac{\partial}{\partial y_j}R_2[w_0]\right]\\
&-|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\epsilon^4\left(\frac{\partial}{\partial x_j}\nabla_x R_0(p^0,q^0)\cdot y\right)w_0^2\\
&+|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\epsilon^4 w_0^2\int(y^t\nabla_x\nabla_z R_0(p^0,p^0)z)2w_0\frac{\partial w_0}{\partial z_j}\,dz\left(\int w_0^2\,dz\right)^{-1}
\end{aligned} \tag{4.30}$$

Using (4.27) and

$$\begin{aligned}
&R_2\left[\frac{\partial w_0}{\partial y_1}\right] - \frac{\partial}{\partial y_1}R_2[w_0]\\
&=\left(y_1\frac{\partial^2 K}{\partial x_1^2}(p^0) + y_2\frac{\partial^2 K}{\partial x_1\partial x_2}(p^0)\right)\left(y_1\frac{\partial w}{\partial y_1} + y_2\frac{\partial w}{\partial y_2}\right)\\
&+\left(y_2\frac{\partial K}{\partial x_2^2}(p^0) + y_1\frac{\partial^2 K}{\partial x_1\partial x_2}(p^0)\right)\left(-y_1\frac{\partial w}{\partial y_2} + y_1\frac{\partial w}{\partial y_1}\right)\\
&+\frac{\partial^2 K}{\partial x_1^2}(p^0)\left(\frac{y_1^2-y_2^2}{2}\frac{\partial w}{\partial y_1} + y_1y_2\frac{\partial w}{\partial y_2}\right)\\
&+\frac{\partial^2 K}{\partial x_1\partial x_2}(p^0)\left(\frac{y_2^2-y_1^2}{2}\frac{\partial w}{\partial y_2} + y_1y_2\frac{\partial w}{\partial y_1}\right)\\
&=\left(\frac{\partial}{\partial x_1}\nabla K(p^0)\cdot y\right)(rw') + 0 + \frac{1}{2}\left(\frac{\partial^2 K}{\partial x_1^2}(p^0)\right)y_1(rw') + \frac{1}{2}\left(\frac{\partial^2 K}{\partial x_1\partial x_2}(p^0)\right)y_2(rw')\\
&=\frac{3}{2}\left(\frac{\partial}{\partial x_1}\nabla K(p^0)\cdot y\right)(rw')
\end{aligned}$$

for $j=1$ (the case $j=2$ is handled with minor change), the term in (4.30) simplifies to

$$\begin{aligned}
&\frac{\epsilon^4}{4}\left(\frac{\partial}{\partial x_j}\nabla K(p^0)\cdot y\right)(rw_0')\\
&-|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{\epsilon^4}{2}\left(\frac{\partial}{\partial x_j}\nabla_x R(p^0)\cdot y\right)w_0^2.
\end{aligned} \tag{4.31}$$

Using (4.21) and

$$r^2(3Q-4P)\left[\frac{\partial w_0}{\partial y_j}\right] + \left[\frac{\partial}{\partial y_j}\right](r^3w_0')$$

$$= -r^2 P\left[\frac{\partial w_0}{\partial y_j}\right] + 3\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial w_0}{\partial y_j}\right) + \left[\frac{\partial}{\partial y_j}\right](r^3 w_0')$$
$$= -r^3 w_0'' \cos\theta + 3r^2 w_0' \cos\theta + (r^3 w_0')' = 0, \tag{4.32}$$

we get

$$\epsilon^4 \Delta \frac{\partial w_3^0}{\partial y_j} - \epsilon^4 \frac{\partial w_3^0}{\partial y_j}$$
$$+\frac{1}{45}K^2(0)|y|^2\epsilon^4\left(3Q\left[\frac{\partial w_0}{\partial y_j}\right] - 4P\left[\frac{\partial w_0}{\partial y_j}\right]\right)$$
$$+2\epsilon^4 w_0 \frac{\partial w_3^0}{\partial y_j} + 2\epsilon^4 w_3^0 \frac{\partial w_0}{\partial y_j} - 2\epsilon^4 w_0 \frac{\partial w_0}{\partial y_j}\frac{\int 2w_0 w_3^0}{\int w_0^2}$$
$$= \frac{1}{45}K^2(0)|y|^2\epsilon^4\left(3Q\left[\frac{\partial w_0}{\partial y_j}\right] - 4P\left[\frac{\partial w_0}{\partial y_j}\right]\right) + \frac{1}{45}K^2(0)\epsilon^4\frac{\partial}{\partial y_j}(r^3 w_0') = 0. \tag{4.33}$$

Putting together the contributions of w_0, w_2^0, w_1^0, w_3^0 given in (4.26) (vanishing), (4.29), (4.31), (4.33) (vanishing), respectively, we get

$$L_g\left[\left(\frac{\partial w_0}{\partial y_j} + \epsilon^3\frac{\partial w_2^0}{\partial y_j} + \epsilon^4\frac{\partial w_1^0}{\partial y_j} + \epsilon^4\frac{\partial w_3^0}{\partial y_j}\right)\chi_{\delta_0}\right] = \frac{\epsilon^3}{3}\frac{\partial K}{\partial x_j}(p^0)(rw_0') - |\mathcal{S}|\frac{\beta^2}{\epsilon^2}\epsilon^3\left(\frac{\partial}{\partial x_j}R(p^0)\right)w_0^2$$
$$+\frac{\epsilon^4}{4}\left(\frac{\partial}{\partial x_j}\nabla K(p^0)\cdot y\right)(rw_0')$$
$$-|\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{\epsilon^4}{2}\left(\frac{\partial}{\partial x_j}\nabla R(p^0)\cdot y\right)w_0^2 + O(\epsilon^5). \tag{4.34}$$

Step 1.

Substituting the eigenfunction expansion given in (4.23) into the linear operator L_g, we get

$$L_g\left[\sum_{k=1}^{2} a_k^\epsilon\left(\frac{\partial w_0}{\partial y_k} + \epsilon^3\frac{\partial w_2^0}{\partial y_k} + \epsilon^4\frac{\partial w_1^0}{\partial y_k} + \epsilon^4\frac{\partial w_3^0}{\partial y_k}\right)\chi_{\delta_0} + \phi^\perp\right] = \lambda_\epsilon\left(\sum_{k=1}^{2} a_k^\epsilon\frac{\partial w_0}{\partial y_k}\chi_{\delta_0} + \phi^\perp\right) + O(\epsilon^5). \tag{4.35}$$

Therefore $\phi^\perp$ satisfies the equation

$$L_g[\phi^\perp] - \lambda_\epsilon\phi^\perp = \lambda_\epsilon\sum_{k=1}^{2} a_k^\epsilon\frac{\partial w_0}{\partial y_k}\chi_{\delta_0}$$
$$+\sum_{k=1}^{3} a_k^\epsilon\left(-\frac{\epsilon^3}{3}\frac{\partial}{\partial x_k}K(p^0)(rw_0') + |\mathcal{S}|\frac{\beta^2}{\epsilon^2}\epsilon^3\left(\frac{\partial}{\partial x_k}R(p^0)\right)w_0^2\right)\chi_{\delta_0}$$
$$+\sum_{k=1}^{3} a_k^\epsilon\left(-\frac{\epsilon^4}{4}\left(\frac{\partial}{\partial x_k}\nabla K(p^0)\cdot y\right)(rw_0') + |\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{\epsilon^4}{2}\left(\frac{\partial}{\partial x_k}\nabla R(p^0)\cdot y\right)w_0^2\right)\chi_{\delta_0} + O(\epsilon^5).$$

Note that the operator $L_g - \lambda_\epsilon$ is invertible with uniformly bounded inverse for ϵ small enough if domain and codomain consist of those functions in $H_\epsilon^2(\mathcal{S})$ and $L_\epsilon^2(\mathcal{S})$ which are orthogonal to $\tilde{K}_\epsilon$ and the analogously defined cokernel $\tilde{C}_\epsilon$, respectively.

Therefore Liapunov-Schmidt reduction can be applied as in Proposition 3.5.

The terms on the r.h.s. of order ϵ^3 are rotationally symmetric and so they are orthogonal to the cokernel. This implies

$$\phi^\perp = \epsilon^3\phi_2^0 + O(\epsilon^4 + |\lambda_\epsilon|) \quad \text{in } H_\epsilon^2(\mathcal{S}),$$

where ϕ_2^0 is a rotationally symmetric function.

Step 2.

We multiply (4.35) by $\frac{\partial w_0}{\partial y_l}\chi_{\delta_0}$ and integrate, using the fact that $\int \phi^\perp \frac{\partial w_0}{\partial y_j}\chi_{\delta_0}\,dy = 0$.

This gives

$$\sum_{k=1}^{2} a_k^\epsilon \int_{B(0,\delta_0/\epsilon)} L_g\left[\frac{\partial w_0}{\partial y_k} + \epsilon^3\frac{\partial w_2^0}{\partial y_j} + \epsilon^4\frac{\partial w_1^0}{\partial y_j} + \epsilon^4\frac{\partial w_3^0}{\partial y_j}\right]\frac{\partial w_0}{\partial y_l}\,dy + \int_{B(0,\delta_0/\epsilon)} L_g\left[\phi^\perp\right]\frac{\partial w_0}{\partial y_l}\,dy$$
$$= \lambda_\epsilon a_l^\epsilon \int_{B(0,\delta_0/\epsilon)}\left(\frac{\partial w_0}{\partial y_l}\right)^2 dy + O(\epsilon^5) \tag{4.36}$$

Using (4.34), we first compute for the first term on r.h.s. in (4.36)

$$\int_{B(0,\delta_0/\epsilon)} L_g\left[\frac{\partial w_0}{\partial y_k} + \epsilon^3\frac{\partial w_2^0}{\partial y_j} + \epsilon^4\frac{\partial w_1^0}{\partial y_j} + \epsilon^4\frac{\partial w_3^0}{\partial y_j}\right]\frac{\partial w_0}{\partial y_l}\,dy$$
$$= \frac{\epsilon^4}{4}\left(\frac{\partial}{\partial x_k}\frac{\partial K}{\partial x_l}(p^0)\right)\int_{\mathbb{R}^2} y_l\frac{\partial w_0}{\partial y_l}(rw_0')\,dy - |\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{\epsilon^4}{2}\left(\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}R(p^0)\right)\int_{\mathbb{R}^2} y_l\frac{\partial w_0}{\partial y_l}w_0^2\,dy + O(\epsilon^5)$$
$$= \frac{\epsilon^4\pi}{4}\left(\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}K(p^0)\right)\int_0^\infty (w_0')^2 r^3\,dr + |\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{\epsilon^4}{6}\left(\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}R_0(p^0,p^0)\right)\int_{\mathbb{R}^2} w_0^3\,dy + O(\epsilon^5).$$

Note that the terms of order ϵ^3 vanish because of symmetry.

The l.h.s. in (4.36) gives

$$\lambda_\epsilon a_l^\epsilon \int_{\mathbb{R}^2}\left(\frac{\partial w_0}{\partial y_1}\right)^2 dy = \lambda_\epsilon a_l^\epsilon \pi \int_0^\infty (w')^2 r\,dr.$$

The following error estimate for the second term on the r.h.s. of (4.36) is derived using the structure of $\phi^\perp$:

Integration by parts gives

$$\int_{B(0,\delta_0/\epsilon)} L\phi^\perp \frac{\partial w_0}{\partial y_k}\chi_{\delta_0}\,dy = \int_{\mathbb{R}^2}(L_0\phi^\perp)\frac{\partial w_0}{\partial y_k}\,dy - 2\frac{\int w_0\phi^\perp}{\int w_0^2}\int_{\mathbb{R}^2} w_0^2\frac{\partial w_0}{\partial y_k}\,dy + O(\epsilon^5)$$
$$= \int_{\mathbb{R}^2} L_0\left[\frac{\partial w_0}{\partial y_k}\right]\phi^\perp\,dy - 2\frac{\int w_0\phi^\perp}{\int w_0^2}\int_{\mathbb{R}^2} w_0^2\frac{\partial w_0}{\partial y_k}\,dy + O(\epsilon^5) = O(\epsilon^5)$$

since $\frac{\partial w_0}{\partial y_k}$ belongs to the kernel of L_0.

It remains to estimate the difference between $L_g\phi^\perp$ and $L\phi^\perp$:

$$\left|\int_{B(0,\delta_0/\epsilon)}(L_g\phi^\perp - L\phi^\perp)\frac{\partial w_0}{\partial y_k}\chi_{\delta_0}\,dy\right|$$
$$\le C(\|A_\epsilon - \xi_\epsilon w_0\|_{H^2_\epsilon(\mathcal{S})})\|\phi^\perp\|_{H^2_\epsilon(\mathcal{S})} = O(\epsilon^2)(O(\epsilon^3) + O(|\lambda_\epsilon|)) = O(\epsilon^5 + \epsilon^2|\lambda_\epsilon|).$$

This implies the estimate $\int L_g[\phi^\perp]\frac{\partial w_0}{\partial y_k}\,dy = O(\epsilon^5)$ for the second term on the r.h.s. of (4.36).

Putting all the contributions for (4.36) together, we get

$$\lambda_\epsilon a_l^\epsilon = \sum_{k=1}^{2}\epsilon^4 m_{kl}\left(\int\left(\frac{\partial w_0}{\partial y_l}\right)^2 dy\right)^{-1} + O(\epsilon^2|\lambda_\epsilon| + \epsilon^5), \tag{4.37}$$

where

$$m_{kl} = \frac{\pi}{4}\left(\frac{\partial^2 K}{\partial x_k\partial x_l}(p^0)\right)\int_0^\infty (w')^2 r^3\,dr + |\mathcal{S}|\frac{\beta^2}{\epsilon^2}\frac{1}{6}\left(\frac{\partial^2 R}{\partial x_k\partial x_l}(p^0)\right)\int_{\mathbb{R}^2} w^3\,dy.$$

We summarize the result as follows: If $\lambda_\epsilon \to 0$, then $\lambda_\epsilon \sim \frac{\epsilon^4}{\pi\int_0^\infty (w')^2\,r\,dr}\sigma_0$, where σ_0 is an eigenvalue of the matrix $\mathcal{M}$. Further, $a^\epsilon = (a_1^\epsilon, a_2^\epsilon)$ is a corresponding eigenvector of $\mathcal{M}(p^0)$, i.e. the eigenfunction is given by

$$\phi^\epsilon = \sum_{k=1}^{2} a_k^\epsilon\left(\frac{\partial w_0}{\partial y_k} + \epsilon^3\frac{\partial w_2^0}{\partial y_k} + \epsilon^4\frac{\partial w_1^0}{\partial y_k} + \epsilon^4\frac{\partial w_3^0}{\partial y_k}\right)\chi_{\delta_0} + \phi^\perp + O(\epsilon^5).$$

Completion of the proof of Theorem 1.2:

Theorem 1.2 now follows from the results in this section.

□

5. Appendix A: Expansion of the Laplace-Beltrami Operator

In this appendix, we start from a well-known power series expansion of the metric tensor for Riemannian manifolds in normal coordinates (see for e.g. [1]) and adapt it to our special case of compact manifolds to finally obtain an expansion of the Laplace-Beltrami operator which will be central to our analysis.

The expansion involves the Gaussian curvature and its derivatives in different terms and they together capture essential geometrical information critical to the existence and stability of a single spike solution.

We first derive a local expansion of the metric.

Let $p \in S$ be fixed. Then, in the normal neighborhood $B_g(p, \delta_0)$, where δ_0 is independent of ϵ and p, let us denote $x = (x_1, x_2)$ to be geodesic normal coordinates about p (i.e. $x \to q = X_p^{-1}(x) \in B_g(p, \delta_0)$). Then, instead of redeveloping a formula from scratch, we learn from [1] (Corollary 2.9), that the metric tensor has the following local expansion up to the quartic term:

$$\begin{aligned} & g_{ij}(X_p^{-1}(x)) \\ = \; & \delta_{ij} - \frac{1}{3}\sum_{k,l} R_{ikjl}(0)x_k x_l - \frac{1}{6}\sum_{k,l,t} R_{ikjl,t}(0)x_k x_l x_t - \frac{1}{20}\sum_{k,l,s,t} R_{isjt,kl}(0)x_k x_l x_s x_t \\ & + \frac{2}{45}\sum_{k,l,s,t}\left(\sum_m R_{iklm}(0)R_{jstm}(0)x_k x_l x_s x_t\right) + O(|x|^5). \end{aligned} \tag{5.1}$$

For simplicity, we will subsequently write $g_{ij}(x)$ for $g_{ij}(X_p^{-1}(x))$ and similarly for all other functions.

The sectional curvature, by definition, has a relation with the curvature tensor expressible by:

$$R_{ijij} = K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)(g_{ii}g_{jj} - g_{ji}g_{ij}).$$

Since we consider two-dimensional manifolds, the only two-dimensional subspace of $T_p\mathcal{S}$, trivially, is itself, and so we have only one sectional curvature, which coincides with the classical Gaussian curvature. Thus one can apply Bianchi identities to obtain

$$R_{ikjl} = K(g_{ij}g_{lk} - g_{il}g_{jk}),$$

where K now denotes the Gaussian curvature on the manifold, which is independent of the choice of basis of the tangent plane.

We now begin our computations.

First, note that by the compatibility equations, we always have $\nabla_m g_{ij} = 0$. Hence we can calculate in turn:

For order $O(|x|^2)$,

$$\begin{aligned}\sum_{k,l} R_{ikjl}(0)x_k x_l &= K(0)\sum_{k,l} \left.(g_{ij}g_{lk} - g_{il}g_{jk})\right|_0 x_k x_l \\ &= K(0)\sum_{k,l}(\delta_{ij}\delta_{lk} - \delta_{il}\delta_{jk})x_k x_l \\ &= K(0)a_{ij},\end{aligned}$$

where $(a_{ij}) = \begin{pmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{pmatrix}$.

For order $O(|x|^3)$,

$$\begin{aligned}\sum_{k,l,t} R_{ikjl,t}(0)x_k x_l x_t &= \sum_{k,l,t} \nabla_t \left.[K(g_{ij}g_{lk} - g_{il}g_{jk})]\right|_0 x_k x_l x_t \\ &= \sum_{k,l,t}\left\{\frac{\partial K}{\partial x_t}(0)\ \left.(g_{ij}g_{lk} - g_{il}g_{jk})\right|_0 x_k x_l x_t\right\} \\ &= \left(\sum_t \frac{\partial K}{\partial x_t}(0)x_t\right)\left(\sum_{k,l} \left.(g_{ij}g_{lk} - g_{il}g_{jk})\right|_0 x_k x_l\right) \\ &= (\nabla K(0)\cdot x)a_{ij},\end{aligned}$$

where $\nabla K = (\frac{\partial K}{\partial x_1}, \frac{\partial K}{\partial x_2})$.

For order $O(|x|^4)$, the first term is

$$\begin{aligned}&\sum_{k,l,s,t} R_{isjt,kl}(0)x_k x_l x_s x_t \\ =& \sum_{k,l,s,t} \nabla_l\nabla_k \left.[K(g_{ij}g_{ts} - g_{it}g_{js})]\right|_0 x_k x_l x_s x_t \\ =& \sum_{k,l,s,t} \frac{\partial^2 K}{\partial x_l \partial x_k}(0)\ \left.(g_{ij}g_{ts} - g_{it}g_{js})\right|_0 x_k x_l x_s x_t \\ =& \left\{\sum_{k,l}\frac{\partial^2 K}{\partial x_l \partial x_k}(0)x_k x_l\right\}\left\{\sum_{s,t}(g_{ij}g_{ts} - g_{it}g_{js})x_s x_t\right\} \\ =& (x^t\nabla^2 K(0)x)a_{ij},\end{aligned}$$

where $\nabla^2 K = \begin{pmatrix} \frac{\partial^2 K}{\partial x_1\partial x_1} & \frac{\partial^2 K}{\partial x_1\partial x_2} \\ \frac{\partial^2 K}{\partial x_2\partial x_1} & \frac{\partial^2 K}{\partial x_2\partial x_2} \end{pmatrix}$. The second term is

$$\begin{aligned}&\sum_{k,l,s,t}\left(\sum_m R_{iklm}(0)R_{jstm}(0)x_k x_l x_s x_t\right) \\ =& K^2(0)\sum_{k,l,s,t}\left(\sum_m \left.(g_{il}g_{mk} - g_{im}g_{lk})(g_{jt}g_{ms} - g_{jm}g_{ts})\right|_0 x_k x_l x_s x_t\right) \\ =& K^2(0)\sum_{k,l,s,t}\left(\sum_m(\delta_{il}\delta_{mk} - \delta_{im}\delta_{lk})(\delta_{jt}\delta_{ms} - \delta_{jm}\delta_{ts})x_k x_l x_s x_t\right) \\ =& K^2(0)\sum_m\left(\sum_{k,l}(\delta_{il}\delta_{mk} - \delta_{im}\delta_{lk})x_k x_l\right)\left(\sum_{s,t}(\delta_{jt}\delta_{ms} - \delta_{jm}\delta_{ts})x_s x_t\right) \\ =& K^2(0)|x|^2 a_{ij}\end{aligned}$$

because $(a_{ij})^2 = \begin{pmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{pmatrix}^2 = |x|^2(a_{ij})$.

Therefore, (5.1) can be simplified as follows to give our desired local expansion of the metric

$$\begin{aligned} g_{ij}(x) &= \delta_{ij} - \left[\frac{1}{3}K(0) + \frac{1}{6}(\nabla K(0)\cdot x) + \frac{1}{20}(x^t\nabla^2 K(0)x)\right. \\ &\quad \left. -\frac{2}{45}K^2(0)|x|^2\right] a_{ij} + O(|x|^5). \end{aligned}$$

Second, we derive a local expansion of the Laplace-Beltrami operator.

The Laplace-Beltrami operator in local coordinates is given by

$$\triangle_g = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\right),$$

where $|g| := \det(g_{ij})$. We also write $\partial_1 = \frac{\partial}{\partial x_1}$ and $\partial_2 = \frac{\partial}{\partial x_2}$. Moreover, we indicate the variable, with respect to which the differentials operators are defined, by a subscript.

By straightforward calculations we get

$$\begin{aligned} |g| &= 1 - |x|^2\left[\frac{1}{3}K(0) + \frac{1}{6}(\nabla_x K(0)\cdot x) + \frac{1}{20}\left(x^t\nabla^2 K(0)x\right)\right] \\ &\quad +\frac{2}{45}|x|^4 + O(|x|^5), \end{aligned}$$

$$\begin{aligned} \sqrt{|g|} &= 1 - \frac{|x|^2}{2}\left[\frac{1}{3}K(0) + \frac{1}{6}(\nabla K(0)\cdot x) + \frac{1}{20}\left(x^t\nabla^2 K(0)x\right)\right] \\ &\quad +\frac{1}{120}K^2(0)|x|^4\epsilon^4 + O(|x|^5), \end{aligned} \tag{5.2}$$

$$\begin{aligned} \frac{1}{\sqrt{|g|}} &= 1 + \frac{|x|^2}{2}\left[\frac{1}{3}K(0) + \frac{1}{6}(\nabla K(0)\cdot x) + \frac{1}{20}\left(x^t\nabla^2 K(0)x\right)\right] \\ &\quad +\frac{7}{360}K^2(0)|x|^4 + O(|x|^5), \end{aligned}$$

$$\begin{aligned} g^{ij} &= \delta^{ij} + \left[\frac{1}{3}K(0) + \frac{1}{6}(\nabla K(0)\cdot x) + \frac{1}{20}\left(x^t\nabla^2 K(0)x\right) + \frac{1}{15}K^2(0)|x|^2\right] a^{ij} \\ &\quad +O(|x|^5), \end{aligned}$$

where $(g^{ij}) := (g_{ij})^{-1}$, $\delta^{ij} := \delta_{ij}$ and $a^{ij} := a_{ij}$.

Now, since $\triangle_g = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\right) = g^{ij}\partial_i\partial_j + \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\right)\partial_j$, we calculate in turn

$$\begin{aligned} g^{ij}\partial_i\partial_j &= \triangle_x + \left[\frac{1}{3}K(0) + \frac{1}{6}(\nabla K(0)\cdot x) + \frac{1}{20}\left(x^t\nabla^2 K(0)x\right)\right. \\ &\quad \left. +\frac{1}{15}K^2(0)|x|^2\right]\left(a^{ij}\partial_i\partial_j\right) + O(|x|^5), \end{aligned}$$

where $\triangle_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, and

$$\begin{aligned} \sqrt{|g|}g^{ij} &= \delta^{ij} + \left[\frac{1}{3}K(0) + \frac{1}{6}(\nabla K(0)\cdot x) + \frac{1}{20}\left(x^t\nabla^2 K(0)x\right)\right]\left(a^{ij} - \frac{|x|^2}{2}\delta^{ij}\right) \\ &\quad +\frac{1}{120}K^2(0)|x|^4\delta^{ij} - \frac{1}{18}K^2(0)|x|^2a^{ij} + \frac{1}{15}K^2(0)|x|^2a^{ij} + O(|x|^5). \end{aligned}$$

Define $(b^{ij}) = \left(a^{ij} - \frac{|x|^2}{2}\delta^{ij}\right) = \begin{pmatrix} \frac{x_2^2-x_1^2}{2} & -x_1x_2 \\ -x_1x_2 & \frac{x_1^2-x_2^2}{2} \end{pmatrix}$. Then differentiate and group terms to obtain

$$\begin{aligned} & \partial_i\left(\sqrt{|g|}g^{ij}\right)\partial_j \\ = & \left[\frac{1}{3}K(0) + \frac{1}{6}\left(\nabla K(0)\cdot x\right) + \frac{1}{20}\left(x^t\nabla^2 K(0)x\right)\right]\left(\partial_i a^{ij}\partial_j - x_i\delta^{ij}\partial_j\right) \\ & + \frac{1}{90}K^2(0)|x|^2\left(\partial_i a^{ij}\partial_j\right) \\ & + \left[\frac{1}{6}\frac{\partial K}{\partial x_i}(0) + \frac{1}{10}\left(\frac{\partial^2 K}{\partial x_i^2}(0)x_i + \frac{\partial^2 K}{\partial x_i\partial x_{3-i}}(0)x_{3-i}\right)\right]b^{ij}\partial_j \\ & + \frac{1}{30}K^2(0)|x|^2\epsilon^4\left(x_i\delta^{ij}\partial_j\right) + \frac{1}{45}K^2(0)\left(x_i a^{ij}\partial_j\right) + O(|x|^5). \end{aligned}$$

Now substitute $\partial_i a^{ij}\partial_j = -x_i\delta^{ij}\partial_j$ and $x_i a^{ij}\partial_j = 0$ and group the differentials to get

$$\begin{aligned} & \partial_i\left(\sqrt{|g|}g^{ij}\right)\partial_j \\ = & -2\left[\frac{1}{3}K(0) + \frac{1}{6}\left(\nabla K(0)\cdot x\right) + \frac{1}{20}\left(x^t\nabla^2 K(0)x\right)\right]\left(x_i\delta^{ij}\partial_j\right) \\ & + \frac{1}{6}\left(\frac{\partial K}{\partial x_i}(0)b^{ij}\partial_j\right) + \frac{1}{10}\left(\frac{\partial^2 K}{\partial x_i^2}(0)x_i b^{ij}\partial_j\right) + \frac{1}{10}\left(\frac{\partial^2 K}{\partial x_i\partial x_{3-i}}(0)x_{3-i}b^{ij}\partial_j\right) \\ & + \frac{1}{45}K^2(0)|x|^2\left(x_i\delta^{ij}\partial_j\right) + O(|x|^5). \end{aligned}$$

Finally, focusing on the coefficient of $K^2(0)$, we find

$$\begin{aligned} \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\right)\partial_j = & -2\left[\frac{1}{3}K(0) + \frac{1}{6}\left(\nabla K(0)\cdot x\right) + \frac{1}{20}\left(x^t\nabla^2 K(0)x\right)\right]\left(x_i\delta^{ij}\partial_j\right) \\ & + \frac{1}{6}\epsilon^3\left(\frac{\partial K}{\partial x_i}(0)b^{ij}\partial_j\right) + \frac{1}{10}\left(\frac{\partial^2 K}{\partial x_i^2}(0)x_i b^{ij}\partial_j\right) \\ & + \frac{1}{10}\left(\frac{\partial^2 K}{\partial x_i\partial x_{3-i}}(0)x_{3-i}b^{ij}\partial_j\right) \\ & - \frac{4}{45}K^2(0)|x|^2\left(x_i\delta^{ij}\partial_j\right) + O(|x|^5). \end{aligned}$$

We now write out the differentials explicitly

$$\begin{aligned} a^{ij}\partial_i\partial_j &= x_2^2\partial_1^2 - 2x_1x_2\partial_1\partial_2 + x_1^2\partial_2^2, \\ x_i\delta^{ij}\partial_j &= x_1\partial_1 + x_2\partial_2, \\ \frac{\partial K}{\partial x_i}(0)b^{ij}\partial_j &= \frac{x_2^2-x_1^2}{2}\left(\frac{\partial K}{\partial x_1}(0)\partial_1 - \frac{\partial K}{\partial x_2}(0)\partial_2\right) \\ & \quad -x_1x_2\left(\frac{\partial K}{\partial x_2}(0)\partial_1 + \frac{\partial K}{\partial x_1}(0)\partial_2\right), \\ \frac{\partial^2 K}{\partial x_i^2}(0)x_i b^{ij}\partial_j &= \frac{x_2^2-x_1^2}{2}\left(x_1\frac{\partial^2 K}{\partial x_1^2}(0)\partial_1 - x_2\frac{\partial^2 K}{\partial x_2^2}(0)\partial_2\right) \\ & \quad -x_1x_2\left(x_2\frac{\partial^2 K}{\partial x_2^2}(0)\partial_1 + x_1\frac{\partial^2 K}{\partial x_1^2}(0)\partial_2\right), \\ \frac{\partial^2 K}{\partial x_i\partial x_{3-i}}(0)x_{3-i}b^{ij}\partial_j &= \frac{x_2^2-x_1^2}{2}\left(x_2\frac{\partial^2 K}{\partial x_1\partial x_2}(0)\partial_1 - x_1\frac{\partial^2 K}{\partial x_2\partial x_1}(0)\partial_2\right) \\ & \quad -x_1x_2\left(x_1\frac{\partial^2 K}{\partial x_2\partial x_1}(0)\partial_1 + x_2\frac{\partial^2 K}{\partial x_1\partial x_2}(0)\partial_2\right). \end{aligned}$$

We switch to the rescaled coordinate y by setting $x = \epsilon y$, then $\frac{\partial}{\partial x_i} = \frac{1}{\epsilon}\frac{\partial}{\partial y_i}$. So, for a function u in rescaled coordinates y, the Laplace-Beltrami operator applied on u has the following expansion:

$$\begin{aligned}\epsilon^2 \triangle_g u(x) &= \triangle_y u(y) \\ &+ \left[\frac{1}{3}K(0)\epsilon^2 + \frac{1}{6}(\nabla K(0)\cdot y)\epsilon^3 + \frac{1}{20}(y\nabla^2 K(0) y^t)\epsilon^4\right](Q[u] - 2P[u]) \\ &+ \frac{1}{45}K^2(0)|y|^2\epsilon^4 (3Q[u] - 4P[u]) \\ &+ \frac{1}{6}\epsilon^3 R_1[u] + \frac{1}{10}\epsilon^4 R_2[u], \end{aligned} \tag{5.3}$$

where $\triangle_y = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}$ and

$$Q[u](y) := y_2^2 \frac{\partial^2 u}{\partial y_1^2} - 2y_1 y_2 \frac{\partial^2 u}{\partial y_1 \partial y_2} + y_1^2 \frac{\partial^2 u}{\partial y_2^2}, \tag{5.4}$$

$$P[u](y) := y_1 \frac{\partial u}{\partial y_1} + y_2 \frac{\partial u}{\partial y_2}, \tag{5.5}$$

$$\begin{aligned} R_1[u](y) :&= \frac{y_2^2 - y_1^2}{2}\left(\frac{\partial K}{\partial x_1}(0)\frac{\partial u}{\partial y_1} - \frac{\partial K}{\partial x_2}(0)\frac{\partial u}{\partial y_2}\right) \\ &-y_1 y_2\left(\frac{\partial K}{\partial x_2}(0)\frac{\partial u}{\partial y_1} + \frac{\partial K}{\partial x_1}(0)\frac{\partial u}{\partial y_2}\right), \end{aligned} \tag{5.6}$$

$$\begin{aligned} R_2[u](y) :&= \left(\frac{y_2^2 - y_1^2}{2}\frac{\partial u}{\partial y_1} - y_1 y_2 \frac{\partial u}{\partial y_2}\right)\left(y_1 \frac{\partial^2 K}{\partial x_1^2}(0) + y_2 \frac{\partial^2 K}{\partial x_1 \partial x_2}(0)\right) \\ &- \left(\frac{y_2^2 - y_1^2}{2}\frac{\partial u}{\partial y_2} + y_1 y_2 \frac{\partial u}{\partial y_1}\right)\left(y_2 \frac{\partial^2 K}{\partial x_2^2}(0) + y_1 \frac{\partial^2 K}{\partial x_2 \partial x_1}(0)\right). \end{aligned} \tag{5.7}$$

Note that $\nabla K(0) = (\frac{\partial K}{\partial x_1}, \frac{\partial K}{\partial x_2})(0)$ and $\nabla^2 K(0) = \begin{pmatrix} \frac{\partial^2 K}{\partial x_1 \partial x_1} & \frac{\partial^2 K}{\partial x_1 \partial x_2} \\ \frac{\partial^2 K}{\partial x_2 \partial x_1} & \frac{\partial^2 K}{\partial x_2 \partial x_2} \end{pmatrix}(0)$ are not rescaled.

6. Appendix B: Some Technical Calculations

In this appendix, we compute values of several integrals needed in the proofs of existence and stability of a single spike steady state. We transform rectangular coordinates to polar coordinates by $y = (y_1, y_2) = (r\cos\theta, r\sin\theta)$. Note that if w is radially symmetric, then $\nabla w = (\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}) = (w'\cos\theta, w'\sin\theta)$, where $w' := \frac{dw}{dr}$.

Lemma 6.1. *If w is a twice differentiable, radially symmetric function on $\mathbb{R}^2$. Then*

$$Q[w] = P[w] = rw'$$

in polar coordinates (r, θ),

Proof. From the definitions, $P[w] = y_1 \frac{\partial w}{\partial y_1} + y_2 \frac{\partial w}{\partial y_2} = r\frac{\partial w}{\partial r} = rw'$, so $P[w] = rw'$.

Then note that $\frac{\partial w}{\partial \theta} = y_2 \frac{\partial w}{\partial y_1} - y_1 \frac{\partial w}{\partial y_2}$ and consider

$$\begin{aligned}
0 = \frac{\partial^2 w}{\partial \theta^2} &= y_2 \frac{\partial}{\partial y_1}\left(y_2 \frac{\partial w}{\partial y_1} - y_1 \frac{\partial w}{\partial y_2}\right) - y_1 \frac{\partial}{\partial y_2}\left(y_2 \frac{\partial w}{\partial y_1} - y_1 \frac{\partial w}{\partial y_2}\right) \\
&= y_2^2 \frac{\partial^2 w}{\partial y_1^2} - y_2 \frac{\partial w}{\partial y_2} - y_2 y_1 \frac{\partial^2 w}{\partial y_2 \partial y_1} - y_1 y_2 \frac{\partial^2 w}{\partial y_1 \partial y_2} + y_1 \frac{\partial^2 w}{\partial y_2^2} - y_1 \frac{\partial w}{\partial y_1} \\
&= y_2^2 \frac{\partial^2 w}{\partial y_1^2} - 2 y_1 y_2 \frac{\partial^2 w}{\partial y_1 \partial y_2} + y_1 \frac{\partial^2 w}{\partial y_2^2} - y_1 \frac{\partial w}{\partial y_1} - y_2 \frac{\partial w}{\partial y_2} \\
&= Q[w] - P[w].
\end{aligned}$$

□

Lemma 6.2. *If w is a twice differentiable, radially symmetric function on $\mathbb{R}^2$. Then*

$$\begin{aligned}
\int_{\mathbb{R}^2} (Q[w] - 2P[w]) y_j \frac{\partial w}{\partial y_j} dy &= -\pi \int_0^\infty (w')^2 r^3 \, dr, \\
\int_{\mathbb{R}^2} R_1[w] \frac{\partial w}{\partial y_j} dy &= -\frac{\pi}{2} \frac{\partial K}{\partial y_j}(0) \int_0^\infty (w')^2 r^3 \, dr
\end{aligned}$$

for $j = 1, 2$. Hence, $\int_{\mathbb{R}^2} \left((Q[w] - 2P[w]) y_j \frac{\partial K}{\partial y_j}(0) + R_1[w]\right) \frac{\partial w}{\partial y_j} dy = -\frac{3\pi}{2} \frac{\partial K}{\partial y_j}(0)$.

Proof.

We compute for $j = 1$. Using Lemma 6.1, and $y_1 \frac{\partial w}{\partial y_1} = r w' \cos^2 \theta$,

$$\begin{aligned}
\int_{\mathbb{R}^2} (Q[w] - 2P[w]) y_1 \frac{\partial w}{\partial y_1} dy &= \int_0^{2\pi} \int_0^\infty (-r w') r w' \cos^2 \theta r dr d\theta \\
&= -\int_0^{2\pi} \cos^2 \theta d\theta \int_0^\infty r^3 (w')^2 dr \\
&= -\pi \int_0^\infty r^3 (w')^2 dr,
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^2} R_1[w] \frac{\partial w}{\partial y_1} dy &= \int_{\mathbb{R}^2} \frac{y_2^2 - y_1^2}{2} \left(\frac{\partial K}{\partial y_1}(0) \frac{\partial w}{\partial y_1} - \frac{\partial K}{\partial y_2}(0) \frac{\partial w}{\partial y_2}\right) \frac{\partial w}{\partial y_1} dy \\
&\quad - \int_{\mathbb{R}^2} y_1 y_2 \left(\frac{\partial K}{\partial y_2}(0) \frac{\partial w}{\partial y_1} + \frac{\partial K}{\partial y_1}(0) \frac{\partial w}{\partial y_2}\right) \frac{\partial w}{\partial y_1} dy \\
&= \frac{\partial K}{\partial y_1}(0) \left[\int_{\mathbb{R}^2} \frac{y_2^2 - y_1^2}{2} \left(\frac{\partial w}{\partial y_1}\right)^2 dy - \int_{\mathbb{R}^2} y_1 y_2 \frac{\partial w}{\partial y_2} \frac{\partial w}{\partial y_1} dy\right] \\
&= \frac{\partial}{\partial x_1} K(p^0) \left[\int_0^{2\pi} \int_0^\infty r^2 \frac{\sin^2 \theta - \cos^2 \theta}{2} (w')^2 \cos^2 \theta r dr \right. \\
&\quad \left. - \int_0^{2\pi} \int_0^\infty r^2 (w')^2 \sin^2 \theta \cos^2 \theta r dr \right] \\
&= -\frac{1}{2} \frac{\partial K}{\partial x_1}(p^0) \int_0^{2\pi} \cos^2 \theta d\theta \int_0^\infty r^3 (w')^2 dr \\
&= -\frac{\pi}{2} \frac{\partial K}{\partial x_1}(p^0) \int_0^\infty r^3 (w')^2 dr.
\end{aligned}$$

The same calculations work for $j = 2$ with minor change. □

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