On matrix model partition functions for QCD with chemical potential

G. Akemann[1], Y.V. Fyodorov[2,3] and G. Vernizzi[1]

[1]Service de Physique Théorique, CEA/DSM/SPhT Saclay
Unité associée CNRS/SPM/URA 2306
F-91191 Gif-sur-Yvette Cedex, France

[2]Department of Mathematical Sciences, Brunel University
Uxbridge, UB8 3PH, United Kingdom

[3] Petersburg Nuclear Physics Institute
Gatchina 188350, Russia

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Abstract

Partition functions of two different matrix models for QCD with chemical potential are computed for an arbitrary number of quark and complex conjugate anti-quark flavors. In the large-N limit of weak nonhermiticity complete agreement is found between the two models. This supports the universality of such fermionic partition functions, that is of products of characteristic polynomials in the complex plane. In the strong nonhermiticity limit agreement is found for an equal number of quark and conjugate flavours. For a general flavor content the equality of partition functions holds only for small chemical potential. The chiral phase transition is analyzed for an arbitrary number of quarks, where the free energy presents a discontinuity of first order at a critical chemical potential. In the case of nondegenerate flavors there is first order phase transition for each separate mass scale.

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1 Introduction

The idea to use random matrix models as a simple model to study the non-perturbative phenomenon of chiral symmetry breaking in QCD [1] has been very successful. Apart from direct studies through lattice QCD or other effective models, it has become one of the available tools in this area, and we refer to [2] for a review on the matrix model approach. In the broken phase at zero temperature and chemical potential the applicability of a matrix model has been completely understood by rederiving part of its results from the underlying effective chiral Lagrangian picture, that describes the pseudo-Goldstone fields. This has been achieved for the partition function [3], the spectral correlation functions of Dirac operator eigenvalues [4, 5, 6], as well as for individual eigenvalue correlations [7].

A random matrix model that includes the effect of a baryonic chemical potential was introduced in [8], studying the nature of the quenched approximation and the chiral phase transition. The same model was enlarged to include the effect of temperature, and the phase diagram of QCD with two light flavors was predicted [9], including the existence of a tricritical point. The analysis was repeated very recently in [10], distinguishing between baryon and isospin chemical potential. Only the matrix model with isospin chemical potential can be related to an effective chiral Lagrangian of quenched QCD [11] so far.

The unquenched matrix model partition function with chemical potential for a single flavor has been analyzed in great detail, including its analytic solution and the behavior of its zeros [12, 13]. It has been used as a test case in [13, 14] for lattice algorithms for chemical potential. A complete and detailed solution of the unquenched matrix model partition function with several, non degenerate flavors has been lacking so far, apart from a first attempt in [15]. It is one of the purposes of this article to provide such a solution, both for finite-N as well as in the large-N limit. The characteristic feature of the model is the nonhermiticity induced via the chemical potential, which renders the eigenvalues of the Dirac operator complex. Under these conditions one has to distinguish in the large-N limit between the regimes of weak and strong nonhermiticity [16] (see [17] for a review).

In [18] a random matrix model with complex eigenvalues was proposed for the phase with broken chiral symmetry . The resulting spectral correlation functions were computed, and the analytic predictions were confirmed by comparing them with the results from quenched lattice simulations [19], both in the limit of weak and strong nonhermiticity. However, the equivalence in the phase with broken symmetry between the model proposed in [18] and the original model with chemical potential proposed in [8] was only conjectured [18, 15].

In a recent paper [6] the authors managed to derive the spectral density of complex eigenvalues in the regime of weak nonhermiticity directly from the effective chiral Lagrangian for quenched QCD combined with the model [8]. Exploiting a variant of recently suggested exact replica method [20] they arrived to a density profile slightly different from earlier results of [18]. Both results agree asymptotically, and the difference was too small to be distinguished from the lattice data [19] for the values of chemical potential used. Under these circumstances it is conceptually important to be able to prove the universality of results within the random matrix model approach, apart from matching them with first principle lattice data. Without such a universality random matrix models loose much of their predictive power, being deduced from global symmetry arguments alone. When the chemical potential is absent, universality was proved in [21, 22]. For correlation functions involving complex eigenvalues only partial results exist for non-chiral random matrix models [23] at weak nonhermiticity.

One of goals of the present paper is to clarify the issue of equivalence and thus possible universality of the two different random matrix models [8] and [18] for QCD with chemical potential, at the level of the corresponding partition functions. We are going to demonstrate that both models agree at the regime of weak nonhermiticity for any number of quarks and conjugate anti-quarks, and in that sense they are universal. At the regime of strong nonhermiticity the agreement, however, persists only for

an equal number of such flavors. For a general flavor content, including only quarks, the two partition functions [8, 18] agree only to the leading order term in an expansion in small chemical potential. We will also relate our findings to the so far open question of universal spectral eigenvalue correlations, by mapping it to the problem of universality of the so-called bosonic partition functions.

The object of our investigation can be phrased also in more mathematical terms. In the presence of quark flavors the problem of calculation of the random matrix model partition function amounts to computing the expectation value of a product of characteristic polynomials (also known as spectral determinants) in the model with zero flavors. Characteristic polynomials in Hermitian random matrix models have received a lot of attention recently, partly due to their relevance to the behaviour of Riemann zeta-function suggested in [24]. On the other hand, such polynomials find important applications in theory of disordered and chaotic systems, see [25] and discussion and further references in [26]. Different formulas for arbitrary products [27, 28, 29] and ratios [26, 30, 31, 32, 33, 34, 35] of characteristic polynomials have been derived and the universality of these expressions has been shown [33, 36, 37].

Again much less is known for complex matrix models [38, 39, 40]. Here one has to distinguish between characteristic polynomials and their complex conjugates. A closed determinant formula for arbitrary products of characteristic polynomials and their conjugates (of not necessary the same number) for quite a general class of models has been given in [39], see also related objects emerging in the theory of quantum chaotic scattering [41, 17]. Our aim here is thus to compute and compare such products within the two models [8, 18]. These results may also be useful when several sets of replicas are needed in the computation of two- or higher k-point eigenvalue correlation functions, generalizing [6].

We derive a compact, new expression for the partition function of the matrix model [8] at finite-N with arbitrary many quark flavours . It allows us to analyze the chiral phase transition in more detail, in particular concerning the influence on nondegenerate quark masses[1]. We find that the first order phase transition found in [8, 9] persists, and that it is always driven by the flavor with the smallest (or zero) mass. For different mass scales present there is a discontinuity of first order for each different mass. This is due to the fact that roughly speaking the partition function can be written as a determinant over single flavor partition functions. A similar phenomenon of having two first order lines for two flavors was observed in [10], where two different chemical potentials for each flavor were introduced.

The outline of the article is as follows. In the section 2 we define the two matrix model partition functions for QCD with chemical potential and compute them for finite-N. We distinguish between the presence of only quarks in section 2.1 and quarks with additional complex conjugate anti-quarks in section 2.2. Several technical details of the derivation are summarized in the appendices A and B. In section 3 we turn to the large-N limit, where we distinguish between the limit of weak and strong nonhermiticity in sections 3.1 and 3.2. The resulting consequences for the universality of matrix model partition functions are discussed in section 3.3. In section 4 we exploit the results for finite-N from section 2 to investigate the chiral phase transition at a critical chemical potential, for an arbitrary number of quarks. In section 5 we summarize our findings.

[1] We recall that the model [18] is always in the broken phase by definition, it cannot reach the phase transition.

2 Matrix model partition functions for finite-N

2.1 Partition functions with N_f quark flavors

We start with defining the first matrix model for QCD with chemical potential, initially introduced by Stephanov [8]. In the sector of topological charge[2] ν it is given by

$$\mathcal{Z}_I^{(N_f,\nu)}(\mu;\{m_f\}) \equiv \int d\Phi d\Phi^\dagger \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f \mathbf{1}_N & i\Phi + \mu\tilde{\mathbf{1}}_N \\ i\Phi^\dagger + \mu\tilde{\mathbf{1}}_N^\dagger & m_f \mathbf{1}_{N+\nu} \end{pmatrix} \exp\left[-N\langle\bar{q}q\rangle^2 \mathrm{Tr}\Phi\Phi^\dagger\right]. \quad (2.1)$$

Here Φ is a complex matrix of size $N \times (N+\nu)$. Apart from the unity matrix $\mathbf{1}_n$ of size $n \times n$ we have also introduced the rectangular unity matrix of size $N \times (N+\nu)$

$$(\tilde{\mathbf{1}}_N)_{ij} \equiv \begin{cases} \delta_{ij} & i,j = 1,\ldots,N \\ 0 & j = N+1,\ldots,N+\nu \end{cases}. \quad (2.2)$$

Eq. (2.1) contains N_f quark flavors with real masses m_f. The chemical potential μ is added to the Dirac matrix of the usual matrix model [1] in a standard way by shifting $\not{D} \to \not{D} + \gamma_0\mu$. The model has the same global symmetries as QCD with gauge group $SU(N_c \geq 3)$ in the fundamental representation. The Gaussian weight replacing the gauge field average was chosen for simplicity in [8]. It will allow us to exactly solve the partition functions for finite-N. The variance of the random matrix entries is such that the Banks-Casher relation for $\mu = 0$ is satisfied. We will not address the issue of universality by allowing for a more general weight function, $\Phi\Phi^\dagger \to V(\Phi\Phi^\dagger)$, with V being a polynomial, as in [21, 22]. Instead, we compare to a different model given in terms of complex eigenvalues [18] defined below, which reduces to the same model [1] at $\mu = 0$.

In the presence of $\mu \neq 0$ a diagonalization necessary to obtain complex eigenvalues of the Dirac operator does not any longer amounts to a simple procedure. More precisely, the angular variables used in the singular value decomposition of the matrix Φ in the form $\Phi = U_1 \Lambda U_2$,with $U_{1,2}$ being unitary, no longer decouple in the matrix integral. Although a Schur decomposition to an upper triangular form $\phi = U(Z+R)U^\dagger$ remains possible, it does not reveal a natural relevant degrees of freedom in the matrix integral. In particular, proceeding in this way one retains the eigenvalues in Z complex even after setting $\mu = 0$, which is not a natural choice of integration variables.

The standard fermionic approach to compute the partition function remains a viable alternative. It consists in replacing the determinants by equivalent Grassmann integrals, and further integrating out the matrix Φ explicitly, and finally performing a Hubbard-Stratonovich transformation. The details are given in the appendix A, and include a more general case of additional conjugate anti-quarks, to be addressed below. For real quark masses the result simplifies considerably after choosing a polar decomposition, with the Jacobian computed in appendix B. After these manipulations we arrive at the following result

$$\begin{aligned} \mathcal{Z}_I^{(N_f,\nu)}(\mu;\{m_f\}) \quad &\sim \quad \mathrm{e}^{-N\langle\bar{q}q\rangle^2 \mathrm{Tr}\, M^2} \int_0^\infty \prod_{f=1}^{N_f} dr_k \; r_k^{\nu+1}(r_k^2-\mu^2)^N \mathrm{e}^{-N\langle\bar{q}q\rangle^2 r_k^2} \; \Delta_{N_f}(r^2)^2 \\ &\times \int dV \int dU \det[U^\dagger]^\nu \exp\left[N\langle\bar{q}q\rangle^2 \mathrm{Tr}\left(MUV\hat{r}V^\dagger + V\hat{r}V^\dagger U^\dagger M\right)\right]. \quad (2.3) \end{aligned}$$

Here, dU and dV denote the Haar measure over the unitary group of size $N_f \times N_f$. The matrix $M = \mathrm{diag}(m_1,\ldots,m_{N_f})$ contains the quark masses and $\hat{r} = \mathrm{diag}(r_1,\ldots,r_{N_f})$ is the diagonal matrix

[2]Without loss of generality we restrict ourselves to $\nu \geq 0$ throughout the following.

of radial coordinates. $\Delta_{N_f}(r^2) = \prod_{k>l}^{N_f}(r_k^2 - r_l^2)$ denotes the Vandermonde determinant. Due to the unitary invariance we can shift $U \to UV^\dagger$ and then rename $U \to U^\dagger$. The resulting unitary integrals can be performed exactly using [42], together with the fact that $M = M^\dagger$. Our first main result valid for finite-N is thus reading

$$\begin{aligned}\mathcal{Z}_I^{(N_f,\nu)}(\mu;\{m_f\}) &\sim \mathrm{e}^{-N\langle\bar{q}q\rangle^2\sum_{f=1}^{N_f}m_f^2}\int_0^\infty \prod_{f=1}^{N_f} dr_k\; r_k^{\nu+1}(r_k^2-\mu^2)^N \mathrm{e}^{-N\langle\bar{q}q\rangle^2 r_k^2}\;\Delta_{N_f}(r^2) \\ &\times \frac{1}{\Delta_{N_f}(m^2)}\;\det_{i,j=1,\ldots,N_f}\left[I_\nu(2N\langle\bar{q}q\rangle^2 m_i\, r_j)\right]\;. \end{aligned} \tag{2.4}$$

Such a compact expression reducing eq. (2.1) to N_f real integrations that factorize has previously been known only for $N_f = 1$ [12]. The integral representation for $N_f = 2$ in [15] using a Schur decomposition of the matrix $Q = UV\hat{r}V^\dagger$ is more involved. Eq. (2.4) is now amenable to a saddle point computation, both at weak and strong nonhermiticity. Furthermore, it will be useful when investigating the phase transition in section 4.

We now turn to an alternative random matrix model for QCD with chemical potential, introduced in [18] in terms of N complex eigenvalues. This type of model can also be solved exactly at any finite N. Here we do not have to distinguish between real and complex masses. Moreover, due to the powerful technique of orthogonal polynomials the corresponding correlation functions of complex eigenvalues also can be found [18]. The model is defined as

$$\mathcal{Z}_{II}^{(N_f,\nu)}(\tau;\{m_f\}) \equiv \int\prod_{j=1}^{N}\left(d^2z_j\; w(z_j,z_j^*)\prod_{f=1}^{N_f} m_f^\nu\,(z_j^2+m_f^2)\right)\left|\Delta_N(z^2)\right|^2 \tag{2.5}$$

$$w(z,z^*) \equiv |z|^{2\nu+1}\exp\left[-\frac{N}{1-\tau^2}\left(|z|^2-\frac{\tau}{2}(z^2+z^{2\,*})\right)\right]\;. \tag{2.6}$$

The parameter $\tau \in [0,1]$ appearing in the Gaussian weight function $w(z,z^*)$ controls the effective degree of nonhermiticity. It allows to interpolate between models with real and maximally complex eigenvalues for $\tau = 1$ and $\tau = 0$, respectively. The two partition functions eq. (2.1) and eq. (2.5) are of course different in general at finite N. Only in the Hermitian limit $\tau \to 1$, $\mu \to 0$ they are exactly the same under the identification of the masses: $m_{f,II} = m_{f,I}\langle\bar{q}q\rangle\sqrt{2}$. In fact, in this paper we will show a much more interesting relation: two models are also generally equivalent at an appropriate large N limit, under an appropriate correspondence among the relevant parameters (the mapping between the two models actually depends on the way the large N limit is performed).

The massive partition function eq. (2.5) can be evaluated due to the following observation. We can write

$$\mathcal{Z}_{II}^{(N_f,\nu)}(\tau;\{m_f\}) \sim \left\langle\prod_{j=1}^{N}\prod_{f=1}^{N_f} m_f^\nu\,(z_j^2+m_f^2)\right\rangle\;, \tag{2.7}$$

where the expectation value is taken with respect to the zero flavor partition function, $\mathcal{Z}_{II}^{(N_f=0,\nu)}(\tau)$. The relation to a product of characteristic polynomials is evident.

The relevant set of orthogonal polynomials with respect to the weight eq. (2.6) are given by the complex generalization of standard Laguerre polynomials, see [18] :

$$\tilde{P}_k^{(\nu)}(z^2) \equiv (-1)^k k!\left(\frac{2\tau}{N}\right)^k L_k^{(\nu)}\left(\frac{Nz^2}{2\tau}\right)\;, \tag{2.8}$$

which are given here in the monic normalization. Using the theorem proven in [39] the expectation value eq. (2.7) can be conveniently expressed as a determinant of size $N_f \times N_f$, with entries being these orthogonal polynomials. Taking into account the extra factors $\prod_{f=1}^{N_f} m_f^\nu$ we arrive at [39]

$$\mathcal{Z}_{II}^{(N_f,\nu)}(\tau;\{m_f\}) \;\sim\; \frac{\det_{k,l=1,\ldots,N_f}\left[m_l^\nu \tilde{P}_{N+k-1}^{(\nu)}(-m_l^2)\right]}{\Delta_{N_f}(m^2)}\,. \tag{2.9}$$

We would like to note that the partition function is real for real quark masses as it should be. This is not obvious at all when looking at the definition eq. (2.5). The obtained result is exact for finite-N. It can also be continued to complex quark masses without modification.

2.2 Partition functions with quarks and complex conjugate anti- quarks

In this subsection we enlarge the flavor space by adding pairs of complex conjugate quarks to the partition functions eqs. (2.1) and (2.5). Such partition functions were already considered for one such pair in [8] when analyzing the quenched approximation and for n such pairs of degenerate mass in [6] in the replica approach. The special feature of quarks and conjugate anti-quarks occurring in the partition function at the same time is that they may form a non vanishing meson density. This is, in turn, reflected in the existence of an effective chiral Lagrangian with isospin chemical potential as pointed out in [11], linking it to QCD.

Here, we will derive such an effective model in terms of a unitary group integral over the Goldstone manifold directly from the underlying random matrix model. We will allow for any, not necessarily equal number of quarks and conjugate anti-quarks with non degenerate mass. In the first model eq. (2.1) the presence of additional conjugate anti-quarks implies considerably more effort in computing the partition functions. In the procedure we will make use of the results from [6]. In contrast to that in the second model eq. (2.5) the resulting partition functions immediately follow from the theorem proved in [39].

The matrix model partition functions eq. (2.1) with m quarks of mass m_f and n conjugate anti-quarks of mass n_f^* in the sector of topological charge ν is defined as

$$\begin{aligned}
&\mathcal{Z}_I^{(N_f=m+n,\nu)}(\mu;\{m_f\}_m,\{n_g^*\}_n) \;\equiv \qquad (2.10)\\
&\equiv \int d\Phi d\Phi^\dagger \prod_{f=1}^{m} \det \begin{pmatrix} m_f \mathbf{1}_N & i\Phi + \mu\tilde{\mathbf{1}}_N \\ i\Phi^\dagger + \mu\tilde{\mathbf{1}}_N^\dagger & m_f \mathbf{1}_{N+\nu} \end{pmatrix} \prod_{g=1}^{n} \det \begin{pmatrix} n_g^* \mathbf{1}_N & -i\Phi + \mu\tilde{\mathbf{1}}_N \\ -i\Phi^\dagger + \mu\tilde{\mathbf{1}}_N^\dagger & n_g^* \mathbf{1}_{N+\nu} \end{pmatrix} \mathrm{e}^{-N\langle\bar{q}q\rangle^2 \mathrm{Tr}\Phi\Phi^\dagger}.
\end{aligned}$$

Note that in contrast to the previous section we now allow for complex masses, as they may serve as source terms for the complex Dirac operator eigenvalues (see e.g. in [6]). Using the standard fermionization technique we arrive at the following matrix model representation in terms of the complex matrix Q of size $N_f \times N_f$

$$\begin{aligned}
\mathcal{Z}_I^{(N_f,\nu)}(\mu;M) \;\sim\; & \int dQ dQ^\dagger \det[M+Q^\dagger]^\nu \qquad (2.11)\\
& \times \det\left[(M+Q^\dagger)(M+Q) - \mu^2 (M+Q^\dagger)\Sigma_3 (M+Q^\dagger)^{-1}\Sigma_3\right]^N \mathrm{e}^{-N\langle\bar{q}q\rangle^2 \mathrm{Tr}\, Q^\dagger Q},
\end{aligned}$$

with the mass matrix $M = \mathrm{diag}(m_1,\ldots,m_m,-n_1^*,\ldots,-n_n^*)$ and $\Sigma_3 = \mathrm{diag}(\mathbf{1}_m,-\mathbf{1}_n)$ being a generalized Pauli matrix. The details of the derivation are given in appendix A. It is interesting here to work out the limit of real masses, $M = M^\dagger$ explicitly. To this end, we introduce a polar decomposition of

the matrix $Q = U\,R$ with $R = V\hat{r}V^\dagger$ being hermitian with positive eigenvalues r_k, and U, V being unitary. Using eq. (A.9) and the Jacobian calculated in the appendix B we obtain

$$\begin{aligned}\mathcal{Z}_I^{(N_f,\nu)}(\mu;M) &\sim \mathrm{e}^{-N\langle\bar{q}q\rangle^2 \mathrm{Tr}\, M^2}\int_0^\infty \prod_{f=1}^{N_f} dr_k\; r_k^{\nu+1}\mathrm{e}^{-N\langle\bar{q}q\rangle^2 r_k^2}\;\Delta_{N_f}(r^2)^2\int dV\int dU\,\det[U^\dagger]^\nu \\ &\times \mathrm{e}^{N\langle\bar{q}q\rangle^2 \mathrm{Tr} M(UV\hat{r}V^\dagger + V\hat{r}V^\dagger U^\dagger)}\det\left[V\hat{r}^2V^\dagger - \mu^2 V\hat{r}V^\dagger U^\dagger\Sigma_3 U V\hat{r}^{-1}V^\dagger\Sigma_3\right]^N .\end{aligned} \tag{2.12}$$

This expression will be used in the next section.

Let us now consider the second matrix model [18] with pairs of complex conjugate quarks inserted. The corresponding partition function is defined as

$$\begin{aligned}\mathcal{Z}_{II}^{(N_f,\nu)}(\tau;\{m_f\}_m,\{n_g^*\}_n) &\equiv \int\prod_{j=1}^N d^2z_j\; w(z_j,z_j^*)\prod_{f=1}^m m_f^\nu(z_j^2+m_f^2)\prod_{g=1}^n n_g^{*\nu}(z_j^{*2}+n_g^{*2})\left|\Delta_N(z^2)\right|^2 \\ &\sim \left\langle\prod_{j=1}^N\left(\prod_{f=1}^{N_f} m_f^\nu(z_j^2+m_f^2)\prod_{g=1}^n n_g^{*\nu}(z_j^{*2}+n_g^{*2})\right)\right\rangle ,\end{aligned} \tag{2.13}$$

where we again presented this object in a form of an expectation value. In contrast to eq. (2.13) we do not need to distinguish between signs of masses and chemical potential for the quarks and their conjugates. The reason is that the large-N result for eq. (2.13) turns out to be a quadratic function in all masses. Apart from that, in the identification between the nonhermiticity parameters τ and μ appears quadratically. The same weight function eq. (2.6) for all eigenvalues is therefore equally valid for both signs of the chemical potential[3].

The evaluation of eq. (2.13) is again straightforward, due to the general theorem proved in [39]. To write the result in a compact form we need to introduce more notation. In addition to the orthogonal polynomials eq. (2.8) the so-called bare kernel made of these polynomials appears in the corresponding expressions:

$$\begin{aligned}\kappa_N(z^2,u^{*2}) &\equiv \sum_{k=0}^{N-1} P_k^{(\nu)}(z^2)P_k^{(\nu)}(u^{*2}) \\ &= \frac{1}{f^{(\nu)}(\tau)}\sum_{k=0}^{N-1}\frac{\Gamma(\nu+1)\,k!}{\Gamma(\nu+k+1)}\tau^{2k}L_k^{(\nu)}\left(\frac{Nz^2}{2\tau}\right)L_k^{(\nu)}\left(\frac{Nu^{*2}}{2\tau}\right) .\end{aligned} \tag{2.14}$$

The kernel contains a sum over the orthonormalized polynomials $P_k^{(\nu)}(z^2)$,

$$P_k^{(\nu)}(z^2) \equiv h_k^{-\frac{1}{2}}\tilde{P}_k^{(\nu)}(z^2) , \tag{2.15}$$

with norms given by

$$h_k \equiv \int d^2z\; w(z,z^*)\tilde{P}_k^{(\nu)}(z^2)\tilde{P}_k^{(\nu)}(z^{*2}) = f^{(\nu)}(\tau)\left(\frac{2}{N}\right)^{2k}\frac{\Gamma(\nu+1+k)\,k!}{\Gamma(\nu+1)} . \tag{2.16}$$

[3]In this sense the model does not distinguish between isospin and baryon chemical potential.

Here we have introduced the functions

$$f^{(\nu)}(\tau) \equiv \int d^2z\ w(z,z^*) = \pi\Gamma\left(\nu+\frac{3}{2}\right)(1-\tau^2)^{\frac{\nu}{2}+\frac{3}{4}}\ \mathcal{P}_{\nu+\frac{1}{2}}\left(\frac{1}{\sqrt{1-\tau^2}}\right)\ , \tag{2.17}$$

where $\mathcal{P}_\gamma(x)$ stands for the Legendre function. The full kernel $K_N(z,u^*)$ as it appears in the expression for the eigenvalue correlation functions is then obtained by multiplying the bare kernel with the weight functions, $K_n(z,u^*) \equiv [w(z,z^*)w(u,u^*)]^{\frac{1}{2}}\kappa_N(z^2,u^{*\,2})$. Following [39] we immediately obtain

$$\mathcal{Z}_{II}^{(N_f,\nu)}(\tau;\{m_f\}_m,\{n_g^*\}_n) \sim \frac{\prod_{f=1}^m m_f^\nu\ \prod_{g=1}^n n_g^{*\,\nu}}{\Delta_m(m^2)\Delta_n(n^{*\,2})}\left(\prod_{k=N}^{N+m-1}h_k\right)\det_{i,j=1,\ldots,n}[\mathcal{B}(m_i,n_j^*)]\ , \tag{2.18}$$

$$\mathcal{B}(m_i,n_j^*) \equiv \begin{cases} \kappa_{N+m}(-m_i^2,-n_j^{*\,2}) & i=1,\ldots,m \\ \tilde{P}^{(\nu)}_{N+i-1}(-n_j^{*\,2}) & i=m+1,\ldots,n \end{cases}\ . \tag{2.19}$$

where we may assume that $n \geq m$, without loss of generality (the case $n<m$ follows from complex conjugation). This result is exact for finite-N. We note that the model eq. (2.13) (and eq. (2.5)) is always in the phase with broken chiral symmetry [18].

3 The large-N limit

3.1 The weak nonhermiticity limit

The limit of weak nonhermiticity [16] is defined by taking simultaneously the large-N limit $N\to\infty$ and the hermitian limit $\tau\to 1$ or $\mu\to 0$ in such a way that the following product is kept constant:

$$\lim_{N\to\infty,\,\tau\to 1} N(1-\tau^2) \equiv \alpha^2 \tag{3.1}$$

or, equivalently

$$\lim_{N\to\infty,\,\mu\to 0} 2N\langle\bar{q}q\rangle^2\mu^2 = \tilde{\alpha}^2\ . \tag{3.2}$$

In this limit the macroscopic spectral density has support only on the real line and is given by a semicircle for both models eqs. (2.1) and (2.5), in the limit $\tau\to 1$ and $\mu\to 0$ respectively [18]. In contrast to that the microscopic correlation functions differ from those on the real line. They still extend into the complex plane and depend explicitly on the parameter α or $\tilde{\alpha}$. To identify them for the two models a relation between τ and μ, or α and $\tilde{\alpha}$, has to be imposed, which we will find by comparing the two partition functions.

We will see that also the partition functions at weak nonhermiticity contain important information, and this has already been exploited for example in [6] when computing the microscopic density. It turns out that the partition functions with quarks alone differ from the partition functions at $\mu=0$ only by an overall prefactor $\exp[-N_f\alpha/2]$. When adding conjugate anti-quarks the situations however changes and the partition functions differ from their $\mu=0$ values nontrivially[4].

The microscopic rescaling of the quark masses (and complex eigenvalues) at weak nonhermiticity is defined as

$$\begin{aligned} \zeta_f &\equiv 2N\langle\bar{q}q\rangle m_f\ , f=1,\ldots,m \\ \xi_g^* &\equiv 2N\langle\bar{q}q\rangle n_g^*\ , g=1,\ldots,n \end{aligned} \tag{3.3}$$

[4]Obviously, for only conjugate anti-quarks alone we are back to the situation of only quarks.

where $2N\langle\bar{q}q\rangle \sim \rho(0)$ is the macroscopic density at the origin of the model eq. (2.1), after taking the limit $\mu \to 0$.

We begin with the partition functions containing only quarks from subsection 2.1. First of all due to the rescaling of the masses eq. (3.3) the constant prefactor $\mathrm{e}^{-N\langle\bar{q}q\rangle^2 \mathrm{Tr}\, M^2}$ reduces to unity. Taking the weak limit eq. (3.2) we can replace the factor

$$(r_k^2 - \mu^2)^N \;\to\; r_k^{2N} \exp\left[-N\mu^2 r_k^{-2}\right] \tag{3.4}$$

inside the integral eq. (2.4) over the radial coordinates. After rescaling the arguments of the Bessel functions through eq. (3.3) a saddle point evaluation in the variables r_k leads to the value

$$\hat{r}|_{\mathrm{sp}} \;=\; \mathbf{1}_{N_f}\langle\bar{q}q\rangle^{-1} \; . \tag{3.5}$$

Since the determinant in eq. (2.4) is degenerate at the saddle point we have to take into account the Gaussian fluctuations. Performing the calculation, we get

$$\mathcal{Z}_I^{(N_f,\nu)}(\mu;\{\zeta_f\})|_{\mathrm{weak}} \;\sim\; \exp\left[-N_f\langle\bar{q}q\rangle^2 N\mu^2\right] \; \frac{\det_{k,l=1,\ldots,N_f}\left[\zeta_l^{k-1} I_\nu^{(k-1)}(\zeta_l)\right]}{\Delta_{N_f}(\zeta^2)} \; . \tag{3.6}$$

With $I_\nu^{(j)}(x)$ we denote the $j-$th derivative of the modified Bessel functions. This final expression for the partition function generalizes previous results which were known only in the sector of topological charge $\nu = 0$ with $N_f = 1$ [12] and $N_f = 2, 3$ flavors [15]. Eq. (3.6) is exactly the same as for $\mu = 0$ [3, 43], apart from the exponential prefactor $\exp[-N_f\tilde{\alpha}^2/2]$. This implies in particular that the partition functions obey the same consistency conditions [44] as those for $\mu = 0$. More generally speaking, they belong to the same Toda lattice hierarchy as it was already exploited in [6].

We turn to the second partition function eq. (2.9). In order to read off the proper rescaling defined in eq. (3.3) we first take the *Hermitian* limit $\tau \to 1$ of the weight eq. (2.6) in order to determine the corresponding variance. This procedure is based on the macroscopic spectral density at weak nonhermiticity being given by taking the Hermitian limit[5]:

$$\begin{aligned}
\lim_{\tau\to 1} w(z,z^*) \;&=\; \lim_{\tau\to 1} (x^2+y^2)^{\nu+\frac{1}{2}} \exp\left[-\frac{N}{1+\tau}x^2 - \frac{N}{1-\tau}y^2\right] \\
&=\; x^{2\nu+1}\sqrt{\frac{\pi\alpha^2}{2N^2}}\,\delta(y)\exp\left[-\frac{1}{2}Nx^2\right] \; ,
\end{aligned} \tag{3.7}$$

where we have inserted $z = x + iy$. Therefore we have to use the value $\langle\bar{q}q\rangle = 1/\sqrt{2}$ in eq. (3.3), leading to the correspondence $\zeta_f = \sqrt{2}Nm_f$ and similar for the conjugate. Such a rescaling which is different for the two models is precisely the mapping between the two sets of different mass parameters mentioned after eq. (2.6).

We could have also introduced a variable $\langle\bar{q}q\rangle$ in the model eq. (2.9) by rescaling the eigenvalues. However, since the parameter drops out after microscopic rescaling we kept $\langle\bar{q}q\rangle^2 = 1/2$ for simplicity here. In order to perform the weak nonhermiticity limit in eq. (2.9) we first extract the powers of τ used in the definition of the polynomials (2.8) from the corresponding determinant. Using the definition $\tau^2 = 1 - \alpha^2/N$ we obtain the prefactor

$$\lim_{N\to\infty\,\tau\to 1} \tau^{N_f N + N_f(N_f-1)/2} \;=\; \exp\left[-\frac{1}{2}N_f\alpha^2\right] \; . \tag{3.8}$$

[5]The corresponding weak nonhermiticity limit for the weight function is given in section 3.3 eq. (3.59), but we do not need it here.

The remaining determinant of Laguerre polynomials can be evaluated in the same way as for $\mu = 0$ using eq. (3.3)

$$\lim_{N\to\infty,\tau\to 1} m^\nu L_N^{(\nu)}\left(\frac{-Nm^2}{2\tau}\right) \sim I_\nu(\zeta) , \tag{3.9}$$

and differentiating due to degeneracy. Ignoring constant factors and powers of N we thus obtain

$$\mathcal{Z}_{II}^{(N_f,\nu)}(\alpha;\{\zeta_f\})|_{\rm weak} \sim \exp\left[-\frac{1}{2}N_f\alpha^2\right] \frac{\det_{k,l=1,\ldots,N_f}\left[\zeta_l^{k-1}I_\nu^{(k-1)}(\zeta_l)\right]}{\Delta_{N_f}(\zeta^2)} . \tag{3.10}$$

We immediately see that the two partition functions (3.6) and (3.10) agree upon identifying

$$(1-\tau^2) = 2\langle\bar{q}q\rangle^2\mu^2 \tag{3.11}$$

or, equivalently $\alpha = \tilde{\alpha}$. Such an identification can be conveniently interpreted in terms of equating the two macroscopic densities in the complex plane for small μ [18]. In principle, partition functions can always be multiplied by an overall constant, making them agree. The nontrivial statement is that in both cases the rescaled nonhermiticity parameter factors out.

We thus proved the equivalence of the two partition functions is the limit of weak nonhermiticity, for an arbitrary number of flavors N_f of nondegenerate masses m_f for any given topological sector ν. This equivalence was claimed in [15] and verified there for degenerate masses and sector $\nu = 0$, up to and including $N_f = 3$ flavors.

Next we investigate the weak nonhermiticity limit of the partition function with quarks and conjugate anti-quarks, eq. (2.11). In this case, all powers higher than linear in M and μ^2 in eq. (2.11) are dropped. After expanding also the determinant to the power N in powers of M, we have

$$\begin{aligned}\mathcal{Z}_I^{(N_f,\nu)}(\mu;M) \sim & \int dQdQ^\dagger \det[Q^\dagger]^\nu e^{-N\langle\bar{q}q\rangle^2 {\rm Tr}\, Q^\dagger Q} \\ & \times\exp\Big[N{\rm Tr}\ln(Q^\dagger Q) + N{\rm Tr}\left(M(Q^{-1}+Q^{\dagger\,-1}) - \mu^2 Q^{-1}\Sigma_3 Q^{\dagger\,-1}\Sigma_3\right) + \mathcal{O}(1/N)\Big].\end{aligned} \tag{3.12}$$

By using again the polar decomposition $Q = U\,R$ and the respective Jacobian, we obtain

$$\begin{aligned}\mathcal{Z}_I^{(N_f,\nu)}(\mu;M) \sim & \int_0^\infty \prod_{f=1}^{N_f} dr_k\; r_k^{\nu+1} e^{-N\langle\bar{q}q\rangle^2 r_k^2 + N\ln r_k^2}\;\Delta_{N_f}(r^2)^2 \int dV \int dU \det[U^\dagger]^\nu \\ & \times \exp\left[N{\rm Tr}\left(M(V\hat{r}^{-1}V^\dagger U^\dagger + UV\hat{r}^{-1}V^\dagger)\right) - \mu^2 N{\rm Tr}\left(V\hat{r}^{-1}V^\dagger U^\dagger\Sigma_3 UV\hat{r}^{-1}V^\dagger\Sigma_3\right)\right] .\end{aligned} \tag{3.13}$$

In this form the integral is amenable to a saddle point approximation at large-N in the variables r_k, which will lead to the chiral Lagrangian picture [11]. In the previous case, with only quarks included the μ-dependence factorizes out as we discussed above. Now the dependence will be less trivial, making it a real check for the equivalence of the two models. Due to the rescaling eqs. (3.3) and (3.2) the exponents inside the unitary integral are of the order of unity and the the saddle point value is taken at

$$\hat{r}|_{\rm sp} = \mathbf{1}_{N_f}\langle\bar{q}q\rangle^{-1} , \tag{3.14}$$

as previously in eq. (3.5). Since the unitary integral is non vanishing at $\hat{r}|_{\rm sp} \sim \mathbf{1}_{N_f}$, we have

$$\mathcal{Z}_I^{(N_f,\nu)}(\mu;M)|_{\rm sp} \sim \int dU \det[U^\dagger]^\nu \exp\left[N\langle\bar{q}q\rangle{\rm Tr}(M(U^\dagger+U)) - \mu^2 N\langle\bar{q}q\rangle^2{\rm Tr}(U^\dagger\Sigma_3 U\Sigma_3)\right] . \tag{3.15}$$

This is precisely (a zero-dimensional version of) the chiral effective theory for isospin chemical potential as derived in [11], with replacements $2N \longleftrightarrow V$ and $\mu^2\langle\bar{q}q\rangle^2 \longleftrightarrow \mu^2 F_\pi^2$. The manifestation of the zero-dimensional nature is that the parameter F_π, the pion decay constant entering in the chiral Lagrangian [11], is not contained in the present matrix model partition function, as it comes from the non-zero momentum modes of the Goldstone bosons. If we wish to write the $\mu-$dependent term as $\frac{1}{2}\mathrm{Tr}[U^\dagger,\Sigma_3][U,\Sigma_3]$ as it appears when deriving it from a gauge principle, we have to multiply a factor $\exp\left[N\langle\bar{q}q\rangle^2\mu^2 N_f\right]$ to the partition function. By doing so also in the previous section it would remove the trivial $\mu-$dependence, that leads to an unphysical negative quark number density in the broken phase (see e.g. in [12] for $N_f = 1$). Finally we wish to mention that the usual mass term in the unquenched, effective QCD Lagrangian is $\mathrm{Tr}(M^\dagger U^\dagger + MU)$ instead. There complex masses are needed to locate the dependence of the $\theta-$angle in the action for example, leaving the partition function real.

We are now ready to evaluate the unitary integral in eq. (3.15) exactly, following closely [6]. There, the following parameterization of the unitary matrix $U \in U(N_f = m+n)$ has been suggested[6]

$$U = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}\Lambda\begin{pmatrix} v_1^\dagger & 0 \\ 0 & v_2^\dagger \end{pmatrix}, \tag{3.16}$$

$$\Lambda \equiv \begin{pmatrix} \hat{\lambda} & \sqrt{\mathbf{1}_m - \hat{\lambda}^2} & 0 \\ \sqrt{\mathbf{1}_m - \hat{\lambda}^2} & -\hat{\lambda} & 0 \\ 0 & 0 & -\mathbf{1}_{n-m} \end{pmatrix}. \tag{3.17}$$

Here, we denote $\hat{\lambda} \equiv \mathrm{diag}(\lambda_1,\ldots,\lambda_m)$ with $\lambda_k \in [0,1]$ for $k = 1,\ldots,m$. The unitary submatrices are $u_1, v_1 \in U(m)$, $u_2 \in U(n)$ and $v_2 \in \tilde{U}(n) \equiv U(n)/(U(1)^m \times U(n-m))$, where $n \geq m$ has been chosen without loss of generality. The Jacobian for this transformation was computed in [6],

$$J(\{\lambda_k\}) \equiv \prod_{k>l}^{m}(\lambda_k^2 - \lambda_l^2)^2 \prod_{j=1}^{m} 2\lambda_k(1-\lambda_k^2)^{n-m}. \tag{3.18}$$

For the trace containing Σ_3 it follows

$$\mathrm{Tr}(U^\dagger\Sigma_3 U\Sigma_3) = n - 3m + 4\sum_{k=1}^{m}\lambda_k^2. \tag{3.19}$$

With these steps taken we can insert the above parameterization in eq. (3.15), arriving at the following factorized group integrals

$$\begin{aligned}\mathcal{Z}_I^{(N_f,\nu)}(\mu;\{m_f\}_m,\{n_g^*\}_n)\,|_{\mathrm{weak}} \quad &\sim \exp\left[-(n-3m)N\langle\bar{q}q\rangle^2\mu^2\right]\int_0^1\prod_{k=1}^{m}d\lambda_k J(\{\lambda_k\})\mathrm{e}^{-4\langle\bar{q}q\rangle^2 N\mu^2\lambda_k^2} \\ &\times \int_{U(m)}du_1\int_{U(m)}dv_1\,\det[u_1]^\nu\exp\left[N\langle\bar{q}q\rangle\mathrm{Tr}(u_1^\dagger v_1^\dagger\hat{m}\,v_1\hat{\lambda} + v_1^\dagger\hat{m}\,v_1u_1\hat{\lambda})\right] \\ &\times \int_{U(n)}du_2\int_{\tilde{U}(n)}dv_2\,\det[u_2]^\nu\exp\left[N\langle\bar{q}q\rangle\mathrm{Tr}(u_2^\dagger v_2^\dagger\hat{n}^*v_2\hat{\lambda}_- + v_2^\dagger\hat{n}^*v_2u_2\hat{\lambda}_-)\right].\end{aligned} \tag{3.20}$$

By splitting into blocks we introduced the following obvious notation, denoting $\hat{m} \equiv \mathrm{diag}(m_1\ldots,m_m)$, $\hat{n}^* \equiv \mathrm{diag}(-n_1^*\ldots,-n_n^*)$, and by $\hat{\lambda}_- \equiv \mathrm{diag}(-\lambda_1,\ldots,-\lambda_m,-1,\ldots,-1)$ a matrix of size $n\times n$. Performing the additional transformations $u_i \to v_i^\dagger u_i$ and renaming $u_i \to u_i^\dagger$ for $i = 1,2$, we can bring

[6] In addition to [6] we have performed an extra rotation $u_i \to v_i u_i v_i^\dagger$ for $i = 1,2$, interchanging the first 2 matrices.

both double unitary integrals to the form

$$\int_U du \int_U dv \det[u\,v]^\nu \exp\left[\frac{1}{2}\mathrm{Tr}(u^\dagger \hat{a}\, v^\dagger \hat{b} + v\hat{a}\, u\hat{b})\right] = \frac{1}{\Delta(a^2)\Delta(b^2)} \det_{i,j}[I_\nu(a_i b_j)]\ , \qquad (3.21)$$

with $U = U(l)$ for $l = m$ or n, respectively[7]. Here we have used the result [45] which gives this integral also in the case when the matrices $\hat{a}$ and $\hat{b}$ are not hermitian. In our special case the a_i^2 and b_i^2 are the complex eigenvalues of the diagonal matrices $\hat{a}^2$ and $\hat{b}^2$ respectively. In the second double unitary group integral we still have to take limits due to the $(n-m)$-fold degeneracy in the matrix $\hat{\lambda}_-$. Both integrals together then cancel the Jacobian $J(\{\lambda_k\})$ up to the factor $\prod_{k=0}^m 2\lambda_k$, leading to

$$\mathcal{Z}_I^{(N_f,\nu)}(\mu;\{\zeta_f\}_m,\{\xi_g^*\}_n)|_{\rm weak} \sim \frac{\exp[-(n-3m)\langle\bar{q}q\rangle^2 N\mu^2]}{\Delta_m(\zeta^2)\Delta_n(\xi^{*2})} \int_0^1 \prod_{k=1}^m d\lambda_k \lambda_k \mathrm{e}^{-4\langle\bar{q}q\rangle^2 N\mu^2\lambda_k^2}$$
$$\times \det_{k,l=1,\ldots,m}[I_\nu(\lambda_k\zeta_l)]\ \det\begin{pmatrix} I_\nu(\xi_1^*\lambda_1) & \cdots & I_\nu(\xi_1^*\lambda_m) & I_\nu(\xi_1^*) & \cdots & \xi_1^{*\,n-m-1} I_\nu^{(n-m-1)}(\xi_1^*) \\ \vdots & & \vdots & \vdots & & \vdots \\ I_\nu(\xi_n^*\lambda_1) & \cdots & I_\nu(\xi_n^*\lambda_m) & I_\nu(\xi_n^*) & \cdots & \xi_n^{*\,n-m-1} I_\nu^{(n-m-1)}(\xi_n^*) \end{pmatrix}. \qquad (3.22)$$

The above equation can be further simplified. First we multiply the two determinants of Bessel functions, $\det(B)\det(A) = \det(AB)$. Because of the different size of the two matrices we have to add a block of unity $\mathbf{1}_{n-m}$ to the first matrix $I_\nu(\lambda_k\zeta_l)$ of size $m \times m$. Each entry in the first m columns of the resulting product matrix AB contains a sum of m terms, $\sum_{j=1}^m I_\nu(\xi_k^*\lambda_j) I_\nu(\lambda_j\zeta_l)$. Due to the symmetry of the integrand under permutations of the variables λ_k and due to the invariance properties of determinants the sums can be reduced to single terms, $I_\nu(\xi_k^*\lambda_k) I_\nu(\lambda_k\zeta_l)$, and we only sketch the procedure briefly. We begin with the first column. As determinants differing only by one column can be added, we expand the determinant as a sum of m terms, which only differ from each other by the labeling of λ_k in the first column. Due to the invariance with respect to permuting λ_k all terms can be written as m times the same determinant, with label λ_1 in the first column. Next we can eliminate all the terms with label λ_1 in the remaining columns, by successively subtracting the first column times an appropriate factor. Next we process the second column in the same way, keeping only the label λ_2 there, and so forth.

We finally arrive at the following expression

$$\mathcal{Z}_I^{(N_f=m+n,\nu)}(\mu;\{\zeta_f\}_m,\{\xi_g^*\}_n)|_{\rm weak} \sim \frac{\exp\left[-(n-3m)\langle\bar{q}q\rangle^2 N\mu^2\right]}{\Delta_m(\zeta^2)\Delta_n(\xi^{*2})} \times$$
$$\det\begin{pmatrix} \int_0^1 d\lambda_1 \lambda_1 \mathrm{e}^{-4N\langle\bar{q}q\rangle^2\mu^2\lambda_1^2} I_\nu(\xi_1^*\lambda_1) I_\nu(\lambda_1\zeta_1) & \cdots & \int_0^1 d\lambda_1 \lambda_1 \mathrm{e}^{-4N\langle\bar{q}q\rangle^2\mu^2\lambda_1^2} I_\nu(\xi_n^*\lambda_1) I_\nu(\lambda_1\zeta_1) \\ \vdots & & \vdots \\ \int_0^1 d\lambda_m \lambda_m \mathrm{e}^{-4N\langle\bar{q}q\rangle^2\mu^2\lambda_m^2} I_\nu(\xi_1^*\lambda_m) I_\nu(\lambda_m\zeta_m) & \cdots & \int_0^1 d\lambda_m \lambda_m \mathrm{e}^{-4N\langle\bar{q}q\rangle^2\mu^2\lambda_m^2} I_\nu(\xi_n^*\lambda_m) I_\nu(\lambda_m\zeta_m) \\ I_\nu(\xi_1^*) & \cdots & I_\nu(\xi_n^*) \\ \vdots & & \vdots \\ \xi_1^{*\,n-m-1} I_\nu^{(n-m-1)}(\xi_1^*) & \cdots & \xi_n^{*\,n-m-1} I_\nu^{(n-m-1)}(\xi_n^*) \end{pmatrix} \qquad (3.23)$$

[7] In order to apply the result [45] we have to promote v_2 to the full unitary group $U(n)$. We first multiply by the additional integrations $(\int_{U(1)} dw_i \det(w_i)^\nu)^m \int_{U(n-m)} dw_0 \det(w_0)^\nu$ and shift $u_2 \to \mathrm{diag}(w_1,\ldots,w_m,w_0)u_2$. Due to the cyclicity of the trace and the $(n-m)-$fold degeneracy of $\hat{\lambda}_-$ which makes it commute with $\mathrm{diag}(w_1,\ldots,w_m,w_0)$ we obtain a matrix $v_2\mathrm{diag}(w_1,\ldots,w_m,w_0)$ that parameterizes the full group $U(n)$.

suppressing the symmetry factor $m!$ as well as taking the transpose of the matrix. At the last step the m integrations over the λ_k have been taken inside the rows of the determinant. Eq. (3.23) generalizes the results obtained in [6] for an equal number $m = n$ of quarks and conjugate anti-quarks of degenerate, complex conjugate mass $\hat{m} = z\mathbf{1}_n$ and $\hat{n^*} = z^*\mathbf{1}_n$ each. In [6] the partition function is given as well for $n \geq m$, for real degenerate masses $\hat{m} = x\mathbf{1}_m$ and $\hat{n}^* = y\mathbf{1}_n$.

We can now compare to the second model by taking the weak nonhermiticity limit of eq. (2.18). To achieve this we first multiply all powers m_f^ν into the first m rows and all powers $n_g^{*\nu}$ into all n columns of the determinant $\mathcal{B}(m_i, n_j^*)$. The large-N limit of the norms h_k eq. (2.16) and the bare kernel eq. (2.14) have already been taken in [18]. Ignoring all factors of N the product of norms will lead to

$$\lim_{N\to\infty\,\tau\to 1} \prod_{k=N}^{N+m-1} h_k \;\sim\; \alpha^m \;. \tag{3.24}$$

The weak nonhermiticity limit of the bare kernel is given by [18]

$$\lim_{N\to\infty\,\tau\to 1} m_f^\nu n_g^{*\nu} \kappa_{N+m}(-m_f^2, -n_g^{*2}) \;\sim\; \frac{1}{\alpha}\int_0^1 d\lambda\lambda \mathrm{e}^{-\alpha^2\lambda^2} I_\nu(\lambda\zeta_f) I_\nu(\lambda\xi_g^*) \;, \tag{3.25}$$

where we have continued to negative arguments of the kernel. We have used again the value $\langle\bar{q}q\rangle = 1/\sqrt{2}$ in eq. (3.3) corresponding to our model. Since all matrix elements of kernels have different arguments there is no degeneracy in this part of the determinant. Multiplying with eq. (3.24) cancels all inverse powers of α from the kernel inside the determinant. The part of the determinant containing only polynomials $\tilde{P}_k^{(\nu)}(-n^{*2})$ can be dealt with as previously, using eqs. (3.8) and (3.9). First we take out all powers of τ, leading to

$$\lim_{N\to\infty\,\tau\to 1} \tau^{(n-m)(2N+n+m-1)\frac{1}{2}} \;\sim\; \exp\left[-\frac{1}{2}(n-m)\alpha^2\right] \;. \tag{3.26}$$

The rows of polynomials become degenerate after taking the limit eq. (3.9), leading again to differentiations as in eq. (3.10). Performing the manipulations, we arrive at the following result

$$\begin{aligned}
\mathcal{Z}_{II}^{(N_f=m+n,\nu)}(\alpha;\{\zeta_f\}_m,\{\xi_g^*\}_n)|_{\text{weak}} \;&\sim\; \frac{\exp\left[-\frac{1}{2}(n-m)\alpha^2\right]}{\Delta_m(\zeta^2)\Delta_n(\xi^{*2})} \\
\times \det \begin{pmatrix}
\int_0^1 d\lambda\lambda\, \mathrm{e}^{-\alpha^2\lambda^2} I_\nu(\lambda\zeta_1) I_\nu(\lambda\xi_1^*) & \cdots & \int_0^1 d\lambda\lambda\, \mathrm{e}^{-\alpha^2\lambda^2} I_\nu(\lambda\zeta_1) I_\nu(\lambda\xi_n^*) \\
\vdots & & \vdots \\
\int_0^1 d\lambda\lambda\, \mathrm{e}^{-\alpha^2\lambda^2} I_\nu(\lambda\zeta_m) I_\nu(\lambda\xi_1^*) & \cdots & \int_0^1 d\lambda\lambda\, \mathrm{e}^{-\alpha^2\lambda^2} I_\nu(\lambda\zeta_m) I_\nu(\lambda\xi_n^*) \\
I_\nu(\xi_1^*) & \cdots & I_\nu(\xi_n^*) \\
\vdots & & \vdots \\
\xi_1^{*n-m-1} I_\nu^{(n-m-1)}(\xi_1^*) & \cdots & \xi_n^{*n-m-1} I_\nu^{(n-m-1)}(\xi_n^*)
\end{pmatrix}.
\end{aligned} \tag{3.27}$$

The equivalence of the two partition functions can be established as follows. If we multiply eq. (3.27) by the overall constant $\exp[+\frac{1}{4}N_f\alpha^2]$, with $N_f = n + m$, and then identify

$$(1-\tau^2) \;=\; 4\langle\bar{q}q\rangle^2\mu^2 \tag{3.28}$$

or equivalently $\alpha = 2\tilde{\alpha}$ the two partition functions (3.23) and (3.27) agree. We conclude that both models can be mapped onto each other as a function of rescaled masses and rescaled nonhermiticity

parameter, for an arbitrary number of quarks and conjugate anti-quarks of different, nondegenerate masses. In the present case the dependence on α is much less trivial as it enters all the integrals inside the determinant. We are not able to provide a simple explanation for the fact that the two identifications eqs. (3.11) and (3.28) differ only by a factor of two. However, our main point here is that there exists a mapping of parameters making the two partition functions the same. We will find yet another mapping of parameters in the limit of strong nonhermiticity below.

3.2 The strong nonhermiticity limit

The limit of strong nonhermiticity is defined by keeping μ^2 or $\tau \in [0,1]$ fixed, independent of N when performing the large-N limit. Furthermore, the quark masses (and the complex Dirac eigenvalues) have to be rescaled with a different power in N [18, 15][8]

$$\begin{aligned} \zeta_f &\equiv \sqrt{2N}\langle\bar{q}q\rangle m_f \ , f = 1, \ldots, m \\ \xi_g^* &\equiv \sqrt{2N}\langle\bar{q}q\rangle n_g^* \ , g = 1, \ldots, n \ . \end{aligned} \tag{3.29}$$

In this limit complex eigenvalues of the Dirac operator fill in a truly two dimensional domain in the complex plane and the corresponding spectral density is nonvanishing there. With these definitions at hand we can again make a saddle point approximation for the partition functions in question.

We start with the first model eq. (2.4) which contains only quarks. Due to the rescaling of the masses the arguments of the Bessel functions now become large. But even when replacing them with their asymptotic value, $I_\nu(x) \sim \exp[x]/\sqrt{x}$, they will not contribute to the saddle point, as the argument is of order $\mathcal{O}(\sqrt{N})$ and is small compared to terms $\mathcal{O}(N)$ in the exponent for radial variables. The saddle point is thus given by conditions

$$\frac{2r_k}{r_k^2 - \mu^2} \ - \ 2r_k\langle\bar{q}q\rangle^2 \ = \ 0, \quad k = 1, \ldots, N_f \ . \tag{3.30}$$

and we choose the positive value

$$\hat{r}|_{\rm sp} \ = \ \mathbf{1}_{N_f}\sqrt{\langle\bar{q}q\rangle^{-2} + \mu^2} \ . \tag{3.31}$$

as the relevant solution. For a more detailed analysis, in particular, on the connection of the saddle point solution at zero to the phase transition we refer to section 4 below. Taking the usual limit of a degenerate matrix at $\hat{r}|_{\rm sp} \sim \mathbf{1}_{N_f}$ we arrive at

$$\begin{aligned} \mathcal{Z}_I^{(N_f,\nu)}(\mu;\{\zeta_f\})|_{\rm strong} &\sim \exp[-NN_f(1 + \langle\bar{q}q\rangle^2\mu^2)] \ \exp\left[-\frac{1}{2}\sum_{k=1}^{N_f}\zeta_k^2\right] \\ &\times \ \frac{\det_{k,l=1,\ldots,N_f}\left[\zeta_l^{k-1} I_\nu^{(k-1)}\left(\sqrt{2N}\zeta_l\sqrt{1 + \langle\bar{q}q\rangle^2\mu^2}\right)\right]}{\Delta_{N_f}(\zeta^2)} \ . \end{aligned} \tag{3.32}$$

Now we treat the second model by looking at eq. (2.9) at finite-N. The strong nonhermiticity limit is most easily taken here by using an integral representation of the Laguerre polynomials, following [15] for $N_f = 1$,

$$\begin{aligned} x^\nu L_N^{(\nu)}(x^2) &\sim \mathrm{e}^{x^2}\int_0^\infty ds \ \mathrm{e}^{-Ns} s^{N+\frac{\nu}{2}} J_\nu(2\sqrt{sN}x) \\ &\sim \mathrm{e}^{x^2}\mathrm{e}^{-N} J_\nu(2\sqrt{N}x) \ . \end{aligned} \tag{3.33}$$

[8] For the constant proportionality factor we have kept $\langle\bar{q}q\rangle$ as in the weak limit, although the macroscopic spectral density will in general no longer be constant but rather depend on μ. For a $\mu-$dependent rescaling we refer to the discussion after eq. (3.36) and after eq. (3.47).

At the second step we have made a saddle point approximation, taking into account that $x^2 = -Nm^2/(2\tau)$ is fixed and finite. Using the rescaling of the masses eq. (3.29) with the chosen value $\langle\bar{q}q\rangle = 1/\sqrt{2}$, corresponding to $\sqrt{N}m = \zeta$, and taking care of the degeneracy of the determinant we obtain

$$\mathcal{Z}_{II}^{(N_f,\nu)}(\tau;\{\zeta_f\})|_{\text{strong}} \sim \exp[-NN_f]\ \tau^{N_f(N+\frac{1}{4}(N_f-1)+\frac{1}{2}\nu)}\ \exp\left[-\frac{1}{2\tau}\sum_{k=1}^{N_f}\zeta_k^2\right] \times \frac{\det_{k,l=1,\ldots,N_f}\left[\zeta_l^{k-1}I_\nu^{(k-1)}\left(\sqrt{2N}\zeta_l\tau^{-\frac{1}{2}}\right)\right]}{\Delta_{N_f}(\zeta^2)}\ . \tag{3.34}$$

Using now the relation eq. (3.11),

$$\tau = \sqrt{1-2\langle\bar{q}q\rangle^2\mu^2} \tag{3.35}$$

we see that the $\mu-$dependence does not match the previous case in general, unless we expand for small μ^2. To the leading order we obtain

$$\mathcal{Z}_{II}^{(N_f,\nu)}(\mu;\{\zeta_f\})|_{\text{strong}} \sim \exp[-NN_f(1+\langle\bar{q}q\rangle^2\mu^2)+\mathcal{O}(\mu^4)]\exp\left[-\frac{1}{2}(1+\langle\bar{q}q\rangle^2\mu^2+\mathcal{O}(\mu^4))\sum_{k=1}^{N_f}\zeta_k^2\right] \times \frac{\det_{k,l=1,\ldots,N_f}\left[\zeta_l^{k-1}I_\nu^{(k-1)}\left(\sqrt{2N}\zeta_l(1+\frac{1}{2}\langle\bar{q}q\rangle^2\mu^2+\mathcal{O}(\mu^4))\right)\right]}{\Delta_{N_f}(\zeta^2)}\ . \tag{3.36}$$

The two expressions eqs. (3.32) and (3.36) match to the leading order terms in the determinant and in the first factor. However, the coupling between masses ζ_k and μ introduced in the second model in the second exponential prefactor does not have an analogue in the first model. Here the matching only holds to terms of zeroth order.

It is worth mentioning that even apart from the $\mu-$dependent exponential suppression factor $\exp[-NN_f\langle\bar{q}q\rangle^2\mu^2]$ a proper large-N limit of the partition functions in terms of rescaled masses ζ_k does not exist. The Bessel functions still depend on $\sqrt{N}$ in the argument. In the asymptotic limit however, the exponential suppression wins. Our microscopic rescaling of the masses eq. (3.29) cannot be modified to achieve an N-independent result. Such a rescaling is, in fact, dictated by corresponding rescaling of the complex eigenvalues [18] necessary to obtain a smooth limiting eigenvalue correlation functions. This type of rescaling in N at the regime of strong nonhermiticity was also found earlier in models without chiral symmetry [38].

One also may wonder if it is possible to match the two partition functions eqs. (3.32) and (3.34) beyond an expansion in μ, by introducing a different, μ- and τ-dependent rescaling of the masses in eq. (3.29) respectively. However, a little thought shows that due to essentially different τ- and μ-dependence of the masses of the two models such a procedure is impossible.

In order to take the large-N limit at strong nonhermiticity of the first model with quarks and conjugate anti-quarks we go back to the expressions for finite-N of the partition function, eqs. (2.11) and (2.12). At the regime of weak nonhermiticity it was crucial to expand the determinant $\det[\cdots]^N$ to be able to perform exactly the unitary group integrals arising from the parameterization of Q. However, at the regime of strong nonhermiticity such an expansion is no longer possible with respect to the parameter μ since the latter no longer scales with N. For the sake of simplicity we will restrict ourselves to real quark and conjugate anti-quarks masses, $M = M^\dagger$, eq. (2.12). In general the radial and unitary degrees of freedom no longer decouple. However, if we assume that the permutation

symmetry makes all radial coordinates r_k to take the same saddle point value as it was the case previously, see eq. (3.5), that means

$$\hat{r}|_{\rm sp} \;=\; r_{\rm sp}\mathbf{1}_{N_f}\;, \tag{3.37}$$

with the value $r_{\rm sp}$ to be determined, the integral eq. (2.12) simplifies considerably. The unitary matrix V drops out from the integrand and we obtain

$$\begin{aligned}\mathcal{Z}_I^{(N_f,\nu)}(\mu;M) \;&\sim\; {\rm e}^{-N\langle\bar{q}q\rangle^2{\rm Tr}\,M^2}{\rm e}^{-N\langle\bar{q}q\rangle^2N_f r_{\rm sp}^2}\int dU\,\det[U^\dagger]^\nu \\ &\times\; \exp\Big[N\langle\bar{q}q\rangle^2 r_{\rm sp}{\rm Tr}M(U+U^\dagger)\Big]\det\Big[r_{\rm sp}^2\mathbf{1}_{N_f}-\mu^2U^\dagger\Sigma_3U\Sigma_3\Big]^N. \end{aligned}\tag{3.38}$$

This expression can be regarded as the effective partition function at the regime of strong nonhermiticity, and serves as generalization of the effective Lagrangian from chiral perturbation theory [11], eq. (3.15) which was valid at weak nonhermiticity. The two expressions only agree after expanding to the first order in μ^2 (for real masses). Note that while in the chiral Lagrangian picture higher powers terms $(\mu^2U^\dagger\Sigma_3U\Sigma_3)^k$ can be excluded from power counting in U, they cannot be excluded in the matrix model. We will see that in order to describe a phase transition to the symmetric phase it will be important to keep all powers in μ, see section 4 below, also cf.[12].

In the following we also restrict ourselves to an equal number of quarks and their conjugate partners, $n = m$. The reason is that in the general case $n > m$ one naturally expects a result to be of the mixed form (compare eq. (2.18) or eq. (3.23), containing both limiting "kernels" and "polynomials"[9]. However, we have just seen that the quantities corresponding to the polynomials do not have a proper large-N limit. Since we are looking for partition functions that do possess such a limit at strong nonhermiticity (as we will find for the second model below) we choose $n = m$ to ensure that the result contains only kernel terms.

In eq. (3.38) we can again employ the parameterization eqs. (3.16) and (3.17), where it is instructive to first look at the determinant alone. We obtain

$$\begin{aligned}\det\Big[r_{\rm sp}^2\mathbf{1}_{N_f}-\mu^2U^\dagger\Sigma_3U\Sigma_3\Big]^N \;&=\; \det\begin{pmatrix} r_{\rm sp}^2\mathbf{1}_n-\mu^2v_1(2\hat{\lambda}^2-\mathbf{1}_n)v_1^\dagger & 2\mu^2v_1\hat{\lambda}\sqrt{\mathbf{1}_n-\hat{\lambda}^2})v_2^\dagger \\ -2\mu^2v_2\hat{\lambda}\sqrt{\mathbf{1}_n-\hat{\lambda}^2})v_1^\dagger & r_{\rm sp}^2\mathbf{1}_n-\mu^2v_2(2\hat{\lambda}^2-\mathbf{1}_n)v_2^\dagger\end{pmatrix}^N \\ &=\; \det\Big[r_{\rm sp}^4\mathbf{1}_n-2r_{\rm sp}^2\mu^2(2\hat{\lambda}^2-\mathbf{1}_n)+\mu^4\mathbf{1}_n\Big]^N \\ &=\; (r_{\rm sp}^2+\mu^2)^{2nN}\prod_{i=1}^n\left(1-\frac{4r_{\rm sp}^2\mu^2}{(r_{\rm sp}^2+\mu^2)^2}\lambda_i^2\right)^N, \end{aligned}\tag{3.39}$$

where all angular-variable dependence has dropped out. The integrals over the unitary subgroups $u_{1,2}$ and $v_{1,2}$ of the mass dependent exponential $\exp[N\langle\bar{q}q\rangle^2r_{\rm sp}{\rm Tr}M(U+U^\dagger)]$ can be performed as in the weak nonhermiticity limit eq. (3.20), using the integral eq. (3.21). In fact we can literally repeat all the following simplifying steps there after eq. (3.22), leading to

$$\begin{aligned}\mathcal{Z}_I^{(N_f,\nu)}(\mu;\{\zeta_f\}_n,\{\xi_g\}_n) \;&\sim\; \frac{1}{\Delta_n(\zeta^2)\Delta_n(\xi^2)}\,{\rm e}^{-\frac{1}{2}\sum_{k=1}^n(\zeta_k^2+\xi_k^2)}\,r_{\rm sp}^{N_f(\nu+1)}{\rm e}^{-N\langle\bar{q}q\rangle^2N_fr_{\rm sp}^2}\,(r_{\rm sp}^2+\mu^2)^{2nN} \\ &\times\det_{i,j=1,\ldots,n}\left[\int_0^1 d\lambda\lambda\left(1-\frac{4r_{\rm sp}^2\mu^2}{(r_{\rm sp}^2+\mu^2)^2}\lambda^2\right)^N I_\nu\Big(\sqrt{2N}\langle\bar{q}q\rangle r_{\rm sp}\lambda\zeta_i\Big)\,I_\nu\Big(\sqrt{2N}\langle\bar{q}q\rangle r_{\rm sp}\lambda\xi_j\Big)\right]. \end{aligned}\tag{3.40}$$

Here we have also inserted the microscopic scaling of the masses eq. (3.29). We still have to determine the saddle point value $r_{\rm sp}$ and take the large-N limit. In the latter the integral can be computed by

[9]For the interpretation of the partition function in terms of kernels and polynomials we refer to section 3.3.

making the change of variables $s = 4r_{\rm sp}^2\mu^2\, N\lambda^2/(r_{\rm sp}^2+\mu^2)^2$ and using that $\lim_{N\to\infty}(1-s/N)^N = \exp[-s]$. We also employ the following integral,

$$\int_0^\infty ds \mathrm{e}^{-s} J_\nu\left(\sqrt{2s}\zeta\right) J_\nu\left(\sqrt{2s}\xi\right) \;=\; \exp\left[-\frac{1}{2}(\zeta^2+\xi^2)\right] I_\nu(\zeta\xi)\;, \tag{3.41}$$

after analytically continuing in the masses[10]. Taking out common factors of the determinant we arrive at

$$\begin{aligned}\mathcal{Z}_I^{(N_f,\nu)}(\mu;\{\zeta_f\}_n,\{\xi_g\}_n) \;&\sim\; \frac{1}{\Delta_n(\zeta^2)\Delta_n(\xi^2)}\,\mathrm{e}^{-\frac{1}{2}\sum_{k=1}^n(\zeta_k^2+\xi_k^2)}\mathrm{e}^{-N\langle\bar{q}q\rangle^2 N_f r_{\rm sp}^2}\,(r_{\rm sp}^2+\mu^2)^{2nN}\\ &\times\left(\frac{(r_{\rm sp}^2+\mu^2)^2}{4Nr_{\rm sp}^2\mu^2}\right)^n \mathrm{e}^{\frac{1}{2}\langle\bar{q}q\rangle^2(r_{\rm sp}^2+\mu^2)^2\sum_{k=1}^n\frac{(\zeta_k^2+\xi_k^2)}{4\mu^2}}\det_{i,j=1,\ldots,n}\left[I_\nu\left(\frac{\langle\bar{q}q\rangle^2(r_{\rm sp}^2+\mu^2)^2\zeta_i\xi_j}{4\mu^2}\right)\right].\end{aligned} \tag{3.42}$$

The value for $r_{\rm sp}$ can finally be read off as

$$(r_{\rm sp}^2+\mu^2) \;=\; \langle\bar{q}q\rangle^{-2}\;, \tag{3.43}$$

which allows to simplify the final result down to

$$\begin{aligned}\mathcal{Z}_I^{(N_f,\nu)}(\mu;\{\zeta_f\}_n,\{\xi_g\}_n) \;&\sim\; \frac{1}{\Delta_n(\zeta^2)\Delta_n(\xi^2)}\frac{1}{(4\langle\bar{q}q\rangle^2\mu^2)^n}\,\mathrm{e}^{-2nN(\langle\bar{q}q\rangle^2\mu^2-1)}\\ &\times\exp\left[\frac{1}{2}(1-4\langle\bar{q}q\rangle^2\mu^2)\sum_{k=1}^n\frac{(\zeta_k^2+\xi_k^2)}{4\langle\bar{q}q\rangle^2\mu^2}\right]\det_{i,j=1,\ldots,n}\left[I_\nu\left(\frac{\zeta_i\xi_j}{4\langle\bar{q}q\rangle^2\mu^2}\right)\right].\end{aligned} \tag{3.44}$$

Our conclusion is therefore that the large-N limit of the partition function (as a functions of the masses) with an equal number of quarks and conjugate anti-quarks is well-defined, in contrast to that for quarks alone, eq. (3.32). The μ-dependence can be almost entirely absorbed by redefining the rescaling of the masses eq. (3.29) to $\zeta_f \to \zeta_f/(2\langle\bar{q}q\rangle\mu)$ and $\xi_g \to \xi_g/(2\langle\bar{q}q\rangle\mu)$, giving:

$$\mathcal{Z}_I^{(N_f,\nu)}(\mu;\{\zeta_f\}_n,\{\xi_g\}_n) \;\sim\; \frac{\mathrm{e}^{-2nN(\langle\bar{q}q\rangle^2\mu^2-1)}}{\Delta_n(\zeta^2)\Delta_n(\xi^2)}\frac{1}{(4\langle\bar{q}q\rangle^2\mu^2)^n}\,\mathrm{e}^{\frac{1}{2}(1-4\langle\bar{q}q\rangle^2\mu^2)\sum_{k=1}^n(\zeta_k^2+\xi_k^2)}\det_{i,j=1,\ldots,n}\left[I_\nu\left(\zeta_i\xi_j\right)\right].$$

The strong nonhermiticity limit of the second model with quarks and conjugate quarks is again performed easily, without being restricted to real masses. Looking at the expression eq. (2.18) we need the limiting normalization factors, kernel and polynomials. The latter have been analyzed by us already, and we start with the normalization factors. Omitting all constant factors we obtain

$$\prod_{k=N}^{N+m-1} h_k \;\sim\; (f^{(\nu)}(\tau))^m\;, \tag{3.45}$$

which contains all $\tau-$dependence. Before taking the strong nonhermiticity limit for the kernel (and polynomials) we multiply all the prefactors m_f^ν and $n_g^{*\nu}$ into the determinant $\det[\mathcal{B}(m_i,n_j^*)]$ in eq. (2.18), as in the weak limit before. The asymptotic kernel has been already evaluated in [18], reading

$$m^\nu n^{*\nu}\kappa_N(-m^2,-n^{*2}) \;\sim\; \frac{1}{(1-\tau^2)}\frac{1}{f^{(\nu)}(\tau)}\;\exp\left[\frac{\tau}{2(1-\tau^2)}(\zeta^2+\xi^{*2})\right] I_\nu\left(\frac{\zeta\xi^*}{1-\tau^2}\right). \tag{3.46}$$

[10]The integral in eq. (3.40) is also convergent after change of variables and taking the large-N limit as it stands, without continuing to imaginary masses.

Multiplying all factors of $f^{(\nu)}(\tau)$ from eq. (3.45) into the first m rows of the determinant cancels the $f^{(\nu)}(\tau)$-dependence of the kernel. Using eq. (3.33) for the asymptotic polynomials results in the full expression for the partition function reading

$$\mathcal{Z}_{II}^{(N_f=m+n,\nu)}(\tau;\{\zeta_f\}_m,\{\xi_g^*\}_n)|_{\text{strong}} \sim \frac{\mathrm{e}^{-N(n-m)}}{\Delta_m(\zeta^2)\Delta_n(\xi^{*\,2})}\,\frac{1}{(1-\tau^2)^m}\,\tau^{(n-m)(N+\frac{1}{4}(n+3m-1+2\nu))}$$
$$\times\det\begin{pmatrix} \mathrm{e}^{\frac{\tau}{2(1-\tau^2)}(\xi_1^{*\,2}+\zeta_1^2)}I_\nu(\frac{\xi_1^*\zeta_1}{1-\tau^2}) & \cdots & \mathrm{e}^{\frac{\tau}{2(1-\tau^2)}(\xi_n^{*\,2}+\zeta_1^2)}I_\nu(\frac{\xi_n^*\zeta_1}{1-\tau^2}) \\ \vdots & & \vdots \\ \mathrm{e}^{\frac{\tau}{2(1-\tau^2)}(\xi_1^{*\,2}+\zeta_m^2)}I_\nu(\frac{\xi_1^*\zeta_m}{1-\tau^2}) & \cdots & \mathrm{e}^{\frac{\tau}{2(1-\tau^2)}(\xi_n^{*\,2}+\zeta_m^2)}I_\nu(\frac{\xi_n^*\zeta_m}{1-\tau^2}) \\ \mathrm{e}^{\frac{-\xi_1^{*\,2}}{2\tau}}I_\nu(\sqrt{2N}\xi_1^*\tau^{-\frac{1}{2}}) & \cdots & \mathrm{e}^{\frac{-\xi_n^{*\,2}}{2\tau}}I_\nu(\sqrt{2N}\xi_n^*\tau^{-\frac{1}{2}}) \\ \vdots & & \vdots \\ \mathrm{e}^{\frac{-\xi_1^{*\,2}}{2\tau}}\xi_1^{*\,n-m-1}I_\nu^{(n-m-1)}(\sqrt{2N}\xi_1^*\tau^{-\frac{1}{2}}) & \cdots & \mathrm{e}^{\frac{-\xi_n^{*\,2}}{2\tau}}\xi_n^{*\,n-m-1}I_\nu^{(n-m-1)}(\sqrt{2N}\xi_n^*\tau^{-\frac{1}{2}}) \end{pmatrix}. \tag{3.47}$$

In order to compare with eq. (3.45) we look at the particular case of equal number of quarks and conjugate anti-quarks, $n=m$,

$$\mathcal{Z}_{II}^{(N_f=2n,\nu)}(\tau;\{\zeta_f\}_n,\{\xi_g^*\}_n)|_{\text{strong}} \sim \frac{1}{\Delta_n(\zeta^2)\Delta_n(\xi^{*\,2})}\,\frac{1}{(1-\tau^2)^n}\,\exp\left[\frac{\tau}{2}\sum_{k=1}^n\frac{\xi_k^{*\,2}+\zeta_k^2}{1-\tau^2}\right]$$
$$\times\det_{i,j=1,\ldots,n}\left[I_\nu\left(\frac{\xi_i^*\zeta_j}{1-\tau^2}\right)\right]. \tag{3.48}$$

This expression has a finite limit at large N as a function of τ and the masses. We note that here the masses always appear in the same $\tau-$dependent combination $1/\sqrt{1-\tau^2}$. If we define a $\tau-$dependent microscopic rescaling $m\sqrt{2N/(1-\tau^2)}=\zeta$ instead of eq. (3.29) that square-root factor can be absorbed:

$$\mathcal{Z}_{II}^{(N_f=2n,\nu)}(\tau;\{\zeta_f\}_n,\{\xi_g^*\}_n)|_{\text{strong}} \sim \frac{1}{\Delta_n(\zeta^2)\Delta_n(\xi^{*\,2})}\,\frac{1}{(1-\tau^2)^n}\,\mathrm{e}^{\frac{\tau}{2}\sum_{k=1}^n(\xi_k^{*\,2}+\zeta_k^2)}\det_{i,j=1,\ldots,n}\left[I_\nu\left(\xi_i^*\zeta_j\right)\right]. \tag{3.49}$$

We note that $\rho(0)=\frac{1}{2\pi(1-\tau^2)}$ is the constant macroscopic density corresponding to uniform filling of an ellipse. The scaling defined above is very reminiscent to the scaling on the real line used to get read of the mean level spacing. The two partition functions eq. (3.45) and (3.49) can now be identified upon the mapping

$$\tau = 1-4\langle\bar{q}q\rangle^2\mu^2\ . \tag{3.50}$$

It is yet different from the two previous mappings, the main point of our analysis being the very existence of a correspondence between the two functions in terms of their arguments.

3.3 Universality

After having demonstrated several cases of agreement between the two partition function with a given flavors content let us use those results for discussing the issue of universality of the random matrix theory in the complex plane. For matrix models with eigenvalues on the real line universality is a well established principle, both on the base of heuristic, physical arguments [21, 22, 36] as well on firm mathematical grounds (see [46, 33] and references therein; we mainly discuss works relevant for

QCD applications). On the other hand, for complex eigenvalues the subject has so far attracted only little attention, apart from the discussion in [23]. As we already have mentioned several times, it is important to distinguish between the limits of weak and strong nonhermiticity [16]. As is well known, the eigenvalue correlations in the weak nonhermiticity limit interpolate between those typical for real eigenvalues and those known for complex eigenvalues. More precisely, sending the rescaled nonhermiticity parameter α from eq. (3.2) to 0 or ∞ allows to recover correlation functions typical for eigenvalues of Hermitian or, respectively, complex strongly nonhermitian (Ginibre-like) matrices. This was explicitly checked in several cases, as for example in [16] for the unitary (UE) and in [18] for the chiral unitary ensemble (chUE).

In that sense the weak nonhermiticity limit is closely related to real, universal correlations, and it is natural to expect it should maintain at least some universality features. In [23] a certain class of deformations of measures of the complex UE were studied with the conclusion, that the asymptotic polynomials, kernels and correlation functions remain universal.

It therefore came as a surprise that for the complex extension of the chUE two slightly different results were obtained for the microscopic spectral density. Indeed, the second model eq. (2.5) with complex eigenvalues was solved in the paper [18] by analytical continuation of the Laguerre polynomials, and a comparison to quenched QCD lattice data with chemical potential confirmed the obtained results [19]. On the other hand, the original model eq. (2.1) based on a random matrix representation as a starting point was solved recently for the microscopic spectral density in the weak limit, using an exact replica approach, see [6]. The obtained density, although having a very similar structure, was slightly different, and the difference was confirmed by simulating numerically the underlying random matrix model, eq. (2.1). We remark however, that for the range of parameters μ used in the analysis of the lattice data [19] the difference was too small to be detected. In view of the *agreement* between partition functions of the two models found in the preceding sections and in [15] we will try to shed some more light on the origin of the discrepancy, and thus on the issue of universality.

In the random matrix theory several different objects can be tested for the property of universality. Let us start with recalling some basic facts for models that can be solved with the technique of orthogonal polynomials, such as eq. (2.5) or the chUE. The solution for the eigenvalue correlation functions can be written as

$$\begin{aligned}\rho_k(z_1,\ldots,z_k) &= \det_{i,j=1,\ldots,k}[K_N(z_i,z_j^*)] \;=\; \prod_{l=1}^{k} w(z_l,z_l^*)\;\det_{i,j=1,\ldots,k}[\kappa_N(z_i,z_j^*)] \\ &= \prod_{l=1}^{k} w(z_l,z_l^*)\;\det_{i,j=1,\ldots,k}\left[\sum_{k=0}^{N-1} h_k^{-1}\tilde{P}_k^{(\nu)}(z_i)\tilde{P}_k^{(\nu)}(z_j^*)\right]\;, \end{aligned} \tag{3.51}$$

where we use the notation introduced in eqs. (2.14) and (2.15).

It is evident, that three different objects can be analyzed from the point of view of asymptotic universality in the large-N limit: the polynomials $\tilde{P}_N^{(\nu)}(z)$, the bare kernel $\kappa_N(z_i,z_j^*)$ and the weight function $w(z,z^*)$ itself. All these objects have a direct relation to partition functions as we will see below. For the universality of the spectral correlations eq. (3.51) we need that both the weight and the bare kernel are universal, as the full kernel is given by $K_N(z_i,z_j^*)=[w(z_i,z_i^*)w(z_j,z_j^*)]^{\frac{1}{2}}\kappa_N(z_i,z_j^*)$. We also note that for real eigenvalues the universality of the bare kernel follows directly from that of the polynomials[11] due to the Christoffel-Darboux formula

$$\kappa_N(x,y) \;=\; h_{N-1}^{-1}\;\frac{\tilde{P}_N(x)\tilde{P}_{N-1}(y)-\tilde{P}_N(y)\tilde{P}_{N-1}(x)}{x-y}\;. \tag{3.52}$$

[11] We note however, that the universality of the norms h_k is a separate issue, see e.g. in [21].

For orthogonal polynomials in the complex plane this relation does not hold and thus the universality of polynomials does not necessarily imply the universality of the bare kernel, and vise versa.

There exists an alternative method for computing spectral correlation functions from working with characteristic polynomials. An advantage of the method is that it retains its validity also in a general case when orthogonal polynomials are unavailable, for example, when the partition function cannot be represented in terms of eigenvalues. In the simplest case the resolvent $G(z)$ can be generated from differentiating a single ratio of characteristic polynomials,

$$G(z) \;\equiv\; \left\langle \mathrm{Tr}\frac{1}{z-H}\right\rangle_N \;=\; \partial_z \left\langle \frac{\det(z-H)}{\det(u-H)}\right\rangle_N\Big|_{u=z} \;. \tag{3.53}$$

Here $H = H^\dagger$ denotes a hermitian random matrix of size $N \times N$ averaged over a Gaussian or more general weight function. Knowing the resolvent, the spectral density follows by taking the discontinuity along the support or the antiholomorphic derivative, for real or complex eigenvalues, respectively. For getting access to higher order correlation functions more ratios of characteristic polynomials can be used as source terms.

Apart from being a generator for spectral correlation functions characteristic polynomials can be regarded as interesting objects in their own right, and it is therefore natural to ask about their universality. In fact, it is possible to relate characteristic polynomials to orthogonal polynomials in such models where the latter are known. First of all, the expectation value of a single characteristic polynomial directly gives the orthogonal polynomial for the corresponding model in monic normalization:

$$\langle \det(z-H)\rangle_N \;=\; \tilde{P}_N(z) \;. \tag{3.54}$$

Second, the expectation value of the product of two characteristic polynomials yields directly the bare kernel (see e.g. in [47])

$$\langle \det(z-H)\det(u-H)\rangle_N \;=\; h_N\; \kappa_{N+1}(z,u) \;. \tag{3.55}$$

Third, for real eigenvalues the expressions involving the inverse of a characteristic polynomial are given in terms of the Cauchy transform of the orthogonal polynomials, as was recently discovered in [34], and further developed in [33, 35, 32, 36]. For example:

$$\left\langle \frac{1}{\det(z-H)}\right\rangle_N \;=\; \frac{1}{2\pi i}\int dx \frac{w(x)}{z-x}\tilde{P}_N(x) \;\equiv\; \vartheta(z) \;, \tag{3.56}$$

where to ensure that the object is well-defined, the poles have to be suitably avoided by giving an imaginary part to z, $\Im\mathrm{m}(z) \neq 0$. Various formulas expressing arbitrary products [27, 29, 28, 38, 39] and arbitrary ratios [34, 33, 35] of characteristic polynomials in terms of orthogonal polynomials, their Cauchy transforms and bare kernels containing one or both of these have been proved recently.

A natural question which immediately arises is whether the universality also holds for arbitrary ratios of characteristic polynomials, especially in view of their relation to orthogonal polynomials and bare kernels just mentioned. We would like to point out that because of the presence of the Cauchy transform the universality of such objects is in general not a simple consequence of the known universality of the kernels and polynomials. For the unitary ensembles with real eigenvalues this question has been completely answered. For the standard UE the universality was rigorously proved in [33]. Arbitrary ratios of characteristic polynomials for the chUE have been computed in [30, 32, 31] and proved to be universal in various regimes in [36, 37]. Let us mention that all what we have said immediately applies to the matrix model partition functions of QCD, as the insertion of massive flavors is nothing else than the insertion of N_f characteristic polynomials. In fact, from the point of

view of massive partition functions the universality of arbitrary products has been proved previously in [22] for the UE and chUE.

For characteristic polynomials of matrices with complex eigenvalues much less is known. First of all we have to distinguish between the characteristic polynomial and its complex conjugate, just as we have distinguished between quarks and conjugate anti-quarks in our previous considerations. For arbitrary products of characteristic polynomials and a different number of complex conjugate characteristic polynomials an expression in terms of polynomials and bare kernels has been derived in [39] (see eq. (2.18) which we have already used). From what has been said in this section it is clear, that using products alone we will not be able to deduce the universality complex eigenvalue correlations.

We can thus conclude the following. We have studied two different models, eqs. (2.1) and (2.5), which are apriori not the same, in particular as the former does not admit a simple eigenvalue representation. We have found that in the large-N limit at weak nonhermiticity the two partition functions agree for an arbitrary and different number of quarks and conjugate anti-quarks, showing that they both belong to the same universality class. This gives a strong argument in favor of universality for arbitrary products of characteristic polynomials in the regime of weak nonhermiticity . For complex models without chiral symmetry this follows from [23] by proving universality of the kernel and polynomials there.

At strong nonhermiticity we found that the partition functions of the two models can be mapped onto each other, provided we consider an equal number of quarks and conjugate anti-quarks. This correspondence indicates possible universality of some strongly nonhermitian partition functions. Another argument could be that for the second model its weakly nonhermitian (universal) partition function can be matched to the strongly nonhermitian one by taking the limit $\alpha \to \infty$. However, we found that two partition functions with a general flavor content, for example containing only quarks, disagree in general. Despite the fact that they have a very similar structure, only the leading order term in expansion with respect to μ can be put in correspondence. This indicated that universality at strong nonhermiticity regime may be more subtle. When discussing such a disagreement some caution has to be added. As we have already remarked, a large-N limit of the partition functions with general flavor content as functions of the masses does not exist, properly speaking.

Let us come back to the question of universality of correlation functions at weak nonhermiticity. In [6] the corresponding microscopic spectral density was computed for the first model eq. (2.1) at $\nu = 0$. There, an exact replica method was used, that expresses the density in terms of a product and a ratio of characteristic polynomials,

$$\rho_I|_{\rm weak}(\zeta) \;=\; \frac{1}{2}|\zeta|^2 \mathcal{Z}_I^{(N_f=-1-1,0)}(\mu;\zeta,\zeta^*)\; \mathcal{Z}_I^{(N_f=1+1,0)}(\mu;\zeta,\zeta^*)\;, \tag{3.57}$$

written in terms of the rescaled variable $\zeta = 2N\langle\bar{q}q\rangle z$. Here by a negative number $N_f = -2$ of flavors we indicate the corresponding number of inverse powers of determinants $m = n = 1$ in eq. (2.10). The latter objects in the present context are also frequently called the bosonic partition functions. Let us compare this expression to the result of [18]

$$\begin{aligned}\rho_{II}|_{\rm weak}(\zeta) \;&=\; \lim_{N\to\infty\tau\to 1} w(z,z^*)\;\kappa_N(z,z^*) \\ &=\; \lim_{N\to\infty\,\tau\to 1} w(z,z^*)\; h_{N-1}^{-1}\mathcal{Z}_{II}^{(N_f=1+1,\nu)}(\tau;z,z^*)\;,\end{aligned} \tag{3.58}$$

where in the second step we have used eq. (2.18) for $n = m = 1$, corresponding to eq. (3.55) in the complex plane. As we know, the two fermionic partition functions, that is corresponding to positive value $N_f = 2$, $\mathcal{Z}_{I,II}^{(N_f,\nu)}$ agree for any value of ν. We therefore can compare the weight function of

the second model in the weak nonhermiticity limit with $\frac{1}{2}|\zeta|^2 \mathcal{Z}_{II}^{(N_f=-1-1,0)}$ that plays the same role. While for the weight factor we have

$$\lim_{N\to\infty\,\tau\to 1} w(z,z^*)\; h_N^{-1} \;=\; \frac{1}{\alpha}|\zeta|^{2\nu+1}\exp\left[-\frac{1}{\alpha^2}(\Im\mathrm{m}\zeta)^2\right] \tag{3.59}$$

in terms of the rescaled variable, for the bosonic partition functions taken from [6] it follows[12]

$$|\zeta|^2\mathcal{Z}_I^{(N_f=-1-1,0)}(\alpha;\zeta,\zeta^*) \;\sim\; \frac{|\zeta|^2}{\alpha^2}\exp\left[\frac{1}{4\alpha^2}(\zeta^2+\zeta^{*\,2})\right]K_0\left(\frac{|\zeta|^2}{2\alpha^2}\right)\;. \tag{3.60}$$

In the limit of small α or large $|\zeta|^2$ we can expand the K-Bessel function $K_0(x)\sim \mathrm{e}^{-x}/\sqrt{x}$ and both expressions eqs. (3.59) and (3.60) will coincide for $\nu=0$. However in general they disagree, leaving different possibilities for the universality of spectral correlation functions at weak nonhermiticity. We are left with two possibilities, one being that the models are in different universality classes for the spectral correlations, despite agreeing for all correlations of products of characteristic polynomials. Or, alternatively, there is no spectral universality at weak nonhermiticity at all, in the sense that such a universality only holds for characteristic polynomials and not for the weight function itself.

To further illustrate these possibilities let us point out a major difference from the Hermitian large-N limit. At zero chemical potential the weight function corresponding $\exp[-NV(x)]$, with $V(x)$ an even polynomial, is reduced to a trivial unity factor in the microscopic scaling limit $Nx = const$. For that reason the distinction between bare and full kernel becomes immaterial in that limit. Therefore the agreement between the two models for the product of two characteristic polynomials leading to the kernel, eq. (3.55) implies the same agreement for the spectral correlations. However, at weak nonhermiticity this is not the case any longer. As we see in eq. (3.59) the weight function remains different from unity in that limit. In fact it has to be a function of the imaginary part $\Im\mathrm{m}(\zeta)$ and the rescaled chemical potential α, such that it reduces to a $\delta-$function in $\Im\mathrm{m}(\zeta)$ in the limit $\alpha\to 0$[13]. Otherwise the model would not reduce to the chUE as a model of QCD [1].

Finally we would like to mention that there exists a matrix model [48] different from the two models studies in this paper. It is given in terms of two independent complex matrices and can be diagonalized to give a complex eigenvalue representation. The weight function of the eigenvalues of this model is precisely given by eq. (3.60). The solution of the model [48] by the method of orthogonal polynomials provides an very interesting third alternative model for QCD with chemical potential, having simultaneously a matrix and complex eigenvalue representation. While the microscopic density at weak nonhermiticity agrees with that found in [6] the strong limit has not been studied so far.

Another approach to the open problem of universality would be to compute the bosonic partition function in the second model $\mathcal{Z}_{II}^{(N_f=-2,0)}$. Thus it remains to be seen if the universality found for products breaks down for inverse powers of characteristic polynomials.

4 Phase transitions

The aim of this section is to study the chiral phase transition at a critical value μ_c. Since the second model eq. (2.5) is always in the broken phase [18], with a constant macroscopic density on an ellipse for all allowed values of $\tau\in[0,1]$, we will only study the first model eq. (2.1). Furthermore, we will restrict ourselves to the presence of quarks alone. This is mainly because only in this case a very compact expression eq. (2.4) is available (compared to eqs. (2.11) or (2.12)).

[12] Note that in [6] the matrices are chosen to be antihermitian.

[13] We would like to mention that in the analysis [23] only such deformations of the weight functions were studied which keep a Gaussian representation of the $\delta-$function. Such deformations still do enter in the macroscopic density $\rho(0)$.

The same model has already been studied for one flavor $N_f = 1$ in great detail [12]. The virtue of the expression eq. (2.4) is that most of this analysis carries over to several flavors. An extended version of the model eq. (2.1) including temperature has been studied to predict the phase diagram of QCD for two light flavors [9]. However, in order to be able to solve the model rather strong assumption had to be made on the nature of the saddle point. One of the main goals of our present analysis is to see if such a consideration for zero temperature can be checked and further extended, with an additional number of not necessarily degenerate flavors included into the model. More recently the model with temperature and chemical potential [9] has been extended to study the different effect of baryon and isospin chemical potential for two flavors [10]. The latter corresponds to having a pair of a quark and its conjugate, and the authors find a doubling of the critical line as compared with [9].

Here there is an important difference from the previous section in the large-N limit. We are interested in finding discontinuities of the partition function eq. (2.4) as a function of masses and chemical potential μ when N becomes large. In contrast to the weak and strong nonhermiticity limit considered before, we therefore will not assume any scaling of the masses m_f with N. This will modify the respective saddle point eqs. (3.5) and (3.31) as the Bessel function will now make a nontrivial contribution to it. It introduces a mass dependence into the saddle point that makes an analysis of the effect of light versus heavy flavors possible.

Before taking the saddle point limit we further compactify the expression eq. (2.4), by multiplying the Vandermonde determinant of the radial coordinates, $\Delta_{N_f}(r^2) = \det_{i,j}[r_i^{2(j-1)}]$, with the determinant of Bessel functions. The resulting matrix elements (i,j) of the single determinant read $\sum_{k=1}^{N_f} r_k^{2(j-1)} I_\nu(m_i r_k)$. Due to the symmetry of the partition function under permutations of the r_k we can follow the same steps as described after eq. (3.22). This reduces the determinant to $N_f!$ times a simpler determinant with elements $r_j^{2(j-1)} I_\nu(m_i r_j)$. We can now take the integration over each dr_j into the $j-$th column, and obtain up to the symmetry factor $N_f!$

$$\begin{aligned} \mathcal{Z}_I^{(N_f,\nu)}(\mu;\{m_f\}) &= \mathrm{e}^{-N\langle\bar{q}q\rangle^2 \sum_{f=1}^{N_f} m_f^2} \; \frac{1}{\Delta_{N_f}(m^2)} \\ &\quad \times \det_{i,j=1,\ldots,N_f} \left[\int_0^\infty dr\, r^{\nu+1+2(j-1)} (r^2-\mu^2)^N \mathrm{e}^{-N\langle\bar{q}q\rangle^2 r^2} I_\nu(2N\langle\bar{q}q\rangle^2 m_i\, r) \right] . \end{aligned} \tag{4.1}$$

In this form valid at finite-N the partition function looks exactly as a determinant over finite-N partition functions of a *single* flavor, apart for the difference between the index of the Bessel function ν and the different power in r to $(\nu + 1 + 2(j-1))$. At the saddle point this difference will be of course subleading and allows for the analysis of discontinuities of the N_f flavors in terms of a one-flavor partition function elaborated in [12]. We can now evaluate the saddle points of the integrals individually. Taking the asymptotic limit for the Bessel function $I_\nu(x) \sim \mathrm{e}^x/\sqrt{x}$ we obtain for each row

$$\frac{r}{r^2-\mu^2} \; - \; r\langle\bar{q}q\rangle^2 \; + \; \langle\bar{q}q\rangle^2 m_i \; = \; 0 \;\; , \; i = 1,\ldots,N_f \; . \tag{4.2}$$

In order to lift the resulting degeneracy of the determinant we have to differentiate the Bessel functions as usual. Furthermore, we note that since it is only their asymptotic exponential behavior that enters the saddle point equation we interchange the differentiation and the saddle point procedures. The result is

$$\mathcal{Z}_I^{(N_f,\nu)}(\mu;\{m_f\}) \;=\; \frac{\mathrm{e}^{-N\langle\bar{q}q\rangle^2 \sum_{f=1}^{N_f} m_f^2}}{\Delta_{N_f}(m^2)} \det_{i,j=1,\ldots,N_f} \left[m_i^{j-1} \partial_{m_i}^{j-1} z_I^{(N_f=1,\nu)}(\mu;m_i)|_{\mathrm{sp}} \right] , \tag{4.3}$$

where we have defined the partition function with its trivial exponential mass dependence removed,

$$z_I(\mu;m) \;\equiv\; \mathrm{e}^{+N\langle\bar{q}q\rangle^2 m^2} \; \mathcal{Z}_I^{(N_f=1,\nu)}(\mu;m) \; . \tag{4.4}$$

The large-N partition function is thus given by a determinant of (differentiated) single flavor partition functions at their saddle point value. This enables us to draw some general conclusions.

A first order phase transition occurs if the first logarithmic derivative of the partition function is discontinuous at some value. We look for a discontinuity with respect to μ so we define the quark number density

$$n_q \equiv \frac{1}{N_f}\partial_\mu \ln \mathcal{Z}_I^{(N_f,\nu)}(\mu;\{m_f\}) \ . \tag{4.5}$$

Applying this to our result eq. (4.3),

$$n_q = \mathrm{Tr}\ln\left[\frac{\partial_\mu A}{A}\right] \ , \quad (A)_{ij} \equiv m_i^{j-1}\partial_{m_i}^{j-1} z_I(\mu;m_i)|_{\rm sp} \ , \tag{4.6}$$

we observe that a discontinuity occurs whenever an individual differentiated matrix element becomes singular. Thus, the phase transition arises when the one-flavor partition function $\mathcal{Z}_I^{(1,\nu)}(\mu;m_i)$ with the smallest value μ_c becomes discontinuous. Since μ_c is a functions of the mass m_i, we have to compare the m_i-dependence of the saddle point equation (4.2) with its corresponding value $\mu_c(m_i)$. We will find that it is always the smallest mass (which may be zero) that has the smallest value of μ_c and thus drives the transition.

The analysis of the saddle point solution and the corresponding partition function for $N_f = 1$ has been made already in great detail in [12], and we follow it closely. The saddle point equation (4.2) is of third order and thus may have up to three real solution. We begin with the simplest, massless case:

$$\frac{r}{r^2-\mu^2} - r\langle\bar{q}q\rangle^2 = 0 \ \Rightarrow \ r|_{\rm sp} = \begin{cases} 0 & = r_r(m=0) & \text{restored} \\ +\sqrt{\langle\bar{q}q\rangle^{-2}+\mu^2} & = r_b(m=0) & \text{broken} \\ -\sqrt{\langle\bar{q}q\rangle^{-2}+\mu^2} & & \notin \ [0,\infty) \end{cases} . \tag{4.7}$$

It is easy to see that the solution with $r|_{\rm sp} > 0$ belongs to the broken phase (b) with an exponentially suppressed partition functions, while the solution with $r|_{\rm sp} = 0$ corresponds to the restored phase (r) [12]. The negative solution is rejected being outside the integration domain. If we switch on a mass m the signature of the saddle point will remain the same[14], with solutions being $0 < r_r(m) < r_b(m)$ as we will see below.

The equation that determines the critical value μ_c is given by the requirement of partition functions at two competing saddle points being equal,

$$(r_b^2-\mu^2)\mathrm{e}^{\langle\bar{q}q\rangle^2(2mr_b-r_b^2)} = (\mu^2-r_r^2)\mathrm{e}^{\langle\bar{q}q\rangle^2(2mr_r-r_r^2)} \ . \tag{4.8}$$

In the massless case $m = 0$ this leads, after inserting the solutions (4.7), to the following equation for the critical line:

$$1+\langle\bar{q}q\rangle^2\mu_c^2+\ln[\langle\bar{q}q\rangle^2\mu_c^2] = 0 \ , \ \Rightarrow \ \langle\bar{q}q\rangle\mu_c \approx 0,527\ldots \quad , \tag{4.9}$$

where we have given the approximate numerical value for its real solution (see also fig 1 below).

Let us now determine how this value shifts if we include a small mass. We first present a perturbative analysis to the leading order in the mass[15] and then come back to the full solution below. First we determine the shift of the saddle point solution due to the mas term,

$$\begin{aligned} r_r(m)-r_r(0) \equiv \delta_r(m) &= \frac{m\langle\bar{q}q\rangle^2\mu^2}{1+\langle\bar{q}q\rangle^2\mu^2} \\ r_b(m)-r_b(0) \equiv \delta_b(m) &= \frac{m}{2(1+\langle\bar{q}q\rangle^2\mu^2)} \ , \end{aligned} \tag{4.10}$$

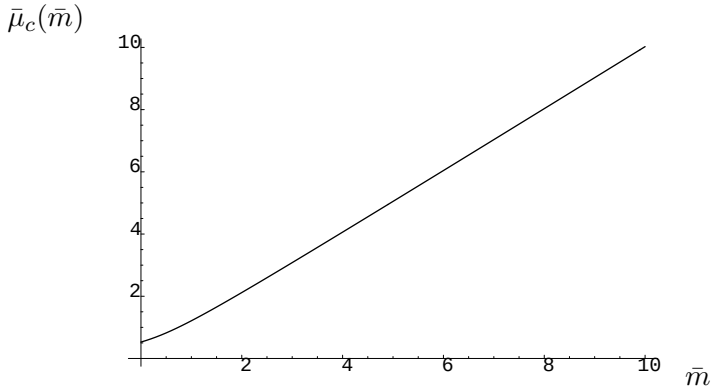

Figure 1: The critical potential as a function of mass, both given in units of $\langle\bar{q}q\rangle$. The critical value at $\bar{m}=0$ is $\bar{\mu}_c(0)=0.527\ldots$.

where we retained only the leading order linear behavior in m.

Next we compute the resulting shift in the critical line, determining $\mu_c(m)$:

$$1+\langle\bar{q}q\rangle^2\mu_c^2+\ln[\langle\bar{q}q\rangle^2\mu_c^2] \;-\; 2m\langle\bar{q}q\rangle^2\sqrt{\langle\bar{q}q\rangle^{-2}+\mu_c^2}+\mathcal{O}(m^2) \;=\; 0\ . \tag{4.11}$$

To the leading linear order in m both $\delta_{r,b}$ dropped out. We thus obtain for the shift in μ_c^2 defined as

$$\mu_c^2(m) \;-\; \mu_c^2(0)\equiv\ \gamma(m)\ , \tag{4.12}$$

the positive quantity

$$\gamma(m) \;=\; \frac{2\langle\bar{q}q\rangle m\langle\bar{q}q\rangle^2\mu_c(0)}{\sqrt{1+\langle\bar{q}q\rangle^2\mu_c(0)}} \;+\; \mathcal{O}(m^2)\ , \tag{4.13}$$

after inserting it into eq. (4.11). It is again given to leading order only. From this result we can deduce the following. If we have massless and massive flavors present in the partition function eq. (4.3) the critical value for the massless single flavor partition functions will be reached before that of the massive flavors, as $\mu_c(0)<\mu_c(m\neq 0)$. Second, if only massive flavors are present it is the critical $\mu_c(m)$ of the lightest flavor which is reached first, as for $m_1<m_2$ holds $\mu_c^2(m_1)<\mu_c^2(m_2)$. In both cases it is the lightest or zero mass that triggers the phase transition. Since it is of first order for a single flavor [12] it is thus first order for any combination of massive and massless flavors, as follows from eq. (4.6). Moreover, for each different mass there is separate transition, leading in principle to a sequence of transitions[16]. A similar feature of two distinct first order lines was found in [10] for two flavors.

We have also convinced ourselves that the monotonic behavior of the critical chemical potential $\mu_c(m)$ persists beyond the linear approximation, eqs. (4.13) and (4.12). This can be done by inserting the two positive real solutions for the third order saddle point equation into the condition eq. (4.8) and solving it numerically. If we measure all quantities in units of $\langle\bar{q}q\rangle$, $\bar{r}=\langle\bar{q}q\rangle r$, $\bar{\mu}=\langle\bar{q}q\rangle\mu$ and $\bar{m}=\langle\bar{q}q\rangle m$ (or set $\langle\bar{q}q\rangle=1$) we have

$$\bar{r}_r \;=\; \frac{2}{3}\sqrt{3(1+\bar{\mu}^2)+\bar{m}^2}\cos[\varphi/3+4\pi/3]+\frac{\bar{m}}{3}$$

[14]This can be seen from the discriminant for all real values of the masses.

[15]A similar analysis was made in [10] for $N_f=2$ with a different chemical potential for each flavor.

[16]We wish to emphasize that in general the flavor dependence of matrix models is too weak. In QCD the sign of the $\beta-$function changes for sufficiently many flavors, and the finite temperature transition changes from second to first order when changing from 2 to 3 flavors, which is not reproduced from the matrix model [9]. For that reason our conclusion should be taken seriously for small N_f only.

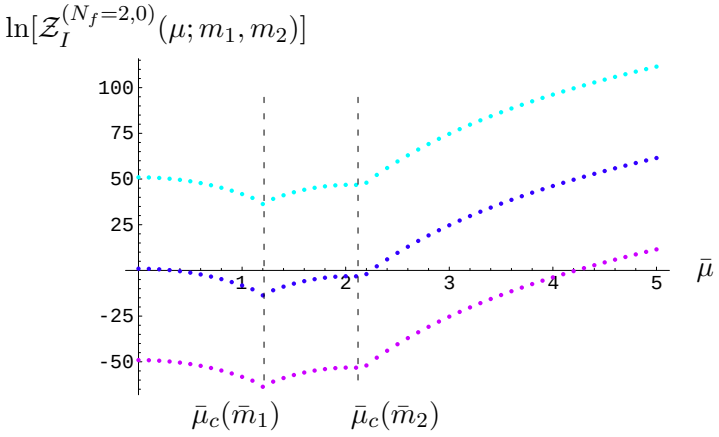

Figure 2: The logarithm of the partition function for N=10 (top), N=20 (middle), and N=30 (lower line). The vertical lines indicate the positions of the of the corresponding critical values $\mu_c(m_{1,2})$.

$$\begin{aligned} \bar{r}_b &= \frac{2}{3}\sqrt{3(1+\bar{\mu}^2)+\bar{m}^2}\cos[\varphi/3] + \frac{\bar{m}}{3} \\ \cos[\varphi] &= \frac{\bar{m}(\bar{m}^2 - 9\bar{\mu}^2 + \frac{9}{2})}{[3(1+\bar{\mu}^2)+\bar{m}^2]^{\frac{3}{2}}} \ , \end{aligned} \tag{4.14}$$

as the two positive solutions of the saddle point equation. At $m = 0$ ($\varphi = \pi/2$) the first solution $\bar{r}_r$ vanishes as it should. We have plotted above the numerical solution of eq. (4.8), after inserting into it the full solution eq. (4.14). It can be seen that $\bar{\mu}_c(\bar{m})$ is a monotonous function beyond the regime of small masses $\bar{m}$.

In order to check our findings we have examined numerically the partition function for two flavors $N_f = 2$ with two nondegenerate masses, in the representation of eq. (4.1). We have chosen the parameters $m_1 = 1$ and $m_2 = 2$ in units of $\langle\bar{q}q\rangle$ and $\nu = 0$. The resulting logarithm of the partition function is plotted above for three different values of N as a function of μ (figure 2). Even for the smallest value of N the points where the derivative of the partition function and thus the free energy becomes discontinuous are very well visible. The positions coincide with the critical values of μ for each mass scale, $\bar{\mu}_c(\bar{m}_1) = 1.2119\ldots$ and $\bar{\mu}_c(\bar{m}_2) = 2.119\ldots$. (from fig. 1).

After having made a general statement we give an explicit example for the partition function eq. (4.3) below and above the transition. Here we can again make use of analysis [12] where we briefly repeat some of the results. The single flavor partition function (modulo its mass prefactor eq. (4.4)) can be conveniently written as

$$\begin{aligned} z_I(\mu; m) &= \int_0^{\mu^2} dr\, r(r^2-\mu^2)^N \mathrm{e}^{-Nr^2} I_0(2Nm\,r) \ + \ \int_{\mu^2}^{\infty} dr\, r(r^2-\mu^2)^N \mathrm{e}^{-Nr^2} I_0(2Nm\,r) \\ &\equiv z_r(\mu; m) \ + \ z_b(\mu; m) \ . \end{aligned} \tag{4.15}$$

We used [12] that the restored symmetry solution satisfies $r_r \in [0, \mu^2]$ and the broken symmetry solution is $r_b \in [\mu^2, \infty)$ (see also eq. (4.14)). Here and in the following we set $\langle\bar{q}q\rangle = 1$ and $\nu = 0$ as it is done in [12]. The two different solutions can be written as a double sum at finite-N,

$$\begin{aligned} z_r(\mu; m) &= \mu^{2(N+1)} \mathrm{e}^{-n\mu^2} \sum_{l=0}^{\infty}\sum_{s=0}^{\infty} \frac{(N+s)!}{l!s!(N+l+s+1)!} (Nm^2)^l (N\mu^2)^{l+s} \ , \\ z_b(\mu; m) &= \frac{1}{N^{N+1}} \mathrm{e}^{-n\mu^2} \sum_{l=0}^{\infty}\sum_{s=0}^{l} \frac{(N+s)!}{l!s!(l-s)!} (Nm^2)^l (N\mu^2)^{l-s} \ . \end{aligned} \tag{4.16}$$

These formulas easily generate the massless partition function and its derivatives, $z^{(l)}_{r,b}(\mu; m = 0)$, as we need them in eq. (4.3). We note that only even derivatives contribute and give the examples necessary for $N_f = 2$ massless flavors. They read in the restored phase

$$\begin{aligned} z_r(\mu; m = 0) &\sim \mu^{2(N+1)} \\ z_r''(\mu; m)|_{m=0} &\sim \mu^{2(N+1)} 2N\mu^2 \ , \end{aligned} \tag{4.17}$$

and in the broken phase

$$\begin{aligned} z_b(\mu; m = 0) &\sim \mathrm{e}^{-N\mu^2} \\ z_b''(\mu; m)|_{m=0} &\sim \mathrm{e}^{-N\mu^2} 2N^2(1 + \mu^2 + 1/N) \ . \end{aligned} \tag{4.18}$$

Taking the limit of two degenerate, massless flavors in eq. (4.3) we obtain for large N

$$\mathcal{Z}_I^{(N_f=2,\nu=0)}(\mu; m = 0) = \det \begin{pmatrix} z_I(\mu; 0) & 0 \\ 0 & z_I''(\mu; 0) \end{pmatrix} \sim \begin{cases} \mu^{4N} & \text{restored} \\ \mathrm{e}^{-2N\mu^2}(1 + \mu^2) & \text{broken} \end{cases} . \tag{4.19}$$

This compares to the single flavor result [12]

$$\mathcal{Z}_I^{(N_f=1,\nu=0)}(\mu; m = 0) \sim \begin{cases} \mu^{2N} & \text{restored} \\ \mathrm{e}^{-N\mu^2} & \text{broken} \end{cases} . \tag{4.20}$$

We see that the behavior is very similar, with the quark number density in the two flavor case being twice as high.

As we have mentioned already there is more than one transition in the presence of several mass scales. In particular this opens the possibility to have for example two transitions for two massless flavors and one massive flavor, as it is often used in a simple model for QCD. For $\mu_c(0) < \mu < \mu_c(m)$, the massless building blocks are already in the restored phase while the massive ones are still in the broken phase. This is an intermediate regime before full symmetry restoration. It would be very interesting to use the exact solution of our model as a further testing ground for lattice algorithms as for example in [14]. For the discussion of phenomenological consequences we refer to [10] where a similar phenomenon is observed.

5 Summary

We have computed and compared two different matrix model partition functions for QCD with chemical potential. While the first model has only a matrix representation and no eigenvalue representation the situation is the reverse for the second model. We give very compact, new expressions for finite-N for the first model in terms of integrals over the radial coordinates of the underlying sigma-model-like representation. They hold for an arbitrary number of quarks or both quarks and conjugate anti-quarks. By taking the large-N limit these expressions are then compared to the corresponding results from the second model expressed in terms of orthogonal polynomials in the complex plane and their kernel.

In the limit of weak nonhermiticity we find a complete agreement, with a different identification of the nonhermiticity parameters μ and τ and mass parameters in the cases of only quarks present, or both quarks and conjugate anti-quarks.

The matching between the two models in the limit of strong nonhermiticity holds only for an equal number of quarks and conjugate anti-quarks. For such a matching the masses have to be rescaled with respect to the μ- or τ-dependent macroscopic density proportional to the level spacing. For a generic number of quarks and conjugate anti-quarks, including the case of only quarks, a mapping

can be achieved when perturbatively expanding in small chemical potential μ, despite the very similar structure of the two partition functions. However, in this case a proper large-N limit of the two partition functions does not exist.

From these finding we concluded that at weak nonhermiticity and also in a special case at strong nonhermiticity the expressions for the fermionic partition functions are universal. The issue of universality of eigenvalue correlation functions is left open, as it involves also the computation of bosonic partition functions.

In the last part we have investigated the phase structure of the first matrix model, which is known to possess a first order phase transition for a single or several degenerate flavours (the second model is always in the symmetry broken phase by construction). We have analyzed the model for an arbitrary number of nondegenerate quark flavours, exploiting the compact expressions derived for finite-N. In the large-N limit they give the multi-flavour partition function as a determinant of single flavor partition functions of different masses, and their derivatives. Consequently for each nondegenerate flavor a separate first order transition occurs, where the critical μ increases monotonically in mass. We have checked our findings numerically for two flavors with nonvanishing, nondegenerate masses.

It would be very interesting to compute also the corresponding bosonic partition functions and compare them, in particular in the light of their relation to complex eigenvalue correlations and the related question of universality.

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A The sigma-model representation of the partition function

Here we derive the sigma-model representation for both the partition functions eqs. (2.5) and (2.1). The latter can be obtained by omitting all conjugate flavors. To this aim we first unify notation, by writing eq. (2.10) as

$$\mathcal{Z}_I^{(N_f,\nu)}(\mu;\{\omega_f\}) = (-1)^{n(2N+\nu)} \int d\Phi d\Phi^\dagger \prod_{f=1}^{N_f} \det \begin{pmatrix} \omega_f \mathbf{1}_N & i\Phi + \mu_f \tilde{\mathbf{1}}_N \\ i\Phi^\dagger + \mu_f \tilde{\mathbf{1}}_N^\dagger & \omega_f \mathbf{1}_{N+\nu} \end{pmatrix} \exp[-N\langle\bar{q}q\rangle^2 \mathrm{Tr}\Phi\Phi^\dagger], \tag{A.1}$$

denoting the complex masses with

$$\omega_f \equiv \begin{cases} +m_f & f = 1,\ldots,m \\ -n_f^* & f = m+1,\ldots,N_f = m+n \end{cases}, \tag{A.2}$$

as well as choosing signs for the chemical potentials $\mu_f = +\mu$ for $f = 1,\ldots,m$, and $\mu_f = -\mu$ for $f = m+1,\ldots,N_f = m+n$. After introducing two sets of complex Grassmann vectors χ_A and χ_B of size N and $N+\nu$ respectively, we can express the determinant in eq. (2.1) as

$$\begin{aligned} \mathcal{Z}_I^{(N_f,\nu)}(\mu;\{\omega_f\}) \;\sim\; & \int d^2\chi_A d^2\chi_B \; \mathrm{e}^{-\sum_{f=1}^{N_f}\left(\omega_f(\chi_{A\,f}^\dagger\chi_{A\,f}+\chi_{B\,f}^\dagger\chi_{B\,f}) \,+\, \mu_f(\chi_{A\,f}^\dagger\tilde{\mathbf{1}}_N\chi_{B\,f}+\chi_{B\,f}^\dagger\tilde{\mathbf{1}}_N^\dagger\chi_{A\,f})\right)} \\ & \times \left\langle \mathrm{e}^{-i\sum_{f=1}^{N_f}(\chi_{A\,f}^\dagger\Phi\chi_{B\,f}+\chi_{B\,f}^\dagger\Phi^\dagger\chi_{A\,f})} \right\rangle_\Phi . \end{aligned} \tag{A.3}$$

We define the expectation value over the ensemble amounting to the integration over matrices Φ as

$$\left\langle \mathrm{e}^{-i\sum_{f=1}^{N_f}(\chi_{A\,f}^\dagger \Phi \chi_{B\,f} + \chi_{B\,f}^\dagger \Phi^\dagger \chi_{A\,f})} \right\rangle_\Phi \equiv \int d\Phi d\Phi^\dagger \mathrm{e}^{-i\sum_{f=1}^{N_f}(\chi_{A\,f}^\dagger \Phi \chi_{B\,f} + \chi_{B\,f}^\dagger \Phi^\dagger \chi_{A\,f})}\ \mathrm{e}^{-N\langle\bar{q}q\rangle^2 \mathrm{Tr}\Phi^\dagger\Phi}. \tag{A.4}$$

It can be further rewritten as

$$\begin{aligned}
\left\langle \mathrm{e}^{-i\sum_{f=1}^{N_f}(\chi_{A\,f}^\dagger \Phi \chi_{B\,f} + \chi_{B\,f}^\dagger \Phi^\dagger \chi_{A\,f})} \right\rangle_\Phi &= \left\langle \mathrm{e}^{+i\,\mathrm{Tr}(\Phi\sum_{f=1}^{N_f}\chi_{B\,f}\otimes\chi_{A\,f}^\dagger\ +\ \Phi^\dagger\sum_{f=1}^{N_f}\chi_{A\,f}\otimes\chi_{B\,f}^\dagger)} \right\rangle_\Phi \\
&= \exp\left[-\frac{1}{N\langle\bar{q}q\rangle^2}\sum_{f,g=1}^{N_f}\mathrm{Tr}(\chi_{A\,f}\otimes\chi_{B\,f}^\dagger)(\chi_{B\,g}\otimes\chi_{A\,g}^\dagger)\right] \\
&= \exp\left[-\frac{1}{N\langle\bar{q}q\rangle^2}\sum_{f,g=1}^{N_f}(\chi_{A\,g}^\dagger\chi_{A\,f})(\chi_{B\,f}^\dagger\chi_{B\,g})\right] \\
&= \int dQdQ^\dagger \mathrm{e}^{\sum_{f,g=1}^{N_f}\left((\chi_{A\,f}^\dagger\chi_{A\,g})Q^{gf}+(\chi_{B\,f}^\dagger\chi_{B\,g})(Q^\dagger)^{gf}\right)}\mathrm{e}^{-N\langle\bar{q}q\rangle^2\mathrm{Tr}\,Q^\dagger Q}
\end{aligned} \tag{A.5}$$

where we have integrated out the matrices Φ, performed the trace and made a Hubbard-Stratonovich transformation. The auxiliary matrices Q are now quadratic of size $N_f \times N_f$. The integrations over the Grassmann vectors χ_A and χ_B in eqs. (A.3) and (A.5) are now all Gaussian, leading to the following determinant

$$\begin{aligned}
\mathcal{Z}_I^{(N_f,\nu)}(\mu;M) &\sim \int dQdQ^\dagger \det\begin{pmatrix} (M+Q)^T\otimes\mathbf{1}_N & \mu\Sigma_3\otimes\tilde{\mathbf{1}}_N \\ \mu\Sigma_3\otimes\tilde{\mathbf{1}}_N^\dagger & (M+Q^\dagger)^T\otimes\mathbf{1}_{N+\nu}\end{pmatrix}\mathrm{e}^{-N\langle\bar{q}q\rangle^2\mathrm{Tr}\,Q^\dagger Q} \\
&= \int dQdQ^\dagger \det[M+Q^\dagger]^\nu \\
&\quad \times\det\left[(M+Q^\dagger)(M+Q)-\mu^2(M+Q^\dagger)\Sigma_3(M+Q^\dagger)^{-1}\Sigma_3\right]^N \mathrm{e}^{-N\langle\bar{q}q\rangle^2\mathrm{Tr}\,Q^\dagger Q}.
\end{aligned} \tag{A.6}$$

Here we have introduced the mass matrix $M = \mathrm{diag}(\omega_1,\ldots,\omega_{N_f})$ and the generalized Pauli matrix of size $N_f \times N_f$

$$\Sigma_3 \equiv \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix}. \tag{A.7}$$

Furthermore we made use of the following property of determinants for invertible matrices D

$$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \det[D]\ \det[A\ -\ B\,D^{-1}C]\ . \tag{A.8}$$

For real valued masses, $M = M^\dagger$, we can further simplify eq. (A.6) by shifting $Q + M \to Q$,

$$\begin{aligned}
\mathcal{Z}_I^{(N_f,\nu)}(\mu;M) &= \mathrm{e}^{-N\langle\bar{q}q\rangle^2\mathrm{Tr}\,M^2}\int dQdQ^\dagger \det[Q^\dagger]^\nu \det\left[Q^\dagger Q-\mu^2 Q^\dagger\Sigma_3 Q^{\dagger\,-1}\Sigma_3\right]^N \\
&\quad\times\mathrm{e}^{-N\langle\bar{q}q\rangle^2\mathrm{Tr}\left(Q^\dagger Q\ -\ M(Q+Q^\dagger)\right)}.
\end{aligned} \tag{A.9}$$

In the absence of conjugate anti-quarks, $\Sigma_3 = \mathbf{1}_{N_f}$, the second determinant further simplifies, leaving $\mu^2\mathbf{1}_{N_f}$ only. In order to arrive at a unitary group integral we introduce the "polar decomposition" $Q = U\,R$, where $U^\dagger = U^{-1}$ is an $N_f \times N_f$ unitary matrix and $R^\dagger = R$ is hermitian and positive. The Jacobian for the transformation is derived in the following appendix. An alternative is the Schur decomposition of the matrix Q used in [15] which quickly becomes cumbersome for more than $N_f = 3$ flavors. The advantage of our present method is that we can conveniently exploit the saddle point approximation for the integrals over positive eigenvalues r_i of the matrix R.

B Jacobian for the polar decomposition

A classical result of linear algebra states that any $N_f \times N_f$ complex matrix Q can be written as $Q = U\,R$ where $U^\dagger = U^{-1}$ is an $N_f \times N_f$ unitary matrix and $R^\dagger = R$ is $N_f \times N_f$ hermitian positive matrix. That is the matrix-generalization of the "polar decomposition" of a complex number $q = re^{i\theta}$ Clearly enough the number of real degrees of freedom of the matrix Q (i.e. $2N_f^2$) matches with the total number of real degrees of freedom of U (i.e. N_f^2) and R (i.e. N_f^2). Under this matrix change of variables, the integration measure produces a Jacobian:

$$dQ\,dQ^\dagger = J(R)\,d\mu(U)\,dR\,, \tag{B.1}$$

where $d\mu(U)$ is the Haar measure on the unitary group, dR is the measure on the space of Hermitian positive matrices and $J(R)$ is the Jacobian that we are going to calculate in this appendix. For calculating the Jacobian[17], we differentiate:

$$dQ = dU\,R + U dR\,. \tag{B.2}$$

As R is an Hermitian matrix, it can be diagonalized by an unitary matrix V: $R = V\hat{r}V^{-1}$ with $\hat{r} = \mathrm{diag}(r_1, \ldots, r_n)$ having positive entries. If we left-multiply eq. (B.2) by $V^{-1}U^{-1}$ and right-multiply by V, we get

$$V^{-1}U^{-1}\,dQ\,V = V^{-1}U^{-1}\,dU\,V\,r + V^{-1}\,dR\,V\,. \tag{B.3}$$

This equation can be written as $d\tilde{Q} = d\tilde{U}\,r + d\tilde{R}$ with $d\tilde{Q} \equiv V^{-1}U^{-1}\,dQ\,V$, $d\tilde{U} \equiv V^{-1}U^{-1}\,dU\,V$ and $d\tilde{R} \equiv V^{-1}\,dR\,V$. The matrix $d\tilde{U}$ is anti-hermitian, since $(d\tilde{U})^\dagger = V^\dagger (dU)^\dagger UV = -V^{-1}U^{-1}\,dU\,V = -d\tilde{U}$ (we used $dU^\dagger U = -U^{-1}\,dU$). Moreover, the matrix $d\tilde{R}$ is hermitian, since $d\tilde{R}^\dagger = (V^{-1}\,dR\,V)^\dagger = \tilde{R}$. Therefore, the relation between the differentials of the independent complex variables is:

$$\begin{cases} (d\tilde{Q})_{ii} = d\tilde{U}_{ii}\,r_i + d\tilde{R}_{ii} & i = 1, \ldots, N_f \\ (d\tilde{Q})_{ij} = d\tilde{U}_{ij}\,r_j + d\tilde{R}_{ij} & \\ (d\tilde{Q})_{ji} = d\tilde{U}^*_{ij}\,r_i + d\tilde{R}^*_{ij} & i < j \end{cases} . \tag{B.4}$$

Let us introduce a real parameterization as follows:

$$\begin{aligned} (d\tilde{Q})_{ij} &= dx_{ij} + i dy_{ij}\,, \quad (d\tilde{R})_{ij} = \begin{cases} (d\tilde{R})_{ij} = dp_{ij} + i dq_{ij} & i < j\,, \\ (d\tilde{R})_{ii} = dp_{ii} & i = 1, \ldots, N_f \end{cases} \\ (d\tilde{U})_{ij} &= \begin{cases} (d\tilde{U})_{ij} = dv_{ij} + i dw_{ij} & i < j\,, \\ (d\tilde{U})_{ii} = i dw_{ii} & i = 1, \ldots, N_f \end{cases} . \end{aligned}$$

By matching the real and imaginary parts we have:

1) $dx_{ii} = dp_{ii}$ and $dy_{ii} = r_i dw_{ii}$, for $i = 1, \ldots, N_f$. The corresponding Jacobian is $J_1 = \prod_{i=1}^{N_f} r_i$.

2) $dx_{ij} = dv_{ij} r_j + dp_{ij}$ and $dx_{ji} = -dv_{ij} r_i + dp_{ij}$ for any pair $i < j$. The corresponding Jacobian is $(J_2)_{ij} = \det \frac{\partial(x_{ij}, x_{ji})}{\partial(v_{ij}, p_{ij})} = \det \begin{vmatrix} r_j & 1 \\ -r_i & 1 \end{vmatrix} = (r_i + r_j)$, and therefore $J_2 = \prod_{i<j}^{N_f} (r_i + r_j)$.

3) $dy_{ij} = dw_{ij} r_j + dq_{ij}$ and $dy_{ji} = dw_{ij} r_i - dq_{ij}$ for any pair $i < j$. The corresponding Jacobian is $J_3 = \prod_{i<j}^{N_f} \det \begin{vmatrix} r_j & 1 \\ r_i & -1 \end{vmatrix} = \prod_{i<j}^{N_f} (r_i + r_j)$.

[17] For general methods of evaluating Jacobians, see [49] [50].

Therefore the full Jacobian is $J = \prod_{i=1}^{N_f} r_i \prod_{i<j}^{N_f} (r_i + r_j)^2$. Going back to the original variables $d\tilde{U} \to dU$, $d\tilde{R} \to dR$ and $d\tilde{Q} \to dQ$ does not produce any additional factor in the final Jacobian (up to an overall sign) as they differ only by unitary transformations. We thus obtain the final result

$$\begin{aligned} dQ\, dQ^\dagger &= \prod_{i=1}^{N_f} r_i \prod_{i<j}^{N_f} (r_i + r_j)^2 \, d\mu(U)\, dR \\ &\propto \prod_{i=1}^{N_f} r_i \prod_{i<j}^{N_f} (r_i + r_j)^2 \prod_{i<j}^{N_f} (r_i - r_j)^2 \, d\mu(U) d\mu(V) \prod_{i}^{N_f} dr_i \\ &= \prod_{i=1}^{N_f} r_i \, dr_i \prod_{i<j}^{N_f} = (r_i^2 - r_j^2)^2 d\mu(U) d\mu(V) \ , \end{aligned} \tag{B.5}$$

where we used the standard fact that $R = V \hat{r} V^{-1}$ implies $dR \propto \prod_{i<j}^{N_f} (r_i - r_j)^2 \prod_i^{N_f} dr_i \, d\mu(V)$. We have thus shown eq. (B.5) to be the expression for the Jacobian for the polar decomposition $Q = U\, R$.

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