Massive partition functions and complex eigenvalue correlations in Matrix Models with symplectic symmetry

G. Akemann[1] and F. Basile[1,2]

[1]Department of Mathematical Sciences & BURSt Research Centre
School of Information Systems, Computing and Mathematics
Brunel University West London
Uxbridge UB8 3PH, United Kingdom

[2]Dipartimento di Fisica dell'Universitá di Pisa and INFN
Via Buonarroti, 56127 Pisa, Italy

Abstract

We compute all massive partition functions or characteristic polynomials and their complex eigenvalue correlation functions for non-Hermitean extensions of the symplectic and chiral symplectic ensemble of random matrices. Our results are valid for general weight functions without degeneracies of the mass parameters. The expressions we derive are given in terms of the Pfaffian of skew orthogonal polynomials in the complex plane and their kernel. They are much simpler than the corresponding expressions for symplectic matrix models with real eigenvalues, and we explicitly show how to recover these in the Hermitean limit. This explains the appearance of three different kernels as quaternion matrix elements there in terms of derivatives of a single kernel here.

1 Introduction

Random Matrix Models with complex eigenvalues have received much attention recently. This is due to both the development of new techniques leading to a wealth of new analytical results, as well as to many interesting new applications, and we refer to [1, 2] for recent reviews and references. On the technical side new results were obtained on orthogonal polynomials in the complex plane such as Hermite [3] and Laguerre [4, 5] polynomials, the solution of complex Matrix Models (MM) using Replicas [6] and their relation to the Toda hierarchy [7], as well as sigma-model and supersymmetry methods [7, 8]. All 3 approaches lead to the same results when applicable, and the two former [9] and the two latter have [10] been shown to be equivalent.

Our main motivation is the application of complex matrix models to Quantum Chromodynamics (QCD) and related theories in the presence of a quark chemical potential μ. Analytical MM predictions can be compared to numerical simulations in Lattice gauge theory. Here gauge group $SU(2)$ or the adjoint representation have the virtue that the Dirac spectrum remains real even in the presence of a chemical potential, and standard Monte-Carlo techniques can be applied. For $SU(2)$ the sign of the Dirac operator determinant fluctuates and one has to restrict oneself to an even number of flavours.

In [5] a MM was introduced that describes the symmetry class for theories in the adjoint representation with chemical potential. It is given by a two-MM generalising the chiral Symplectic Ensemble (chSE) [11] and was solved for two-fold degenerate quark masses [5]. Its predictions were compared to quenched [12] and unquenched Lattice simulations [13][1] for 2-colour QCD using staggered fermions. Because of this success we expect that this MM is equivalent to the corresponding chiral Lagrangian in the epsilon-regime [15]. In order to be able to prove this conjecture we complete here the solution of [5] by allowing for arbitrary non-degenerate quark masses. This will permit a detailed comparison to group integrals of the chiral Lagrangian in the epsilon-regime in the future. We also compute all massive partition functions (or characteristic polynomials) and complex eigenvalue correlations functions with arbitrary mass insertions for another symmetry class, the complex extension of the Symplectic Ensemble (SE) [16] for general weight functions [17].

The expressions we obtain for the complex SE and chSE are very similar in structure to the known results for the complex Unitary (UE) and chiral Unitary Ensemble (chUE), replacing determinants by Pfaffian's. However, our computations are more difficult as we have to use skew orthogonal instead of orthogonal polynomials in the complex plane. This is due to the Jacobian of symplectic matrices [16, 5]. The difficulty is reflected in the fact that we can only compute products of characteristic polynomials, extending [18]. It is not clear how to extend the results for ratios in the unitary case [19, 20] to symplectic ensembles. The proof for ratios in the SE with real eigenvalues [21] is based on a discretisation of the ensemble which is not obvious to extend to the complex case.

Our second main result concerns the link between symplectic MM with real and complex eigenvalues. We use the fact that there exist weight functions in the complex plane that allow to take the Hermitean limit leading to real eigenvalue correlations, of the SE and chSE respectively. In this way we explicitly recover the results [22, 23, 24, 25, 26] for real eigenvalue correlation functions and characteristic polynomials for an arbitrary weight function. Moreover, from the Hermitean limit of our simple result for complex eigenvalue correlation functions in terms of a Pfaffian of a single kernel we can derive why the known results for the chSE and SE are written in terms of a quaternion matrix of *three different* kernels, that are related by differentiation. This follows very naturally from a Taylor expansion of our single kernel in the Hermitean limit, due to the resulting degeneracies of the variables z with their complex conjugate z^*.

Our paper is organised as follows. In the next section 2 we give the definitions of our MM and

[1]In fact all simulations were unquenched using the code of [14]. The closest eigenvalues to the origin were then quenched (unquenched) using large (small) masses respectively.

their complex eigenvalue correlations in subsection 2.1, and present the new results we obtain for massive correlation functions in subsection 2.2. Our results in the Hermitean limit are summarised in subsection 2.3. The derivation of our findings are given in section 3 where we prove two theorems. Here, the chiral and non-chiral case are treated in parallel. At the end of this section we give examples for weight functions in either case that allow to take the Hermitean limit.

2 Summary of Results

In this section we present our main results for complex extensions of the SE and chSE to be defined below, with an arbitrary number of masses or characteristic polynomials inserted. In the Hermitean limit we recover the known results for the SE and chSE with real eigenvalues. We also briefly compare these findings to known results for the unitary complex MM.

2.1 Definitions

The massive partition function of the complex extension of the SE is defined as

$$\mathcal{Z}_N^{(M)}(\{m\}) \sim \int d\Phi \prod_{f=1}^{M} \det[m_f \mathbf{1}_N - \Phi] \exp[-\text{Tr}NV(\Phi, \Phi^\dagger)] \ , \tag{2.1}$$

where Φ is an $N \times N$ matrix with quaternion real elements without further symmetries, and $\mathbf{1}_N$ is the quaternion unity element. The integral runs over all independent matrix elements Φ_{ij}. The M mass parameters or arguments of the characteristic polynomials m_f are taken to be complex and pairwise distinct, $m_i \neq m_j$.

To proceed we only consider harmonic potentials of the form $V(\Phi, \Phi^\dagger) = \Phi \cdot \Phi^\dagger + V_1(\Phi) + V_1(\Phi^\dagger)$. The reason is that this choice in eq. (2.1) allows to go to a complex eigenvalue basis by a Schur decomposition [22] $\Phi = U(Z + T)U^\dagger$. Here the diagonal matrix Z with quaternion matrix element Z_{ii} =diag(z_i, z_i^*) as a complex 2×2 matrix contains the complex eigenvalues in complex conjugate pairs. The upper triangular matrix T drops out in the potentials V_1 and decouples in $\text{Tr}\Phi\Phi^\dagger = \text{Tr}(ZZ^\dagger + TT^\dagger)$. Thus it can be integrated out being Gaussian, as well as the symplectic matrix U. For a detailed discussion of harmonic potentials we refer to [2]. After these manipulations we arrive at a complex eigenvalue integral given by

$$\mathcal{Z}_N^{(M)}(\{m\}) \equiv \int \prod_{i=1}^{N} d^2 z_i \ w(z_i, z_i^*) \prod_{f=1}^{M} (m_f - z_i)(m_f - z_i^*) \ \mathcal{J}(\{z, z^*\}) \ , \tag{2.2}$$

with Jacobian [16]

$$\mathcal{J}(\{z, z^*\}) \equiv \prod_{k>l}^{N} |z_k - z_l|^2 \ |z_k - z_l^*|^2 \prod_{h=1}^{N} |z_h - z_h^*|^2 \ . \tag{2.3}$$

Here $d^2z = dxdy$ with $z = x + iy$ denotes an integration over the full complex plane. In all the following we will take eq. (2.2) as a starting point. Furthermore we will only require from the weight function $w(z, z^*) = \exp[-V(z, z^*)]$ that i) it is real and symmetric $w(z, z^*) = w(z^*, z)$ and that ii) all complex moments exist: $\int d^2 z w(z, z^*) z^k z^{*\,l} < \infty$, being more general than in eq. (2.1). We note that the Jacobian considerably differs from the one of the SE given by the Vandermonde determinant $\Delta_N(\{x\}) \equiv \prod_{k>l}^{N}(x_k - x_l)$ of real eigenvalues to the $4th$ power, $\mathcal{J}_{SE} = \Delta_N(\{x\})^4$. A complex extension of the SE with a similar Jacobian can be constructed from normal matrices, $\mathcal{J}_{norm} = |\Delta_N(\{z\})|^4$ [27]. However, we have not been able to obtain results for correlation functions in that case, even without mass insertions this model is currently unsolved at finite-N (for the spectral correlations from a saddle point approximation at $N \to \infty$ see [2]).

Similar to eq. (2.2) we define the complex extension of the chiral SE as

$$\mathcal{Z}_{N\,ch}^{(M)}(\{m\}) \equiv \int \prod_{i=1}^{N} d^2 z_i \; w(z_i, z_i^*) \prod_{f=1}^{M} m_f^{2\nu}(m_f^2 - z_i^2)(m_f^2 - z_i^{*\,2}) \; \mathcal{J}_{ch}(\{z, z^*\}) \;, \tag{2.4}$$

with Jacobian [5]

$$\mathcal{J}_{ch}(\{z, z^*\}) \equiv \prod_{k>l}^{N} |z_k^2 - z_l^2|^2 \; |z_k^2 - z_l^{*\,2}|^2 \prod_{h=1}^{N} |z_h^2 - z_h^{*\,2}|^2 \;. \tag{2.5}$$

Note that in applications to QCD the mass terms are usually taken to be positive, shifting $m \to im$. Eq. (2.4) only differs from the non-chiral ensemble by replacing its eigenvalues by squares $z \to z^2$. While the complex SE has a repulsion of eigenvalues from the real axis [17], as can be seen from the terms $|z_i - z_i^*|^2 = 4y_i^2$ in the Jacobian eq. (2.3), the occurrence of squared eigenvalues here leads to a repulsion from both the real and imaginary axis [5]: $|z_i^2 - z_i^{*\,2}|^2 = 16x_i^2 y_i^2$. In addition we have added a factor $\prod_{f=1}^{M} m_f^{2\nu}$, ensuring a finite limit when $m_f \to 0$.

The complex chSE enjoys a representation as a Gaussian two-matrix model of rectangular $N \times (N+\nu)$ matrices with real quaternion entries [5]. In this case the transformation to eigenvalues can be carried out explicitly, and we refer to [5] for details. The outcome is that the resulting weight w factorises into two parts: w_V which results from inserting the eigenvalue matrix Z into the harmonic potential. This part is non-universal. The second part w_U comes about as follows. In the two-MM we initially have 2 sets of complex eigenvalues. Since we are only interested in the Dirac operator eigenvalues given by their product we change variables and integrate out one set. The resulting part w_U is not the exponential of the potential, for an explicit example see eq. (3.52). This factor is expected to be universal as for the unitary ensembles it relates to bosonic partition functions [7, 9] (for a detailed discussion see [28]).

In the following we will allow for a general weight w in eq. (2.4) in terms of complex eigenvalues, with the only requirement of convergent moments as before. Expectation values of characteristic polynomials, which are proportional to massive partition functions, are defined as

$$\left\langle \prod_{j=1}^{N} \prod_{f=1}^{M} (m_f - z_j)(m_f - z_j^*) \right\rangle_{\mathcal{Z}_N^{(0)}} \equiv \frac{\mathcal{Z}_N^{(M)}(\{m\})}{\mathcal{Z}_N^{(0)}} \;, \tag{2.6}$$

and similarly for the chiral ensemble in terms of squared variables, $m_f \to m_f^2$, $z_j^{(*)} \to z_j^{(*)2}$. The characteristic polynomials also enjoy a matrix representation $\langle \prod_{f=1}^{M} \det[m_f \mathbf{1} - \Phi] \rangle_{\mathcal{Z}_N^{(0)}}$ as in eq. (2.1). We define the k-point complex eigenvalues correlation functions in the presence of M masses as

$$R_{N,k}^{(M)}(z_1, \ldots, z_k; \{m\}) \equiv \frac{N!}{(N-k)!} \frac{1}{\mathcal{Z}_N^{(M)}(\{m\})} \int \prod_{j=k+1}^{N} d^2 z_j \prod_{i=1}^{N} w(z_i, z_i^*) \prod_{f=1}^{M} (m_f - z_i)(m_f - z_i^*) \mathcal{J}(\{z, z^*\}) \tag{2.7}$$

where we integrate out all eigenvalues z_l with $l \geq (k+1)$. Obviously the k-point function also depends on the complex conjugate arguments $z_1^*, \ldots, z_k^*$ which we have suppressed in the notation, but not on the complex conjugate masses. In the chiral expression we again have squared variables.

2.2 Results for correlation functions

We obtain the following new result for arbitrary characteristic polynomials

$$\left\langle \prod_{j=1}^{N} \prod_{l=1}^{M} (m_l - z_j)(m_l - z_j^*) \right\rangle_{\mathcal{Z}_N^{(0)}} = \frac{(-)^{[M/2]}}{\Delta_M(\{m\})} \mathrm{Pf}_{\ i,j=1,\ldots,M} [\Theta_{N+[M/2]}(\{m\})] \ . \tag{2.8}$$

Here we have to distinguish between even and odd M

$$\Theta_{N+[M/2]}(\{m\}) \equiv \begin{cases} \left(\kappa_{N+[M/2]}(m_i, m_j)\right)_{i,j=1,\ldots,M} & \text{if } M \text{ is even} \\ \begin{pmatrix} \kappa_{N+[M/2]}(m_i, m_j)_{i,j=1,\ldots,M} & q_{2N+M-1}(m_i) \\ -q_{2N+M-1}(m_j) & 0 \end{pmatrix} & \text{if } M \text{ is odd,} \end{cases} \tag{2.9}$$

where $[M/2]$ denotes the integer part of $M/2$. Thus for M odd the $M \times M$ matrix Θ has 1 extra last row and column. The polynomials $q_k(m)$ are the skew-orthogonal polynomials with respect to the following antisymmetric scalar product

$$\langle f, g \rangle_S \ \equiv \ \int d^2z \ w(z, z^*)(z^* - z)[f(z)g(z)^* - f(z)^* g(z)]. \tag{2.10}$$

In the chiral case we simply modify the factor $(z^* - z) \to (z^{*\,2} - z^2)$ in the scalar product. They satisfy

$$\begin{aligned} \langle q_{2k+1}, q_{2l} \rangle_S &= -\langle q_{2l}, q_{2k+1} \rangle_S = r_k \ \delta_{kl} \ , \\ \langle q_{2k+1}, q_{2l+1} \rangle_S &= \quad \langle q_{2l}, q_{2k} \rangle_S \quad = 0 \ . \end{aligned} \tag{2.11}$$

In all the following we will chose them in monic normalisation, $q_k(z) = z^k + \mathcal{O}(z^{k-1})$. From these polynomials the second ingredient in eq. (2.9) is constructed, the anti-symmetric kernel[2]

$$\kappa_N(z_1, z_2^*) \ \equiv \ \sum_{k=0}^{N-1} \frac{1}{r_k} \left(q_{2k+1}(z_1) q_{2k}(z_2^*) - q_{2k+1}(z_2^*) q_{2k}(z_1) \right) \ . \tag{2.12}$$

If we multiply eq. (2.8) by the normalisation

$$\mathcal{Z}_N^{(0)} = N! \prod_{i=0}^{N-1} r_i \tag{2.13}$$

we obtain the massive partition functions $\mathcal{Z}_N^{(M)}(\{m\})$.

Our second new result is for correlation functions with arbitrary masses:

$$R_{N,k}^{(M)}(z_1, \ldots, z_k; \{m\}) = \prod_{h=1}^{k} w(z_h, z_h^*)(z_h^* - z_h) \frac{\mathrm{Pf}_{\ 1,\ldots,2k+M} \left[\Theta_{N+[M/2]}(\{u\})\right]}{\mathrm{Pf}_{\ 1,\ldots,M} \left[\Theta_{N+[M/2]}(\{m\})\right]} \ , \tag{2.14}$$

where the set of variables in the numerator $\{u\} = \{z_1, z_1^*, \ldots, z_k, z_k^*, m_1, \ldots, m_M\}$ runs through all masses, and all un-integrated complex eigenvalues including their complex conjugates. In general Θ is not the complex representation of a quaternion matrix. The only modification of eq. (2.14) in the chiral case are squared arguments in the Vandermonde in eq. (2.8), $\Delta_M(\{m\}) \to \Delta_M(\{m^2\})$ and in the prefactor in eq. (2.14), $(z_h^* - z_h) \to (z_h^{*2} - z_h^2)$.

[2] This is usually called pre-kernel as it does not include the weight function.

In the case without masses $M = 0$ eq. (2.14) reduces to the known result [17], as we will show explicitly in the derivation in section 3 below. When M is even and we choose the masses to appear in complex conjugate pairs we recover the results of [5]. In this case eqs. (2.8) and (2.14) can be expressed entirely in terms of k-point eigenvalue correlation functions without mass insertions $R^{(0)}_{N,k}$ with $k = M$ and $k + M$ respectively [5][3].

Let us give some examples. In the simplest case of a single mass or characteristic polynomial we have

$$\left\langle \prod_{j=1}^{N} (m - z_j)(m - z_j^*) \right\rangle_{\mathcal{Z}_N^{(0)}} = q_{2N}(m) , \tag{2.15}$$

giving the subset of even skew-orthogonal polynomials in monic normalisation. This relation was already noted in [17] in the complex SE and in [5] for the complex chSE, and we can also write it as an expectation value of a determinant as in eq. (2.1). A similar relation holds for the odd skew-orthogonal polynomials as the expectation value of a single determinant times the trace $\mathrm{Tr}(m\mathbf{1}_N + \Phi)$ [17, 5] (up to a constant), see eq. (3.42) below. Note that in the general expression eq. (2.8) only the even skew-orthogonal polynomials appear explicitly (for M odd) while the odd ones only occur through the kernel.

The second simplest example contains two characteristic polynomials,

$$\left\langle \prod_{j=1}^{N} (m_1 - z_j)(m_1 - z_j^*)(m_2 - z_j)(m_2 - z_j^*) \right\rangle_{\mathcal{Z}_N^{(0)}} = \frac{r_N}{m_2 - m_1} \, \kappa_{N+1}(m_2, m_1) . \tag{2.16}$$

This new relation gives the anti-symmetric kernel, the second building block to all characteristic polynomials or eigenvalue correlation functions. From eqs. (2.15) and (2.16) we could thus rewrite our general results eqs. (2.8) and (2.14) entirely in terms of expectations values of one and two determinants. A similar structure has been revealed for MM with real eigenvalues in all 3 classical Wigner-Dyson ensembles [21] (see also refs. in [29]).

It is very instructive to compare eqs. (2.15) and (2.16) to the known results [18] for the complex extension of the UE and chUE: both equations hold almost identically with the modification that we obtain *all* orthogonal polynomials $P_k(z)$ with respect to $\int d^2z \; w(z, z^*) P_k(z) P_l(z^*) = h_k \delta_{kl}$, for both even and odd k [18]:

$$\langle \det[m - \psi] \rangle_{UE} = P_N(m) . \tag{2.17}$$

Here ψ is a complex non-Hermitean matrix and the polynomials are in monic normalisation $P_k(z) = z^k + \mathcal{O}(z^{k-1})$. The Hermitean kernel of the polynomials $P_k(z)$ exactly equals the expectation value of a determinant times its complex conjugate [18]:

$$\langle \det[m_1 - \psi] \det[m_2^* - \psi^\dagger] \rangle_{UE} = h_N \; K_{N+1}(m_1, m_2^*) , \tag{2.18}$$

where $K_{N+1}(m_1, m_2^*) = \sum_{k=0}^{N} h_k^{-1} P_k(m_1) P_k(m_2)$. In order to underline the similarity in structure between the results for the complex unitary and symplectic ensembles we give the result corresponding to eqs. (2.8) and (2.14) as well. For products of characteristic polynomials of the UE we have [18]

$$\left\langle \prod_{i=1}^{K} \det[m_i - \psi] \prod_{j=1}^{L} \det[n_j^* - \psi^\dagger] \right\rangle_{UE} = \frac{\prod_{j=N}^{N+L-1} h_j}{\Delta_K(\{m\}) \Delta_L(\{n^*\})} \det_{1 \le i,j \le K+L} \left[K_{N+L}(m_i, n_j^*) \ldots P_{N+L-j}(m_i) \right] \tag{2.19}$$

[3]For this relation to be exact we have to have an N-independent weight function.

where $K \geq L$ without loss of generality. For $K = L$ this is a determinant only made of kernels, as in eq. (2.8) for M even. In the symplectic ensemble the pairing of complex conjugated eigenvalues is automatic, and eq. (2.8) resembles the square root of eq. (2.19) at $m_i = n_i$. For $K > L$ there are $K - L$ extra polynomials inside the determinant. For the massive eigenvalue correlations in the complex UE we have [30]

$$R^{(M)}_{N,k\ UE}(z_1,\ldots,z_k;\{m\}) = \prod_{h=1}^{k} w(z_h,z_h^*)(z_h^* - z_h)\frac{\det_{1,\ldots,2k+M}\left[K_N(u_i,u_j)\right]}{\det_{1,\ldots,M}\left[K_N(m_i,m_j)\right]} \ . \tag{2.20}$$

Its structure completely agrees with eq. (2.14) upon replacing the Pfaffian with a determinant.

2.3 Results in the Hermitean limit

Suppose the weight function in the complex plane $w(z,z^*)$ depends on a non-Hermiticity parameter τ and permits a Hermitean limit $\tau \to 1$ that projects $z = x + iy$ onto the real x-axis:

$$\lim_{\tau\to 1} |z^* - z|^2 w(z,z^*) = \delta(y)\,\overline{w}(x) \ , \tag{2.21}$$

where $\delta(y)$ is the Dirac delta-distribution[4]. The main difference to the UE is that at any value $\tau < 1$ the left hand side appearing inside the integrals is identically zero for $y = 0$, the signature of the symplectic ensembles. The weight $\overline{w}(x)$ is the projected weight function on $\mathbb{R}$, and we denote the projected skew orthogonal polynomials by $\overline{q}_j(x)$, their norms by $\overline{r}_j$ and their kernel by $\overline{\kappa}_N(x_1,x_2)$, respectively.

An example for such a weight function is the Gaussian weight of the complex SE

$$\begin{aligned} w_{GSE}(z,z^*) &= \frac{N^{\frac{3}{2}}\exp\left(-\frac{N}{1-\tau^2}\left(|z|^2 - \frac{\tau}{2}(z^2+z^{*2})\right)\right)}{2\sqrt{\pi}(1-\tau)^{\frac{3}{2}}} \\ &= \frac{N^{\frac{3}{2}}}{2\sqrt{\pi}(1-\tau)^{\frac{3}{2}}}\exp\left(-\frac{Ny^2}{1-\tau}\right)\exp\left(-\frac{Nx^2}{1+\tau}\right) \ , \quad \tau\in[0,1) \ , \end{aligned} \tag{2.22}$$

with a resulting projected Gaussian weight $\overline{w}(x) = \exp(-Nx^2/2)$. Before the Hermitean limit $\tau \to 1$ is taken the corresponding skew orthogonal polynomials are given in terms of Hermite polynomials in the complex plane [17]. After the projection they are given by the ordinary skew orthogonal polynomials of the Gaussian SE in terms of Hermite polynomials on the real line [22]. A similar example exists for the chiral ensemble in terms of Laguerre polynomials in the complex plane [5], and we refer to subsection 3.2 for more details. The important point is that the Hermitean limit maps our previous results eqs. (2.8) and (2.14) to the known results of the symplectic ensembles with real eigenvalues.

More explicitly the formula (2.8) for the characteristic polynomials we trivially replace the polynomials and kernel by their projected quantities inside the matrix Θ in eq. (2.9):

$$\lim_{\tau\to 1}\left\langle \prod_{j=1}^{N}\prod_{l=1}^{M}(m_l - z_j)(m_l - z_j^*)\right\rangle_{\mathcal{Z}_N^{(0)}} = \frac{(-)^{[M/2]}}{\Delta_M(\{m\})}\text{Pf}_{\ 1,\ldots,M}[\overline{\Theta}_{N+[M/2]}(\{m\})] \tag{2.23}$$

The limit for correlation functions is more involved, and we obtain the following non-trivial result:

$$\lim_{\tau\to 1} R^{(M)}_{N,k}(z_1,\ldots,z_k;\{m\}) = \prod_{h=1}^{k}\delta(y_h)\overline{w}(x_h)\frac{\text{Pf}_{\ 1,\ldots,2k+M}\left[\overline{\Omega}_{N+[M/2]}(x_1,x_2,\ldots,x_k,m_1,\ldots,m_M)\right]}{\text{Pf}_{\ 1,\ldots,M}\left[\overline{\Omega}_{N+[M/2]}(\{m\})\right]} \ . \tag{2.24}$$

[4] In [5] the limit was denoted by $-y\delta(y)'$ instead, which is equivalent after integration by parts.

To compare to real eigenvalue correlation functions we have to drop the delta-functions, or formally integrate over the remaining imaginary parts $y_1, \ldots, y_k$. The projected matrix $\overline{\Omega}_{N+[M/2]}$ is obtained as

$$\overline{\Omega}_R \equiv \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \partial_{x_i}\partial_{x_j}\overline{\kappa}_R(x_i,x_j) & \partial_{x_i}\overline{\kappa}_R(x_i,x_j) & \cdots & \partial_{x_i}\overline{\kappa}_R(x_i,m_f) & \cdots & \partial_{x_i}\overline{q}_{2R}(x_i) \\ \cdots & \partial_{x_j}\overline{\kappa}_R(x_i,x_j) & \overline{\kappa}_R(x_i,x_j) & \cdots & \overline{\kappa}_R(x_i,m_f) & \cdots & \overline{q}_{2R}(x_i) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \partial_{x_j}\overline{\kappa}_R(m_g,x_j) & \overline{\kappa}_R(m_g,x_j) & \cdots & \overline{\kappa}_R(m_g,m_f) & \cdots & \overline{q}_{2R}(m_g) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -\partial_{x_j}\overline{q}_{2R}(x_j) & -\overline{q}_{2R}(x_j) & \cdots & -\overline{q}_{2R}(m_f) & \cdots & \cdots \end{pmatrix} \tag{2.25}$$

and depends on the projected kernel and its first and second derivatives. Here $R = N + [M/2]$ and $2R = 2k + M - 1$. If we now identify

$$\begin{aligned} I_N(x,t) &= \sqrt{\overline{w}(x)\overline{w}(t)}\ \overline{\kappa}_N(x,t) \\ S_N(x,t) &= \sqrt{\overline{w}(x)\overline{w}(t)}\ \partial_t\overline{\kappa}_N(t,x) \ = \ -\sqrt{\overline{w}(x)\overline{w}(t)}\ \partial_t\overline{\kappa}_N(x,t) \\ D_N(x,t) &= -\sqrt{\overline{w}(x)\overline{w}(t)}\ \partial_x\partial_t\overline{\kappa}_N(x,t) \end{aligned} \tag{2.26}$$

with the standard notation for the matrix elements of the quaternion valued kernel [22, 26] we recover the following known results for symplectic matrix models on the real line:

i) the result eqs. (2.23) was derived in [21] for the Gaussian SE in terms of expectation values of two characteristic polynomials, or equivalently in terms of the $I_N(x,t)$-kernel [26]. In taking the Hermitean limit we now understand why only the I- and not the S- and D-kernels appear. Moreover, the result from [21] is strictly valid only for a Gaussian weight function, whereas we can allow for an arbitrary weight here (for details see the derivation below)[5]. For the chiral ensembles we recover the results of [24, 25].

ii) In eq. (2.24) we recover the well known result [22] for correlation functions of the Gaussian SE in the absence of masses, which are given equivalently in terms of a quaternion determinant. We also re-obtain the massive correlation functions from [26], and in the chiral case the correlation functions [11] and [24, 25], respectively. Before taking the Hermitean limit our results are not only simpler, but they also offers an explanation why in the Hermitean limit three different kernels appear that are the first and second derivative of the kernel $\overline{\kappa}_N(x,t)$.

The fact that the 3 kernels in eq. (2.26) are related by differentiation was of course known previously, but without a deeper reasoning behind. So why does the complexification $x_i \to z_i$ simplify? The answer is that it lifts the following degeneracy: when replacing the Jacobian for real eigenvalues $\Delta(\{x\})^4$ by a determinant one has to use polynomials and their derivatives [22]. Our Jacobians eqs. (2.3) and (2.5) are proportional to a single Vandermonde determinant and can thus be expressed in terms of polynomials only, see eq. (3.4) below. Furthermore we are able to build a full 2×2 matrix depending on a single kernel of two arguments u_i and u_j and their complex conjugates. For real eigenvalues this is not possible.

As a final remark we comment on the other ensembles. For the Hermitian limit of the complex UE and chUE some interesting identities follow, and we refer to [18]. On the other hand for complex extensions of the orthogonal ensemble in terms of real non-symmetric matrices we do not expect any simplification to happen. To date no closed formula is known for their correlation functions, even without mass insertions.

[5]In [24] the authors restrict themselves to a Gaussian weight. A closer analysis of the derivation that follows from an earlier paper [23] shows that this restriction can be lifted.

3 Derivation of Results

3.1 Complex eigenvalue integrals

In this subsection we prove the following theorem in terms of a Pfaffian for $(N-k)$-fold integrals over complex eigenvalues with M mass insertions.

THEOREM 1 *Let $z_i \in \mathbb{C}, i=1,\ldots N$ be complex variables and $m_f \in \mathbb{C}, f=1,\ldots M$ be complex mass parameters. Given a real weight function $w(z,z^*)$ defined in the whole complex plane such that all moments exits, $\int d^2z\; w(z,z^*)z^k z^{*\,l}<\infty$, and a set of skew orthogonal polynomials $q_k(z)$ satisfying eq. (2.11) with scalar product (2.10). Then the following integral can be computed as*

$$\begin{aligned}
\mathcal{A}^{(M)}_{k,N}(z_1,\ldots,z_k;m_1,\ldots,m_M) &\equiv \int \prod_{j=k+1}^{N} d^2z_j \prod_{i=1}^{N} w(z_i,z_i^*) \prod_{f=1}^{M}(m_f-z_i)(m_f-z_i^*)\mathcal{J}(\{z,z^*\}) \\
&= \frac{(-)^{[M/2]}(N-k)!}{\Delta_M(\{m\})} \prod_{h=1}^{k} w(z_h,z_h^*)(z_h^*-z_h) \prod_{h=0}^{N+[M/2]-1} r_h \\
&\quad \times \mathrm{Pf}_{\,1,\ldots,2k+M}[\Theta_{N+[M/2]}(\{u\})]\ , \qquad (3.1)
\end{aligned}$$

with the set $\{u\}=\{z_1,z_1^*,\ldots,z_k,z_k^*,m_1,\ldots,m_M\}$, $\mathcal{J}(\{z,z^*\})$ defined by eq. (2.3) and the matrix Θ defined by eqs. (2.9) and (2.12) respectively. For the corresponding chiral integral with Jacobian eq. (2.5) we obtain

$$\begin{aligned}
\mathcal{A}^{(M)}_{k,N\,ch}(z_1,\ldots,z_k;m_1,\ldots,m_M) &\equiv \int \prod_{j=k+1}^{N} d^2z_j \prod_{i=1}^{N} w(z_i,z_i^*) \prod_{f=1}^{M}(m_f^2-z_i^2)(m_f^2-z_i^{*\,2})\mathcal{J}_{ch}(\{z,z^*\}) \\
&= \frac{(-)^{[M/2]}(N-k)!}{\Delta_M(\{m^2\})} \prod_{h=1}^{k} w(z_h,z_h^*)(z_h^{*\,2}-z_h^2) \prod_{h=0}^{N+[M/2]-1} r_h \\
&\quad \times \mathrm{Pf}_{\,1,\ldots,2k+M}[\Theta_{N+[M/2]}(\{u^2\})]\ , \qquad (3.2)
\end{aligned}$$

using the skew orthogonal polynomials $q_k(z)$ and kernel of the corresponding chiral scalar product.

Note that the integrals $\mathcal{A}$ defined above depend also on the complex conjugated eigenvalues $z_1^*,\ldots,z_k^*$, but not on the conjugated masses.

PROOF: The main ingredient of the proof is the availability of a skew orthogonal basis. For simplicity we will give the proof only for the non-chiral case, eq. (3.1). The chiral case trivially follows along the very same lines, inserting squared variables and using the chiral skew orthogonal product.

The first step is to write the integrand of (3.1) in terms of a Vandermonde determinant. For this aim we explicitly write out the Jacobian eq. (2.3)

$$\begin{aligned}
&\prod_{i=1}^{N}\prod_{f=1}^{M}(m_f-z_i)(m_f-z_i^*)\prod_{k>l}^{N}|z_k-z_l|^2|z_k-z_l^*|^2\prod_{j=1}^{N}|z_j-z_j^*|^2 = \\
&= (-)^N\prod_{k>l}^{N}(m_k-m_l)^{-1}\prod_{j=1}^{N}(z_j^*-z_j)\Delta_{2N+M}(z_1,z_1^*,\ldots,z_N,z_N^*,m_1,\ldots,m_M)\ , \qquad (3.3)
\end{aligned}$$

where we have used that the masses m_f are pairwise distinct. Using the standard trick a Vandermonde determinant may be written in terms of any set of monic polynomials $p_{j-1}(v_i)$ of degree $j-1$:

$$\Delta_n(v_1,\ldots,v_n) = \det_{i,j=1,\ldots,n}[v_i^{j-1}] = \det_{i,j=1,\ldots,n}[p_{j-1}(v_i)] \tag{3.4}$$

Here we chose the set of complex skew orthogonal polynomials $q_i(z)$ that satisfies the skew orthogonality eq. (2.11) with respect to our weight function. Defining the set of $2N+M$ variables $\{v_1,\ldots,v_{2N+M}\} = \{z_1,z_1^*,\ldots,z_N,z_N^*,m_1,\ldots,m_M\}$ we can write (suppressing the arguments of $\mathcal{A}$)

$$\begin{aligned}
\mathcal{A}_{k,N}^{(M)} &= \frac{(-)^N}{\Delta_M(\{m\})}\int\prod_{h=k+1}^{N} d^2z_h \prod_{j=1}^{N} w(z_j,z_j^*)\ (z_j^*-z_j) \det_{i,j=1,\ldots,2N+M}[q_{j-1}(v_i)] &(3.5)\\
&= \frac{(-)^N}{\Delta_M(\{m\})}\sum_{\{a\}}\sigma(a)\int\prod_{h=k+1}^{N} d^2z_h \prod_{j=1}^{N} w(z_j,z_j^*)\ (z_j^*-z_j)\prod_{j=0}^{2N+M-1} q_{a_j}(v_{j+1}). &(3.6)
\end{aligned}$$

The summation runs over all the possible permutations of $\{0,1,\ldots,2N+M-1\}$ with sign $\sigma(a)$. The integrand is symmetric with respect to exchanging $z_h \leftrightarrow z_h^*$ for all integrated variables $z_{k+1},\ldots,z_N$. Without loss of generality we can therefore arrange the sum over all permutations such that it always holds $a_{2j} < a_{2j+1}\ \forall j:\ k \le j < N$. We denote this rearrangement as $\sum_{\{a\}'}$. After this manipulation we can write

$$\begin{aligned}
\mathcal{A}_{k,N}^{(M)} &= \frac{(-)^N}{\Delta_M(\{m\})}\sum_{\{a\}'}\sigma(a)\prod_{h=1}^{k} w(z_h,z_h^*)(z_h^*-z_h)\prod_{i=0}^{2k-1} q_{a_i}(v_{i+1})\prod_{l=2N}^{2N+M-1} q_{a_l}(v_{l+1})\\
&\times\int\prod_{j=k}^{N-1} d^2z_{j+1}w(z_{j+1},z_{j+1}^*)(z_{j+1}^*-z_{j+1})(q_{a_{2j}}(z_{j+1})q_{a_{2j+1}}(z_{j+1}^*)-q_{a_{2j+1}}(z_{j+1})q_{a_{2j}}(z_{j+1}^*))\\
&= \frac{(-)^N}{\Delta_M(\{m\})}\sum_{\{a\}'}\sigma(a)\prod_{h=1}^{k} w(z_h,z_h^*)(z_h^*-z_h)\prod_{i=0}^{2k-1} q_{a_i}(v_{i+1})\prod_{l=2N}^{2N+M-1} q_{a_l}(v_{l+1})\prod_{j=k}^{N-1}\langle q_{a_{2j}}|q_{a_{2j+1}}\rangle_S
\end{aligned} \tag{3.7}$$

taking out the un-integrated variables. Using the properties (2.11) we can see that those permutations giving a non-zero value will be the ones satisfying:

$$a_{2j}\ +\ 1 = a_{2j+1}\ \ \textit{and}\ \ a_{2j}\ \text{is even}\ \forall\ j: k\le j<N\ . \tag{3.8}$$

Hence for all other indices a_l with $l \notin \{2k+1,\ldots,2N\}$ of the non-integrated polynomials only the following configurations contribute. The a_l are coupled in pairs as for even indices a_l their successor $a_l\ +\ 1$ and for odd indices a_l their predecessor $a_l\ -\ 1$ cannot belong to the integrated polynomials:

- a_l is even *and* $a_l \neq 2N+M-1\ \Rightarrow \exists\ l' \notin \{2k+1,\ldots,2N\}$ such that $a_{l'} = a_l\ +\ 1$
- a_l is odd $\ \Rightarrow \exists\ l' \notin \{2k+1,\ldots,2N\}$ such that $a_{l'}\ +\ 1 = a_l$.

There is only one possibility that a_l is un-coupled:

- M is odd and a_l assumes the maximum value[6], $a_l = 2N+M-1$

[6]We have an odd number of variables here.

This coupling into pairs is thus invoked from the scalar products in the last term in eq. (3.7).

We rename the set of the $2k+M$ non-integrated variables $\{u_j\} = \{z_1, z_1^*, \ldots, z_k, z_k^*, m_1, \ldots, m_M\}$. Eq. (3.7) can be seen as the sum of configurations anti-symmetrised with respect to these variables $\{u\}$ - we call such permutations η with sign $\sigma(\eta)$ - and moving the indices of subsequent non-integrated polynomials $q_{2j}(\ )q_{2j+1}(\ )$ along the line $j = 0, \ldots, N+[M/2]$. In order to count terms only once we must divide by $L!$, with $L \equiv k + [M/2]$, as these are the number of possible permutations to put L variables into pairs $q_{2j}(\ )q_{2j+1}(\ )$.

In the next step we show how these sums can be written in terms of the kernel eq. (2.12). First of all there are $(N-k)!$ permutations of the arguments of the integrated variables inside the scalar products that all give the same contribution. If we multiply and divide by all the remaining norms r_h we can write

$$\mathcal{A}_{k,N}^{(M)} = \frac{(-)^N}{\Delta_M(\{m\})L!} \prod_{h=1}^{k} w(z_h, z_h^*)(z_h^* - z_h)(N-k)! \prod_{h=0}^{N+[M/2]-1} (-r_h) \tag{3.9}$$

$$\times \sum_{\{\eta\}} \sigma(\eta) \sum_{\substack{h_1,\ldots,h_L=0 \\ h_l \neq h_j}}^{N+[M/2]-1} \prod_{i=1}^{L} \frac{1}{(-r_{h_i})} q_{2h_i}(u_{\eta_{2i-1}}) q_{2h_i+1}(u_{\eta_{2i}}) \begin{cases} 1 & M \text{ even} \\ q_{2N+M-1}(u_{\eta_{2L+1}}) & M \text{ odd} \end{cases}$$

$$= \frac{(-)^{[M/2]}}{\Delta_M(\{m\})2^L L!} \prod_{h=1}^{k} w(z_h, z_h^*)(z_h^* - z_h)(N-k)! \prod_{h=0}^{N+[M/2]-1} r_h$$

$$\times \sum_{\{\eta\}} \sigma(\eta) \prod_{i=1}^{L} \kappa_{N+[M/2]}(u_{\eta_{2i-1}}, u_{\eta_{2i}}) \begin{cases} 1 & M \text{ even} \\ q_{2N+M-1}(u_{\eta_{2L+1}}) & M \text{ odd} \end{cases} . \tag{3.10}$$

To go from (3.9) to (3.10) we have used the antisymmetry to generate all the terms in the sum of the kernel, giving a factor of $1/2^L$. Furthermore, we have used that where $h_l = h_j$ the terms drop out due to the antisymmetry in the $\{u\}$.

In a final step we show that the second line of (3.10) may be written in terms of a Pfaffian. For that we distinguish between even and odd M.

M even: We define the antisymmetric matrix of size $2L = 2k + M$

$$\Theta_R(\{u\}) \equiv \begin{pmatrix} 0 & \kappa_R(u_1,u_2) & \cdots & \kappa_R(u_1,u_{2L}) \\ \kappa_R(u_2,u_1) & 0 & \cdots & \kappa_R(u_2,u_{2L}) \\ \vdots & & \ddots & \vdots \\ \kappa_R(u_{2L},u_1) & \kappa_R(u_{2L},u_2) & \cdots & 0 \end{pmatrix} = (\kappa_R(u_i,u_j))_{i,j=1\ldots 2L} . \tag{3.11}$$

For this matrix the Pfaffian written as an ordered expansion [22] is given by

$$\mathrm{Pf}_{\ i,j=1\ldots 2L}[\kappa_R(u_i,u_j)] = \frac{1}{2^L L!} \sum_{\{\eta\}} \sigma(\eta) \prod_{j=1}^{L} \kappa_R(u_{\eta_{2j-1}}, u_{\eta_{2j}}) . \tag{3.12}$$

For $R = N + [M/2]$ this is exactly the desired result.

M odd: We define the antisymmetric matrix of size $2L+2=2k+M+1$

$$\begin{aligned}
\Theta_R(\{u\}) &= \begin{pmatrix} 0 & \kappa_R(u_1,u_2) & \cdots & \kappa_R(u_1,u_{2L+1}) & q_{2R}(u_1) \\ \kappa_R(u_2,u_1) & 0 & \cdots & \kappa_R(u_2,u_{2L+1}) & q_{2R}(u_2) \\ \vdots & & \ddots & \vdots & \vdots \\ \kappa_R(u_{2L+1},u_1) & \kappa_R(u_{2L+1},u_2) & \cdots & 0 & q_{2R}(u_{2L+1}) \\ -q_{2R}(u_1) & -q_{2R}(u_2) & \cdots & -q_{2R}(u_{2L+1}) & 0 \end{pmatrix} \\
&= \begin{pmatrix} (\kappa_R(u_i,u_j))_{i,j=1\dots 2L+1} & q_{2R}(u_i) \\ -q_{2R}(u_j) & 0 \end{pmatrix} . \qquad (3.13)
\end{aligned}$$

Consider the Pfaffian of this matrix

$$\mathrm{Pf}\;_{i,j=1\dots 2L+2}[\Theta_R(\{u\})] = \frac{1}{2^{L+1}(L+1)!}\sum_\eta \sigma(\eta)\prod_{j=1}^{L+1}\Theta_R(\{u\})_{\eta_{2j-1},\eta_{2j}} . \qquad (3.14)$$

In every permutation there exists exactly one index i such that $\eta_i = 2L+1$. The idea is to manipulate the permutations η over $2L+2$ elements to obtain a sum of permutations η' over $2L$ elements, leading to eq. (3.10).

As a first step we verify the invariance of the product $\sigma(\eta)\prod_{j=1}^{L+1}\Theta_R(\{u\})_{\eta_{2j-1},\eta_{2j}}$ under the following elementary transformations:

i) exchange of indices of a single factor $\Theta_R(\{u\})_{\eta_{2j-1},\eta_{2j}}$: $\eta_{2j-1} \longleftrightarrow \eta_{2j}$

ii) exchange of 2 pairs of indices among two factors: $\eta_{2j-1},\eta_{2j} \longleftrightarrow \eta_{2j+1},\eta_{2j+2}$

In the first transformation the change in sign is compensated by $\sigma(\eta)$ and in the second transformation the sign remains. Composing these two elementary steps we can always achieve that the factor containing the index $\eta_i = 2L+2$ is commuted to the end:

- if i is even: $\sigma(\eta)\prod_{j=1}^{L+1}\Theta_R(\{u\})_{\eta_{2j-1},\eta_{2j}} = \sigma(\eta)\prod_{j=1}^{L}\Theta_R(\{u\})_{\eta_{2j-1},\eta_{2j}}\Theta_R(\{u\})_{\eta_{2j+1},\eta_i=2L+2}$
- if i is odd: $\sigma(\eta)\prod_{j=1}^{m+1}\Theta_R(\{u\})_{\eta_{2j-1},\eta_{2j}} = -\sigma(\eta)\prod_{j=1}^{L}\Theta_R(\{u\})_{\eta_{2j-1},\eta_{2j}}\Theta_R(\{u\})_{\eta_{2j+1},\eta_i=2L+2}$.

In the first case we have used only ii), in the second we used both i) and ii). In each case we can thus write the result as a permutation η' of only $2L$ elements:

$$\sigma(\eta)\prod_{j=1}^{L+1}\Theta_R(\{u\})_{\eta_{2j-1},\eta_{2j}} = \sigma(\eta')\prod_{j=1}^{L}\Theta_R(\{u\})_{\eta'_{2j-1},\eta'_{2j}}\Theta_R(\{u\})_{\eta'_{2j+1},2L+2} . \qquad (3.15)$$

Here $2(L+1)$ permutations η will be mapped to the same permutation η': there are $(L+1)$ ways to choose the second index corresponding to η_i, and it can be in the first (i even) or second place (i odd) of the index pair. Together we obtain from eq. (3.14) that

$$\mathrm{Pf}\;_{i,j=1\dots 2L+2}[\Theta_R(\{u\})] = \frac{1}{2^{L+1}(L+1)!}2(L+1)\sum_{\{\eta'\}}\sigma(\eta')\prod_{j=1}^{L}\kappa_R(y_{\eta'_{2j-1}},y_{\eta'_{2j}})q_{2R}(y_{\eta'_{2L+1}}) , \qquad (3.16)$$

where η' runs over $2L$ elements and we have inserted the explicit matrix elements from eq. (3.13). That is the desired result for $R=N+[M/2]$, $2R=2N+M-1$.

To summarise we have shown

$$\mathcal{A}_{k,N}^{(M)}(z_1,\ldots,z_k;m_1,\ldots,m_M)=\frac{(N-k)!(-)^{[M/2]}}{\Delta_M(\{m\})}\prod_{h=1}^{k}w(z_h,w_h^*)(z_h^*-z_h)\prod_{h=0}^{N+[M/2]-1}r_h\,\mathrm{Pf}\left[\Theta_{N+[M/2]}(\{u\})\right] \tag{3.17}$$

where $\Theta_{N+[M/2]}(\{u\})$ is defined in (3.11) for even M and in (3.13) for odd M, as announced in eq. (2.9). □

Let us point out how all the results in subsection 2.2 follow from Theorem 1. For $k=M=0$ the Pfaffian and prefactors are absent and we obtain the normalisation of the massless partition function eq. (2.13)

$$\mathcal{Z}_N^{(0)}=\mathcal{A}_{k=0,N}^{(M=0)}=N!\prod_{i=0}^{N-1}r_i\ . \tag{3.18}$$

For $k=0$ and $M>0$ we obtain the massive partition functions $\mathcal{Z}_N^{(M)}(\{m\})$

$$\mathcal{Z}_N^{(M)}(\{m\})=\mathcal{A}_{k=0,N}^{(M)}(m_1,\ldots,m_M)\ , \tag{3.19}$$

and after dividing by $\mathcal{Z}_N^{(0)}$ the characteristic polynomials eq. (2.8). Finally the complex eigenvalues correlation functions are obtained as

$$R_{N,k}^{(M)}(z_1,\ldots,z_k;\{m\})=\frac{N!}{(N-k)!}\frac{1}{\mathcal{Z}_N^{(M)}(\{m\})}\mathcal{A}_{k,N}^{(M)}(z_1,\ldots,z_k;m_1,\ldots,m_M)\ . \tag{3.20}$$

The Vandermonde determinant of the mass parameters cancels as well as all factorials and the product over norms r_h, leading to the ratio of Pfaffian in eq. (2.14). In the same way all the results for the chiral ensemble follow, where for the partition functions we have to multiply in the factor $\prod_{f=1}^{M}m_f^{2\nu}$.

As a check we can recover the results of the non-chiral model without mass insertion [17][7] expressed in term of quaternion determinants [31]:

$$\begin{aligned}
R_{N,k}^{(0)}(z_1,\ldots,z_k) &= (-)^k\prod_{h=1}^{k}w(z_h,z_h^*)(z_h^*-z_h)\ \mathrm{Qdet}\left[\begin{pmatrix}\kappa_N(z_i^*,z_j) & -\kappa_N(z_i^*,z_j^*)\\ \kappa_N(z_i,z_j) & -\kappa_N(z_i,z_j^*)\end{pmatrix}_{i,j=1\ldots k}\right]\\
&= (-)^k\prod_{h=1}^{k}w(z_h,z_h^*)(z_h^*-z_h)\ \mathrm{Pf}\ \left[\begin{pmatrix}\kappa_N(z_i,z_j) & -\kappa_N(z_i,z_j^*)\\ -\kappa_N(z_i^*,z_j) & \kappa_N(z_i^*,z_j^*)\end{pmatrix}_{i,j=1\ldots k}\right]\\
&= \prod_{h=1}^{k}w(z_h,z_h^*)(z_h^*-z_h)\ \mathrm{Pf}\ \left[\begin{pmatrix}\kappa_N(z_i,z_j) & \kappa_N(z_i,z_j^*)\\ \kappa_N(z_i^*,z_j) & \kappa_N(z_i^*,z_j^*)\end{pmatrix}_{i,j=1\ldots k}\right]\\
&= \prod_{h=1}^{k}w(z_h,z_h^*)(z_h^*-z_h)\ \mathrm{Pf}\ \left[\Theta_N(z_1,z_1^*,\ldots,z_k,z_k^*)\right]\ .
\end{aligned} \tag{3.21}$$

Here we have used two properties of the Pfaffian, one is the well known identity by Dyson [31]

$$\mathrm{Qdet}\,[A]=\mathrm{Pf}\ \left[C\,(\Sigma\otimes\mathbf{1}_N)\cdot C(A)\right] \tag{3.22}$$

[7] The result is the same up to an overall factor. The proof in [17] follows the one in [33], that is derived for the squared quantities, hence the result is true up to an overall sign.

where A is an $N \times N$ quaternion matrix, $\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $C(A)$ is the $2N \times 2N$ complex representation of A. The other identity is that for a matrix B and an anti-symmetric matrix A, both of the same size, the following holds [32]

$$\mathrm{Pf}[B\,A\,B^T] \;=\; \mathrm{Pf}[A]\;\det[B]\;. \tag{3.23}$$

Using for $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathbf{1}_k$ with determinant $\det[B] = (-)^k$ we can equate the second and third line above in eq. (3.21). In the last step we used the definition eq. (2.9).

It is relatively easy to see that for pairs of complex conjugated masses the correlation functions in eq. (2.14) can be expressed in terms of ratios of correlation functions without mass insertions [5].

3.2 The Hermitean limit

Theorem 2 (Hermitean limit) *Given the real weight function from Theorem 1 is depending on a non-Hermiticity parameter τ, $w = w(z, z^*; \tau)$, and the following limit exist for any finite number of eigenvalues N*

$$\lim_{\tau\to 1} |z^* - z|^2 w(z, z^*) = \delta(y)\;\overline{w}(x)\;. \tag{3.24}$$

It is called the Hermitean limit and $\overline{w}(x)$ is the projected weight function on $\mathbb{R}$. It then follows that the Hermitean limit of the integrals computed in Theorem 1 exists and is given by

$$\lim_{\tau\to 1} \mathcal{A}^{(M)}_{k,N}(z_1,\ldots,z_k;m_1,\ldots,m_M) = \frac{(-)^{[M/2]}(N-k)!}{\Delta_M(\{m\})} \prod_{h=1}^{k} \delta(y_h)\overline{w}(x_h) \prod_{h=0}^{N+[M/2]-1} \overline{r}_h \mathrm{Pf}\,[\overline{\Omega}_{N+[M/2]}(\{\overline{u}\})] \tag{3.25}$$

where for M even we have

$$\overline{\Omega}_R \equiv \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \partial_{x_i}\partial_{x_j}\overline{\kappa}_R(x_i,x_j) & \partial_{x_i}\overline{\kappa}_R(x_i,x_j) & \cdots & \partial_{x_i}\overline{\kappa}_R(x_i,m_f) & \cdots \\ \cdots & \partial_{x_j}\overline{\kappa}_R(x_i,x_j) & \overline{\kappa}_R(x_i,x_j) & \cdots & \overline{\kappa}_R(x_i,m_f) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \partial_{x_j}\overline{\kappa}_R(m_g,x_j) & \overline{\kappa}_R(m_g,x_j) & \cdots & \overline{\kappa}_R(m_g,m_f) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{3.26}$$

and for M odd

$$\overline{\Omega}_R \equiv \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \partial_{x_i}\partial_{x_j}\overline{\kappa}_R(x_i,x_j) & \partial_{x_i}\overline{\kappa}_R(x_i,x_j) & \cdots & \partial_{x_i}\overline{\kappa}_R(x_i,m_f) & \cdots & \partial_{x_i}\overline{q}_{2R}(x_i) \\ \cdots & \partial_{x_j}\overline{\kappa}_R(x_i,x_j) & \overline{\kappa}_R(x_i,x_j) & \cdots & \overline{\kappa}_R(x_i,m_f) & \cdots & \overline{q}_{2R}(x_i) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \partial_{x_j}\overline{\kappa}_R(m_g,x_j) & \overline{\kappa}_R(m_g,x_j) & \cdots & \overline{\kappa}_R(m_g,m_f) & \cdots & \overline{q}_{2R}(m_g) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -\partial_{x_j}\overline{q}_{2R}(x_j) & -\overline{q}_{2R}(x_j) & \cdots & -\overline{q}_{2R}(m_f) & \cdots & \cdots \end{pmatrix}. \tag{3.27}$$

Here $R = N + [M/2]$. The set $\{\overline{u}\} = \{x_1, x_2, \ldots, x_N, m_1, \ldots, m_M\}$ of $N + M$ parameters contains the real parts of the N complex eigenvalues and the M masses which can in general be complex. The overlined quantities are the norms $\overline{r}_k$, skew orthogonal polynomial $\overline{q}_{2R}(x)$ and kernel $\overline{\kappa}_R(x_1, x_2)$ corresponding to the projected weight $\overline{w}(x)$, where $\langle f|g\rangle_{\overline{S}} = \int dx\,\overline{w}(x)(f(x)'g(x) - f(x)g(x)')$ defines *the projected skew orthogonal product. For the chiral theory, eq. (3.24) remains unchanged. Otherwise the results are the same modulo substituting $\overline{w}(x) \to x \cdot \overline{w}(x)$ in eq. (3.25) and using the chiral skew orthogonal polynomials and kernel.*

We note that the skew orthogonal product obtained in the Hermitean limit is the one of the SE and chSE[8]. We will show below that the identification eq. (2.26) then leads to the know results for the correlation functions of the massive SE and chSE. Examples for weight functions satisfying our theorem will be given after the proof.

Let us also comment on why we keep N finite and fixed above. The first reason is that we need to manipulate polynomials of finite degree and a finite number of integrals. The second reason is more subtle and concerns that the limit $N \to \infty$ is not unique. First one has to distinguish the macroscopic from the microscopic limit, where in the latter the complex eigenvalues are rescaled with respect to the local mean level spacing. There may be different regimes as for real eigenvalues we distinguish the bulk, origin and edge region. But even if we restrict ourselves to a specific region of the spectrum in the microscopic limit there are two different limits possible, the limits of weak and strong non-Hermiticity. The weak non-Hermiticity limit [34] represents a one-parameter deformation that interpolates between real eigenvalue correlations ($\alpha = 0$) and those at strong non-Hermiticity $\alpha = \infty$, the parameter being $\alpha^2 = \lim_{N\to\infty,\tau\to 1} N(1-\tau^2)$. Therefore we conjecture that there exists a class of weight functions such that in the large-N limit at weak non-Hermiticity an analogous theorem to our Theorem 2 can be proven.

PROOF: The proof consist mainly of two steps: the first is to isolate all the terms proportional to $(z_i - z_i^*)^2$ for all i (or $(z_i^2 - z_i^{*\,2})^2$ in the chiral case) in order to take the Hermitean limit. In the second step we determine the limiting skew product and show that the limiting quantities $\overline{r}_k$, $\overline{q}_{2R}$ and $\overline{\kappa}_R$ exist and are the corresponding norms, skew orthogonal polynomials and kernels of the limiting skew product on $\mathbb{R}$. Some details are given in appendix A. We will only present the proof for the non-chiral case, and comment at the places where the proof for the chiral case differs. Let us start from Theorem 1 using the same notation.

Step 1 It can be easily seen that Pf $[\Theta\{u\}]$ is vanishing whenever 1 or several eigenvalues become real, $\exists\ i : z_i = z_i^*$. The idea is to manipulate the Pfaffian via summing or subtracting the first k rows or columns, in order to isolate these vanishing contributions by taking out factors $(z_i - z_i^*)$. We apply the following transformations:

$$\begin{aligned}
&\mathrm{Row}_{2i} \to \mathrm{Row}_{2i} - \mathrm{Row}_{2i+1}\\
&\mathrm{Row}_{2i+1} \to \mathrm{Row}_{2i+1} + \frac{1}{2}\mathrm{Row}_{2i}\\
&\mathrm{Column}_{2j} \to \mathrm{Column}_{2j} - \mathrm{Column}_{2j+1}\\
&\mathrm{Column}_{2j+1} \to \mathrm{Column}_{2j+1} + \frac{1}{2}\mathrm{Column}_{2j}
\end{aligned} \tag{3.28}$$

[8] Special care has to be taken as in the literature sometimes the variable $u = x^2$ is used. In our chiral skew product we denote $' = \partial/\partial x$.

for all $i, j \leq k$, leaving the Pfaffian invariant. We divide the matrix $\Theta\{u\}$ into 4 blocks with the upper left block of size $2k \times 2k$. These operations then imply the following for the upper left block

$$\begin{pmatrix} \kappa(z_i, z_j) & \kappa(z_i, z_j^*) \\ \kappa(z_i^*, z_j) & \kappa(z_i^*, z_j^*) \end{pmatrix} \to \tag{3.29}$$
$$\begin{pmatrix} \kappa(z_i, z_j) - \kappa(z_i, z_j^*) - (\kappa(z_i^*, z_j) - \kappa(z_i^*, z_j^*)) & \frac{1}{2}(\kappa(z_i, z_j) - \kappa(z_i^*, z_j) + \kappa(z_i, z_j^*) - \kappa(z_i^*, z_j^*)) \\ \frac{1}{2}(\kappa(z_i, z_j) - \kappa(z_i, z_j^*) + \kappa(z_i^*, z_j) - \kappa(z_i^*, z_j^*)) & \frac{1}{4}(\kappa(z_i, z_j) + \kappa(z_i, z_j^*) + \kappa(z_i^*, z_j) + \kappa(z_i^*, z_j^*)) \end{pmatrix}$$

where we suppressed the subscript here and in the following lines. For the upper right part of $\Theta\{u\}$ only the manipulation of rows matter and we obtain for M odd

$$\begin{pmatrix} \kappa(z_i, m_f) & q(z_i) \\ \kappa(z_i^*, m_f) & q(z_i^*) \end{pmatrix} \to \begin{pmatrix} (\kappa(z_i, m_f) - \kappa(z_i^*, m_f)) & q(z_i) - q(z_i^*) \\ \\ \frac{1}{2}(\kappa(z_i, m_f) + \kappa(z_i, m_f)) & \frac{1}{2}(q(z_i) + q(z_i^*)) \end{pmatrix}. \tag{3.30}$$

For M even the last column of polynomials $q(z)$ is absent. Similarly in the lower left part of $\Theta\{u\}$ only the manipulation of columns matter and we obtain for M odd

$$\begin{pmatrix} \kappa(m_g, z_j) & \kappa(m_g, z_j^*) \\ -q(z_i) & -q(z_i^*) \end{pmatrix} \to \begin{pmatrix} (\kappa(m_g, z_j) - \kappa(m_g, z_j^*)) & \frac{1}{2}(\kappa(m_g, z_j) + \kappa(m_g, z_j^*)) \\ -(q(z_i) - q(z_i^*)) & -\frac{1}{2}(q(z_i) + q(z_i^*)) \end{pmatrix}, \tag{3.31}$$

again dropping the row containing $q(z)$ for M even. The lower right block remains untouched by the change of rows and columns. Both the kernels and skew orthogonal polynomials are sums of polynomials in each variable. Therefore we can expand the following differences in the limit of vanishing imaginary parts y of $z = x + iy$ to leading order in $(z - z^*)$:

$$\begin{aligned} \lim_{y \to 0} (q(z) - q(z^*)) - (z - z^*) \left. \frac{\partial}{\partial z} q(z) \right|_{z=x} &= O((z - z^*)^2) \\ \lim_{y \to 0} (\kappa(z, u) - \kappa(z^*, u)) - (z - z^*) \left. \frac{\partial}{\partial z} \kappa(z, u) \right|_{z=x} &= O((z - z^*)^2) \end{aligned} \tag{3.32}$$

and similar for the second argument of the kernel. In this expansion we obtain

$$\begin{aligned} &\lim_{\forall i: y_i \to 0} \mathrm{Pf} \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \kappa_R(z_i, z_j) & \kappa_R(z_i, z_j^*) & \cdots & \kappa_R(z_i, m_f) & \cdots & q_{2R}(z_i) \\ \cdots & \kappa_R(z_i^*, z_j) & \kappa_R(z_i^*, z_j^*) & \cdots & \kappa_R(z_i^*, m_f) & \cdots & q_{2R}(z_i^*) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \kappa_R(m_g, z_j) & \kappa_R(m_g, z_j^*) & \cdots & \kappa_R(m_g, m_f) & \cdots & q_{2R}(m_g) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -q_{2R}(z_j) & -q_{2R}(z_j^*) & \cdots & -q_{2R}(m_f) & \cdots & \cdots \end{pmatrix} \\ &= \prod_{h=1}^{k} (z_h - z_h^*) \ \mathrm{Pf} \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \partial_{x_i}\partial_{x_j}\kappa_R(x_i, x_j) & \partial_{x_i}\kappa_R(x_i, x_j) & \cdots & \partial_{x_i}\kappa_R(x_i, m_f) & \cdots & \partial_{x_i} q_{2R}(x_i) \\ \cdots & \partial_{x_j}\kappa_R(x_i, x_j) & \kappa_R(x_i, x_j) & \cdots & \kappa_R(x_i, m_f) & \cdots & q_{2R}(x_i) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \partial_{x_j}\kappa_R(m_g, x_j) & \kappa_R(m_g, x_j) & \cdots & \kappa_R(m_g, m_f) & \cdots & q_{2R}(m_g) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -\partial_{x_j} q_{2R}(x_j) & -q_{2R}(x_j) & \cdots & -q_{2R}(m_f) & \cdots & \cdots \end{pmatrix} \end{aligned} \tag{3.33}$$

plus higher orders in $(z_h - z_h^*)$. From each second row $2i$ we get a factor $(z_i - z_i^*)$ and from each second column $2j$ we get a factor $(z_j - z_j^*)$ for $i, j \leq k$, which we have taken out of the Pfaffian. We have again given the expression for M odd, for M even the last row and column is absent. It can be seen easily that all rows and columns are linearly independent.

In the chiral case the same manipulations can be performed, replacing $z \to z^2$, $x \to x^2$. One modification is needed because the differentiations are now with respect to the quadratic arguments: $\frac{\partial}{\partial z^2} = \frac{1}{2z}\frac{\partial}{\partial z}$. Hence the prefactor is[9]

$$\prod_{h=1}^{k} \left(\frac{z_h^2 - z_h^{*2}}{2z_h} \right) = \prod_{h=1}^{k} (z_h - z_h^*) \frac{(z_h + z_h^*)}{2z_h} \tag{3.34}$$

Now that we have isolated all the vanishing terms in the Pfaffian we can take the Hermitean limit $\tau \to 1$. This leads to a vanishing of all imaginary parts. Taking the limit eq. (3.24) together with our expansion eq. (3.33) we arrive at eq. (3.25) where we have replaced the matrix in eq. (3.33) with the overlined quantities, as in eqs. (3.26) and (3.27). This step of course assumes that the limit of the polynomials and kernels exists which we will show now in the next step.

Step 2 First of all we determine the limiting skew product from the limit of the weight eq. (3.24). We therefore write down the original skew product on $\mathbb{C}$ for monomials given by the quantity $W_{s,t}$ (see also appendix A). For simplicity we fix $s < t$ ($t > s$ follows by complex conjugation):

$$\begin{aligned} W_{s,t} &\equiv \int d^2z \; w(z,z^*)(z^* - z) \left[z^s z^{*t} - z^{*s} z^t \right] \\ &= \int d^2z \; w(z,z^*)(z^* - z)^2 |z|^{2s} \left[z^{*t-s-1} + z^{*t-s-2} z + \cdots + z^{t-s-2} z^* + z^{t-s-1} \right]. \end{aligned} \tag{3.35}$$

Here we have isolated all the contributions that vanish for $z = z^*$. Using the Hermitean limit (3.24) we obtain

$$\overline{W}_{s,t} \equiv \lim_{\tau \to 1} W_{s,t} = -\int dx \; dy \delta(y) \; \overline{w}(x) |x + iy|^{2s} \left[(x+iy)^{*t-s-1} + \ldots \right] \tag{3.36}$$

$$= -\int dx \; \overline{w}(x) \; x^{2s} x^{t-s-1} (t-s), \tag{3.37}$$

where the minus comes from $(z^* - z)^2 = -|z^* - z|^2$. We can therefore define the following limiting skew product on the real line, that coincides with the standard one of the SE

$$\langle x^s | x^t \rangle_{\overline{S}} \equiv \int dx \; \overline{w}(x) \; \left[(x^s)' x^t - x^s (x^t)' \right] = \overline{W}_{s,t} \; . \tag{3.38}$$

From this consideration we can in principle evaluate the limiting skew product on any set of polynomials. However, we need to make sure that the coefficients of the polynomials, which depend on τ in the case of our $q_k(z)$ converge. For a basis independent representation of the kernel only on terms of matrix W we refer to appendix A.

Our strategy is as follows. We start with the limit of the partition function as a normalisation constant, and then use integral representations of the polynomials $q_j(z)$ to show their convergence. They can then be used to construct the kernel including the norms r_k. The limit of the partition

[9] Because of the vanishing diagonal all such derivatives commute.

function eq. (2.2) without mass insertion is easily seen to exist:

$$\begin{aligned}\overline{\mathcal{Z}}_N^{(0)} &\equiv \lim_{\tau\to 1} \mathcal{Z}_N^{(0)} = \lim_{\tau\to 1} \int \prod_{i=1}^{N} d^2 z_i \ |z_i - z_i^*|^2 w(z_i, z_i^*) \prod_{k>l}^{N} |z_k - z_l|^2 \ |z_k - z_l^*|^2 \\ &= \int \prod_{i=1}^{N} dx_i dy_i \ \delta(y_i) \overline{w}(x_i) \prod_{k>l}^{N} (x_k - x_l)^4 \ = \ \prod_{k=0}^{N-1} \overline{r}_k \ . \end{aligned} \tag{3.39}$$

Using the delta function the integration dy can be dropped and we obtain the usual SE partition function. Here we have introduced the normalisation constants $\overline{r}_k$ of the limiting skew orthogonal polynomials with respect to the limiting skew product on $\mathbb{R}$ eq. (3.38)

$$\begin{aligned}\langle \overline{q}_{2k+1}, \overline{q}_{2l} \rangle_{\overline{S}} &= -\langle \overline{q}_{2l}, \overline{q}_{2k+1} \rangle_{\overline{S}} \ = \ \overline{r}_k \ \delta_{kl} \ , \\ \langle \overline{q}_{2k+1}, \overline{q}_{2l+1} \rangle_{\overline{S}} &= \quad \langle \overline{q}_{2l}, \overline{q}_{2k} \rangle_{\overline{S}} \quad = \ 0 \ . \end{aligned} \tag{3.40}$$

Note the relative minus sign in the definition of the skew product eq. (3.38) and in eq. (3.40) compared to [22]. They compensate each other to give the same defining equations for skew orthogonal polynomials. The fact that in eq. (3.39) $\forall$ N, $\lim_{\tau\to1}(\mathcal{Z}_N^{(0)} = \prod_{k=0}^{N-1} r_k) = \prod_{k=0}^{N-1} \overline{r}_k$ implies that the $\lim_{\tau\to1} r_k = \overline{r}_k$ holds individually for all k.

Next we show that the limit of the $q_k(z)$ exists and that they satisfy the relations eq. (3.40). We start with the even ones. The $q_{2j}(z)$ on $\mathbb{C}$ enjoy an integral representation eq. (2.15), and the limit of it is easily taken:

$$\begin{aligned}\lim_{\tau\to1} q_{2j}(z) &= \lim_{\tau\to1} \frac{1}{\mathcal{Z}_j^{(0)}} \int \prod_{i=1}^{j} d^2 z_i \ |z_i - z_i^*|^2 w(z_i, z_i^*)(z - z_i)(z - z_i^*) \prod_{k>l}^{N} |z_k - z_l|^2 \ |z_k - z_l^*|^2 \\ &= \frac{1}{\overline{\mathcal{Z}}_j^{(0)}} \int \prod_{i=1}^{N} dx_i dy_i \ \delta(y_i) \overline{w}(x_i)(z - x_i)^2 \prod_{k>l}^{N} (x_k - x_l)^4 \\ &= \overline{q}_{2j}(z) \ . \end{aligned} \tag{3.41}$$

In the last step we do the integration dy and use the known integral representation of the $\overline{q}_k(z)$ on $\mathbb{R}$ [35]. Hence we have not only shown the existence of the limit but also determined its limiting function. The same step can be done for the odd polynomials, with the following representation [17]

$$\begin{aligned}\lim_{\tau\to1} q_{2j+1}(z) &= \lim_{\tau\to1} \frac{1}{\mathcal{Z}_j^{(0)}} \int \prod_{i=1}^{j} d^2 z_i \ |z_i - z_i^*|^2 w(z_i, z_i^*) \\ &\quad \times (z - z_i)(z - z_i^*) \left(z + \sum_{k=1}^{j} z_k + z_k^* \right) \prod_{k>l}^{N} |z_k - z_l|^2 \ |z_k - z_l^*|^2 \\ &= \frac{1}{\overline{\mathcal{Z}}_j^{(0)}} \int \prod_{i=1}^{N} dx_i dy_i \ \delta(y_i) \overline{w}(x_i)(z - x_i)^2 \left(z + \sum_{k=1}^{j} 2x_k \right) \prod_{k>l}^{N} (x_k - x_l)^4 \\ &= \overline{q}_{2j+1}(z) \ . \end{aligned} \tag{3.42}$$

This proves both the existence of the limit as well as its value $\overline{q}_{2j+1}(z)$ through the known integral representation [35][10]. There is one subtlety to be mentioned here. The definition of the odd skew

[10]The limit above $q_j(x) \to \overline{q}_j(x)$ can be proven without using the integral representation of skew orthogonal polynomials, but only using the property (3.35).

orthogonal is not unique, we can redefine $q_{2j+1} \to q_{2j+1} + c \cdot q_{2j}$ for any possibly τ-dependent constant c. We have set the constant to zero here to avoid problems in the Hermitean limit for an ill chosen c. We can thus finally take the limit of the kernel as well to arrive at

$$\lim_{\tau\to 1} \kappa_N(z,v) = \overline{\kappa}_N(z,v) = \sum_{j=0}^{N-1} \frac{\overline{q}_{2j+1}(z)\overline{q}_{2j}(v) - \overline{q}_{2j}(z)\overline{q}_{2j+1}(v)}{\overline{r}_j} . \tag{3.43}$$

It is proportional to *one* of the kernels of the SE, $I_N(z,v)$, the two other kernels are obtained by differentiation of $\overline{\kappa}_N(z,v)$, see eq. (2.26). In the appendix A we give another representation of the kernel and its limit, that is independent of the choice of basis for the polynomials and contains only the matrix W and its limit $\overline{W}$ in eq. (3.36) above.

Let us add a few remarks on the chiral case. In the prefactor mentioned at the end of step 1, eq. (3.34), the first factor contributes to $|z_h - z_h^*|^2 w(z_h, z_h^*)$ to give delta-functions, $\delta(y_h)\overline{w}(x_h)$, in the limit eq. (3.24). From these delta-functions the second factor in eq. (3.34) reduces to unity. In step 2 all variables z and z^* have to be replaced by their squares, leading to a projected skew product in squared variables as well. The determination of the partition function, norms and both even and odd skew orthogonal polynomials follows along the same limes. We note that both are polynomials in variables z^2. This can be seen from their integral representation corresponding to eqs. (3.41) and (3.42) [5], and their Hermitean limit follows along the same lines. This ends the proof of the Hermitean limit. □

Let us now show how to recover the known results for the SE and chSE. First we consider the SE without mass insertions. Using the same algebra of eq. (3.21) we verify that the result is the same as the one in [22]:

$$\begin{aligned}
\overline{R}_{N,k}^{(0)}(x_1,\ldots,x_k) &= \mathrm{Qdet}\left[\begin{pmatrix} S_N(x_i,x_j) & D_N(x_i,x_j) \\ I_N(x_i,x_j) & S_N(x_j,x_i) \end{pmatrix}_{ij}\right] \\
&= \prod_{h=1}^{k} w(x_h)\ \mathrm{Qdet}\left[\begin{pmatrix} \partial_{x_i}\kappa_N(x_i,x_j) & -\partial_{x_i}\partial_{x_j}\kappa_N(x_i,x_j) \\ \kappa_N(x_i,x_j) & -\partial_{x_j}\kappa_N(x_i,x_j) \end{pmatrix}_{ij}\right] \\
&= \prod_{h=1}^{k} w(x_h)\ \mathrm{Pf}\left[\begin{pmatrix} \kappa_N(x_i,x_j) & -\partial_{x_j}\kappa_N(x_i,x_j) \\ -\partial_{x_i}\kappa_N(x_i,x_j) & \partial_{x_i}\partial_{x_j}\kappa_N(x_i,x_j) \end{pmatrix}_{ij}\right] \\
&= \prod_{h=1}^{k} w(x_h)\ \mathrm{Pf}\ \left[\overline{\Omega}(x_1,\ldots,x_k)\right] .
\end{aligned} \tag{3.44}$$

In the first line we have used the usual notation with functions I_N, S_N,and D_N defined as in eq. (2.26). In the second line we have used again the identity eq. (3.23), with $B = C(\Sigma \otimes \mathbf{1}_k)$ having $\det[B] = 1$.

Second, we immediately obtain from eqs. (2.23) and (3.43) the result of [26] for the massive partition functions. Our result following from the Hermitean limit also explains why it is given solely in terms of the kernel $\overline{\kappa}_N(z,v) \sim I_N(u,v)$ and not the other two, as no degeneracies occur and hence no Taylor expansions have to be made.

Finally we also obtain the following general result for the massive correlation functions:

$$\overline{R}_{N,k}^{(M)}(x_1,\ldots,x_k;\{m\}) = \prod_{h=1}^{k} \overline{w}(x_k) \frac{\mathrm{Pf}\ \left[\overline{\Omega}_{N+[M/2]}(x_1,\ldots,x_k;m_1,\ldots,m_M)\right]}{\mathrm{Pf}\ \left[\overline{\Omega}_{N+[M/2]}(m_1,\ldots,m_M)\right]} \tag{3.45}$$

We can check our result with the one in [26], where we consider only the case of M odd, the even one follows easily:

$$\overline{R}^{(M)}_{N,k}(x_1,\ldots,x_k;\{m\}) = (-)^{\frac{k(k-1)}{2}} \frac{\mathrm{Pf}\begin{bmatrix} -I(m_f,m_g) & q(m_f) & -I(m_f,x_i) & S(m_f,x_i) \\ -q(m_g) & 0 & -q(x_i) & -\partial_{x_i}q(x_i) \\ I(m_g,x_j) & q(x_j) & -I(x_j,x_i) & S(x_j,x_i) \\ -S(m_g,x_j) & \partial_{x_j}q(x_j) & -S(x_i,x_j) & D(x_j,x_i) \end{bmatrix}}{\begin{bmatrix} -I(m_f,m_g) & q(m_f) \\ -q(m_g) & 0 \end{bmatrix}} \ . \quad (3.46)$$

Here $i,j = 1,\ldots,k$, $f,g = 1,\ldots,M$, where f and j label the columns, g and i label the rows. For simplicity we suppress the subscript in what follows. The two equations look slightly different, but after some manipulation we can see that they coincide. We have to rearrange the elements of the matrices, and we show how to do so only for the numerator. We use the eq. (3.23) with

$$B = \begin{pmatrix} 0 & 0 & 1\!\!|_k \\ 0 & 1\!\!|_k & 0 \\ 1\!\!|_{M+1} & 0 & 0 \end{pmatrix} \ , \quad (3.47)$$

where $\det[B] = (-)^k$, and $1\!\!|_k$ being the ordinary unity matrix of size k (and not the quaternion one). Hence we obtain for the denominator

$$\begin{aligned} &(-)^k \mathrm{Pf}\begin{bmatrix} D(x_j,x_i) & -S(x_i,x_j) & -S(m_g,x_j) & \partial_{x_j}q(x_j) \\ S(x_j,x_i) & -I(x_j,x_i) & I(m_g,x_j) & q(x_j) \\ S(m_f,x_i) & -I(m_f,x_i) & -I(m_f,m_g) & q(m_f) \\ -\partial_{x_i}q(x_i) & -q(x_i) & -q(m_j) & 0 \end{bmatrix} \\ &= (-)^{\frac{M-1}{2}} \mathrm{Pf}\begin{bmatrix} -D(x_j,x_i) & S(x_i,x_j) & S(m_g,x_j) & \partial_{x_j}q(x_j) \\ -S(x_j,x_i) & I(x_j,x_i) & I(x_j,m_g) & q(x_j) \\ -S(m_f,x_i) & I(m_f,x_i) & I(m_f,m_g) & q(m_f) \\ -\partial_{x_j}q(x_i) & -q(x_i) & -q(m_j) & 0 \end{bmatrix} \ . \end{aligned} \quad (3.48)$$

In the last equation D, S, and I are matrices of dimension $k\times k$, $k\times M$, $M\times k$ or $M\times M$. In contrast to that in eq. (2.25) we have a 2×2 matrix block structure. We can map the two as follows. Every time we swap 1 pair of rows and columns in a Pfaffian we gain a minus sign, see eq. (3.23). Thus we gain an overall factor of $(-)^{\frac{k(k-1)}{2}}$ when transforming eq. (3.48) into the form of eq. (2.25). By using the equations (2.26) we complete the proof.

Matching the chiral results in the Hermitean limit is a mere repetition of the above manipulations, as can also be seen from comparing references [24] and [26].

We close this section by giving explicit examples for weight functions satisfying Theorem 2, including their sets of skew orthogonal polynomials.

Non-Hermitean Gaussian SE As a first example we consider the case of the non-Hermitean Gaussian SE as it was introduced in [17]. Its weight function (times a proper normalisation function) was already mentioned in eq. (2.22):

$$w_{GSE}(z,z^*) = \frac{1}{2\sqrt{\pi}}\left(\frac{N}{1-\tau^2}\right)^{3/2} \exp\left(-\frac{N}{1-\tau^2}\left(|z|^2 - \frac{\tau}{2}(z^2+z^{*2})\right)\right) \ , \quad \tau\in[0,1) \ . \quad (3.49)$$

The normalisation factor is chosen in order to fulfil the condition mentioned in Theorem 2:

$$\lim_{\tau\to 1} |z-z^*|^2 w_{GSE}(z,z^*) = \delta(y)\ \exp[-\frac{N}{2}x^2] \ . \quad (3.50)$$

The projected weight is that of the Gaussian SE with real eigenvalues [22]. The skew orthogonal polynomials and their norms in the non-Hermitean case are [17]:

$$\begin{aligned} q_{2k+1}(z) &= \left(\frac{\tau}{2N}\right)^{k+1/2} H_{2k+1}\left(z\sqrt{\frac{N}{2\tau}}\right) , \\ q_{2k}(z) &= \left(\frac{2}{N}\right)^{k} k! \sum_{j=0}^{k} \left(\frac{\tau}{2}\right)^{j} \frac{1}{(2j)!!} H_{2j}\left(z\sqrt{\frac{N}{2\tau}}\right) , \\ r_k &= \sqrt{\pi}(1+\tau)^{\frac{1}{2}} \frac{(2k+1)!}{N^{2k+1/2}} . \end{aligned} \tag{3.51}$$

Performing the Hermitean limit by setting $\tau = 1$ we obtain the usual Gaussian SE skew orthogonal polynomials of weight $\overline{w}(x) = \exp(-Nx^2/2)$ [22].

Non-Hermitean Gaussian chSE As a second example we give the non-Hermitean extension of the chiral ensemble [5]. The non-Hermiticity parameter is given here by $\mu \in (0, 1]$, with the Hermitean limit given by $\mu \to 0$. The properly normalised the weight function reads

$$\begin{aligned} w^{\nu}_{chGSE}(z, z^*) &= \frac{\sqrt{N}}{\mu\sqrt{\pi}} \frac{1}{2\sqrt{\pi}} \left(\frac{N(1-\mu^2)}{\mu^2}\right)^{3/2} |z|^{4\nu+2} K_{2\nu}\left(\frac{N(1+\mu^2)}{2\mu^2}|z|^2\right) \\ &\quad \times \exp\left(\frac{N(1-\mu^2)}{4\mu^2}(z^2 + z^{*2})\right) . \end{aligned} \tag{3.52}$$

The first line represents the universal part w_U mentioned at the end of subsection 2.1, whereas the second line giving w_V comes from the Gaussian potential (see [5] for details of the derivation). The normalisation factor is chosen in order to fulfil the condition in Theorem 2,

$$\lim_{\mu\to 0} |z - z^*|^2 \frac{1}{2\sqrt{\pi}} \left(\frac{N(1-\mu^2)}{\mu^2}\right)^{3/2} \exp\left(\frac{N(1-\mu^2)}{4\mu^2}(z - z^*)^2\right) = \delta(y) . \tag{3.53}$$

The projected weight function can be read off from

$$\lim_{\mu\to 0} \frac{\sqrt{N}}{\mu\sqrt{\pi}} |z|^{4\nu+2} K_{2\nu}\left(\frac{N(1+\mu^2)}{2\mu^2}|z|^2\right) \exp\left(\frac{N(1-\mu^2)}{2\mu^2}|z|^2\right) = |z|^{4\nu+1} \exp(-N|z|^2) , \tag{3.54}$$

leading to $\overline{w}(x) = x^{4\nu+1} \exp[-Nx^2]$ of the Gaussian chSE with rectangular matrices of size $N \times (N+\nu)$. The skew orthogonal polynomials and their norms for the non-Hermitean case are [5]:

$$\begin{aligned} q_{2k+1}(z) &= -\,(2k+1)! \left(\frac{1-\mu^2}{N}\right)^{2k+1} L^{2\nu}_{2k+1}\left(\frac{Nz^2}{1-\mu^2}\right) , \\ q_{2k}(z) &= k!\, \Gamma(k+\nu+1) \frac{2^{2k}(1+\mu^2)^{2k}}{N^{2k}} \sum_{j=0}^{k} \frac{(1-\mu^2)^{2j}}{(1+\mu^2)^{2j}} \frac{(2j)!}{2^{2j} j!\, \Gamma(j+\nu+1)} L^{2\nu}_{2j}\left(\frac{Nz^2}{1-\mu^2}\right) , \\ r_k &= 4(2k+1)!(2k+2\nu+1)! \frac{(1-\mu^2)^{3/2}(1+\mu^2)^{4k+2\nu}}{N^{4k+2\nu+2}} . \end{aligned} \tag{3.55}$$

When taking the Hermitean limit by setting $\mu = 0$ we recover the results in [23].

Let us conclude on the following remark. In both examples given above, eqs. (3.49) and (3.52), the weight functions can be decomposed into a "radial" part w_R depending only on the modulus $|z|$, and a part w_y that depends only on the imaginary part $\Im\text{m}(z) = y$. This is possible due to the fact that the holomorphic and anti-holomorphic parts of the measure are Gaussian, $V_1(\Phi) = \Phi^2$:

$$w(z, z^*) = w_R(|z|)\ w_I\left(\frac{z - z^*}{2if(\tau)}\right)\ . \tag{3.56}$$

In the Hermitean limit $\tau \to 1$ the radial part $w_R(|z|)$ remains a non-degenerate, positive definite measure over $\mathbb{C}$. Moreover we have $\forall\, \tau \in [0, 1)$: $\int dy\, y^2\, w_I\left(\frac{z-z^*}{2if(\tau)}\right) = 1$, and the limit $\lim_{\tau\to1} f(\tau) = 0$ implies $w_I\left(\frac{z-z^*}{2if(\tau)}\right) \to \delta(y)$. The latter squeezes the eigenvalues onto the real axis and projects $|z|$ to x in $w_R(|z|)$.

4 Conclusions

We have computed all expectation values of products of characteristic polynomials (or massive partition functions) and all complex eigenvalue correlation functions in the presence of such characteristic polynomials (or mass terms), without imposing any degeneracies on their arguments. Our results hold for complex matrix models with symplectic symmetry generalising the symplectic and chiral symplectic ensemble, for general weight functions only restricted by convergence.

All formulas are given in terms of a Pfaffian, containing a single kernel of skew orthogonal polynomials in the complex plane, as well as the even skew orthogonal polynomials for an odd number of mass insertions. Our findings are thus much simpler than the corresponding results for symplectic matrix models with real eigenvalues. We have exploited this fact by taking the Hermitean limit of our results, and we have given explicit examples for weights allowing such a limit. We do not only recover easily all known results for real eigenvalues, but we also provide an explanation for the structure of symplectic real eigenvalue correlations in terms of three kernel elements of a quaternion matrix given by derivatives of our single kernel element.

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A Appendix: Representation of the kernel

In the present section we show that the kernel may be written in terms of monomials as

$$\kappa_N(z, v) = \sum_{m,n=0}^{2N-1} z^m \left(W^{-1}\right)_{m,n} v^n. \tag{A.1}$$

Here we introduce the matrix of the skew product of monomials $W_{m,n} \equiv \langle z^m | z^{*\,n}\rangle_S$, with the hypothesis that $\det[W] \neq 0$. We borrow the general definition of kernels [33, 17] in the complex plane

$$\kappa_N(z, v) \equiv \sum_{k,l=0}^{2N-1} p_k(z) \left(M^{-1}\right)_{k,l} p_l(v)\ , \tag{A.2}$$

where $p_k(z)$ is a basis of polynomials of degree $k \leq 2N - 1$ with real coefficients $p_k(z^*) = p_k(z)^*$ due to our real weight $w(z, z^*)$. The matrix M is defined by the skew product of these polynomials

$$M_{k,l} \equiv \langle p_k | p_l \rangle_S, \tag{A.3}$$

being nonsingular as well. We can write the linear transformation from z^k to $p_k(z)$ as

$$p_k(z) \equiv \sum_{j=0}^{2N-1} P_{k,j}\ z^j \tag{A.4}$$

where P is an $2N \times 2N$ matrix. The $\{p_k(z)\}$ being a basis we know that $\det[P] \neq 0$. With this notation we can rewrite (A.3) and (A.2) in terms of P:

$$M_{k,l} = P_{k,m} P_{l,n} \langle z^m | z^{*\,n} \rangle_S \equiv P_{k,m} W_{m,n} P^T_{n,l}. \tag{A.5}$$

Consequently the kernel can be written independently of the basis $\{p_k(z)\}$ chosen, as we have claimed above:

$$\kappa_N(z,v) = \sum_{k,l,m,n=0}^{2N-1} z^m P^T_{m,k} \left(M^{-1}\right)_{k,l} P_{l,n} v^n = \sum_{m,n=0}^{2N-1} z^m \left(W^{-1}\right)_{m,n} v^n. \tag{A.6}$$

In this form the Hermitean limit can be most easily taken as we have shown the limiting matrix $\overline{W}$ to exists in eq. (3.36).

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