

A new Chiral Two-Matrix Theory for Dirac Spectra with Imaginary Chemical Potential

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Abstract

We solve a new chiral Random Two-Matrix Theory by means of biorthogonal polynomials for any matrix size N . By deriving the relevant kernels we find explicit formulas for all (n, k) -point spectral (mixed or unmixed) correlation functions. In the microscopic limit we find the corresponding scaling functions, and thus derive all spectral correlators in this limit as well. We extend these results to the ordinary (non-chiral) ensembles, and also there provide explicit solutions for any finite size N , and in the microscopic scaling limit. Our results give the general analytical expressions for the microscopic correlation functions of the Dirac operator eigenvalues in theories with imaginary baryon and isospin chemical potential, and can be used to extract the tree-level pion decay constant from lattice gauge theory configurations. We find exact agreement with previous computations based on the low-energy effective field theory in the two special cases where comparisons are possible.

1 Introduction

In this paper we propose a chiral Random Matrix Theory of two coupled hermitian matrices, and solve the theory exactly at finite matrix size N . In the $N \rightarrow \infty$ limit we find the associated microscopic scaling regime, and derive closed expressions for all spectral correlation functions. It has been shown recently [1, 2] that these spectral correlation functions describe correlations between the lowest-lying Dirac operator eigenvalues in the $\beta = 2$ Dyson universality class of spontaneous chiral symmetry breaking [3], once subjected to an external vector field. The effect of the external vector source on the chiral condensate was considered in ref. [4] and very recently spectral sum rules have been derived directly from the effective partition functions [5]. The chiral Random Two-Matrix Theory generalizes the original one-matrix problem of chiral Random Matrix Theory [6, 7, 8, 9] (the microscopic hard-edge problem with a certain determinant measure) and is also related to the chiral non-Hermitian Random Two-Matrix Theory [10] for QCD with a chemical potential by setting the chemical potential to be purely imaginary.

The solution of the chiral Random Two-Matrix Theory is of interest in its own right, but it is also of importance in the context of lattice gauge theory. This is due to the fact that the spectral correlation functions of the chiral two-matrix problem can be used to measure the pion decay constant F_π in a manner analogous to how the chiral condensate Σ can be extracted by means of the original microscopic solution of the chiral Random Matrix Theory of just one chiral matrix [11]. The idea is to compare the analytic finite-volume scaling functions of Dirac operator eigenvalues to lattice simulation data. Fits to these data yield accurate determinations of the two low-energy constants of QCD, Σ and F_π .

Before we describe the relevant chiral Random Two-Matrix Theory, we will briefly outline how this two-matrix problem arises in the context of particle physics and how F_π enters the game. The coupled chiral two-matrix problem turns out to describe the microscopic limit of QCD Dirac operator spectra close to the origin. The inclusion of one additional matrix in the way we shall describe below allows us to determine the correlations between Dirac operators with different imaginary chemical potentials. In detail, we shall consider two different Dirac operators,

$$\begin{aligned} D_1 \psi_1^{(n)} &\equiv [\not{D}(A) + i\mu_1 \gamma_0] \psi_1^{(n)} = i\lambda_1^{(n)} \psi_1^{(n)} \\ D_2 \psi_2^{(n)} &\equiv [\not{D}(A) + i\mu_2 \gamma_0] \psi_2^{(n)} = i\lambda_2^{(n)} \psi_2^{(n)}. \end{aligned} \tag{1.1}$$

The interest in studying these Dirac operators has already been hinted at above: The correlation functions between eigenvalues of Dirac operators at different imaginary chemical potential is extremely sensitive to the value of F_π . In the so-called ϵ -regime of QCD this dependence on F_π is uniquely determined by the pattern of chiral symmetry breaking. As we will show below the microscopic correlations between the two different sets of eigenvalues λ_1 and λ_2 only depends on the difference $\mu_2 - \mu_1$. One can then for convenience set $\mu_2 = -\mu_1 = \mu$ and study the theory with an imaginary isospin chemical potential μ . Alternatively we will also consider the case where $\mu_1 = 0$ and $\mu_2 \neq 0$. Then D_1 is the usual Dirac operator of a given gauge configuration $A_\mu(x)$ and is what is typically used in lattice QCD simulations. This allows one to extract the low energy constants from existing simulations.

A powerful way to extract information about the lowest-lying Dirac operator eigenvalues is through the effective low-energy theory of Goldstone bosons, the associated chiral Lagrangian [12]. In the ϵ -regime this effective low-energy field theory receives, to leading order, only contributions from momentum zero modes of the Goldstone bosons. In addition, the topological index ν of the given gauge field configuration now plays a crucial role, and it is advantageous to consider the partition function in a sector of fixed topological charge ν . For a theory of N_f quark flavors, we let B denote the $N_f \times N_f$

diagonal matrix of imaginary chemical potentials, i.e. for the two Dirac operators of eq. (1.1) a diagonal matrix of μ_1 's and/or μ_2 's. In the effective $SU(N_f)$ low-energy Lagrangian the fundamental field is taken as $U(x) \in SU(N_f)$. The translation table from quark couplings to couplings in the effective theory is well known. The quark chemical potentials is of vector-kind, and only in the time direction. The coupling to external imaginary chemical potentials described by the matrix B is then achieved by replacing the time-derivative,

$$\partial_0 U(x) \rightarrow \nabla_0 U(x) = \partial_0 U(x) - i[B, U(x)]. \quad (1.2)$$

In both the more conventional “ p -regime” and the ϵ -regime the leading-order effective Lagrangian is

$$\mathcal{L} = \frac{F_\pi}{4} \text{Tr} \left[\nabla_0 U(x) \nabla_0 U^\dagger(x) + \partial_i U(x) \partial_i U^\dagger(x) \right] - \frac{\Sigma}{2} \text{Tr} \left[\mathcal{M} U^\dagger(x) + \mathcal{M}^\dagger U(x) \right], \quad (1.3)$$

where \mathcal{M} is the (diagonal) quark mass matrix of size $N_f \times N_f$. This effective theory simplifies even further in the ϵ -regime, where to leading order only momentum zero modes play a role. At fixed gauge field topology ν the partition function then reads

$$\mathcal{Z}_\nu = \int_{U(N_f)} dU (\det U)^\nu e^{\frac{1}{4} V F_\pi^2 \text{Tr}[U, B][U^\dagger, B] + \frac{1}{2} \Sigma V \text{Tr}(\mathcal{M}^\dagger U + \mathcal{M} U^\dagger)}, \quad (1.4)$$

where the integration is performed over the full unitary group. Note that F_π enters in the dimensionless combination $V F_\pi^2 \mu_f^2$. In the case where all chemical potentials are equal (imaginary baryon chemical potential), B is proportional to the identity and all μ and F_π -dependence trivially drops out of the effective low-energy Lagrangian as expected since the Goldstone bosons do not carry baryon charge. However, when different chemical potentials enters there is a unique dependence on F_π . The correlation function between the spectra of D_1 and D_2 have a dramatic dependence on F_π . In [1, 2] the two point function was obtained for zero and for two flavors directly from the low-energy effective chiral Lagrangian. Here we generalize these results using a chiral Random Matrix Theory. The chiral Random Matrix Theory approach to the ϵ -regime in this case is a two-matrix problem. We need two matrices because we wish to describe two different kinds of operators, D_1 and D_2 . With this extension of the original chiral Random Matrix Theory we will derive all (n, k) -point correlation functions of the operators D_1 and D_2 . The advantage of the chiral Random Two-Matrix Theory approach over the direct calculation using the low-energy effective chiral Lagrangian lies in the ease with which the calculations generalize. We will verify explicitly that the chiral Random Two-Matrix Theory reproduces, in two special cases, the results obtained previously through the low-energy effective chiral Lagrangian [1, 2].

In this paper we will also find the corresponding solution of the non-chiral Random Matrix Theory with two coupled matrices generalizing the results of Eynard and Mehta [13] to include determinants (matrix analogues of Dirac determinants). This is of relevance for Dirac operator spectra of the analogous 3-dimensional gauge theory [14, 15]. It may have applications in condensed matter physics as well [16].

We round off this introduction with a few remarks on the case of real chemical potential. The spectral properties of the Dirac operator change drastically if we were to consider instead ordinary real baryon chemical potential. While both Dirac operators (1.1) are anti-hermitian (and all $\lambda_i^{(n)}$ thus real), this is not the case for real chemical potential. The eigenvalues in that case lie in the complex plane. The precise manner in which the eigenvalues spread into the complex plane when a real baryon chemical potential is turned on has been computed in the ϵ -regime [10] by a random matrix approach and subsequently verified by a low-energy effective calculation [17].

A Random One-Matrix Theory describing real chemical potential in the low-energy limit was first suggested by Stephanov [18]. As shown in [10], the way to find an appropriate eigenvalue representation is to start out with a *two-matrix* extension, of which one matrix is anti-hermitian, and the other is hermitian. The eigenvalues of the combination are complex, and their microscopic distributions coincide with the low-energy Dirac operator eigenvalues at real baryon chemical potential. Much work has recently gone into the understanding of this low-lying Dirac spectrum [19, 20, 21, 10, 22, 17, 23] (for numerical evidence from lattice gauge theory simulations see, *e.g.*, ref. [24]). As stated earlier the Random Matrix Theory at real baryon chemical potential introduced in [10] is related to the chiral Random Two-Matrix Theory for imaginary chemical potential we propose in this paper. We will point out the similarities and differences below.

Our paper is organized as follows. In the next section we introduce the chiral Random Two-Matrix Theory, and proceed to derive an eigenvalue representation. This leads us to the use of biorthogonal polynomials, which we identify explicitly for finite size of the matrices in section 3. There we present our results for all spectral correlation functions in both the quenched case, the partially quenched case, and the full case (appropriate definitions of these terms will be given below). We also derive the microscopic scaling limit in all three cases. In section 4 we turn to the non-chiral variant of this theory, and find also there the explicit solution both at finite N and in the analogous scaling limit. Section 5 contains our conclusions. Several technical details are collected in 3 Appendices.

2 The Chiral Two-Matrix Problem

The chiral Two-Matrix Theory with imaginary chemical potential is described by the partition function

$$\mathcal{Z}_\nu^{(N_f)}(\{m\}) \sim \int d\Phi d\Psi \exp \left[-N \text{Tr} \left(\Phi^\dagger \Phi + \Psi^\dagger \Psi \right) \right] \prod_{f=1}^{N_f} \det[\mathcal{D}(\mu_f) + m_f] \quad (2.1)$$

where the anti-Hermitian operator \mathcal{D} in the determinant is given by

$$\mathcal{D}(\mu_f) = \begin{pmatrix} 0 & i\Phi + i\mu_f\Psi \\ i\Phi^\dagger + i\mu_f\Psi^\dagger & 0 \end{pmatrix}. \quad (2.2)$$

Both Φ and Ψ are complex rectangular matrices of size $N \times (N + \nu)$, where N and ν are integers. As shown in (2.1) we restrict ourselves to Gaussian weight functions for both Φ and Ψ in this paper. The partition function [10] at real chemical potential is obtained by taking all μ_f purely imaginary. Because this changes the Hermiticity properties of the Dirac operator the eigenvalue correlation functions at real and imaginary chemical potential are *not* related by a simple analytic continuation $\mu_f \rightarrow i\mu_f$.

We stress already here that μ_f and m_f are (dimensionless) quantities that a priori have no relation to either chemical potential or quark masses in QCD. However, in the large- N scaling limit that we shall discuss below they do translate into the familiar low-energy parameters of QCD by means of $\hat{\mu}_f^2 = 2\mu_f^2 N \rightarrow \mu_f^2 F_\pi^2 V$ and $\hat{m} = 2mN \rightarrow m\Sigma V$. With this in mind we refer to N_f as the number of flavors (here we consider only $N_f \geq 0$).

We will consider the eigenvalues of the two matrices $\mathcal{D}_1 \equiv \mathcal{D}(\mu_1)$ and $\mathcal{D}_2 \equiv \mathcal{D}(\mu_2)$ for arbitrary (real) values of μ_1 and μ_2 . The case of imaginary isospin chemical potential can be obtained from this by setting $\mu_2 = -\mu_1 = \mu$. To obtain the eigenvalues we first change integration variables to

$$\begin{aligned} \Phi_1 &\equiv \Phi + \mu_1\Psi \\ \Phi_2 &\equiv \Phi + \mu_2\Psi. \end{aligned} \quad (2.3)$$

We will restrict ourselves to the case where all determinants in (2.1) contain \mathcal{D}_1 or \mathcal{D}_2 . Their numbers are denoted by N_1 and N_2 , with masses m_{f_1} and m_{f_2} , respectively:

$$\mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\}) \sim \int d\Phi d\Psi \prod_{f_1=1}^{N_1} \det[\mathcal{D}_1 + m_{f_1}] \prod_{f_2=1}^{N_2} \det[\mathcal{D}_2 + m_{f_2}] e^{-N\text{Tr}(\Phi^\dagger \Phi + \Psi^\dagger \Psi)} \quad (2.4)$$

with $N_f = N_1 + N_2$. After the change of variables (2.3) the partition function becomes

$$\mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\}) \sim \int d\Phi_1 d\Phi_2 \prod_{f_1=1}^{N_1} \det[\mathcal{D}_1 + m_{f_1}] \prod_{f_2=1}^{N_2} \det[\mathcal{D}_2 + m_{f_2}] e^{-N\text{Tr}V(\Phi_1, \Phi_2)} \quad (2.5)$$

where each of the two operators are of the standard chiral form,

$$\mathcal{D}_1 = \begin{pmatrix} 0 & i\Phi_1 \\ i\Phi_1^\dagger & 0 \end{pmatrix} \quad (2.6)$$

and

$$\mathcal{D}_2 = \begin{pmatrix} 0 & i\Phi_2 \\ i\Phi_2^\dagger & 0 \end{pmatrix}. \quad (2.7)$$

However the two matrices Φ_1 and Φ_2 are now coupled:

$$V(\Phi_1, \Phi_2) = c_1 \Phi_1^\dagger \Phi_1 + c_2 \Phi_2^\dagger \Phi_2 - d (\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1) \quad (2.8)$$

with the abbreviations

$$\begin{aligned} c_1 &= (1 + \mu_2^2)/\delta^2 \\ c_2 &= (1 + \mu_1^2)/\delta^2 \\ d &= (1 + \mu_1 \mu_2)/\delta^2 \\ \delta &= \mu_2 - \mu_1, \end{aligned} \quad (2.9)$$

used in the following. This is the chiral analogue of the Two-Matrix problem introduced in the classic papers of refs. [25, 26]. We solve this chiral theory by the biorthogonal polynomial method, analogous to what is described in ref. [26]. At this stage, *after* changing to variables $\Phi_{1,2}$ we could in principle allow for the following more general class of weight functions:¹

$$V(\Phi_1, \Phi_2) \rightarrow v_1 (\Phi_1^\dagger \Phi_1) + v_2 (\Phi_2^\dagger \Phi_2) - d (\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1), \quad (2.10)$$

where $v_{1,2}$ are polynomials. It remains an interesting open challenge to establish universality of the present results in the large- N limit. A generalization of the universality proof in [8, 9] is non-trivial due to the different structure of recursion relations for biorthogonal polynomials in the two-matrix theory.

The first step now consists in finding an eigenvalue representation for the partition function (2.5). To this end we follow the standard procedure [6] and write

$$\begin{aligned} \Phi_1 &= U_1 X V_1^\dagger \\ \Phi_2 &= U_2 Y V_2^\dagger, \end{aligned} \quad (2.11)$$

¹Switching to higher order polynomials before the change of variables leads to higher order terms in $\Phi_1^\dagger \Phi_2$ and its conjugate.

where $U_{1,2}$ and $V_{1,2}$ are unitary matrices (with $V_{1,2}$ restricted to $U(N+\nu)/[U(1)]^{N+\nu}$ [7]), and X and Y are diagonal matrices with real and non-negative entries x_i respectively y_i . Under this change of variables the measure (including the determinants) becomes

$$\begin{aligned} d\Phi_1 \prod_{f1=1}^{N_1} \det[\mathcal{D}_1 + m_{f1}] &= dU_1 dV_1 \prod_{i=1}^N \left(dx_i^2 x_i^{2\nu} \prod_{f1=1}^{N_1} (x_i^2 + m_{f1}^2) \right) \Delta_N(\{x^2\})^2 \\ d\Phi_2 \prod_{f2=1}^{N_2} \det[\mathcal{D}_2 + m_{f2}] &= dU_2 dV_2 \prod_{i=1}^N \left(dy_i^2 y_i^{2\nu} \prod_{f2=1}^{N_2} (y_i^2 + m_{f2}^2) \right) \Delta_N(\{y^2\})^2 \end{aligned} \quad (2.12)$$

where we have left out the factors of m_f^ν coming from the zero modes of the determinants. Here $U_{1,2}$ and $V_{1,2}$ are integrated over the Haar measure, and Δ_N is the Vandermonde determinant. Because of the coupling between Φ_1 and Φ_2 the integration over $U_{1,2}$ and $V_{1,2}$ is not trivial². Using left and right invariance of the Haar measure the required integral reduces to a form that has been evaluated in ref. [27]:

$$\int dU dV \exp [dN \text{Tr}(V X U Y) + \text{h.c.}] = \frac{\det[I_\nu(2dN x_i y_j)]}{\prod (x_i y_i)^\nu \Delta_N(\{x^2\}) \Delta_N(\{y^2\})}, \quad (2.13)$$

up to an irrelevant normalization factor. Here $I_\nu(x)$ is a modified Bessel function.

Putting everything together we thus finally arrive at the eigenvalue representation

$$\begin{aligned} \mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\}) &= \mathcal{N}(\{m_{1,2}\}) \int_0^\infty \prod_{i=1}^N \left(dx_i dy_i (x_i y_i)^{\nu+1} \prod_{f1=1}^{N_1} (x_i^2 + m_{f1}^2) \prod_{f2=1}^{N_2} (y_i^2 + m_{f2}^2) \right) \\ &\times \Delta_N(\{x^2\}) \Delta_N(\{y^2\}) \det [I_\nu(2dN x_i y_j)] e^{-N \sum_i c_1 x_i^2 + c_2 y_i^2} \end{aligned} \quad (2.14)$$

where the factors of m_f^ν from the zero modes have now been absorbed in the normalization constant \mathcal{N} . In contrast to the one-matrix theory we cannot extend the integration range to $-\infty$, as the integrand is no longer even in x_i and y_i . The resulting expression (2.14) has a quite striking resemblance to the non-hermitian theory describing the Dirac operator spectrum in the presence of real baryon chemical potential [10]. In that case the explicit Bessel function in the measure is of the kind K_ν rather than the I_ν of eq. (2.14). In the eigenvalue representation (2.14) we can apply the technique of biorthogonal polynomials. For that purpose we seek two sets of monic polynomials $P_n^{(N_f)}(x^2)$ and $Q_n^{(N_f)}(y^2)$ that are biorthogonal,

$$\int_0^\infty dx dy w^{(N_f)}(x, y) P_n^{(N_f)}(x^2) Q_k^{(N_f)}(y^2) = h_n^{(N_f)} \delta_{nk}, \quad (2.15)$$

with respect to the weight function

$$w^{(N_f)}(x, y) \equiv (xy)^{\nu+1} \prod_{f1=1}^{N_1} (x^2 + m_{f1}^2) \prod_{f2=1}^{N_2} (y^2 + m_{f2}^2) I_\nu(2dN xy) e^{-N(c_1 x^2 + c_2 y^2)}. \quad (2.16)$$

Note that the order of arguments in $w^{(N_f)}(x, y)$ is important, as x only appears in the N_1 mass terms, while y only appears in the N_2 mass terms. Only in the special case of $\mu_2 = -\mu_1$, $N_1 = N_2$ and masses pairwise matched, $m_{f1} = m_{f2}$, is the measure symmetric in x and y . This is in particular true for the quenched case with an imaginary isospin chemical potential.

After the usual rewriting of the Vandermonde determinants,

$$\Delta_N(\{x^2\}) = \det [x_i^{2(j-1)}] = \det [P_{j-1}^{(N_f)}(x_i^2)]$$

²Of course this is only true for $\mu_{1,2} \neq 0$.

$$\Delta_N(\{y^2\}) = \det [y_i^{2(j-1)}] = \det [Q_{j-1}^{(N_f)}(y_i^2)] \quad (2.17)$$

in terms of polynomials in squared variables one can use the biorthogonality condition (2.15) to perform the integral (2.14) in terms of the norms $h_n^{(N_f)}$ giving $\mathcal{Z}_\nu^{(N_f)} = N! \prod_{n=0}^{N-1} h_n^{(N_f)}$. The result can be expressed in terms of the kernel $K_N(x, y)$ of the quenched polynomials $P_k^{(0)}(x^2)$ and $Q_k^{(0)}(y^2)$, defined in eq. (2.20) below. Details of the derivation following [20] are given in appendix C leading to

$$\begin{aligned} \mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\}) &= N! \prod_{f1=1}^{N_1} m_{f1}^{\nu} \prod_{f2=1}^{N_2} m_{f2}^{\nu} \frac{\prod_{j=0}^{N+N_2-1} h_j^{(0)}}{\Delta_{N_1}(\{m_1^2\}) \Delta_{N_2}(\{m_2^2\})} \\ &\times \det \left[K_{N+N_2}^{(0)}(im_{k1}, im_{l2}) P_{N+N_2}^{(0)}((im_{k1})^2) \dots P_{N+N_1-1}^{(0)}((im_{k1})^2) \right], \end{aligned} \quad (2.18)$$

for $N_1 \geq N_2$. Here we have specified the normalization leading to $\mathcal{N}(\{m_{1,2}\}) = \prod_{f1=1}^{N_1} m_{f1}^{\nu} \prod_{f2=1}^{N_2} m_{f2}^{\nu}$.

We are not only interested in the partition function itself, but mostly in the spectral correlation functions. The biorthogonal polynomial technique has been adopted to this purpose in refs. [28, 13, 29]. We review it briefly here, and in particular we make use of the very compact and general formulation of [13]. The measure considered in that paper is of the ordinary unitary theory which through the Itzykson-Zuber integral [25] gives rise to an exponential coupling between the two sets of eigenvalues. In our case this is replaced by the coupling through the Bessel function I_ν as in eq. (2.15). However one can straightforwardly follow through the same steps as in [13] with this replacement. In general four different kernels are required. Three of the kernels also depend on the generalized Bessel transforms

$$\begin{aligned} \chi_k^{(N_f)}(y) &\equiv \int_0^\infty dx w^{(N_f)}(x, y) P_k^{(N_f)}(x^2) \\ \hat{\chi}_k^{(N_f)}(x) &\equiv \int_0^\infty dy w^{(N_f)}(x, y) Q_k^{(N_f)}(y^2). \end{aligned} \quad (2.19)$$

Note that in general the two functions are not polynomials. In particular, they are proportional to the part of the weight which is not integrated.

The four required kernels in terms of the biorthogonal polynomials and their transforms are:

$$K_N^{(N_f)}(y, x) = \sum_{k=0}^{N-1} \frac{Q_k^{(N_f)}(y^2) P_k^{(N_f)}(x^2)}{h_k}, \quad (2.20)$$

$$H_N^{(N_f)}(x_1, x_2) = \sum_{k=0}^{N-1} \frac{\hat{\chi}_k^{(N_f)}(x_1) P_k^{(N_f)}(x_2^2)}{h_k} = \int_0^\infty dy w^{(N_f)}(x_1, y) K_N^{(N_f)}(y, x_2) \quad (2.21)$$

$$\hat{H}_N^{(N_f)}(y_1, y_2) = \sum_{k=0}^{N-1} \frac{Q_k^{(N_f)}(y_1^2) \chi_k^{(N_f)}(y_2)}{h_k} = \int_0^\infty dx w^{(N_f)}(x, y_2) K_N^{(N_f)}(y_1, x) \quad (2.22)$$

$$M_N^{(N_f)}(x, y) = \sum_{k=0}^{N-1} \frac{\hat{\chi}_k^{(N_f)}(x) \chi_k^{(N_f)}(y)}{h_k} = \int_0^\infty dudv w^{(N_f)}(x, u) w^{(N_f)}(v, y) K_N^{(N_f)}(u, v). \quad (2.23)$$

Note that also here the order of arguments is important. We have chosen the notation x_i and y_i to indicate the nature of the arguments so that x_i is an eigenvalue of associated with \mathcal{D}_1 , while y_i is an eigenvalue associated with \mathcal{D}_2 .

All spectral correlation functions can be derived from the four kernels above. In the special case of a symmetric weight, $w^{(N_f)}(x, y) = w^{(N_f)}(y, x)$, the polynomials coincide, $P_k^{(N_f)} = Q_k^{(N_f)}$, leading to coinciding transforms $\chi_k^{(N_f)} = \hat{\chi}_k^{(N_f)}$ and thus to a matching of the kernels $\hat{H}_N^{(N_f)}(x, y) = H_N^{(N_f)}(y, x)$.

We call the correlation function of a set containing n eigenvalues of \mathcal{D}_1 and k eigenvalues of \mathcal{D}_2 an (n, k) -correlation function

$$R_{(n,k)}^{(N_f)}(\{x\}_n, \{y\}_k) \equiv \frac{N!^2}{(N-n)!(N-k)!} \frac{1}{\mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\})} \quad (2.24)$$

$$\times \int_0^\infty \prod_{i=n+1}^N dx_i \prod_{j=k+1}^N dy_j \det \left[w^{(N_f)}(x_i, y_j) \right] \Delta_N(\{x^2\}) \Delta_N(\{y^2\}).$$

Note for later comparison that these include both the connected and disconnected parts.

Eynard and Mehta [13] have proven a generalized ‘‘Dyson Theorem’’ for all correlation functions of random multi-matrix theories coupled in a chain. Adapted to our chiral Two-Matrix Theory it can be stated as follows:

$$R_{(n,k)}^{(N_f)}(\{x\}_n, \{y\}_k) = \det_{\substack{1 \leq i_1, i_2 \leq n; \\ 1 \leq j_1, j_2 \leq k}} \begin{bmatrix} H_N^{(N_f)}(x_{i_1}, x_{i_2}) & M_N^{(N_f)}(x_{i_1}, y_{j_2}) - w^{(N_f)}(x_{i_1}, y_{j_2}) \\ K_N^{(N_f)}(y_{j_1}, x_{i_2}) & \hat{H}_N^{(N_f)}(y_{j_1}, y_{j_2}) \end{bmatrix}. \quad (2.25)$$

The simplest example, the spectral density itself equals

$$R_{(1,0)}^{(N_f)}(x) = H_N^{(N_f)}(x, x), \quad (2.26)$$

and similar for $R_{(0,1)}^{(N_f)}(y)$. In the symmetric case (isospin) the densities are the same for the two operators $\mathcal{D}(\mu)$ and $\mathcal{D}(-\mu)$ as is obvious from the symmetry $\Psi \leftrightarrow -\Psi$ in the partition function (2.1).

More examples will be given in the next section. If all eigenvalues of one kind are integrated out this reduces to the known result for the one-matrix theory (up to a μ dependent rescaling), as can be seen by generalizing the argument from [28] for the unquenched non-chiral theory:

$$R_{(n,k=0)}^{(N_f)}(\{x\}_n) = \det \left[H_N^{(N_f)}(x_i, x_j) \right], \quad (2.27)$$

and similarly for $R_{(n=0,k)}^{(N_f)}(\{y\}_k)$.

3 Explicit Computations

In this section we explicitly determine the biorthogonal polynomials fulfilling the condition (2.15). It is instructive to consider first the quenched case $N_f = 0$ where there are no flavor determinants in the integration measure. This is not only because matters simplify then, but because the general solution can be expressed in terms of quenched quantities. Thus after having gone through the quenched case in detail, we present the general solution for any N_f . We also take the large- N scaling limit, and thus make contact to the field theory calculations of refs. [1, 2].

3.1 THE QUENCHED CASE

The analog of quenching in the Random Matrix Theory formulation is easily achieved: We simply remove the explicit determinant factors in the measure. The theory remains chiral, and we still perform the change of variables (2.3). The weight function (2.16) then simplifies considerably and the orthogonal polynomials can be readily obtained as shown in Appendices A and B. The result is

$$\begin{aligned} P_n(x^2) &= n!(-N\tau c_1)^{-n} L_n^{(\nu)}(N\tau c_1 x^2) \\ Q_n(y^2) &= n!(-N\tau c_2)^{-n} L_n^{(\nu)}(N\tau c_2 y^2) \end{aligned} \quad (3.1)$$

with

$$\tau = 1 - d^2/(c_1 c_2) \quad (3.2)$$

from Appendix B and the normalization constant h_k is given in (B.11). Here and in the following we drop the superscript ($N_f = 0$) for the quenched quantities. The transforms (2.19) can also be found from (B.4) along with (B.9) and (B.10)

$$\begin{aligned} \chi_k(y) &= \frac{k!}{2Nc_1(-N\tau c_2)^k} \left(\frac{d}{c_1}\right)^{2k+\nu} y^{2\nu+1} e^{-N\tau c_2 y^2} L_k^{(\nu)}(N\tau c_2 y^2) \\ \hat{\chi}_k(x) &= \frac{k!}{2Nc_2(-N\tau c_1)^k} \left(\frac{d}{c_2}\right)^{2k+\nu} x^{2\nu+1} e^{-N\tau c_1 x^2} L_k^{(\nu)}(N\tau c_1 x^2). \end{aligned} \quad (3.3)$$

These are proportional to the polynomials in eq. (3.1).

Having the required functions, we can now write down the related kernels. These can be obtained from (B.5) which gives

$$\begin{aligned} K_N(x, y) &= \frac{4N(N\tau c_1 c_2)^{\nu+1}}{d^\nu} \sum_{k=0}^{N-1} \frac{k!}{(k+\nu)!} \left(\frac{c_1 c_2}{d^2}\right)^k L_k^{(\nu)}(N\tau c_1 x^2) L_k^{(\nu)}(N\tau c_2 y^2), \\ H_N(x_1, x_2) &= 2(N\tau c_1)^{\nu+1} x_1^{2\nu+1} e^{-N\tau c_1 x_1^2} \sum_{k=0}^{N-1} \frac{k!}{(k+\nu)!} L_k^{(\nu)}(N\tau c_1 x_1^2) L_k^{(\nu)}(N\tau c_1 x_2^2), \\ \hat{H}_N(y_1, y_2) &= 2(N\tau c_2)^{\nu+1} y_2^{2\nu+1} e^{-N\tau c_2 y_2^2} \sum_{k=0}^{N-1} \frac{k!}{(k+\nu)!} L_k^{(\nu)}(N\tau c_2 y_1^2) L_k^{(\nu)}(N\tau c_2 y_2^2), \\ M_N(x, y) &= \tau(N\tau d)^\nu (xy)^{2\nu+1} e^{-N\tau(c_1 x^2 + c_2 y^2)} \sum_{k=0}^{N-1} \frac{k!}{(k+\nu)!} \left(\frac{d^2}{c_1 c_2}\right)^k L_k^{(\nu)}(N\tau c_1 x^2) L_k^{(\nu)}(N\tau c_2 y^2). \end{aligned} \quad (3.4)$$

We now take the microscopic scaling limit where $N \rightarrow \infty$ and $\hat{\mu}_f^2 = 2\mu_f^2 N$ and $\hat{m} = 2mN$ are kept fixed. We define microscopic variables by

$$\begin{aligned} \hat{x} &\equiv 2xN \\ \hat{y} &\equiv 2yN. \end{aligned} \quad (3.5)$$

Making use of the asymptotic behavior of Laguerre polynomials,

$$\lim_{n \rightarrow \infty} L_n^{(a)}(x^2) \sim n^a (nx^2)^{-a/2} J_a(2\sqrt{nx}) \quad (3.6)$$

where $J_k(x)$ is a Bessel function, we can turn sums into integrals $1/N \sum_{k=0}^{N-1} \rightarrow \int_0^1 dt$ with $t = k/N$

$$\sum_{k=0}^{N-1} \frac{k!}{(k+\nu)!} \left(\frac{c_1 c_2}{d^2}\right)^{sk} L_k^{(\nu)}(N\tau c_x x^2) L_k^{(\nu)}(N\tau c_y y^2) \sim \frac{1}{(\hat{x}\hat{y})^\nu} \int_0^1 dt e^{s\hat{\delta}^2 t/2} J_\nu(\hat{x}\sqrt{t}) J_\nu(\hat{y}\sqrt{t}). \quad (3.7)$$

Here s can have the values $0, \pm 1$ and c_x and c_y can be either of c_1 or c_2 . We have also used

$$\left(\frac{c_1 c_2}{d^2}\right)^{sk} = \left(\frac{(1+\mu_1^2)(1+\mu_2^2)}{(1+\mu_1\mu_2)^2}\right)^{sk} \approx \left(1 + \frac{(\hat{\mu}_1 - \hat{\mu}_2)^2}{2N}\right)^{sk} \rightarrow \exp\left(\frac{s}{2} \hat{\delta}^2 t\right) \quad (3.8)$$

with $\hat{\delta} = \hat{\mu}_2 - \hat{\mu}_1$. In this scaling limit the kernels therefore take the form

$$\lim_{N \rightarrow \infty} K_N\left(x = \frac{\hat{x}}{2N}, y = \frac{\hat{y}}{2N}\right) = 2^{2\nu+4} N^{2\nu+4} (\hat{x}\hat{y})^{-\nu} \hat{\delta}^{-2} \mathcal{I}^+(\hat{x}, \hat{y}) \quad (3.9)$$

$$\begin{aligned} \lim_{N \rightarrow \infty} H_N \left(x = \frac{\hat{x}}{2N}, y = \frac{\hat{y}}{2N} \right) &= 2N \hat{x}^{\nu+1} \hat{y}^{-\nu} \mathcal{I}^0(\hat{x}, \hat{y}) \\ &= \lim_{N \rightarrow \infty} \hat{H}_N \left(y = \frac{\hat{y}}{2N}, x = \frac{\hat{x}}{2N} \right) \end{aligned} \quad (3.10)$$

$$\lim_{N \rightarrow \infty} M_N \left(x = \frac{\hat{x}}{2N}, y = \frac{\hat{y}}{2N} \right) = 2^{-2\nu-2} N^{-2\nu-2} (\hat{x}\hat{y})^{\nu+1} \hat{\delta}^2 \mathcal{I}^-(\hat{x}, \hat{y}) \quad (3.11)$$

where we introduced the abbreviated notation

$$\begin{aligned} \mathcal{I}^0(\hat{x}, \hat{y}) &\equiv \frac{1}{2} \int_0^1 dt J_\nu(\hat{x}\sqrt{t}) J_\nu(\hat{y}\sqrt{t}) = \frac{\hat{x} J_{\nu+1}(\hat{x}) J_\nu(\hat{y}) - \hat{y} J_{\nu+1}(\hat{y}) J_\nu(\hat{x})}{\hat{x}^2 - \hat{y}^2} \\ \mathcal{I}^\pm(\hat{x}, \hat{y}) &\equiv \frac{1}{2} \int_0^1 dt e^{\pm \hat{\delta}^2 t/2} J_\nu(\hat{x}\sqrt{t}) J_\nu(\hat{y}\sqrt{t}) . \end{aligned} \quad (3.12)$$

The weight becomes

$$\hat{w}^{(0)}(\hat{x}, \hat{y}) = \lim_{N \rightarrow \infty} w^{(0)}\left(\frac{\hat{x}}{2N}, \frac{\hat{y}}{2N}\right) = \left(\frac{\hat{x}\hat{y}}{4N^2}\right)^{\nu+1} \exp\left(-\frac{\hat{x}^2 + \hat{y}^2}{2\hat{\delta}^2}\right) I_\nu\left(\frac{\hat{x}\hat{y}}{\hat{\delta}^2}\right) , \quad (3.13)$$

using the abbreviation

$$\tilde{\mathcal{I}}^-(\hat{x}, \hat{y}) \equiv \frac{1}{\hat{\delta}^2} \exp\left(-\frac{\hat{x}^2 + \hat{y}^2}{2\hat{\delta}^2}\right) I_\nu\left(\frac{\hat{x}\hat{y}}{\hat{\delta}^2}\right) - \mathcal{I}^-(\hat{x}, \hat{y}) . \quad (3.14)$$

Notice that the μ 's only appear through their difference. Despite the relation (B.6) between the weight and the kernel M we will derive in appendix B our large- N results of the two *differ*. The reason for this is that the large- N limit is not uniform: while we have rescaled the chemical potentials with N , also called ‘‘weak non-Hermiticity’’ in the context of real μ , another limit exists where the chemical potentials are not scaled, including a different rescaling of the eigenvalues with N . In this limit called ‘‘strong non-Hermiticity’’ for real μ , which can also be obtained by letting $\delta \rightarrow \infty$, the large- N results of weight and M will match, as suggested by relation (B.6). For more details on this limit we refer to [19].

The correlation functions in the microscopic scaling limit are defined as follows for general N_f :

$$\rho_{(n,k)}^{(N_f)}(\{\hat{x}\}_n, \{\hat{y}\}_k) \equiv \lim_{N \rightarrow \infty} \frac{1}{(2N)^{n+k}} R_{(n,k)}^{(N_f)} \left(\{x = \frac{\hat{x}}{2N}\}_n, \{y = \frac{\hat{y}}{2N}\}_k \right) \quad (3.15)$$

Inserting eqs. (3.9) - (3.11) into the definition we arrive at the following result

$$\rho_{(n,k)}(\{\hat{x}\}_n, \{\hat{y}\}_k) = \prod_{i=1}^n \hat{x}_i \prod_{j=1}^k \hat{y}_j \times \det_{\substack{1 \leq i_1, i_2 \leq n; \\ 1 \leq j_1, j_2 \leq k}} \begin{bmatrix} \mathcal{I}^0(\hat{x}_{i_1}, \hat{x}_{i_2}) & -\tilde{\mathcal{I}}^-(\hat{x}_{i_1}, \hat{y}_{j_2}) \\ \mathcal{I}^+(\hat{y}_{j_1}, \hat{x}_{i_2}) & \mathcal{I}^0(\hat{y}_{j_1}, \hat{x}_{j_2}) \end{bmatrix} \quad (3.16)$$

after taking out common factors.

As a special case we note that the (1,0)-correlation function reduces to the known Bessel kernel taken at coincident points, yielding the $\hat{\mu}_f$ -independent

$$\rho_{(1,0)}(\hat{x}) = \frac{\hat{x}}{2} \left[J_\nu^2(\hat{x}) - J_{\nu+1}(\hat{x}) J_{\nu-1}(\hat{x}) \right] , \quad (3.17)$$

which is the known quenched expression from the one-matrix theory [7]. The $\hat{\mu}_f$ -independence of this result is particularly easy to understand from the effective Lagrangian, where the corresponding B -matrix is proportional to the unit matrix, and thus can be shifted away. Similarly, the (2,0)-correlation function of this quenched theory becomes

$$\rho_{(2,0)}(\hat{x}_1, \hat{x}_2) = \rho_{(1,0)}(\hat{x}_1) \rho_{(1,0)}(\hat{x}_2) - \hat{x}_1 \hat{x}_2 \left[\frac{\hat{x}_1 J_{\nu+1}(\hat{x}_1) J_\nu(\hat{x}_2) - \hat{x}_2 J_{\nu+1}(\hat{x}_2) J_\nu(\hat{x}_1)}{\hat{x}_1^2 - \hat{x}_2^2} \right]^2 \quad (3.18)$$

which again is $\hat{\mu}_f$ -independent, and agrees with the standard expression for the $\mu_f = 0$ theory [7]. One easily sees how this generalizes to the the general $(n, 0)$ -correlator in the quenched theory: the usual n -point correlation function of the one-matrix theory [8, 9] is recovered from eq. (2.25), taking out and canceling the prefactors from eq. (3.10) from the determinant.

A particularly important quantity is the mixed $(1,1)$ -correlation function $\rho_{(1,1)}(\hat{x}, \hat{y})$ which reads

$$\rho_{(1,1)}(\hat{x}, \hat{y}) \equiv \rho_{(1,0)}(\hat{x})\rho_{(1,0)}(\hat{y}) + \rho_{(1,1)}^{conn}(\hat{x}, \hat{y}) . \quad (3.19)$$

The connected part is given by

$$\rho_{(1,1)}^{conn}(\hat{x}, \hat{y}) = \hat{x}\hat{y} \mathcal{I}^+(\hat{y}, \hat{x}) \tilde{\mathcal{I}}^-(\hat{x}, \hat{y}) . \quad (3.20)$$

After making the required substitution $\delta^2 \rightarrow 4VF_\pi^2\mu^2$ and the identification $\hat{x} = \lambda_+\Sigma V$, $\hat{y} = \lambda_-\Sigma V$, this agrees exactly with what was previously computed from the effective low-energy QCD Lagrangian [1]. In that framework, however, the general correlation functions $\rho_{(n,k)}^{conn}$ have not been obtained.

Another interesting case is the quenched limit when μ is not scaled microscopically, as discussed above. In this limit the upper right corner of the determinant in eq. (3.16) vanished due to the matching of the weight and large- N kernel M . This leads to a factorisation of all correlation functions into a product of determinants over H and \hat{H} .

3.2 TWO AND MORE FLAVORS

To find the biorthogonal polynomials it is convenient to use the following general expression in the eigenvalue representation

$$\begin{aligned} P_k^{(N_f)}(x^2) &= (-)^k (ix)^{-\nu} \langle \det[\mathcal{D}_1 + ix] \rangle_{\mathcal{Z}_\nu^{(N_f)}} \\ &= \frac{(-)^k \mathcal{N}(\{m_{1,2}\})}{\mathcal{Z}_\nu^{(N_f)}} \int_0^\infty \prod_{i=1}^k \left(dx_i dy_i (x_i y_i)^{\nu+1} (x_i^2 - x^2) \prod_{f1=1}^{N_1} (x_i^2 + m_{f1}^2) \prod_{f2=1}^{N_2} (y_i^2 + m_{f2}^2) \right) \\ &\times \Delta_k(\{x^2\}) \Delta_k(\{y^2\}) \det [I_\nu(2dN x_i y_j)] e^{-N \sum_i c_1 x_i^2 + c_2 y_i^2} \end{aligned} \quad (3.21)$$

generalising the observation made in the appendix of [29]. As also noted there a similar relation holds for

$$Q_k^{(N_f)}(y^2) = (-)^k (iy)^{-\nu} \langle \det \mathcal{D}_2 + iy \rangle_{\mathcal{Z}_\nu^{(N_f)}} . \quad (3.22)$$

The expectation values are taken with respect to the partition function in eq. (2.5) here³. The prefactor $(-)^k (ix)^{-\nu}$ ensures that $P_k^{(N_f)}(x^2)$ is a monic polynomial of order k , and similarly for $Q_k^{(N_f)}(y^2)$. Eq. (3.21) (eq. (3.22)) is recognized as being a partition function of one additional \mathcal{D}_1 -flavor (\mathcal{D}_2 -flavor) with purely imaginary mass ix (iy),

$$\begin{aligned} P_k^{(N_f)}(x^2) &= (-)^k (ix)^{-\nu} \frac{\mathcal{Z}_\nu^{(N_1+1, N_2)}(\{m_1\}, ix, \{m_2\})}{\mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\})} , \\ Q_k^{(N_f)}(y^2) &= (-)^k (iy)^{-\nu} \frac{\mathcal{Z}_\nu^{(N_1, N_2+1)}(\{m_1\}, \{m_2\}, iy)}{\mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\})} \end{aligned} \quad (3.23)$$

indicated explicitly in the superscript. In the large- N scaling limit this gives a particularly useful representation [30] since compact analytical expressions can be obtained from refs. [21, 22], where

³The partition function used here contains k eigenvalues, whereas the explicit factor of N in the exponent remains fixed.

the involved finite-volume partition functions were derived in the effective field theory framework. Alternatively, we can re-derive those expressions from our two-matrix theory by substituting the corresponding large- N expressions from the previous subsection into eq. (3.23), and making use of the general expression for partition functions given in appendix C.

Instead of inserting the relation eq. (3.23) into the definition of the kernel eq. (2.20) a far more useful relation exists, giving the kernel directly in terms of a single partition function:

$$K_{N+1}^{(N_f)}(y, x) = \frac{(-xy)^{-\nu}}{h_N^{(N_f)}} \langle \det[\mathcal{D}_1 + ix] \det[\mathcal{D}_2 + iy] \rangle_{\mathcal{Z}_\nu^{(N_f)}} \quad (3.24)$$

This relation follows in complete analogy to [20] after going through the same steps as in appendix C. The highest polynomials in $K_{N+1}^{(N_f)}(y, x)$ are of degree N , and in order to match we have to divide by their coefficient $h_N^{(N_f)}$, given by [20, 31]

$$h_N^{(N_f)} = (-)^{N_1+N_2} h_{N+N_2} \frac{\det [K_{N+N_2+1}(im_{l_2}, im_{k_1}) P_{N+N_2+1}((im_{k_1})^2) \dots P_{N+N_1}((im_{k_1})^2)]}{\det [K_{N+N_2}(im_{l_2}, im_{k_1}) P_{N+N_2}((im_{k_1})^2) \dots P_{N+N_1-1}((im_{k_1})^2)]} \quad (3.25)$$

for $N_1 \geq N_2$ (for $N_2 > N_1$ simply exchange the 1 with the 2 quantities as in appendix C). In the large- N limit the two determinants will cancel, leaving only the quenched norm. In terms of partition functions with two additional flavors this can be written as

$$K_{N+1}^{(N_f)}(y, x) = (-xy)^{-\nu} \frac{\mathcal{Z}_\nu^{(N_1+1, N_2+1)}(\{m_1\}, ix, \{m_2\}, iy)}{h_N^{(N_f)} \mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\})}. \quad (3.26)$$

Compared to the polynomial we have added a second flavor with chemical potential μ_2 , related to the *first* argument of the kernel. If on the contrary we had added a second flavor of the same kind, $\langle \det[\mathcal{D}_1 + ix_1] \det[\mathcal{D}_1 + ix_2] \rangle_{\mathcal{Z}_\nu^{(N_f)}}$, we would have obtained a determinant of polynomials instead of the kernel (see appendix C). Only when $\mu_1 = \mu_2 = 0$ the two are related through the Christoffel-Darboux identity.

The functions $\chi_k^{(N_f)}(y)$ and $\hat{\chi}_k^{(N_f)}(x)$ can then be computed from their definition (2.19)

$$\begin{aligned} \chi_k^{(N_f)}(y) &= \int_0^\infty dx w^{(N_f)}(x, y) (-)^k (ix)^{-\nu} \frac{\mathcal{Z}_\nu^{(N_1+1, N_2)}(\{m_1\}, ix, \{m_2\})}{\mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\})} \\ \hat{\chi}_k^{(N_f)}(x) &= \int_0^\infty dy w^{(N_f)}(x, y) (-)^k (iy)^{-\nu} \frac{\mathcal{Z}_\nu^{(N_1, N_2+1)}(\{m_1\}, \{m_2\}, iy)}{\mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\})}. \end{aligned} \quad (3.27)$$

Similarly the remaining kernels can be obtained from eq. (3.26) by transforming one or both arguments

$$H_{N+1}^{(N_f)}(x_1, x_2) = \int_0^\infty dy w^{(N_f)}(x_1, y) (-x_2 y)^{-\nu} \frac{\mathcal{Z}_\nu^{(N_1+1, N_2+1)}(\{m_1\}, ix_2, \{m_2\}, iy)}{h_N^{(N_f)} \mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\})} \quad (3.28)$$

$$\hat{H}_{N+1}^{(N_f)}(y_1, y_2) = \int_0^\infty dx w^{(N_f)}(x, y_2) (-x y_1)^{-\nu} \frac{\mathcal{Z}_\nu^{(N_1+1, N_2+1)}(\{m_1\}, ix, \{m_2\}, iy_1)}{h_N^{(N_f)} \mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\})} \quad (3.29)$$

$$M_{N+1}^{(N_f)}(x, y) = \int_0^\infty dudv w^{(N_f)}(x, u) w^{(N_f)}(v, y) (-vu)^{-\nu} \frac{\mathcal{Z}_\nu^{(N_1+1, N_2+1)}(\{m_1\}, iv, \{m_2\}, iu)}{h_N^{(N_f)} \mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\})}. \quad (3.30)$$

Inserting them into eq. (2.25) this provides the general (n, k) -correlation function at finite- N of the theory with arbitrary N_f . Since they are expressed in terms of partition functions the microscopic scaling limit can easily be taken, as we have mentioned above.

We now specialize to the physically most interesting case of two flavors $N_1 = N_2 = 1$, with quark masses m_u and m_d , respectively. We first focus on the $(1,1)$ -correlation function $\rho_{(1,1)}^{(1+1)conn}(\hat{x}, \hat{y})$ in order to compare with the known result [2]. The first building block we need, the unquenched kernel, is given by

$$K_{N+1}^{(1+1)}(y, x) = \frac{h_{N+1}}{h_N^{(N_f)}(x^2 + m_u^2)(y^2 + m_d^2)K_{N+1}(im_d, im_u)} \det \begin{bmatrix} K_{N+1}(im_d, im_u) & K_{N+1}(im_d, x) \\ K_{N+1}(y, im_u) & K_{N+1}(y, x) \end{bmatrix}, \quad (3.31)$$

where we used that the quenched kernel is even in its arguments. The second building block is obtained from that by integrating twice from eq. (2.23)

$$\begin{aligned} M_{N+1}^{(1+1)}(x, y) &= \int_0^\infty dudv w^{(1+1)}(x, u)w^{(1+1)}(v, y)K_{N+1}^{(1+1)}(u, v) \\ &= \frac{h_{N+1}(x^2 + m_u^2)(y^2 + m_d^2)}{h_N^{(N_f)}K_{N+1}(im_d, im_u)} \det \begin{bmatrix} K_{N+1}(im_d, im_u) & \hat{H}_{N+1}(im_d, y) \\ H_{N+1}(x, im_u) & M_{N+1}(x, y) \end{bmatrix}. \end{aligned} \quad (3.32)$$

Taking the integrals inside the determinant leads to the appearance of all different quenched kernels. In order to take the large- N limit we can thus use the quenched results eqs. (3.9) - (3.11). We therefore obtain for the two microscopic building blocks

$$\lim_{N \rightarrow \infty} K_{N+1}^{(1+1)}(y, x) = \frac{(2N)^{2\nu+8}(\hat{x}\hat{y})^{-\nu}\hat{\delta}^{-2}}{(\hat{x}^2 + \hat{m}_u^2)(\hat{y}^2 + \hat{m}_d^2)} \left(\mathcal{I}^+(\hat{y}, \hat{x}) - \frac{\mathcal{I}^+(\hat{y}, i\hat{m}_u)\mathcal{I}^+(i\hat{m}_d, \hat{x})}{\mathcal{I}^+(i\hat{m}_d, i\hat{m}_u)} \right) \quad (3.33)$$

and

$$\lim_{N \rightarrow \infty} M_{N+1}^{(1+1)}(x, y) = \frac{\hat{\delta}^2(\hat{x}\hat{y})^{\nu+1}(\hat{x}^2 + \hat{m}_u^2)(\hat{y}^2 + \hat{m}_d^2)}{(2N)^{2\nu+6}} \left(\mathcal{I}^-(\hat{x}, \hat{y}) - \frac{\mathcal{I}^0(\hat{x}, i\hat{m}_u)\mathcal{I}^0(i\hat{m}_d, \hat{y})}{\mathcal{I}^+(i\hat{m}_u, i\hat{m}_d)} \right) \quad (3.34)$$

along with the weight

$$\lim_{N \rightarrow \infty} w^{(1+1)}(x, y) = \frac{(\hat{x}^2 + \hat{m}_u^2)(\hat{y}^2 + \hat{m}_d^2)}{(2N)^4} \frac{(\hat{x}\hat{y})^{\nu+1}}{(2N)^{2\nu+2}} I_\nu \left(\frac{\hat{x}\hat{y}}{\hat{\delta}^2} \right) e^{-\frac{\hat{x}^2 + \hat{y}^2}{2\hat{\delta}^2}}. \quad (3.35)$$

Here the norms have canceled in the large- N limit, and the explicit mass factors will also drop out when multiplying all together in the next step. Using eqs. (2.25) and (3.15) we first consider the following connected part of the two point function:

$$\begin{aligned} \rho_{(1,1)}^{(1+1)conn}(\hat{x}, \hat{y}) &\equiv \lim_{N \rightarrow \infty} \frac{1}{(2N)^2} K_{N+1}^{(1+1)}(y, x) \left(w^{(1+1)}(x, y) - M_{N+1}^{(1+1)}(x, y) \right) \\ &= \hat{x}\hat{y} \left(\mathcal{I}^+(\hat{y}, \hat{x}) - \frac{\mathcal{I}^+(\hat{y}, i\hat{m}_u)\mathcal{I}^+(i\hat{m}_d, \hat{x})}{\mathcal{I}^+(i\hat{m}_d, i\hat{m}_u)} \right) \left(\tilde{\mathcal{I}}^-(\hat{x}, \hat{y}) + \frac{\mathcal{I}^0(\hat{x}, i\hat{m}_u)\mathcal{I}^0(i\hat{m}_d, \hat{y})}{\mathcal{I}^+(i\hat{m}_u, i\hat{m}_d)} \right). \end{aligned} \quad (3.36)$$

Substituting $\hat{\delta}^2 \rightarrow 4VF_\pi^2\mu^2$ and $\hat{x} = \lambda_+\Sigma V, \hat{y} = \lambda_-\Sigma V$, this again agrees with the unquenched result from effective low-energy QCD [2]. For completeness we also give the other two building blocks for this theory. Following the definition (2.21) we have

$$\begin{aligned} H_{N+1}^{(1+1)}(x_1, x_2) &= \int_0^\infty dy w^{(1+1)}(x_1, y)K_{N+1}^{(1+1)}(y, x_2) \\ &= \frac{h_{N+1}(x_1^2 + m_u^2)}{h_N^{(N_f)}(x_2^2 + m_u^2)K_{N+1}(im_d, im_u)} \det \begin{bmatrix} K_{N+1}(im_d, im_u) & K_{N+1}(im_d, x_2) \\ H_{N+1}(x_1, im_u) & H_{N+1}(x_1, x_2) \end{bmatrix} \end{aligned} \quad (3.37)$$

leading to

$$\lim_{N \rightarrow \infty} H_{N+1}^{(1+1)}(x_1, x_2) = \frac{2N \hat{x}_1^{\nu+1} (\hat{x}_1^2 + \hat{m}_u^2)}{\hat{x}_2^\nu (\hat{x}_2^2 + \hat{m}_u^2)} \left(\mathcal{I}^0(\hat{x}_1, \hat{x}_2) - \frac{\mathcal{I}^0(\hat{x}_1, i\hat{m}_u) \mathcal{I}^+(i\hat{m}_d, \hat{x}_2)}{\mathcal{I}^+(i\hat{m}_d, i\hat{m}_u)} \right). \quad (3.38)$$

For the last block we obtain from eq. (2.22)

$$\begin{aligned} \hat{H}_{N+1}^{(1+1)}(y_1, y_2) &= \int_0^\infty dx w^{(1+1)}(x, y_2) K_{N+1}^{(1+1)}(y_1, x) \\ &= \frac{h_{N+1}(y_2^2 + m_d^2)}{h_N^{(N_f)}(y_1^2 + m_d^2) K_{N+1}(im_d, im_u)} \det \begin{bmatrix} K_{N+1}(im_d, im_u) & \hat{H}_{N+1}(im_d, y_2) \\ K_{N+1}(y_1, im_u) & \hat{H}_{N+1}(y_1, y_2) \end{bmatrix} \end{aligned} \quad (3.39)$$

leading to

$$\lim_{N \rightarrow \infty} \hat{H}_{N+1}^{(1+1)}(y_1, y_2) = \frac{2N \hat{y}_2^{\nu+1} (\hat{y}_2^2 + \hat{m}_d^2)}{\hat{y}_1^\nu (\hat{y}_1^2 + \hat{m}_d^2)} \left(\mathcal{I}^0(\hat{y}_1, \hat{y}_2) - \frac{\mathcal{I}^+(\hat{y}_1, i\hat{m}_u) \mathcal{I}^0(i\hat{m}_d, \hat{y}_2)}{\mathcal{I}^+(i\hat{m}_d, i\hat{m}_u)} \right). \quad (3.40)$$

As a new microscopic result we have the spectral densities

$$\begin{aligned} \rho_{(1,0)}^{(1+1)}(\hat{x}) &= \lim_{N \rightarrow \infty} \frac{1}{2N} H_{N+1}^{(1+1)}(x, x) \\ \rho_{(0,1)}^{(1+1)}(\hat{y}) &= \lim_{N \rightarrow \infty} \frac{1}{2N} \hat{H}_{N+1}^{(1+1)}(y, y), \end{aligned} \quad (3.41)$$

given by eqs. (3.38) and (3.40) at equal arguments divided by $2N$ respectively. Note that the two functions coincide. At $\hat{\mu}_1 = \hat{\mu}_2 = 0$ they reproduce the known result [9] for two different flavors. For a plot of the unquenched eigenvalue density see figure 1.

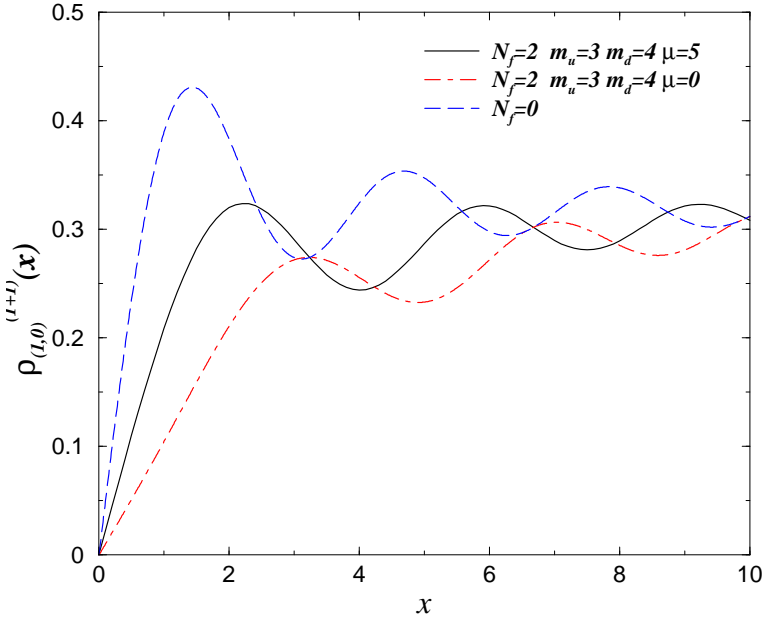


Figure 1: The eigenvalue density for $N_f = 2$ with imaginary isospin chemical potential $\hat{\mu} = \hat{\delta}/2 = 0, 5, \infty$ and quark masses $\hat{m}_u = 3, \hat{m}_d = 4$. For $\hat{\mu} \rightarrow \infty$ the unquenched eigenvalue density approaches the quenched one.

To finish this subsection we give the result for the most general correlation function in the two flavor case. It follows by inserting the above building blocks into eq. (2.25) and taking out common factors:

$$\rho_{(n,k)}^{(1+1)}(\{\hat{x}\}_n, \{\hat{y}\}_k) = \prod_{i=1}^n \hat{x}_i \prod_{j=1}^k \hat{y}_j (\mathcal{I}^+(i\hat{m}_u, i\hat{m}_d))^{-n-k} \quad (3.42)$$

$$\times \det_{\substack{1 \leq i_1, i_2 \leq n \\ 1 \leq j_1, j_2 \leq k}} \begin{bmatrix} \mathcal{I}^+(i\hat{m}_u, i\hat{m}_d) \mathcal{I}^0(\hat{x}_{i_1}, \hat{x}_{i_2}) & -\mathcal{I}^+(i\hat{m}_u, i\hat{m}_d) \tilde{\mathcal{I}}^-(\hat{y}_{j_1}, \hat{x}_{i_2}) \\ -\mathcal{I}^0(\hat{x}_{i_1}, i\hat{m}_u) \mathcal{I}^+(\hat{x}_{i_2}, i\hat{m}_d) & -\mathcal{I}^0(\hat{y}_{j_1}, i\hat{m}_d) \mathcal{I}^0(\hat{x}_{i_2}, i\hat{m}_u) \\ \mathcal{I}^+(i\hat{m}_u, i\hat{m}_d) \mathcal{I}^+(\hat{y}_{j_1}, \hat{x}_{i_2}) & \mathcal{I}^+(i\hat{m}_u, i\hat{m}_d) \mathcal{I}^0(\hat{y}_{j_1}, \hat{y}_{j_2}) \\ -\mathcal{I}^+(\hat{y}_{j_1}, i\hat{m}_u) \mathcal{I}^+(\hat{x}_{i_2}, i\hat{m}_d) & -\mathcal{I}^+(\hat{y}_{j_1}, i\hat{m}_u) \mathcal{I}^0(\hat{y}_{j_2}, i\hat{m}_d) \end{bmatrix}.$$

3.3 PARTIAL QUENCHING

Here we consider a variant of the general N_f -theory of practical importance. We introduce N_f fermions with Dirac operator D_1 and set $\mu_1 = 0$ then seek the spectral correlation functions between eigenvalues of D_1 and D_2 for $\mu_2 \neq 0$. In lattice gauge theory simulations this is a simple modification. The ensemble averages are then taken with respect to a theory which is not coupled to a chemical potential, but the observables (expressed in terms of “valence quarks”) are. In the Random Matrix Theory formulation it can also be introduced straightforwardly.

The corresponding Random Matrix Theory is based on the partition function eq. (2.5) with $N_1 = N_f$ and $N_2 = 0$ and we ask for the mixed correlation function $R_{(1,1)}^{(N_f)}$ between two valence quarks associated with matrices $\mathcal{D}_{1,2}$. For simplicity we here restrict ourselves to the physically most interesting case of $N_f = 2$. Here we have *two* sea quarks with masses m_u and m_d and zero chemical potential $\mu_1 = 0$, while the valence quarks have $\mu_1 = 0$ and $\mu_2 = \delta$.

In the previous subsection the results for the most general $N_f = N_1 + N_2$ flavors content have been given already for the polynomials in eq. (3.23) and the kernel in eq. (3.26) from which we can also compute the transforms (2.19), (2.21), (2.22) and (2.23). We just have to specify our case $N_1 = 2$, $N_2 = 0$ when applying eq. (C.5). We will first focus on $\rho_{(1,1)}^{(2)conn}(\hat{x}, \hat{y})$ in order to pinpoint the difference between the 1 + 1 flavor case eq. (3.36) from the previous subsection, which was also computed in an effective field theory approach [2]. Next, we give the general (n, k) -correlation function for this particular flavor combination.

The first building block for that from eq. (3.26) is given by

$$K_{N+1}^{(2)}(y, x) = \frac{h_N}{h_N^{(N_f)}(x^2 + m_u^2)(x^2 + m_d^2)} \frac{\det \begin{bmatrix} K_{N+1}(y, im_u) & P_{N+1}(-m_u^2) & P_{N+2}(-m_u^2) \\ K_{N+1}(y, im_d) & P_{N+1}(-m_d^2) & P_{N+2}(-m_d^2) \\ K_{N+1}(y, x) & P_{N+1}(x^2) & P_{N+2}(x^2) \end{bmatrix}}{\det \begin{bmatrix} P_N(-m_u^2) & P_{N+1}(-m_u^2) \\ P_N(-m_d^2) & P_{N+1}(-m_d^2) \end{bmatrix}}. \quad (3.43)$$

We observe a different structure of determinants compared to the corresponding previous result eq. (3.31), marking the difference of partial quenching. The microscopic large- N limit is now easily taken using the formulas from the previous subsection. The only difference is that we have to Taylor expand the last columns in numerator and denominator as they become degenerate at large- N . The expansion

in $1/N$ maps into a Taylor expansion with respect to the arguments and we obtain

$$\lim_{N \rightarrow \infty} K_{N+1}^{(2)}(y, x) = \frac{(2N)^{2\nu+8} (\hat{x}\hat{y})^{-\nu} \hat{\delta}^{-2}}{(\hat{x}^2 + \hat{m}_u^2)(\hat{x}^2 + \hat{m}_d^2)} \frac{\det \begin{bmatrix} \mathcal{I}^+(\hat{y}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ \mathcal{I}^+(\hat{y}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \\ \mathcal{I}^+(\hat{y}, \hat{x}) & J_\nu(\hat{x}) & \hat{x} J_{\nu+1}(\hat{x}) \end{bmatrix}}{\det \begin{bmatrix} J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \end{bmatrix}}. \quad (3.44)$$

The result for the second building block of our correlation function is obtained by integrating eq. (3.43) using the definition eq. (2.23)

$$M_{N+1}^{(2)}(x, y) = \frac{h_N}{h_N^{(N_f)}} (x^2 + m_u^2)(x^2 + m_d^2) \frac{\det \begin{bmatrix} H_{N+1}(x, im_u) & P_{N+1}(-m_u^2) & P_{N+2}(-m_u^2) \\ H_{N+1}(x, im_d) & P_{N+1}(-m_d^2) & P_{N+2}(-m_d^2) \\ M_{N+1}(x, y) & \chi_{N+1}(y) & \chi_{N+2}(y) \end{bmatrix}}{\det \begin{bmatrix} P_N(-m_u^2) & P_{N+1}(-m_u^2) \\ P_N(-m_d^2) & P_{N+1}(-m_d^2) \end{bmatrix}}. \quad (3.45)$$

In addition to the asymptotics already computed we also now need one of the transforms χ_N . Being proportional to the polynomials, eq. (B.4), this is an easy task, and we obtain an explicit exponential $\hat{\delta}$ dependence from the prefactors. We thus arrive at

$$\lim_{N \rightarrow \infty} M_{N+1}^{(2)}(x, y) = \frac{(\hat{x}\hat{y})^{\nu+1} (\hat{x}^2 + \hat{m}_u^2)(\hat{x}^2 + \hat{m}_d^2)}{(2N)^{2\nu+6} \hat{\delta}^2} \frac{\det \begin{bmatrix} \mathcal{I}^0(\hat{x}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ \mathcal{I}^0(\hat{x}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \\ \mathcal{I}^-(\hat{x}, \hat{y}) & e^{-\hat{\delta}^2/2} J_\nu(\hat{y}) & e^{-\hat{\delta}^2/2} \hat{y} J_{\nu+1}(\hat{y}) \end{bmatrix}}{\det \begin{bmatrix} J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \end{bmatrix}}. \quad (3.46)$$

Together with the asymptotic of the weight function,

$$\lim_{N \rightarrow \infty} w^{(2)}(x, y) = \frac{(\hat{x}^2 + \hat{m}_u^2)(\hat{x}^2 + \hat{m}_d^2)}{(2N)^4} \frac{(\hat{x}\hat{y})^{\nu+1}}{(2N)^{2\nu+2}} I_\nu \left(\frac{\hat{x}\hat{y}}{\hat{\delta}^2} \right) e^{-\frac{\hat{x}^2 + \hat{y}^2}{2\hat{\delta}^2}} \quad (3.47)$$

we obtain the partially quenched two point function

$$\rho_{(1,1)}^{(2)conn}(\hat{x}, \hat{y}) = \hat{x}\hat{y} \frac{\left| \begin{array}{ccc|ccc} \mathcal{I}^+(\hat{y}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) & -\mathcal{I}^0(\hat{x}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ \mathcal{I}^+(\hat{y}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) & -\mathcal{I}^0(\hat{x}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \\ \mathcal{I}^+(\hat{y}, \hat{x}) & J_\nu(\hat{x}) & \hat{x} J_{\nu+1}(\hat{x}) & \tilde{\mathcal{I}}^-(\hat{x}, \hat{y}) & e^{-\hat{\delta}^2/2} J_\nu(\hat{y}) & e^{-\hat{\delta}^2/2} \hat{y} J_{\nu+1}(\hat{y}) \end{array} \right|}{\left| \begin{array}{cc} J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \end{array} \right|^2}. \quad (3.48)$$

We note that this result is of the structure expected from the replica limit of the Toda lattice equation [21].

It is ideally suited for extracting F_π from lattice simulations. To visualize the effect of partially quenching we compare the partially quenched two point function to that for $N_f = 2$ in figure 2. When the two masses are identical $m_u = m_d = m$ the expression for the partially quenched two point

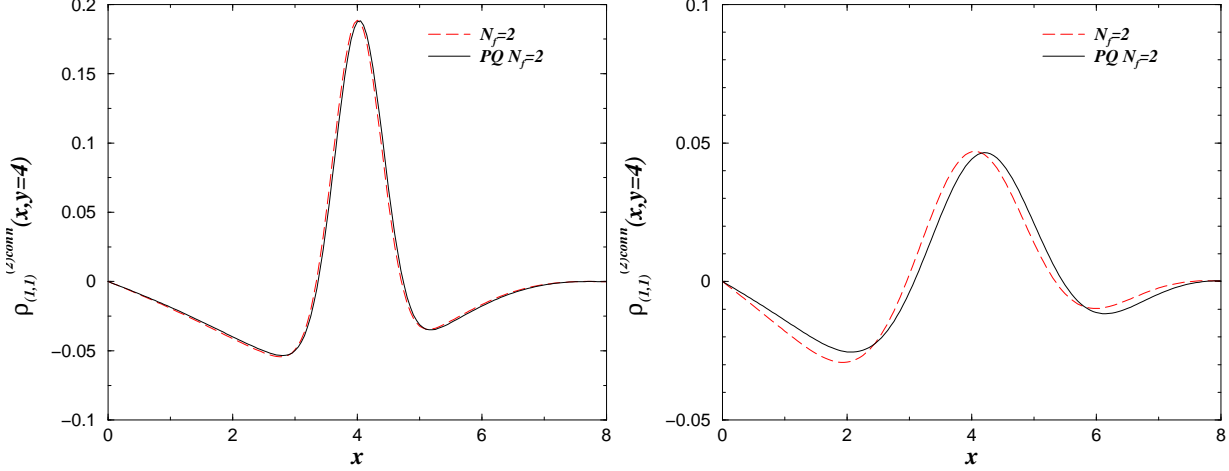


Figure 2: The partially quenched two point function (3.48) compared to the unquenched two point function (3.36). Here plotted as a function of the D_1 eigenvalue keeping the D_2 eigenvalue fixed at $\hat{y} = 4$ and setting $\hat{m}_u = 3$ and $\hat{m}_d = 5$. The left hand figure is for $\hat{\delta} = 0.4$ and the right hand figure is for $\hat{\delta} = 1.0$. Notice the change of scale on the vertical axis.

function becomes

$$\rho_{(1,1)}^{(2) \text{ conn}}(\hat{x}, \hat{y}) = \hat{x}\hat{y} \frac{\begin{vmatrix} \mathcal{I}^+(\hat{y}, i\hat{m}) & J_\nu(i\hat{m}) & i\hat{m}J_{\nu+1}(i\hat{m}) \\ \mathcal{I}^+(\hat{y}, i\hat{m})' & J_\nu(i\hat{m})' & (i\hat{m}J_{\nu+1}(i\hat{m}))' \\ \mathcal{I}^+(\hat{y}, \hat{x}) & J_\nu(\hat{x}) & \hat{x}J_{\nu+1}(\hat{x}) \end{vmatrix} \begin{vmatrix} -\mathcal{I}^0(\hat{x}, i\hat{m}) & J_\nu(i\hat{m}) & i\hat{m}J_{\nu+1}(i\hat{m}) \\ -\mathcal{I}^0(\hat{x}, i\hat{m})' & J_\nu(i\hat{m})' & (i\hat{m}J_{\nu+1}(i\hat{m}))' \\ \tilde{\mathcal{I}}^-(\hat{x}, \hat{y}) & e^{-\hat{\delta}^2/2}J_\nu(\hat{y}) & e^{-\hat{\delta}^2/2}\hat{y}J_{\nu+1}(\hat{y}) \end{vmatrix}}{\begin{vmatrix} J_\nu(i\hat{m}) & i\hat{m}J_{\nu+1}(i\hat{m}) \\ J_{\nu-1}(i\hat{m}) & i\hat{m}J_\nu(i\hat{m}) \end{vmatrix}^2}, \quad (3.49)$$

where the prime indicates the derivative with respect to $i\hat{m}$.

The two remaining building blocks are given as follows. From the definition (2.21) we have from eq. (3.43)

$$H_{N+1}^{(2)}(x_1, x_2) = \frac{h_{N+1}(x_1^2 + m_u^2)(x_1^2 + m_d^2)}{h_N^{(N_f)}(x_2^2 + m_u^2)(x_2^2 + m_d^2)} \frac{\det \begin{bmatrix} H_{N+1}(x_1, im_u) & P_{N+1}(-m_u^2) & P_{N+2}(-m_u^2) \\ H_{N+1}(x_1, im_d) & P_{N+1}(-m_d^2) & P_{N+2}(-m_d^2) \\ H_{N+1}(x_1, x_2) & P_{N+1}(x_2^2) & P_{N+2}(x_2^2) \end{bmatrix}}{\det \begin{bmatrix} P_N(-m_u^2) & P_{N+1}(-m_u^2) \\ P_N(-m_d^2) & P_{N+1}(-m_d^2) \end{bmatrix}}. \quad (3.50)$$

The microscopic limit is readily obtained:

$$\lim_{N \rightarrow \infty} H_{N+1}^{(2)}(x_1, x_2) = \frac{2N\hat{x}_1^{\nu+1}(\hat{x}_1^2 + \hat{m}_u^2)(\hat{x}_1^2 + \hat{m}_d^2)}{\hat{x}_2^\nu(\hat{x}_2^2 + \hat{m}_u^2)(\hat{x}_2^2 + \hat{m}_d^2)} \frac{\det \begin{bmatrix} \mathcal{I}^0(\hat{x}_1, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ \mathcal{I}^0(\hat{x}_1, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \\ \mathcal{I}^0(\hat{x}_1, \hat{x}_2) & J_\nu(\hat{x}_2) & \hat{x}_2 J_{\nu+1}(\hat{x}_2) \end{bmatrix}}{\det \begin{bmatrix} J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \end{bmatrix}}. \quad (3.51)$$

We note that it is $\hat{\mu}_f$ -independent. From eq. (3.43) and the definition (2.22) we have

$$\hat{H}_{N+1}^{(2)}(y_1, y_2) = \frac{h_{N+1}}{h_N^{(N_f)}} \frac{\det \begin{bmatrix} K_{N+1}(y_1, im_u) & P_{N+1}(-m_u^2) & P_{N+2}(-m_u^2) \\ K_{N+1}(y_1, im_d) & P_{N+1}(-m_d^2) & P_{N+2}(-m_d^2) \\ \hat{H}_{N+1}(y_1, y_2) & \chi_{N+1}(y_2^2) & \chi_{N+2}(y_2^2) \end{bmatrix}}{\det \begin{bmatrix} P_N(-m_u^2) & P_{N+1}(-m_u^2) \\ P_N(-m_d^2) & P_{N+1}(-m_d^2) \end{bmatrix}}, \quad (3.52)$$

leading to

$$\lim_{N \rightarrow \infty} \hat{H}_{N+1}^{(2)}(y_1, y_2) = 2N \frac{\hat{y}_2^{\nu+1}}{\hat{y}_1^\nu} \frac{\det \begin{bmatrix} \mathcal{I}^+(\hat{y}_1, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ \mathcal{I}^+(\hat{y}_1, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \\ \mathcal{I}^0(\hat{y}_1, \hat{y}_2) & e^{-\hat{\delta}^2/2} J_\nu(\hat{y}_2) & e^{-\hat{\delta}^2/2} \hat{y}_2 J_{\nu+1}(\hat{y}_2) \end{bmatrix}}{\det \begin{bmatrix} J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \end{bmatrix}}. \quad (3.53)$$

The two densities are given by

$$\rho_{(1,0)}^{(2)}(\hat{x}) = \lim_{N \rightarrow \infty} \frac{1}{2N} H_{N+1}^{(2)}(x, x) \quad (3.54)$$

$$\rho_{(0,1)}^{(2)}(\hat{y}) = \lim_{N \rightarrow \infty} \frac{1}{2N} \hat{H}_{N+1}^{(2)}(y, y). \quad (3.55)$$

They differ because of the asymmetric flavor content, the \hat{x} eigenvalues not coupling to $\hat{\mu}_2$. Eq. (3.54) agrees with the known result [9], as well as eq. (3.55) when setting $\hat{\mu}_2 = 0$.

We are now ready to state the most general correlation function for $N_1 = 2$ and $N_2 = 0$ flavors. Upon using the above results, taking out common factors of the determinant we obtain in terms of the abbreviations eq. (3.12)

$$\rho_{(n,k)}^{(2)}(\{\hat{x}\}_n, \{\hat{y}\}_k) = \prod_{i=1}^n \hat{x}_i \prod_{j=1}^k \hat{y}_j \det \begin{bmatrix} J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \end{bmatrix}^{-n-k} \\ \times \left| \begin{array}{ccc|ccc} \mathcal{I}^0(\hat{x}_{i_1}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) & \mathcal{I}^0(\hat{x}_{i_1}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ \mathcal{I}^0(\hat{x}_{i_1}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) & \mathcal{I}^0(\hat{x}_{i_1}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \\ \mathcal{I}^0(\hat{x}_{i_1}, \hat{x}_{i_2}) & J_\nu(\hat{x}_{i_2}) & \hat{x}_{i_2} J_{\nu+1}(\hat{x}_{i_2}) & -\tilde{\mathcal{I}}^-(\hat{x}_{i_1}, \hat{y}_{j_2}) & e^{-\hat{\delta}^2/2} J_\nu(\hat{y}_{j_2}) & e^{-\hat{\delta}^2/2} \hat{y}_{j_2} J_{\nu+1}(\hat{y}_{j_2}) \end{array} \right| \\ \left| \begin{array}{ccc|ccc} \mathcal{I}^+(\hat{y}_{j_1}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) & \mathcal{I}^+(\hat{y}_{j_1}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ \mathcal{I}^+(\hat{y}_{j_1}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) & \mathcal{I}^+(\hat{y}_{j_1}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \\ \mathcal{I}^+(\hat{y}_{j_1}, \hat{x}_{i_2}) & J_\nu(\hat{x}_{i_2}) & \hat{x}_{i_2} J_{\nu+1}(\hat{x}_{i_2}) & \mathcal{I}^0(\hat{y}_{j_1}, \hat{y}_{j_2}) & e^{-\hat{\delta}^2/2} J_\nu(\hat{y}_{j_2}) & e^{-\hat{\delta}^2/2} \hat{y}_{j_2} J_{\nu+1}(\hat{y}_{j_2}) \end{array} \right|. \quad (3.56)$$

As expected we verify that all correlation functions of eigenvalues \hat{x}_j are $\hat{\mu}$ -independent and agree with the results of [9].

4 The non-chiral Two-Matrix Problem

Having discussed the chiral Unitary Two-Matrix Theory in great detail it is a small step to go through the same analysis in the ordinary Unitary Two-Matrix Theory. As mentioned briefly in the introduction, this ensemble is relevant for QCD in three space-time dimensions, and may have applications in condensed matter physics as well [16].

It is straightforward to consider two arbitrary chemical potentials μ_1 and μ_2 , as in the chiral case treated above. However, for simplicity we restrict ourselves to just imaginary isospin chemical potential, *i.e.* the case where $\mu_1 = -\mu_2 \equiv \mu$. We present the most general case with an even number of flavors, $2N_f = 2N_+ + 2N_-$. The techniques used here can also be applied to the case of an odd number of flavors. The masses are pairwise equal in magnitude but of opposite sign [14, 15]

$$m_{+1}, \dots, m_{+N_+}, -m_{+1}, \dots, -m_{+N_+}, \quad \text{and} \quad m_{-1}, \dots, m_{-N_-}, -m_{-1}, \dots, -m_{-N_-}. \quad (4.1)$$

Both positive and negative mass fermions are coupled to imaginary isospin potential. In intermediate steps also an odd number of flavors will appear. The Dirac operator then remains anti-hermitian as in four dimensions, and the corresponding Random Two-Matrix Theory reads

$$\mathcal{Z}^{(2N_f)}(\{m_+\}, \{m_-\}) \sim \int d\Phi d\Psi \exp \left[-N \text{Tr} \left(\Phi^2 + \Psi^2 \right) \right] \prod_{f=1}^{2N_+} \det[\mathcal{D}_+(\mu) + m_{+f}] \prod_{g=1}^{2N_-} \det[\mathcal{D}_-(\mu) + m_{-g}] \quad (4.2)$$

where both Φ and Ψ now are $N \times N$ hermitian matrices, and

$$\mathcal{D}_\pm(\mu) \equiv i\Phi \pm i\mu\Psi \equiv i\Phi_\pm. \quad (4.3)$$

A related non-hermitian matrix model aimed at real baryon chemical potential was considered in ref. [32]. Here we again restrict ourselves to the Gaussian case for convenience. The same remarks we made about non-Gaussian potentials from the previous section also apply here. For the Riemann-Hilbert problem to be solved in order to prove universality in the quenched case we refer to [33].

We next perform the same change of variables as in eq. (2.3) to obtain

$$\mathcal{Z}^{(2N_f)}(\{m_+\}, \{m_-\}) \sim \int d\Phi_+ d\Phi_- \prod_{f=1}^{2N_+} \det[i\Phi_+ + m_{+f}] \prod_{g=1}^{2N_-} \det[i\Phi_- + m_{-g}] \exp \left[-\frac{N}{4} \text{Tr} V(\Phi_+, \Phi_-) \right] \quad (4.4)$$

with a potential

$$V(\Phi_+, \Phi_-) = \frac{\mu^2 + 1}{\mu^2} (\Phi_+^2 + \Phi_-^2) + 2\frac{\mu^2 - 1}{\mu^2} \Phi_+ \Phi_-. \quad (4.5)$$

Going to an eigenvalue representation can be done exactly as in refs. [25, 26]. Because of the pairing in positive and negative masses the integration measure takes the following form

$$\begin{aligned} \mathcal{Z}^{(2N_f)}(\{m_+\}, \{m_-\}) &= \int_{-\infty}^{\infty} \prod_{i=1}^N \left(dx_i dy_i \prod_{f=1}^{N_+} (x_i^2 + m_{+f}^2) \prod_{g=1}^{N_-} (y_i^2 + m_{-g}^2) \right) \\ &\times \Delta_N(\{x\}) \Delta_N(\{y\}) \det \left[\exp \left(N \frac{1 - \mu^2}{2\mu^2} x_i y_j \right) \right] e^{-N \frac{\mu^2 + 1}{4\mu^2} \sum_i (x_i^2 + y_i^2)}. \end{aligned} \quad (4.6)$$

Due to the presence of determinant factors in the measure this is a generalization of the ordinary (quenched) Gaussian Unitary Random Two-Matrix Model considered in the literature [25, 26, 13, 29]. We seek biorthogonal polynomials satisfying

$$\int_{-\infty}^{\infty} dx dy w^{(2N_f)}(x, y) P_n^{(2N_f)}(x) Q_k^{(2N_f)}(y) = h_n^{(2N_f)} \delta_{nk}, \quad (4.7)$$

with respect to the weight function

$$w^{(2N_f)}(x, y) \equiv \prod_{f=1}^{N_+} (x^2 + m_{+f}^2) \prod_{g=1}^{N_-} (y^2 + m_{-g}^2) e^{N \frac{1 - \mu^2}{2\mu^2} xy} e^{-N \frac{\mu^2 + 1}{4\mu^2} (x^2 + y^2)}. \quad (4.8)$$

The only difference to the previous section is that the I -Bessel function is replaced by an exponential and that we integrate over the full real line. Our polynomials are now polynomials in x and y , and not in their squares. Furthermore all factors $(x^2 + m_f^2)$ are counted as *two flavors* with masses $\pm im_f$, instead of one in the previous section.

Taking into account these differences all definitions of the transforms eq. (2.19) and the four kernels eqs. (2.20) - (2.23) carry through and we shall not repeat them here. The eigenvalue correlation functions defined as

$$\begin{aligned}
R_{(n,k)}^{(2N_f)}(\{x\}_n, \{y\}_k) &\equiv \frac{N!^2}{(N-n)!(N-k)!} \frac{1}{\mathcal{Z}^{(2N_f)}(\{m_+\}, \{m_-\})} \int_{-\infty}^{\infty} \prod_{j=n+1}^N dx_j \prod_{l=k+1}^N dy_l \det \left[e^{N \frac{1-\mu^2}{2\mu^2} x_i y_j} \right] \\
&\times \prod_{i=1}^N \left(\prod_{f=1}^{N_+} (x_i^2 + m_{+f}^2) \prod_{g=1}^{N_-} (y_i^2 + m_{-g}^2) \right) e^{-N \frac{\mu^2+1}{4\mu^2} \sum_i (x_i^2 + y_i^2)} \Delta_N(\{x\}) \Delta_N(\{y\}) .
\end{aligned} \tag{4.9}$$

are given by the same formula eq. (2.25) in terms of the corresponding non-chiral kernels.

4.1 THE QUENCHED CASE

The quenched case corresponds to omitting the explicit factors of determinants from the measure, while still performing the change of variables (4.3). We then get the standard Random Two-Matrix Model, with a coupling between the two matrices Φ_{\pm} as given by eq. (4.5). We will need these results later when expressing the unquenched results in terms of the quenched ones.

The biorthogonality condition (4.7) becomes

$$\int_{-\infty}^{\infty} dx dy e^{N \frac{1-\mu^2}{2\mu^2} xy} e^{-N \frac{\mu^2+1}{4\mu^2} (x^2+y^2)} P_n(x) P_m(y) = h_n \delta_{nm} . \tag{4.10}$$

Because of the weight being symmetric in x and y we have that $Q_n(y) = P_n(y)$. It was noted indirectly in refs. [34] and [28] that this condition is related to Hermite polynomials, although precise details were not given there. This can also be seen along the same lines as in appendix A. In our conventions we find for the monic polynomials

$$P_n(x) = 2^{-n} \left(\frac{\mu^2 + 1}{N} \right)^{\frac{n}{2}} H_n \left(\sqrt{\frac{N}{\mu^2 + 1}} x \right) . \tag{4.11}$$

For the normalization constants we obtain

$$h_n = \frac{\mu\pi}{2^{n-1} N^{n+1}} n! (1 - \mu^2)^n . \tag{4.12}$$

The factorization of the weight observed in Appendix B also applies here. The weight $w(x, y)$ is given by the sum over the biorthogonal polynomials

$$\begin{aligned}
w(x, y) &= \exp \left[-N \frac{\mu^2 + 1}{4\mu^2} (x^2 + y^2) + N \frac{1 - \mu^2}{2\mu^2} xy \right] \\
&= \frac{2\mu}{1 + \mu^2} e^{-\frac{N}{\mu^2+1} (x^2+y^2)} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1 - \mu^2}{2(\mu^2 + 1)} \right)^k H_k \left(\sqrt{\frac{N}{\mu^2 + 1}} x \right) H_k \left(\sqrt{\frac{N}{\mu^2 + 1}} y \right) ,
\end{aligned} \tag{4.13}$$

using the Mehler formula for Hermite polynomials. The individual factors are obviously $w_1(x) = w_2(x) = e^{-\frac{N}{\mu^2+1} x^2}$.

The functions $\chi_n(x)$ can be found explicitly by integrating the polynomials. Alternatively, we could also use the results from the previous section by setting $\nu = \pm\frac{1}{2}$ and using the relation (4.18) below. The I -Bessel function then becomes a hyperbolic sine or cosine and the two terms add up to an integral over the full real line. The result is

$$\chi_n(x) = \frac{\mu\sqrt{\pi}}{2^{n-1}N^{\frac{n+1}{2}}} \frac{(1-\mu^2)^n}{(\mu^2+1)^{\frac{n+1}{2}}} e^{-\frac{N}{\mu^2+1}x^2} H_n\left(\sqrt{\frac{N}{\mu^2+1}}x\right), \quad (4.14)$$

again being proportional to the polynomials. The three different quenched kernels thus read as follows:

$$K_N(y, x) = \sum_{k=0}^{N-1} \frac{N}{\pi\mu} \frac{(\mu^2+1)^k}{k! 2^{k+1} (1-\mu^2)^k} H_k\left(\sqrt{\frac{N}{\mu^2+1}}x\right) H_k\left(\sqrt{\frac{N}{\mu^2+1}}y\right) \quad (4.15)$$

$$\begin{aligned} H_N(x, y) &= \sum_{k=0}^{N-1} \frac{N^{\frac{1}{2}}}{\sqrt{\pi}} \frac{1}{k! 2^k (\mu^2+1)^{\frac{1}{2}}} H_k\left(\sqrt{\frac{N}{\mu^2+1}}x\right) H_k\left(\sqrt{\frac{N}{\mu^2+1}}y\right) e^{-\frac{N}{\mu^2+1}x^2} \\ &= \hat{H}_N(y, x) \end{aligned} \quad (4.16)$$

$$M_N(x, y) = \sum_{k=0}^{N-1} \frac{\mu}{k! 2^{k-1}} \frac{(1-\mu^2)^k}{(\mu^2+1)^{k+1}} H_k\left(\sqrt{\frac{N}{\mu^2+1}}x\right) H_k\left(\sqrt{\frac{N}{\mu^2+1}}y\right) e^{-\frac{N}{\mu^2+1}(x^2+y^2)}. \quad (4.17)$$

With this result we can now obtain all spectral correlation functions via eq. (2.25).

To take the scaling limit, we make use of the relations between Hermite and Laguerre polynomials,

$$\begin{aligned} H_{2n}(x) &= (-1)^n 2^{2n} n! L_n^{(-1/2)}(x^2) \\ H_{2n+1}(x) &= (-1)^n 2^{2n+1} n! x L_n^{(1/2)}(x^2), \end{aligned} \quad (4.18)$$

and the asymptotic behavior of Laguerre polynomials (3.6). This gives

$$\begin{aligned} H_{2n}(X) &\rightarrow (-1)^n 2^{2n} n! n^{-\frac{1}{2}} \frac{1}{\sqrt{\pi}} \cos(\sqrt{4n} X) \\ H_{2n+1}(X) &\rightarrow (-1)^n 2^{2n+1} n! \frac{1}{\sqrt{\pi}} \sin(\sqrt{4n} X). \end{aligned} \quad (4.19)$$

If we rescale $\sqrt{2N}x = \hat{x}$, $\sqrt{2N}y = \hat{y}$ leading to the customary level spacing $\frac{1}{\pi}$ we arrive at the following asymptotic for the kernels:

$$\lim_{N \rightarrow \infty} K_N\left(x = \frac{\hat{x}}{\sqrt{2N}}, y = \frac{\hat{y}}{\sqrt{2N}}\right) = \frac{N^2}{\sqrt{\pi}\hat{\mu}} \mathcal{I}^+(\hat{x}, \hat{y}) \quad (4.20)$$

$$\begin{aligned} \lim_{N \rightarrow \infty} H_N\left(x = \frac{\hat{x}}{\sqrt{2N}}, y = \frac{\hat{y}}{\sqrt{2N}}\right) &= \sqrt{2} N \mathcal{I}^0(\hat{x}, \hat{y}) \\ &= \lim_{N \rightarrow \infty} \hat{H}_N\left(y = \frac{\hat{y}}{\sqrt{2N}}, x = \frac{\hat{x}}{\sqrt{2N}}\right) \end{aligned} \quad (4.21)$$

$$\lim_{N \rightarrow \infty} M_N\left(x = \frac{\hat{x}}{\sqrt{2N}}, y = \frac{\hat{y}}{\sqrt{2N}}\right) = 2\hat{\mu}\sqrt{\pi} \mathcal{I}^-(\hat{x}, \hat{y}), \quad (4.22)$$

where we have introduced the following notation

$$\begin{aligned} \mathcal{I}^\pm(x, y) &\equiv \frac{1}{\pi} \int_0^1 ds e^{\pm\hat{\mu}^2 s^2} \cos(s(x-y)) \\ \mathcal{I}^0(x, y) &\equiv \frac{1}{\pi} \int_0^1 ds \cos(s(x-y)) = \frac{1}{\pi} \frac{\sin(x-y)}{(x-y)}. \end{aligned} \quad (4.23)$$

The arguments can be real or imaginary. In the asymptotic we have replaced the sums by integrals and used again eq. (3.8) with $t = k/N$. For convenience we have kept the scaling $2N\mu^2 = \hat{\mu}^2$ as in the previous section. The weight function has the following limit:

$$\lim_{N \rightarrow \infty} w \left(x = \frac{\hat{x}}{\sqrt{2N}}, y = \frac{\hat{y}}{\sqrt{2N}} \right) = \exp \left(-\frac{(\hat{x} - \hat{y})^2}{8\hat{\mu}^2} \right). \quad (4.24)$$

We will also use the following abbreviation in this section:

$$\tilde{\mathcal{I}}^-(x, y) \equiv \frac{1}{2\hat{\mu}\sqrt{\pi}} \exp \left(-\frac{(x-y)^2}{8\hat{\mu}^2} \right) - \frac{1}{\pi} \int_0^1 ds e^{\pm \hat{\mu}^2 s^2} \cos(s(x-y)). \quad (4.25)$$

Putting all together we obtain from the definition (3.15) with $(2N)^{-n-k}$ replaced by $(\sqrt{2N})^{-n-k}$

$$\rho_{(n,k)}(\{\hat{x}\}, \{\hat{y}\}) = \det \begin{bmatrix} \mathcal{I}^0(\hat{x}_{i_1}, \hat{x}_{i_2}) & -\tilde{\mathcal{I}}^-(\hat{x}_{i_1}, \hat{y}_{j_2}) \\ \mathcal{I}^+(\hat{y}_{j_1}, \hat{x}_{i_2}) & \mathcal{I}^0(\hat{y}_{j_1}, \hat{y}_{j_2}) \end{bmatrix}, \quad (4.26)$$

after taking out common factors from the determinant. As a check we obtain that the following densities are constant and $\hat{\mu}$ -independent:

$$\rho_{(1,0)}(\hat{x}) = \rho_{(0,1)}(\hat{x}) = \frac{1}{\pi}, \quad (4.27)$$

and we also recover the known correlations of the one-matrix model in terms of the sine-kernel for $\rho_{(n,0)}(\hat{x})$ and $\rho_{(0,k)}(\hat{y})$ respectively.

4.2 TWO AND MORE FLAVORS

In this subsection we outline how the insertion of flavors can be reexpressed in terms of quenched quantities as in the last section, and we will also give some examples.

The biorthogonal polynomials enjoy again a determinantal form as in eq. (3.21)

$$\begin{aligned} P_k^{(2N_f)}(x) &= (-i)^k \langle \det[\mathcal{D}_+(\mu) - ix] \rangle_{\mathcal{Z}^{(2N_f)}} \\ &= \frac{(-)^k}{\mathcal{Z}^{(2N_f)}} \int_{-\infty}^{\infty} \prod_{j=1}^k \left(dx_j dy_j (x_j - x) \prod_f^{N_+} (x_j^2 + m_{+f}^2) \prod_g^{N_-} (y_j^2 + m_{-g}^2) \right) \\ &\quad \times \Delta_k(\{x\}) \Delta_k(\{y\}) \det \left[\exp \left(N \frac{1 - \mu^2}{2\mu^2} x_i y_j \right) \right] e^{-N \frac{\mu^2 + 1}{4\mu^2} \sum_i^k (x_i^2 + y_i^2)}, \end{aligned} \quad (4.28)$$

and similarly for

$$Q_k^{(2N_f)}(y) = (-i)^k \langle \det \mathcal{D}_-(\mu) - iy \rangle_{\mathcal{Z}^{(2N_f)}}. \quad (4.29)$$

The kernel of both polynomials is given by

$$\begin{aligned} K_{N+1}^{(2N_f)}(y, x) &= \frac{(-)^k}{h_N^{(2N_f)}} \langle \det[\mathcal{D}_+(\mu) - ix] \det[\mathcal{D}_-(\mu) - iy] \rangle_{\mathcal{Z}^{(2N_f)}} \\ &= \frac{1}{h_N^{(2N_f)} \mathcal{Z}^{(2N_f)}} \int_{-\infty}^{\infty} \prod_i^N \left(dx_i dy_i (x_i - x)(y_i - y) \prod_f^{N_+} (x_i^2 + m_{+f}^2) \prod_g^{N_-} (y_i^2 + m_{-g}^2) \right) \\ &\quad \times \Delta_N(\{x\}) \Delta_N(\{y\}) \det \left[\exp \left(N \frac{1 - \mu^2}{2\mu^2} x_i y_j \right) \right] e^{-N \frac{\mu^2 + 1}{4\mu^2} \sum_i^N (x_i^2 + y_i^2)}, \end{aligned} \quad (4.30)$$

with the norms given in eq. (3.25). We recall that when applying appendix C all different masses in eq. (4.1) are counted as individual flavors, positive and negative, leading to one column or row in eq. (C.5) each. The remaining kernels can be obtained from the definitions eqs. (2.21) - (2.23). The most general correlation function then follows from eq. (2.25).

All these quantities, the transformed polynomials and the remaining kernels can be in principle be expressed in terms of partition functions. They are given by eq. (C.5): eqs. (3.23), (3.26), and by inserting the kernel, (3.26), in the definitions (2.21) - (2.23), with the obvious modification of dropping the ν terms and writing polynomials with arguments of power one. However, there is a subtlety here as the partition function in eq. (4.28) contains an odd number of flavors. There are two such partition functions available [14, 15]

$$\mathcal{Z}_{\text{even}}^{(2N_f+1)} = \int_{U(2N_f+1)} dU \cosh[V\Sigma\text{Tr}(\mathcal{M}U\Gamma_5 U^\dagger)] , \quad (4.31)$$

$$\mathcal{Z}_{\text{odd}}^{(2N_f+1)} = \int_{U(2N_f+1)} dU \sinh[V\Sigma\text{Tr}(\mathcal{M}U\Gamma_5 U^\dagger)] , \quad (4.32)$$

with $\Gamma_5 = \text{diag}(\mathbf{1}_{N_f}, \mathbf{1}_{N_f+1})$, being even and odd in the masses, respectively. Consequently the even polynomials P_{2k} are given by eq. (4.31) whereas the odd ones P_{2k+1} are given by eq. (4.32). Similarly special care has to be taken when adding single unpaired flavors to each of the sectors N_+ and N_- as in the kernel eq. (4.30), and we refer to [15] for the relation between partition functions of chiral and non-chiral theories at $\mu = 0$. Although similar massive partition functions with chemical potential have been computed from complex matrix models [32] and effective field theory [21] we do not further pursue this route. Alternatively we can explicitly compute the large- N limit from our biorthogonal polynomials and kernels, as we will show in the following example.

Let us consider the simplest case with $2N_+ = 2$ flavors of masses $\pm m$, and no other flavors: $2N_- = 0$. The corresponding weight function reads

$$w^{(2)}(x, y) = (x^2 + m^2) e^{N\frac{1-\mu^2}{2\mu^2}xy} e^{-N\frac{\mu^2+1}{4\mu^2}(x^2+y^2)} . \quad (4.33)$$

The kernel is almost identical to the partially quenched one eq. (3.43):

$$K_{N+1}^{(2)}(y, x) = \frac{h_{N+1}}{h_N^{(N_f)}(x-im)(x+im)} \frac{\det \begin{bmatrix} K_{N+1}(y, im) & P_{N+1}(im) & P_{N+2}(im) \\ K_{N+1}(y, -im) & P_{N+1}(-im) & P_{N+2}(-im) \\ K_{N+1}(y, x) & P_{N+1}(x^2) & P_{N+2}(x^2) \end{bmatrix}}{\det \begin{bmatrix} P_N(im) & P_{N+1}(im) \\ P_N(-im) & P_{N+1}(-im) \end{bmatrix}} . \quad (4.34)$$

The large- N limit is easily obtained using the results of the previous subsection, rescaling the mass as the eigenvalues $\sqrt{2}Nm = \hat{m}$:

$$\lim_{N \rightarrow \infty} K_{N+1}^{(2)}(y, x) = \frac{2N^2}{(\hat{x}^2 + \hat{m}^2)} \frac{N^2}{\hat{\mu}\sqrt{\pi}} \frac{\det \begin{bmatrix} \mathcal{I}^+(\hat{y}, i\hat{m}) & \cosh(\hat{m}) & -i \sinh(\hat{m}) \\ \mathcal{I}^+(\hat{y}, -i\hat{m}) & \cosh(\hat{m}) & i \sinh(\hat{m}) \\ \mathcal{I}^0(\hat{y}, \hat{x}) & \cos(\hat{x}) & -\sin(\hat{x}) \end{bmatrix}}{\det \begin{bmatrix} i \sinh(\hat{m}) & \cosh(\hat{m}) \\ -i \sinh(\hat{m}) & \cosh(\hat{m}) \end{bmatrix}} . \quad (4.35)$$

Here we have chosen N to be even, noticing the difference between the asymptotic of P_N and P_{N+2} by a factors of (-1) . Odd N leads to the same result, as can be seen from interchanging the last two columns in numerator and denominator. Similarly the kernel can be seen to be real as it should. The

remaining kernels follow with the same modifications from eqs. (3.45), (3.50) and (3.52). Therefore we just give the results here:

$$\lim_{N \rightarrow \infty} M_{N+1}^{(2)}(x, y) = 2\hat{\mu}\sqrt{\pi} \frac{(\hat{x}^2 + \hat{m}^2)}{2N^2} \frac{\det \begin{bmatrix} \mathcal{I}^0(\hat{y}, i\hat{m}) & \cosh(\hat{m}) & -i \sinh(\hat{m}) \\ \mathcal{I}^0(\hat{y}, -i\hat{m}) & \cosh(\hat{m}) & i \sinh(\hat{m}) \\ \mathcal{I}^-(\hat{x}, \hat{y}) & e^{-\hat{\mu}^2} \cos(\hat{y}) & -e^{-\hat{\mu}^2} \sin(\hat{y}) \end{bmatrix}}{\det \begin{bmatrix} i \sinh(\hat{m}) & \cosh(\hat{m}) \\ -i \sinh(\hat{m}) & \cosh(\hat{m}) \end{bmatrix}}, \quad (4.36)$$

$$\lim_{N \rightarrow \infty} H_{N+1}^{(2)}(x_1, x_2) = \sqrt{2} N \frac{(\hat{x}_1^2 + \hat{m}^2)}{(\hat{x}_2^2 + \hat{m}^2)} \frac{\det \begin{bmatrix} \mathcal{I}^0(\hat{x}_1, i\hat{m}) & \cosh(\hat{m}) & -i \sinh(\hat{m}) \\ \mathcal{I}^0(\hat{x}_1, -i\hat{m}) & \cosh(\hat{m}) & i \sinh(\hat{m}) \\ \mathcal{I}^0(\hat{x}_1, \hat{x}_2) & \cos(\hat{x}_2) & -\sin(\hat{x}_2) \end{bmatrix}}{\det \begin{bmatrix} i \sinh(\hat{m}) & \cosh(\hat{m}) \\ -i \sinh(\hat{m}) & \cosh(\hat{m}) \end{bmatrix}}, \quad (4.37)$$

which is μ -independent, and

$$\lim_{N \rightarrow \infty} \hat{H}_{N+1}^{(2)}(y_1, y_2) = \sqrt{2} N \frac{\det \begin{bmatrix} \mathcal{I}^+(\hat{y}_1, i\hat{m}) & \cosh(\hat{m}) & -i \sinh(\hat{m}) \\ \mathcal{I}^+(\hat{y}_1, -i\hat{m}) & \cosh(\hat{m}) & i \sinh(\hat{m}) \\ \mathcal{I}^0(\hat{y}_1, \hat{y}_2) & e^{-\hat{\mu}^2} \cos(\hat{y}_2) & -e^{-\hat{\mu}^2} \sin(\hat{y}_2) \end{bmatrix}}{\det \begin{bmatrix} i \sinh(\hat{m}) & \cosh(\hat{m}) \\ -i \sinh(\hat{m}) & \cosh(\hat{m}) \end{bmatrix}}. \quad (4.38)$$

We can write the general correlation function as follows:

$$\rho_{(n,k)}^{(2)}(\{\hat{x}\}_n, \{\hat{y}\}_k) = \det \begin{bmatrix} i \sinh(\hat{m}) & \cosh(\hat{m}) \\ -i \sinh(\hat{m}) & \cosh(\hat{m}) \end{bmatrix}^{-n-k} \times \left| \begin{array}{ccc|ccc} \mathcal{I}^0(\hat{x}_{i_1}, i\hat{m}) & \cosh(\hat{m}) & -i \sinh(\hat{m}) & \mathcal{I}^0(\hat{x}_{i_1}, i\hat{m}) & \cosh(\hat{m}) & -i \sinh(\hat{m}) \\ \mathcal{I}^0(\hat{x}_{i_1}, -i\hat{m}) & \cosh(\hat{m}) & i \sinh(\hat{m}) & \mathcal{I}^0(\hat{x}_{i_1}, -i\hat{m}) & \cosh(\hat{m}) & i \sinh(\hat{m}) \\ \mathcal{I}^0(\hat{x}_{i_1}, \hat{x}_{i_2}) & \cos(\hat{x}_{i_2}) & -\sin(\hat{x}_{i_2}) & -\tilde{\mathcal{I}}^-(\hat{x}_{i_1}, \hat{y}_{j_2}) & e^{-\hat{\mu}^2} \cos(\hat{y}_{j_2}) & -e^{-\hat{\mu}^2} \sin(\hat{y}_{j_2}) \\ \mathcal{I}^+(\hat{y}_{j_1}, i\hat{m}) & \cosh(\hat{m}) & -i \sinh(\hat{m}) & \mathcal{I}^+(\hat{y}_{j_1}, i\hat{m}) & \cosh(\hat{m}) & -i \sinh(\hat{m}) \\ \mathcal{I}^+(\hat{y}_{j_1}, -i\hat{m}) & \cosh(\hat{m}) & i \sinh(\hat{m}) & \mathcal{I}^+(\hat{y}_{j_1}, -i\hat{m}) & \cosh(\hat{m}) & i \sinh(\hat{m}) \\ \mathcal{I}^+(\hat{y}_{j_1}, \hat{x}_{i_2}) & \cos(\hat{x}_{i_2}) & -\sin(\hat{x}_{i_2}) & \mathcal{I}^0(\hat{y}_{j_1}, \hat{y}_{j_2}) & e^{-\hat{\mu}^2} \cos(\hat{y}_{j_2}) & -e^{-\hat{\mu}^2} \sin(\hat{y}_{j_2}) \end{array} \right| \quad (4.39)$$

From the density

$$\rho_{(1,0)}^{(2)}(\hat{x}) = \lim_{N \rightarrow \infty} \frac{1}{\sqrt{2}N} H_{N+1}^{(2)}(x, x) \quad (4.40)$$

and from $\rho_{(0,1)}^{(2)}(\hat{y})$ in the limit $\hat{\mu} \rightarrow 0$ we recover the known massive density from the second of ref. [14]. Similarly all higher correlators at $\hat{\mu} = 0$ can be recovered.

Our second example is for $N_+ = N_- = 1$ flavors with masses m_+ and m_- . The structure is very similar to the example in subsection 3.2, but with twice the flavor content: $2N_f = 2 + 2$ here. The four kernels at finite- N are similar to those given by eqs. (3.31), (3.32), (3.37) and (3.39), with 2×2 and 3×3 determinants in denominator and numerator here. For example the first kernel reads

$$\begin{aligned}
K_{N+1}^{(2+2)}(y, x) &= \frac{h_{N+1}}{h_N^{(N_f)}(x^2 + m_+^2)(y^2 + m_-^2)} \frac{1}{\det \begin{bmatrix} K_{N+1}(im_-, im_+) & K_{N+1}(im_-, -im_+) \\ K_{N+1}(-im_-, im_+) & K_{N+1}(-im_-, -im_+) \end{bmatrix}} \\
&\times \det \begin{bmatrix} K_{N+1}(im_-, im_+) & K_{N+1}(im_-, -im_+) & K_{N+1}(im_-, x) \\ K_{N+1}(-im_-, im_+) & K_{N+1}(-im_-, -im_+) & K_{N+1}(-im_-, x) \\ K_{N+1}(y, im_+) & K_{N+1}(y, -im_+) & K_{N+1}(y, x) \end{bmatrix}, \quad (4.41)
\end{aligned}$$

with an asymptotic limit

$$\begin{aligned}
\lim_{N \rightarrow \infty} K_{N+1}^{(2+2)}(y, x) &= \frac{2N^2}{(\hat{x}^2 + \hat{m}_+^2)(\hat{y}^2 + \hat{m}_-^2)} \frac{N^2}{\sqrt{\pi} \hat{\mu}} \frac{1}{\det \begin{bmatrix} \mathcal{I}^+(i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(i\hat{m}_-, -i\hat{m}_+) \\ \mathcal{I}^+(-i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(-i\hat{m}_-, -i\hat{m}_+) \end{bmatrix}} \\
&\times \det \begin{bmatrix} \mathcal{I}^+(i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(i\hat{m}_-, -i\hat{m}_+) & \mathcal{I}^+(i\hat{m}_-, \hat{x}) \\ \mathcal{I}^+(-i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(-i\hat{m}_-, -i\hat{m}_+) & \mathcal{I}^+(-i\hat{m}_-, \hat{x}) \\ \mathcal{I}^+(\hat{y}, i\hat{m}_+) & \mathcal{I}^+(\hat{y}, -i\hat{m}_+) & \mathcal{I}^+(\hat{y}, \hat{x}) \end{bmatrix}. \quad (4.42)
\end{aligned}$$

The remaining kernels can be obtained similarly, and we only give their asymptotic expressions:

$$\begin{aligned}
\lim_{N \rightarrow \infty} M_{N+1}^{(2+2)}(x, y) &= \frac{(\hat{x}^2 + \hat{m}_+^2)(\hat{y}^2 + \hat{m}_-^2)}{2N^2} 2\sqrt{\pi} \hat{\mu} \frac{1}{\det \begin{bmatrix} \mathcal{I}^+(i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(i\hat{m}_-, -i\hat{m}_+) \\ \mathcal{I}^+(-i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(-i\hat{m}_-, -i\hat{m}_+) \end{bmatrix}} \\
&\times \det \begin{bmatrix} \mathcal{I}^+(i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(i\hat{m}_-, -i\hat{m}_+) & \mathcal{I}^0(i\hat{m}_-, \hat{y}) \\ \mathcal{I}^+(-i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(-i\hat{m}_-, -i\hat{m}_+) & \mathcal{I}^0(-i\hat{m}_-, \hat{y}) \\ \mathcal{I}^0(\hat{x}, i\hat{m}_+) & \mathcal{I}^0(\hat{x}, -i\hat{m}_+) & \mathcal{I}^-(\hat{x}, \hat{y}) \end{bmatrix}, \quad (4.43)
\end{aligned}$$

$$\begin{aligned}
\lim_{N \rightarrow \infty} H_{N+1}^{(2+2)}(x_1, x_2) &= \frac{(\hat{x}_1^2 + \hat{m}_+^2)}{(\hat{x}_2^2 + \hat{m}_-^2)} \sqrt{2} N \frac{1}{\det \begin{bmatrix} \mathcal{I}^+(i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(i\hat{m}_-, -i\hat{m}_+) \\ \mathcal{I}^+(-i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(-i\hat{m}_-, -i\hat{m}_+) \end{bmatrix}} \\
&\times \det \begin{bmatrix} \mathcal{I}^+(i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(i\hat{m}_-, -i\hat{m}_+) & \mathcal{I}^+(i\hat{m}_-, \hat{x}_2) \\ \mathcal{I}^+(-i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(-i\hat{m}_-, -i\hat{m}_+) & \mathcal{I}^+(-i\hat{m}_-, \hat{x}_2) \\ \mathcal{I}^0(\hat{x}_1, i\hat{m}_+) & \mathcal{I}^0(\hat{x}_1, -i\hat{m}_+) & \mathcal{I}^0(\hat{x}_1, \hat{x}_2) \end{bmatrix}, \quad (4.44)
\end{aligned}$$

$$\begin{aligned}
\lim_{N \rightarrow \infty} \hat{H}_{N+1}^{(2+2)}(y_1, y_2) &= \frac{(\hat{y}_2^2 + \hat{m}_-^2)}{(\hat{y}_1^2 + \hat{m}_+^2)} \sqrt{2} N \frac{1}{\det \begin{bmatrix} \mathcal{I}^+(i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(i\hat{m}_-, -i\hat{m}_+) \\ \mathcal{I}^+(-i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(-i\hat{m}_-, -i\hat{m}_+) \end{bmatrix}} \\
&\times \det \begin{bmatrix} \mathcal{I}^+(i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(i\hat{m}_-, -i\hat{m}_+) & \mathcal{I}^0(i\hat{m}_-, \hat{y}_2) \\ \mathcal{I}^+(-i\hat{m}_-, i\hat{m}_+) & \mathcal{I}^+(-i\hat{m}_-, -i\hat{m}_+) & \mathcal{I}^0(-i\hat{m}_-, \hat{y}_2) \\ \mathcal{I}^+(\hat{y}_1, i\hat{m}_+) & \mathcal{I}^+(\hat{y}_1, -i\hat{m}_+) & \mathcal{I}^0(\hat{y}_1, \hat{y}_2) \end{bmatrix}. \quad (4.45)
\end{aligned}$$

The general correlation function $\rho_{(n,k)}^{(2+2)}(\{\hat{x}\}_n, \{\hat{y}\}_k)$ can be easily obtained from the above. As a check we have again compared the density

$$\rho_{(1,0)}^{(2+2)}(\hat{x}) = \lim_{N \rightarrow \infty} \frac{1}{\sqrt{2}N} H_{N+1}^{(2+2)}(x, x) \quad (4.46)$$

in the limit $\hat{\mu} \rightarrow 0$ to the known result [14], finding agreement for nondegenerate \hat{m}_+ and \hat{m}_- .

5 Summary

We have introduced a chiral Two-Matrix Gaussian Unitary Ensemble in Random Matrix Theory, and solved it explicitly both at finite matrix size N and in the $N \rightarrow \infty$ scaling limit. The scaling theory describes the Dirac eigenvalue correlations of QCD in the ϵ -regime when subjected to imaginary isospin chemical potential. In QCD language the first non-trivial mixed correlation function has a strong dependence on the pion decay constant F_π . This makes it possible to use such correlation functions to extract F_π from lattice gauge theory simulations [1, 2]. Previously two special cases had been considered from the viewpoint of effective field theory. Here using the Random Matrix Theory formulation we have computed the general correlation functions. We have confirmed that in the two special cases ($N_f = 0, 2$) our general expressions from the present Random Matrix Theory calculation agree with the computations from effective field theory [1, 2].

A particularly useful by-product of the present work is the “partially quenched” case, where the Dirac operators that define the theory do not couple to chemical potentials. One can still measure the eigenvalue correlations of the Dirac operators D_1 and D_2 on such ensembles, and they will again show a very non-trivial dependence on the pion decay constant F_π . From the point of view of a practical implementation in lattice gauge theory simulations this is the ideal set-up. The corresponding calculation in the effective field theory seems technically very difficult without some new short-cut or trick. Here the Random Matrix Theory provides a novel result that is ready to be used to determine the pion decay constant F_π from lattice simulations.

We have also explicitly found the finite- N solution for the ordinary Two-Matrix Unitary Ensemble including mass terms. This extends the existing literature. Our results for the corresponding scaling limit are also new. These results are of relevance for QCD in 3 space-time dimensions. We have also shown directly how to recover the ordinary One-Matrix results from the Two-Matrix theory.

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A Biorthogonal Laguerre polynomials

We wish to show here that for $N_f = 0$ the integral representation eq. (3.21) immediately leads to the fact that the quenched biorthogonal polynomials are given by Laguerre polynomials as stated in eq. (3.1). To that aim we will integrate out one of the matrices being Gaussian to map the polynomial to the standard matrix representation of the orthogonal Laguerre polynomials in the Gaussian chiral *one*-matrix theory. There the following fact is known:

$$\begin{aligned}
 p_k(x^2) &= (-)^k (ix)^{-\nu} \langle \det(\mathcal{D} + ix) \rangle \sim \int d\Phi \det \begin{bmatrix} ix & i\Phi \\ i\Phi^\dagger & ix \end{bmatrix} \exp[-N \text{Tr} \Phi^\dagger \Phi] \quad (\text{A.1}) \\
 &\sim \int_0^\infty \prod_{j=1}^k (dx_j^2 x_j^{2\nu} (x^2 - x_j^2)) \exp \left[-N \sum_{i=1}^k x_i^2 \right] \Delta_k(\{x^2\})^2,
 \end{aligned}$$

leading to

$$p_k(x^2) \sim L_k^{(\nu)}(Nx^2) \quad (\text{A.2})$$

as orthogonal polynomials with respect to the weight $w(x) = x^{2\nu+1} \exp[-Nx^2]$. Here we have omitted all normalisation factors. The matrix Φ is complex and of size $k \times (k + \nu)$.

This can be compared to the representation of our biorthogonal polynomials eq. (3.21):

$$P_k(x^2) = (-)^k (ix)^{-\nu} \frac{1}{\mathcal{Z}_\nu} \int d\Phi_1 d\Phi_2 \det(\mathcal{D}_1 + ix) \exp[-NV(\Phi_1, \Phi_2)] \quad (\text{A.3})$$

with the potential given in eq. (2.8). Since $P_k(x^2)$ is given by the expectation value of a determinant depending on only *one* of the matrices, Φ_1 , the matrix integration over Φ_2 decouples. Completing the square and integrating out Φ_2 we obtain an expectation value only with respect to Φ_1 , with potential

$$V(\Phi_1) = (c_1 - d^2/c_2) \text{Tr}\Phi_1^\dagger \Phi_1 = \frac{1}{1 + \mu_1^2} \text{Tr}\Phi_1^\dagger \Phi_1. \quad (\text{A.4})$$

Thus we have arrived exactly at eq. (A.1) with $N \rightarrow N/(1 + \mu_1^2)$. Consequently our biorthogonal polynomials are Laguerre polynomials as was claimed in eq. (3.1). The same argument immediately leads to $Q_k(y^2)$ being Laguerre polynomials as well, integrating out Φ_1 .

It is interesting to compare this to the orthogonality of the Laguerre polynomials in the complex plane which arises in the solution of the chiral two-matrix theory for a real chemical potential [10]. A proof of the orthogonality was given in [23] by direct computation. An alternative proof can be formulated since the orthogonal polynomials in the complex plane also enjoy a determinantal representation. Even though the determinant contains *both* matrices of the two-matrix theory one can still find a way to shift the variables of integration such that one of the matrices can be easily integrated out leaving a form that again leads to Laguerre polynomials [35]. However the expansion method (see Appendix B) can not be directly applied since the biorthogonal polynomials $p_k(z)$ and $p_k^*(z)$ are not functions of independent variables.

B Biorthogonal polynomials from the expansion of the weight

Consider two sets of polynomials, $p_k(x)$ and $q_k(y)$, orthogonal with respect to the two weights $w_1(x)$ and $w_2(y)$ respectively

$$\begin{aligned} \int dx w_1(x) p_k(x) p_\ell(x) &= f_k \delta_{k,\ell} \\ \int dy w_2(y) q_k(y) q_\ell(y) &= g_k \delta_{k,\ell}. \end{aligned} \quad (\text{B.1})$$

Then it follows automatically that $p_k(x)$ and $q_k(y)$ are *bi*-orthogonal with respect to the weight $w(x, y)$ defined as follows

$$w(x, y) = w_1(x) w_2(y) \sum_{k=0}^{\infty} \frac{h_k}{f_k g_k} p_k(x) q_k(y). \quad (\text{B.2})$$

The corresponding biorthogonality relation is

$$\int dx dy w(x, y) p_k(x) q_\ell(y) = h_k \delta_{k,\ell}. \quad (\text{B.3})$$

While not all weights can be written in the form (B.2) we will show below that the quenched weight (given in (2.16) with $N_f = 0$) in fact can.

Using these conventions one can easily obtain the transformed functions

$$\begin{aligned}\chi_k(y) &= \int dx w(x, y) p_k(x) = (h_k/g_k) w_2(y) q_k(y) \\ \hat{\chi}_k(x) &= \int dy w(x, y) q_k(y) = (h_k/f_k) w_1(x) p_k(x)\end{aligned}\tag{B.4}$$

and all the kernels

$$\begin{aligned}K_N(y, x) &= \sum_{k=0}^{N-1} \frac{1}{h_k} q_k(y) p_k(x) \\ H_N(x_1, x_2) &= w_1(x_1) \sum_{k=0}^{N-1} \frac{1}{f_k} p_k(x_1) p_k(x_2) \\ \hat{H}_N(y_1, y_2) &= w_2(y_2) \sum_{k=0}^{N-1} \frac{1}{g_k} q_k(y_1) q_k(y_2) \\ M_N(x, y) &= w_1(x) w_2(y) \sum_{k=0}^{N-1} \frac{h_k}{f_k g_k} p_k(x) q_k(y).\end{aligned}\tag{B.5}$$

Interestingly we see that the weight function has the same form as the M kernel just with different limits in the sum. We can then write one of the terms appearing in generalized ‘‘Dyson Theorem’’ (2.25) as (for $N_f = 0$)

$$w(x, y) - M_N(x, y) = w_1(x) w_2(y) \sum_{k=N}^{\infty} \frac{h_k}{f_k g_k} p_k(x) q_k(y).\tag{B.6}$$

In order to apply this to the current problem we make use of the known expansion of I_ν in terms of Laguerre polynomials giving (for $N_f = 0$)

$$\begin{aligned}w^{(0)}(x, y) &= (xy)^{\nu+1} e^{-N(c_1 x^2 + c_2 y^2)} I_\nu(2dNxy) \\ &= (Nd)^\nu \tau^{\nu+1} (xy)^{2\nu+1} e^{-N\tau(c_1 x^2 + c_2 y^2)} \sum_{n=0}^{\infty} \frac{n!(1-\tau)^n}{(n+\nu)!} L_n^{(\nu)}(N\tau c_1 x^2) L_n^{(\nu)}(N\tau c_2 y^2)\end{aligned}\tag{B.7}$$

with $\tau = 1 - d^2/(c_1 c_2)$. Defining the monic polynomials

$$\begin{aligned}p_k(x) &= k!(-N\tau c_1)^{-k} L_k^{(\nu)}(N\tau c_1 x^2) \\ q_k(y) &= k!(-N\tau c_2)^{-k} L_k^{(\nu)}(N\tau c_2 y^2)\end{aligned}\tag{B.8}$$

and the weights

$$\begin{aligned}w_1(x) &= x^{2\nu+1} e^{-N\tau c_1 x^2} \\ w_2(y) &= y^{2\nu+1} e^{-N\tau c_2 y^2}\end{aligned}\tag{B.9}$$

with support on the positive real line we get the normalisation of the polynomials

$$\begin{aligned}f_k &= \frac{k!(k+\nu)!}{2(N\tau c_1)^{2k+\nu+1}} \\ g_k &= \frac{k!(k+\nu)!}{2(N\tau c_2)^{2k+\nu+1}}.\end{aligned}\tag{B.10}$$

From eq. (B.7) we can then read off the norm of the biorthogonal polynomials

$$h_k = \frac{k!(k+\nu)!(Nd)^{2k+\nu}}{4(N^2\tau c_1 c_2)^{2k+\nu+1}}.\tag{B.11}$$

The same considerations also apply to the non chiral case in terms of hermite polynomials, cf. eq. (4.13).

C All partition functions

In this appendix we derive the most general partition function, containing a different number of N_1 flavors with masses m_{f1} and N_2 flavors with masses m_{f2} , where $N_f = N_1 + N_2$. This result is particularly useful as both the unquenched polynomials $P_k^{(N_f)}$ and $Q_k^{(N_f)}$ as well as their kernel can be expressed in terms of these partition functions, see eqs. (3.23) and (3.26).

Following the same steps as in section 2 the partition function has the following eigenvalue representation:

$$\begin{aligned} \mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\}) &= \mathcal{N}(\{m_{1,2}\}) \int_0^\infty \prod_{i=1}^N \left(dx_i dy_i (x_i y_i)^{\nu+1} \prod_{f=1}^{N_1} (x_i^2 + m_{f1}^2) \prod_{f=1}^{N_2} (y_i^2 + m_{f2}^2) \right) \\ &\times \Delta_N(\{x^2\}) \Delta_N(\{y^2\}) \det[I_\nu(2dN x_i y_j)] e^{-N \sum_i (c_1 x_i^2 + c_2 y_i^2)}. \end{aligned} \quad (\text{C.1})$$

Because all eigenvalues are integrated out the symmetry of the Vandermonde determinants can be used to bring all $N!$ terms in the sum over permutations from the determinant of the I -Bessel function into diagonal form⁴

$$\begin{aligned} \mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\}) &= \mathcal{N}(\{m_{1,2}\}) N! \int_0^\infty \prod_{i=1}^N \left(dx_i dy_i (x_i y_i)^{\nu+1} \prod_{f=1}^{N_1} (x_i^2 + m_{f1}^2) \prod_{f=1}^{N_2} (y_i^2 + m_{f2}^2) \right) \\ &\times \Delta_N(\{x^2\}) \Delta_N(\{y^2\}) \prod_{i=1}^N I_\nu(2dN x_i y_i) e^{-N \sum_i (c_1 x_i^2 + c_2 y_i^2)}. \end{aligned} \quad (\text{C.2})$$

Thus we can now interpret the I -Bessel function as being part of the weight. In this form we can apply a theorem that was proven in [20] for matrix theories with complex eigenvalues, expressing such partition functions (also called characteristic polynomials) in terms of the quenched polynomials. We can apply this result by formally identifying $z \rightarrow x$ and $z^* \rightarrow y$, thus relating our polynomials to the polynomials \mathcal{P} there as follows: $\mathcal{P}_k(z^2) \rightarrow P_k(x^2)$ and $\mathcal{P}_k(z^2)^* \rightarrow Q_k(y^2)$. Note that unlike in [20], where the two sets of polynomials are related by complex conjugation, our polynomials P and Q are no longer related by a symmetry in general. However, this property was not used in the proof of the theorem we wish to apply.

The quenched kernel made of the monic polynomials without the weight,

$$K_N(y, x) = \sum_{k=0}^{N-1} \frac{Q_k(y^2) P_k(x^2)}{h_k}, \quad (\text{C.3})$$

is obviously self-contractive, due to the biorthogonality:

$$\int_0^\infty dx dy w^{(0)}(x, y) K_N(y, u) K_N(v, x) = K_N(v, u). \quad (\text{C.4})$$

This makes Dyson's Theorem for integrating over determinants of such kernels applicable, which was used in the derivation in [20]. We therefore arrive at

$$\begin{aligned} \mathcal{Z}_\nu^{(N_f)}(\{m_1\}, \{m_2\}) &= \prod_f^{N_1} m_{f1}^\nu \prod_f^{N_2} m_{f2}^\nu \frac{N! \prod_{j=0}^{N+N_2-1} h_j}{\Delta_{N_1}(\{m_1^2\}) \Delta_{N_2}(\{m_2^2\})} \\ &\times \det \left[K_{N+N_2}(im_{l2}, im_{k1}) P_{N+N_2}((im_{k1})^2) \dots P_{N+N_1-1}((im_{k1})^2) \right] \end{aligned} \quad (\text{C.5})$$

⁴This is no longer true in correlation functions $R_{n,k}$ when less than N of each eigenvalues are integrated out.

where we have given the result for $N_1 \geq N_2$. The first N_2 columns are given by kernels and the last $N_1 - N_2$ columns by monic polynomials of increasing degree. The corresponding result for $N_2 > N_1$ then obviously follows by exchanging $N_1 \leftrightarrow N_2$ and $m_{f1} \leftrightarrow m_{f2}$ in the above formula.

The fact that imaginary arguments appear is related to the fact that mass terms are positive, $\prod_i(x_i^2 + m_1^2)$. Polynomials and kernels of real arguments would appear, if we had considered characteristic polynomials, such as $\prod_i(x_i^2 - u^2)$.

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