# A note on the Calculus of Variations

by

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A standard result from the Calculus of Variations is that a necessary condition for a local extremum for the functional

$$I[y] = \int_{\alpha}^{\beta} F(x,y,y')dx, \tag{1}$$

with the boundary conditions,

$$y(\alpha) = A, \quad y(\beta) = B, \tag{2}$$

is that Euler's equation

$$F_y - \frac{d}{dx}\left(F_{y'}\right) = 0, \tag{3}$$

be satisfied with the boundary conditions

$$y(\alpha) = A, \quad y(\beta) = B.$$

It is also well known that if F depends only on y and y' explicitly, then (3) implies that

$$F - y'F_{y'} = C, \quad C \text{ a constant.} \tag{4}$$

The result (4) follows from the fact that the differential equation (3) can be integrated once.

Further if one considers the next obvious generalization of (1):

Extremalize

$$I[y] = \int_{\alpha}^{B} F\left(x,y,y',\ldots.,y^{(m)}\right)dx$$

subject to $y^{(i)}(\alpha) = A_i$, $y^{(i)}(\beta) = B_i$ $i=1,2,\ldots,m$; then it is also well known that y must necessarily satisfy

$$F_y - \frac{d}{dx}\left(F_{y^{(1)}}\right) + \frac{d^2}{dx^2}\left(F_{y^{(2)}}\right) - \ldots. + (-)^m \frac{d^m}{dx^m}\left(F_{y^{(m)}}\right) = 0, \tag{5}$$

and

$$y^{(i)}(\alpha) = A_i, \qquad y^{(i)}(\beta) = B_i. \tag{6}$$

The expression (5) can be written more succinctly as

$$F_y + \sum_{i=1}^{m} (-)^i \frac{d^i}{dx^i} \left[F_{y(i)}\right] = 0 \qquad (7)$$

In a course of lecturs at Brunel the now standard results (3) and (4) were derived. The next obvious generalization (7) was then derived. An enquiring student asked if an equivalent result to (4) could be obtained for (7). I was unable to find any such result on consulting various standard works on the calculus of variations. Apparently it does not seem to be known that the equation (7) can indeed also be integrated once when x does not explicitly appear in F, to give the result that (7) implies

$$F - \sum_{n=1}^{m} y^{(n)} \left[\sum_{i=n}^{m} (-)^{i-n} \frac{d^{i-n}}{dx^{i-n}} \left[F_{y(i)}\right]\right] = C, \quad C \text{ const.} \qquad (8)$$

To show this we differentiate the expression (8) with respect to x giving

$$\frac{d}{dx}\left[F - \sum_{n=1}^{m} y^{(n)} \left[\sum_{i=n}^{m} (-)^{i-n} \frac{d^{i-n}}{dx^{i-n}} \left[F_{y(i)}\right]\right]\right] = 0,$$

$$y^{(1)}F_y + \sum_{n=1}^{m} y^{(n+1)}F_{y(n)} - \sum_{n=1}^{m} y^{(n+1)}\left[\sum_{i=n}^{m} (-)^{i-n} \frac{d^{i-n}}{dx^{i-n}}\left[F_{y(i)}\right]\right]$$

$$- \sum_{n=1}^{m} y^{(n)}\left[\sum_{i=n}^{m} (-)^{i-n} \frac{d^{i-n+1}}{dx^{i-n+1}} \left[F_{y(i)}\right]\right] = 0 ,$$

$$y^{(1)}F_y + \sum_{n=1}^{m} y^{(n+1)} \left[F_{y(n)} - \sum_{i=n}^{m} (-)^{i-n} \frac{d^{i-n}}{dx^{i-n}} \left[F_{y(i)}\right]\right]$$

$$- \sum_{n=1}^{m} y^{(n)}\left[\sum_{i=n}^{m} (-)^{i-n} \frac{d^{i-n+1}}{dx^{i-n+1}} \left[F_{y(i)}\right]\right] = 0 ,$$

$$y^{(1)}\left[F_y + \sum_{i=1}^{m} (-)^i \frac{d^i}{dx^i} \left[F_{y(i)}\right]\right]$$

$$+ \sum_{n=1}^{m} y^{(n+1)}\left[F_{y(n)} - \sum_{i=n}^{m} (-)^{i-n} \frac{d^{i-n}}{dx^{i-n}} \left[F_{y(i)}\right]\right] -$$

$$- \sum_{n=2}^{m} y^{(n)}\left[\sum_{i=n}^{m} (-)^{i-n} \frac{d^{i-n+1}}{dx^{i-n+1}} \left[F_{y(i)}\right]\right] = 0 ,$$

$$y^{(1)}\left[F_y + \sum_{i=1}^{m} (-)^i \frac{d^i}{dx^i} \left[F_{y(i)}\right]\right]$$

$$- \sum_{n=1}^{m-1} y^{(n+1)} \left[ \sum_{i=n}^{m} (-)^{i-n} \frac{d^{i-n}}{dx^{i-n}} \left( F_{y^{(i)}} \right) \right]$$

$$- \sum_{n=1}^{m-1} y^{(n+1)} \left[ \sum_{i=n+1}^{m} (-)^{i-n-1} \frac{d^{i-n}}{dx^{i-n}} \left( F_{y^{(i)}} \right) \right] = 0 \ .$$

Hence

$$y^{(1)} \left[ F_y + \sum_{i=1}^{m} (-)^{i} \frac{d^{i}}{dx^{i}} \left( F_{y^{(i)}} \right) \right] = 0. \qquad (9)$$

**Q.E.D.**

**Note.** In reducing the problem from (7) to (8), an extraneous solution is introduced. The equation (9) shows that (8) implies either (7) or $y'=0$. But in general the solution $y=const$ will not provide an extremal.