A note on Wiener Hopf matrix factorisation

by

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Abstract

In this paper the most general class of (2×2) - matrices is determined, which permit a Wiener-Hopf factorisation by the procedure of Rawlins and Williams [1]. According to this procedure, the factorisation problem is reduced to a matrix Hilbert problem on a half-line, where the matrix involved in the Hilbert problem is required to have zero diagonal elements.

Introduction

In the work of Rawlins and Williams [1] a Wiener-Hopf factorisation of the matrix

$$\underset{\sim}{A}(\alpha) = \begin{pmatrix} F(K) & G(K)F(K) \\ H(K) & -G(K)H(K) \end{pmatrix} , \qquad (1)$$

was carried out. In the expression (1) F, G, and H are analytic functions (except possibly at K = 0) of the variable $K = (k^2 - \alpha^2)^{\frac{1}{2}}$, where α is a complex variable and k a constant with positive real and imaginary parts. The branch of the square root is chosen such that K = k at $\alpha = 0$, with the branch cuts C and C' lying along the half-lines $\alpha = -k - \delta$, and $\alpha = k + \delta$, $\delta \geq 0$, respectively. It was shown in [1] that provided F, G and H do not have any zeros in the cut α-plane and G(K) = − G(−K) then the matrix (1) could be factorised in the form

$$\underset{\sim}{A}(\alpha) = \underset{\sim}{U}(\alpha)\underset{\sim}{L}^{-1}(\alpha) ,$$

where $\underset{\sim}{U}(\alpha)$ and $\underset{\sim}{L}(\alpha)$ are non-singular matrices whose elements are analytic for $\mathrm{Im}(\alpha) > -\mathrm{Im}(k)$, and $\mathrm{Im}(\alpha) < \mathrm{Im}(k)$, respectively.

The crux of the technique of factorisation depended on being able to assume $\underset{\sim}{U}(\alpha)$ to be analytic everywhere except along the branch cut C through $\alpha = -k$ whilst $\underset{\sim}{L}(\alpha)$ to be analytic everywhere except along the branch cut C' through $\alpha = k$, and then to show that

$$\underset{\sim}{A}_{+}(\alpha)\underset{\sim}{A}_{-}^{-1}(\alpha) = \begin{pmatrix} 0 & g(\alpha) \\ h(\alpha) & 0 \end{pmatrix} , \qquad (2)$$

where $g(\alpha)$, $h(\alpha)$ are specific functions, and where the suffices ± denote values evaluated on the upper side and lower side of the branch cut $C : \alpha = -k - \delta$, $\delta \geq 0$.

Professor J. Boersma in his referee report of [1], asked the question as to whether (1) is the most general matrix, with the same branch cuts, for which the matrix product $\underset{\sim}{A}_+(\alpha)\underset{\sim}{A}_-^{-1}(\alpha)$ takes the form (2). He conjectured that it would not be. In this note we confirm his conjecture, and give the most general form of the class of (2×2)-matrices which produce zeros in the diagonal for the Hilbert problem.

We shall show that the most general form is:

$$\underset{\sim}{A}(\alpha) = \begin{pmatrix} a_{11}(\alpha) & a_{11}(\alpha)\{F_1(\alpha) + (k^2-\alpha^2)^{-\frac{1}{2}}F_2(\alpha)\} \\ a_{21}(\alpha) & a_{21}(\alpha)\{F_1(\alpha) - (k^2-\alpha^2)^{-\frac{1}{2}}F_2(\alpha)\} \end{pmatrix}, \quad (3)$$

$$\equiv \begin{pmatrix} a_{11}(\alpha) & a_{11}(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}} \\ a_{21}(\alpha) & -a_{21}(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & F_1(\alpha) \\ 0 & F_2(\alpha) \end{pmatrix},$$

with $a_{11}(\alpha)a_{21}(\alpha)F_2(\alpha) \neq 0$ in the cut plane, where $a_{11}(\alpha), a_{21}(\alpha)$ are analytic functions in the cut plane, (with branch cuts C and C'), and $F_1(\alpha)$ and $F_2(\alpha)$ are analytic in the entire α-plane except possibly along the branch cut C'. If further $\underset{\sim}{A}(\alpha)=\underset{\sim}{A}(-\alpha)$ then $F_1(\alpha)=E_1(\alpha)$, $F_2(\alpha)=E_2(\alpha)$ where $E_1(\alpha)$ and $E_2(\alpha)$ are analytic in the entire α-plane.

Since the post multiplication of $\underset{\sim}{A}(\alpha)$ by an entire or L matrix, will not affect $\underset{\sim}{A}_+(\alpha)\underset{\sim}{A}^{-1}(\alpha)$, the basic general form could be taken to be:

$$\begin{pmatrix} a_{11}(\alpha) & a_{11}(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}} \\ a_{21}(\alpha) & -a_{21}(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}} \end{pmatrix}. \quad (3')$$

This matrix may be post multiplied by an arbitrary L matrix and/or pre multiplied by an arbitrary U matrix yielding a matrix that can also be factorised.

<u>Derivation of the general form (3)</u>

Consider the matrix

$$\underset{\sim}{A}(\alpha) = \begin{pmatrix} a_{11}(\alpha) & a_{12}(\alpha) \\ a_{21}(\alpha) & a_{22}(\alpha) \end{pmatrix},$$

where $a_{11}(\alpha)$, $a_{12}(\alpha)$, $a_{21}(\alpha)$, $a_{22}(\alpha)$ are supposed to be analytic functions in the cut α-plane, and $\det \underset{\sim}{A}(\alpha) \neq 0$ in the cut α-plane.

Then

$$\underset{\sim}{A}_+(\alpha)\underset{\sim}{A}_-^{-1}(\alpha) = \frac{1}{\det \underset{\sim}{A}_-(\alpha)} \begin{pmatrix} \overset{+}{a}_{11}\overset{-}{a}_{22} - \overset{+}{a}_{12}\overset{-}{a}_{21} & \overset{+}{a}_{12}\overset{-}{a}_{11} - \overset{+}{a}_{11}\overset{-}{a}_{12} \\ \overset{+}{a}_{21}\overset{-}{a}_{22} - \overset{+}{a}_{22}\overset{-}{a}_{21} & \overset{+}{a}_{22}\overset{-}{a}_{11} - \overset{+}{a}_{21}\overset{-}{a}_{12} \end{pmatrix}, \quad (4)$$

where $\det \underset{\sim}{A}_-(\alpha) = (a_{11}^- a_{22}^- - a_{12}^- a_{21}^-) \neq 0$. In order that (4) should have the form (2), i,e, zeros on the diagonal, we require

$$a_{11}^+ a_{22}^- = a_{12}^+ a_{21}^- , \quad \text{and } a_{22}^+ a_{11}^- = a_{21}^+ a_{12}^- ,$$

or, ignoring the trivial situation where $a_{11}^{\pm}(\alpha) \equiv 0$, and/or $a_{21}^{\pm}(\alpha) \equiv \hat{0}$,

$$\left(\frac{a_{12}}{a_{11}}\right)^+ - \left(\frac{a_{22}}{a_{21}}\right)^- = 0 , \tag{5}$$

$$\left(\frac{a_{22}}{a_{21}}\right)^+ - \left(\frac{a_{12}}{a_{11}}\right)^- = 0 , \tag{6}$$

where $a_{21}(\alpha) \neq 0$, and $a_{11}(\alpha) \neq 0$ on C .

Adding and subtracting (5) and (6) gives

$$\left(\frac{a_{12}}{a_{11}} + \frac{a_{22}}{a_{21}}\right)^+ - \left(\frac{a_{12}}{a_{11}} + \frac{a_{22}}{a_{21}}\right)^- = 0 , \quad \alpha \in C \tag{7}$$

$$\left(\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}}\right)^+ + \left(\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}}\right)^- = 0 . \quad \alpha \in C \tag{8}$$

Using the fact that $[(k^2-\alpha^2)^{\frac{1}{2}}]^{\pm} = \pm i|k+\alpha|^{\frac{1}{2}}(k-\alpha)^{\frac{1}{2}}$ we can rewrite (8) in the form

$$\left[(k^2-\alpha^2)^{\frac{1}{2}} \left(\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}}\right)\right]^+ - \left[(k^2-\alpha^2)^{\frac{1}{2}} \left(\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}}\right)\right]^- = 0 , \alpha \in C \tag{9}$$

Now provided $a_{11}(\alpha)$ and $a_{21}(\alpha)$ are non-zero in the cut plane and satisfy the conditions

$$\frac{a_{12}}{a_{11}} + \frac{a_{22}}{a_{21}} = O[(k^2-\alpha^2)^{\mu}], \quad \text{as } \alpha \to \pm k , \ 0 \leq \mu < 1 ,$$

$$\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}} = O[(k^2-\alpha^2)^{\nu-\frac{1}{2}}] , \quad \text{as } \alpha \to \pm k ; \ 0 \leq \nu < 1 ,$$

then the most general solution of (7) and (9) which has no pole singularity at $\alpha = \pm k$ and no other singularities in the cut plane except a branch cut along C' is given by (Muskhelishivili [2])

$$\frac{a_{12}}{a_{11}} + \frac{a_{22}}{a_{21}} = 2F_1(\alpha) \tag{10}$$

and

$$\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}} = 2F_2(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}} \tag{11}$$

respectively, where $F_1(\alpha)$ and $F_2(\alpha)$ are analytic in the entire plane except possibly along the branch cut C'. Adding and subtracting (10) and (11) gives

$$a_{12}(\alpha) = a_{11}(\alpha)\{F_1(\alpha) + F_2(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\} ,$$
$$a_{22}(\alpha) = a_{21}(\alpha)\{F_1(\alpha) - F_2(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\} .$$

If $\underset{\sim}{A}(\alpha) = \underset{\sim}{A}(-\alpha)$ then $F_1(\alpha)$ and $F_2(\alpha)$ are analytic in the entire complex plane, as the following analysis will show.

If $\underset{\sim}{A}(\alpha) = \underset{\sim}{A}(-\alpha)$ then $a_{ij}(\alpha) = a_{ij}(-\alpha)$, $i,j = 1,2$, and in an exactly analogous way one obtains similar equations to (7) and (9) on carrying out evaluations on the branch cut C':

$$\left(\frac{a_{12}}{a_{11}} + \frac{a_{22}}{a_{21}}\right)^+ - \left(\frac{a_{12}}{a_{11}} + \frac{a_{22}}{a_{21}}\right)^- = 0 \quad , \quad \alpha \in C' , \tag{7'}$$

$$\left[(k^2-\alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}}\right)\right]^+ - \left[(k^2-\alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}}\right)\right]^- = 0 \quad , \quad \alpha \in C' , \tag{9'}$$

where now $\pm$ corresponds to the lower and upper side of C', respectively. Adding (7) to (7') and (9) to (9') gives

$$\left(\frac{a_{12}}{a_{11}} + \frac{a_{22}}{a_{21}}\right)^+ - \left(\frac{a_{12}}{a_{11}} + \frac{a_{22}}{a_{21}}\right)^- = 0 \quad , \quad \alpha \in C \cup C' \quad , \tag{7''}$$

$$\left[(k^2-\alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}}\right)\right]^+ - \left[(k^2-\alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}}\right)\right]^- = 0 \quad , \quad \alpha \in C \cup C' . \tag{9''}$$

Thus the most general solution of (7") and (9") which has no pole singularity at $\alpha = \pm k$ and no other singularities in the cut α-plane is given by:

$$a_{12}(\alpha) = a_{11}(\alpha)\{E_1(\alpha) + E_2(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\} ,$$
$$a_{22}(\alpha) = a_{21}(\alpha)\{E_1(\alpha) - E_2(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\} ,$$

where $E_1(\alpha)$ and $E_2(\alpha)$ are analytic in the entire α-plane.

If in particular we let $a_{11}(\alpha) = F(K)$, $a_{21}(\alpha) = H(K)$, $E_1(\alpha) = 0$, and $E_2(\alpha) = KG(K)$, (the condition $G(K) = -G(-K)$ ensures that $KG(K)$ is an entire function) we obtain the special form considered in [1] .

Following the procedure outlined in Rawlins and Williams [1] a particular factorisation of the matrix (3), which will be useful in applications, is given by $\underset{\sim}{A}(\alpha) = \underset{\sim}{U}^{(0)}(\alpha)[\underset{\sim}{L}^{(0)}(\alpha)]^{-1}$ where

$$\underset{\sim}{U}^{(0)}(\alpha) = \begin{bmatrix} [W_1(\alpha)]^{\frac{1}{2}}[W_2(\alpha)]^{\frac{1}{2}} & (k+\alpha)^{\frac{1}{2}}[W_1(\alpha)]^{\frac{1}{2}}[W_2(\alpha)]^{\frac{1}{2}} \\ [W_1(\alpha)]^{\frac{1}{2}}/[W_2(\alpha)]^{\frac{1}{2}} & -(k+\alpha)^{\frac{1}{2}}[W_1(\alpha)]^{\frac{1}{2}}/[W_2(\alpha)]^{\frac{1}{2}} \end{bmatrix} ,$$

$W_1(\alpha)$ and $W_2(\alpha)$ are solutions of the standard Hilbert problems on the half-line C:

$$[\ell n W_1(\alpha)]^+ - [\ell n W_2(\alpha)]^- = \ell n[g(\alpha)h(\alpha)] ,$$

$$[(k+\alpha)^{\frac{1}{2}}\ell n W_2(\alpha)]^+ - [(k+\alpha)^{\frac{1}{2}}\ell n W_2(\alpha)]^- = i|k+\alpha|^{\frac{1}{2}}\ell n[g(\alpha)/h(\alpha)] ,$$

where

$$g(\alpha) = (a_{12}^+(\alpha)a_{11}^-(\alpha) - a_{11}^+(\alpha)a_{12}^-(\alpha))/\det \underset{\sim}{A}_-(\alpha) = a_{11}^+(\alpha)/a_{21}^-(\alpha) ,$$

$$h(\alpha) = (a_{21}^+(\alpha)a_{22}^-(\alpha) - a_{22}^+(\alpha)a_{21}^-(\alpha))/\det \underset{\sim}{A}_-(\alpha) = a_{21}^+(\alpha)/a_{11}^-(\alpha) .$$

The set of solutions for $W_1(\alpha), W_2(\alpha)$ is further restricted by the requirement that the factor matrix $\underset{\sim}{L}^{(0)}(\alpha)$ is non-singular at $\alpha = -k$ and its elements should be analytic in the region $\mathrm{Im}(\alpha) < \mathrm{Im}(k)$. It is interesting to note that the functions $F_1(\alpha)$, $F_2(\alpha)$ have dropped out completely. This means that for all matrices of the form (3) the factorisation problem reduces to the same Hilbert Problem! The explanation for this follows from the sentence above the expression (3').

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References

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