ZAMM, Vol. 81, 2001, S983-S984

Mikhailov S.E., Orlik J.

Homogenization in Integral Viscoelasticity

A multi-phase periodic composite subjected to inhomogeneous shrinkage and mechanical loads including prescribed interface jumps of displacements and tractions is considered. The composite components are anisotropic linear viscoelastic and aging (described by the non-convolution Volterra integral operators). The paper presents some results about asymptotic homogenization and 2-scale convergence in appropriate function spaces.

1. Introduction

Assume that a solid Ω has Y-periodic structure with a scaling parameter ε.

Assumption 1 (on the periodic geometry). *Consider mutually disjoint generally non-connected $Y-$periodic domains Ω_l^{per}, $l=1,...,s$ in $\mathbb{R}^n$ with $Y-$periodic boundaries $\partial\Omega_l^{per}\in C^{0,1}$, such that $\cup_{l=1}^s\bar{\Omega}_l^{per}=\mathbb{R}^n$. $Y_l:=\Omega_l^{per}\cap Y$, $l=1,...,s$. We denote by $\Sigma_{lk}^{per}=\partial\Omega_l^{per}\cap\partial\Omega_k^{per}$ the interfaces between the domains Ω_l^{per} and Ω_k^{per}. If Ω_l^{per} and Ω_k^{per} have no common boundary, then $\Sigma_{kl}^{per}=\emptyset$. The net interface is $\Sigma^{per}=\cup_{l=1}^s\cup_{k=l+1}^s\Sigma_{lk}^{per}$. Let $\Omega\subset\mathbb{R}^n$ be an open domain. Suppose $\Omega_l=\varepsilon\Omega_l^{per}\cap\Omega$ are Lipschitz domains. Denote, $\Sigma=\varepsilon\Sigma^{per}\cap\Omega$. Suppose, $\partial\Omega_l\setminus\partial\Omega$ has a positive Lebesgue measure on $\partial\Omega_l$ if $\partial\Omega_l\setminus\partial\Omega\neq\emptyset$. Let $\partial_D\Omega\subset\partial\Omega$ be a subset of the external boundary and $\partial_D\Omega_l=\partial\Omega_l\cap\partial_D\Omega$. Let $\partial_N\Omega=\partial\Omega\setminus\partial_D\Omega$. Suppose the set of points, that belong to boundaries $\partial\Omega_l$ of more than two different subdomains, or two subdomains and the part of boundary $\partial_D\Omega$, has zero measure on each $\partial\Omega_l$.*

Let $i,j,h,k=1,2,...,n$, and summation from 1 to n over the repeating subscripts is assumed hereafter. For the solid Ω, we consider the equilibrium equations with boundary and transmission conditions:

$$\frac{\partial}{\partial x_h}\sigma_{ih}^\varepsilon(x,t)=f_{i\,0}^\varepsilon(x,t),\qquad \sigma_{ih}^\varepsilon(x,t)=\left[\underline{a}_{ihjk}^\varepsilon(x)\frac{\partial u_j^\varepsilon(x,\cdot)}{\partial x_k}\right](t)+\sigma_{ih}'^\varepsilon(x,t),\quad x\in\Omega\setminus\Sigma \tag{1}$$

$$u_i^\varepsilon(x,t)=\chi_i^\varepsilon(x,t),\quad x\in\partial_D\Omega;\qquad \sigma_{ih}^\varepsilon(x,t)n_h(x)=\omega_i^\varepsilon(x,t),\quad x\in\partial_N\Omega, \tag{2}$$

$$u_i^\varepsilon(x,t)\,|_{\Sigma^+}-u_i^\varepsilon(x,t)\,|_{\Sigma^-}=\chi_i^\varepsilon(x,t),\qquad \sigma_{ih}^\varepsilon(x,t)n_h(x)|_{\Sigma^+}+\sigma_{ih}^\varepsilon(x,t)n_h(x)|_{\Sigma^-}=\omega_i^\varepsilon(x,t),\quad x\in\Sigma, \tag{3}$$

holding for any $t\in[0,T]$. Here $\underline{a}_{ihjk}^\varepsilon(x):=a_{ihjk}^{\diamond\varepsilon}(x,t)+a_{ihjk}^\varepsilon(x)\star$, see e.g. [1, Chapter 3]; the out-of-integral term $a_{ihjk}^{\diamond\varepsilon}$ presents the instant elastic coefficients; the Volterra operator $[a_{ihjk}^\varepsilon(x)\star e_{jk}](t):=\int_0^t a_{ihjk}^\varepsilon(x,t,\tau)\cdot e_{jk}(x,\tau)d\tau$ presents the viscosity with ageing (for isotropic materials $\underline{a}_{ihjk}^\varepsilon=\underline{\lambda}^\varepsilon\delta_{hi}\delta_{kj}+\underline{\mu}^\varepsilon\delta_{ij}\delta_{hk}+\underline{\mu}^\varepsilon\delta_{ik}\delta_{hj}$); $-f_{i\,0}^\varepsilon$ are components of a vector of external volume forces; $\sigma_{ih}'^\varepsilon:=-\underline{a}_{ihjk}^\varepsilon e_{jk}'^\varepsilon$ is a shrinkage stress tensor occurring at completely constrained shrinkage, where $e_{jk}'^\varepsilon(x,t)$ is a free shrinkage strain tensor; $\chi_i^\varepsilon(x,t):=\{\{\chi_i^{lk\varepsilon}(x,t)\}_{k=l+1}^s\}_{l=0}^s$, where $\chi_i^{0\varepsilon}(x,t):=\{\chi_i^{0k\varepsilon}(x,t)\}_{k=1}^s$ is a system of boundary values of the displacement vector on the part $\partial_D\Omega$ of the external boundary with $\chi_i^{0k\varepsilon}(x,t)$ given on each $\partial_D\Omega_k$, and $\chi_i^{\Sigma\varepsilon}(x,t):=\{\{\chi_i^{lk\varepsilon}(x,t)\}_{k=l+1}^s\}_{l=1}^s$ is a system of jumps in the displacement vector on the net interface Σ with $\chi_i^{lk\varepsilon}(x,t)$ given on each Σ_{lk}; $\omega_i^\varepsilon(x,t):=\{\{\omega_i^{lk\varepsilon}(x,t)\}_{k=l+1}^s\}_{l=0}^s$, where $\omega_i^{0\varepsilon}(x,t):=\{\omega_i^{0k\varepsilon}(x,t)\}_{k=1}^s$ are components of a vector of boundary traction given on the part $\partial_N\Omega$ of the external boundary; $\omega_i^{\Sigma\varepsilon}(x,t):=\{\{\omega_i^{lk\varepsilon}(x,t)\}_{k=l+1}^s\}_{l=1}^s$ are jumps in the tractions given on the interface Σ. All those functions are supposed to be known. System (1)-(3) is to be solved with respect to the displacement vector $u_i(x,t)$.

2. Two-scale Homogenization.

We will study the asymptotic behavior of solution $u_i^\varepsilon(x)$ to the problem (1)-(3) as $\varepsilon\to0$. We present the dependence of all the known functions on x and ε as dependence on x and $\xi:=x/\varepsilon$. Here dependence on the *slow* variable x describes outer (macro-) effects, while dependence on the *fast* variable ξ describes effects related to the composite structure. Let F_{per}^Y denote the class of Y-periodic functions on $\mathbb{R}^n$; index *per* denote intersection of F_{per}^Y with a corresponding space on the periodicity cell Y or on its part, e.g. $H_{per}^1(Y_l):=H^1(Y_l)\cap F_{per}^Y$; let $H_{per[0]}^1(Y):=\{f\in H_{per}^1(Y):\ \frac{1}{|Y|}\int_Y f(\xi)d\xi=0\}$. Suppose the following conditions are satisfied:

Assumption 2. $\underline{a}_{ihjk}^\varepsilon(x)=\underline{a}_{ihjk}^f(\frac{x}{\varepsilon})$, $a_{ihjk}^{\diamond f}(\xi,t)\in C([0,T],L_{per}^\infty(Y))$ *and* $c_0\eta_{jk}\eta_{jk}\le a_{ihjk}^\diamond(x,t)\eta_{ih}\eta_{jk}\le C_0\eta_{jk}\eta_{jk}$, $\forall\eta_{jk}=\eta_{kj}\in\mathbb{R}$, $t\in[0,T]$, *where constants* $c_0,C_0>0$ *are independent of* ξ *and* t, $a_{ihjk}^f\star\in V(C;[0,T];L_{per}^\infty(Y))$; $f_{i0}^\varepsilon(x,t)=f_{i0}^s(x,t)\in C([0,T],L^2(\Omega))$; $\sigma_{ih}'^\varepsilon(x,t)=\sigma_{ih}'^{sf}(x,\frac{x}{\varepsilon},t)$, $\sigma_{ih}'^{sf}(x,\xi,t)\in C([0,T],L^2(\Omega,L_{per}^2(Y)))$; $\omega_i^{0\varepsilon}(x,t)=\omega_i^{0sf}(x,\frac{x}{\varepsilon},t)$, $\omega_i^{0sf}(x,\xi,t)\in C([0,T],L^2(\partial_N\Omega,L_{per}^2(Y)))$, $\omega_i^{\Sigma\varepsilon}(x,t)=\omega_i^{\Sigma f}(\frac{x}{\varepsilon},t)$, $\omega_i^{\Sigma f}(\xi,t)\in C([0,T],L_{per}^2(\Sigma_Y))$, *furthermore,* $\int_{\Sigma^{per}\cap Y}\omega_i^f(\xi,t)d\xi=0$; $\chi_i^\varepsilon(x,t)=\chi_i^s(x,t)+\varepsilon\chi_i^f(\frac{x}{\varepsilon},t)$ *where* $\chi_i^{\Sigma s}(x,t)=0$,

$\chi_i^s(x,t) \in C([0,T], H^{1/2}(\partial_D\Omega))$, $\chi_i^f(\xi,t) = \{\tilde\chi_i^f(\xi,t)|_{\partial_D(\varepsilon\Omega)},\ \tilde\chi_i^f(\xi,t)|_{\Sigma^+_{Yper}} - \tilde\chi_i^f(\xi,t)|_{\Sigma^-_{Yper}}\}$, *where* $\tilde\chi_i^f(\xi,t) \in C([0,T], H^1_{per}(\cup_{l=1}^s Y_l))$.

Representation 3. *We define a Volterra operator* $\underline{N}_{pjq}(\xi) := N^\diamond_{pjq}(\xi,t) + N_{pjq}(\xi)\star$.

The function $N^\diamond_{pjq} \in C([0,T], H_{per[0]}(Y))$ *is a solution to the uniquely solvable weak problem:*

$\int_Y \left[a^{\diamond f}_{ihjk}(\xi,t)(\frac{\partial}{\partial\xi_k}N^\diamond_{pjq}(\xi,t) + \delta_{kp}\delta_{jq})\right]\frac{\partial}{\partial\xi_h}v_i(\xi)d\xi = 0, \quad \forall v_i \in H_{per[0]}(Y), \quad \forall t \in [0,T],\ p,q = 1,...,n.$

The kernel $N_{pjq} \in C([0,T], L^1([0,T],\ H^1_{per[0]}(Y)))$ *is a solution to the uniquely solvable weak problem:*

$\int_Y \left[\underline{a}^f_{ihjk}(\xi)\frac{\partial}{\partial\xi_k}N_{pjq}(\xi,\cdot,\tau)\right](t)\frac{\partial}{\partial\xi_h}v_i(\xi)d\xi = -\int_Y \left[a^f_{ihjk}(\xi,t,\tau)\left(\frac{\partial}{\partial\xi_k}N^\diamond_{pjq}(\xi,\tau) + \delta_{kp}\delta_{jq}\right)\right]\frac{\partial}{\partial\xi_h}v_i(\xi)d\xi,$
$\forall v_i \in H_{per[0]}(Y), \quad t \in [0,T],\ a.a.\ \tau \in [0,T],\ p,q = 1,...,n.$

The function $y_i = \hat y_i + \tilde\chi_i^f \in C([0,T], L^2(\Omega,\ H^1_{per[0]}(\cup_{l=1}^s Y_l)))$ *where* $\hat y_i \in C([0,T], L^2(\Omega,\ H^1_{per[0]}(Y)))$ *is a solution to the uniquely solvable weak problem (for* $\hat\sigma^{sf}_{ih} := \sigma'^{sf}_{ih} + \underline{a}_{ihjk}\partial\tilde\chi^f_j/\partial\xi_k$*)* $\forall t \in [0,T]$, *a.a.* $x \in \Omega$*:*

$\int_Y \left[\underline{a}^f_{ihjk}(\xi)\frac{\partial}{\partial\xi_k}\hat y_j(x,\xi,\cdot)\right](t)\frac{\partial}{\partial\xi_h}v_i(\xi)d\xi = -\int_Y \hat\sigma^{sf}_{ih}(x,\xi,t)\frac{\partial}{\partial\xi_h}v_i(\xi)d\xi + \int_{\Sigma_Y}\omega^f_i(\xi,t)v_i(\xi)ds \quad \forall v_i \in H^1_{per[0]}(Y).$

The term $u_i^{(0)}(x,t) \in C([0,T], H^1(\Omega))$ *is a solution to the uniquely solvable homogenized problem coincident with (1)-(2) after replacement there* $\chi^\varepsilon(x,t)$ *by* $\chi^{0s}(x,t)$, $\omega^\varepsilon(x,t)$ *by* $\frac{1}{|Y|}\int_Y \omega_i^{0sf}(x,\xi,t)d\xi$, $f^\varepsilon_{i0}(x,t)$ *by* $f^s_{i0}(x,t)$, *and the viscoelastic operator* $\underline{a}^\varepsilon_{ihjk}(x)$, *shrinkage stresses* $\sigma'^\varepsilon_{ih}(x,t)$, *and strain* $e'^\vee_{jk}(x,t)$ *by their homogenized counterparts:*

$\underline{a}^\vee_{ihjk} = \frac{1}{|Y|}\int_Y \underline{a}_{ihqp}(\xi)\left[\delta_{jq}\delta_{kp} + \frac{\partial}{\partial\xi_p}\underline{N}_{kqj}(\xi)\right]d\xi,\ \sigma'^\vee_{ih}(x,t) = \frac{1}{|Y|}\int_Y\left[\sigma'^{sf}_{ih}(x,\xi,t) + \underline{a}_{ihqp}(\xi)\frac{\partial}{\partial\xi_q}y_p(x,\xi,.)\right]d\xi,$

$e'^\vee_{jk}(x,t) = -\{\underline{a}^{\vee-1}_{jkih}\sigma'^\vee_{ih}(x,.)\}(t) = \underline{a}^{\vee-1}_{jkih}\left\{\frac{1}{|Y|}\int_Y \underline{a}_{ihqp}(\xi)\left[e'^{sf}_{qp}(x,\xi,.) - \frac{\partial}{\partial\xi_q}y_p(x,\xi,.)\right]d\xi\right\}(t)$

where $\underline{a}^{\vee-1}_{jkih}$ *is the tensor Volterra operator inverse to* $\underline{a}^\vee_{jkih}$, *i.e.* $\underline{a}^{\vee-1}_{jkih}\underline{a}^\vee_{ihqp} = \underline{a}^\vee_{jkih}\underline{a}^{\vee-1}_{ihqp} = \frac{1}{2}(\delta_{jq}\delta_{kp} + \delta_{jp}\delta_{kq})$.

The function $u_j^{(1)}(x,\xi,t) := [\underline{N}_{pjq}(\xi)\partial u_q^{(0)}(x,\cdot)/\partial x_p](t) + y_j(x,\xi,t) \in C([0,T], L^2(\Omega,\ H^1_{per[0]}(\cup_{l=1}^s Y_l)))$.

The micro-stress tensor $\sigma^0_{ij}(x,\xi,t) := \underline{B}_{ijkl}(\xi)[\sigma^\vee_{kl}(x,.) - \sigma'^\vee_{kl}(x,.)](t) + \sigma'^{sf}_{ij}(x,\xi,t) + \left[\underline{a}_{ijkl}\frac{\partial}{\partial\xi_l}y_k(x,\xi,.)\right](t)$, *where* $\underline{B}_{ijkl}(\xi) = \underline{a}_{ij\gamma\beta}(\xi)\left[\frac{\partial}{\partial\xi_\beta}\underline{N}^q_{\gamma p}(\xi) + \delta_{\beta q}\delta_{\gamma p}\right]\underline{a}^{\vee-1}_{pqkl}$ *is the viscoelastic micro-stress concentration operator, cf. [2].*

Using the technique similar to [3] one can prove the following theorem (proof is to be published elsewhere).

Theorem 4. *Suppose Assumptions 1 and 2 hold. Let, additionally,* $\sigma'^{sf}_{ih}(x,\frac{x}{\varepsilon},t) \in C([0,T], L^2(\Omega))$, $\lim_{\varepsilon\to0}\sup_{t\in[0,T]}\left|\int_\Omega \sigma'^{sf}_{ih}(x,\frac{x}{\varepsilon},t)^2dx - \frac{1}{|Y|}\int_\Omega\int_Y \sigma'^{sf}_{ih}(x,\xi,t)^2d\xi dx\right| = 0$*; and* $\forall\epsilon > 0$, $\exists\ \eta(\epsilon)$ *such that* $\sup_\varepsilon\int_\Omega\left[\sigma'^{sf}_{ih}(x,\frac{x}{\varepsilon},t+\Delta t) - \sigma'^{sf}_{ih}(x,\frac{x}{\varepsilon},t)\right]^2dx \le \epsilon$, $\forall t, t+\Delta t \in [0,T]$, *such that* $|\Delta t| \le \eta$. *Let* $\omega_i^{0sf}(x,\frac{x}{\varepsilon},t) = \omega_i^{0s}(x,t) + \bar\sigma^f_{ij}(\frac{x}{\varepsilon},t)n_j(x)|_{x\in\partial_N\Omega}$, *where* $\omega_i^{0s}(x,t) \in C([0,T], L^2(\partial_N\Omega))$, $\bar\sigma^f_{ij}(\xi,t)$, $\frac{\partial\bar\sigma^f_{ij}(\xi,t)}{\partial\xi_j} \in C([0,T], L^2_{per}(\cup_{l=1}^s Y_l))$ *and* $\frac{\partial\bar\sigma^f_{ij}(\xi,t)}{\partial\xi_j} = 0$, $\xi \in Y_l$, $\bar\sigma^f_{ij}(\xi,t)n_j(\xi)|_{\Sigma^-} + \bar\sigma^f_{ij}(\xi,t)n_j(\xi)|_{\Sigma^+} = \omega^{\Sigma f}(\xi,t)$, $\xi \in \Sigma$, $\forall t \in [0,T]$, *and* $||\bar\sigma^f||_{C([0,T],L^2(Y))} \le C_Y(Y,\Sigma_Y)||\omega^{\Sigma f}||_{C([0,T],L^2(\Sigma_Y))}$. *Then the sequence* u_i^ε *of solutions to (1)-(3) has a subsequence* $u_i^{\varepsilon^*}(x,t)$ *strongly convergent to* $u_i^{(0)}(x,t)$ *in* $C([0,T], L^2(\Omega))$, *such that* $\nabla u_i^{\varepsilon^*}(x,t)$ *two-scale converges to* $\nabla u_i^{(0)}(x,t) + \nabla_\xi u_i^{(1)}(x,\xi,t)$ *and* $\sigma^{\varepsilon^*}_{ij}$ *two-scale converges to* $\sigma^0_{ij}(x,\xi,t)$ *for all* $t \in [0,T]$ *as* $\varepsilon^* \to 0$, *i.e.,*

$$\lim_{\varepsilon^*\to0}\int_\Omega \frac{\partial u_i^{\varepsilon^*}(x,t)}{\partial x_j}v(x,\frac{x}{\varepsilon^*})dx = \frac{1}{|Y|}\int_\Omega\int_Y[\frac{\partial u_i^{(0)}(x,t)}{\partial x_j} + \frac{\partial u_i^{(1)}(x,\xi,t)}{\partial\xi_j}]v(x,\xi)d\xi dx \quad \forall v(x,\xi) \in D(\Omega, C^\infty_{per}(Y))$$

$$\lim_{\varepsilon^*\to0}\int_\Omega \sigma^{\varepsilon^*}_{ij}(x,t)\varphi(x,\frac{x}{\varepsilon^*})dx = \frac{1}{|Y|}\int_\Omega\int_Y \sigma^0_{ij}(x,\xi,t)\varphi(x,\xi)dxd\xi, \quad \forall\varphi \in L^2(\Omega, C_{per}(Y)),$$

Acknowledgements

This research has been partly carried out in the framework of the European CRAFT/BRITE-EURAM Project BES2-2653.

3. References

1 Rabotnov, Yu.N.: Creep Problems in Structural Members, North-Holland Publ., Amsterdam-London, 1969.
2 Pobedrya, B.E.: Mechanics of composite materials, Moscow State University Publication, 1984
3 Allaire, G.: Homogenization and Two-Scale Convergence, SIAM J. Math. Anal., **23** (1992), 1482-1518.

Addresses: Prof. Dr. Sergey E. Mikhailov, Dept. of Mathematics, Glasgow Caledonian Univ., Glasgow, G4 0BA, U.K. (s.mikhailov@gcal.ac.uk)
Dr. Julia Orlik, ITWM, Gottlieb-Daimler-Str. 7^2, D-67663 Kaiserslautern, Germany (orlik@itwm.uni-kl.de)