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Homogenization Methods and Macro-Strength of Composites

A multi-phase periodic composite subjected to inhomogeneous shrinkage or temperature deformation and prescribed mechanical loads is considered. The asymptotic homogenisation is applied for calculation of homogenized macro-stresses. A non-local approximate macro-strength condition, defined on homogenised stress-field, is derived from the micro-strength conditions and their convergence to the approximate macro-strength condition, as the structure period tends to zero, is proved.

1. Statement of problem

The thermo-elasticity problem for a composite materials with a large number of periodically distributed inclusions or pores is given by the equilibrium equations and Hooke's law in the domain $\Omega \in \mathbb{R}^3$,

$$\frac{\partial \sigma^{\varepsilon}_{ij}(x)}{\partial x_j} = f_i(x) \quad x \in \Omega, \quad \sigma^{\varepsilon}_{ij}(x) = a_{ijkl}(\frac{x}{\varepsilon})\frac{\partial u^{\varepsilon}_k(x)}{\partial x_l} + \sigma'^{\varepsilon}_{ij}(x), \tag{1}$$

completed by corresponding boundary and transmission conditions. Here $i, j, k, l = 1, ..., 3$; a_{ijkl} are elastic moduli, $\sigma'^{\varepsilon}_{ij}(x) := -a_{ijkl}(x/\varepsilon)e'^{\varepsilon}_{kl}(x)$ is the thermo- (or shrinkage) stress tensor occuring at completely constrained deformation of each material, where $e'_{kl}(x,t) = T(x)\alpha_{kl}(x/\varepsilon)$ is a free thermo- (or shrinkage) strain tensor of each material; $T(x)$ is the temperature rise; α_{ij} are linear expansion coefficients; f_i are volume forces; ε is a small parameter related to the period of structure. The problem is solved to find displacements u^{ε}_i and stresses $\sigma^{\varepsilon}_{ij}$. Our aim is to derive a macro-strength condition for the composite from the micro-strength conditions, which will allow to estimate the macro-strength in terms of averaged mechanical characteristics and averaged stresses.

2. Elements of strength analysis

For a stress field $\sigma_{ij}(y) \in C(\bar{\Omega})$, any local strength condition for micro-stresses at a point y can be written in the form $\Lambda(\sigma(y), y) < 1$, where $\Lambda \in C(\bar{\Omega}, C_0(\mathbb{R}^{3\times 3}))$ is a normalised equivalent stress function, a material characteristic, which is non-negative and positively homogeneous of the order $+1$ w.r.t. σ.

E x a m p l e 1. *For some materials Λ is associated with the von Mises equivalent stress $\Lambda_M(\sigma(y), y) = \sqrt{[(\sigma_1(y) - \sigma_2(y))^2 + (\sigma_2(y) - \sigma_3(y))^2 + (\sigma_3(y) - \sigma_1(y))^2]/(2\sigma_c^2(y))}$, or with the Tresca equivalent stress $\Lambda_T(\sigma(y), y) = \max_{k,m} |\sigma_k(y) - \sigma_m(y)|/\sigma_c(y)$, where σ_1, σ_2, σ_3 are the principal stresses and σ_c is a known uniaxial strength of material.*

Such local strength conditions, however, are generally not applicable to unbounded stress fields since the conditions will predict fracture under almost any singular stress field. For more general classes of stress fields, e.g. belonging to $L_2(\Omega)$, a (point) non-local strength condition $\underline{\Lambda}(\sigma, y) <$ can be applied. Here $\underline{\Lambda}(\sigma, y)$ is a normalised equivalent stress functional , which is defined on the tensor-functions $\sigma_{ij} \in L_2(\Omega)$ and is non-negative positively homogeneous of the order $+1$ w.r.t. σ, see [3].

Particularly $\underline{\Lambda}$ can be connected with some kind of weighted averaging of $\sigma_{ij}(x)$, $x \in \Omega$ in some surrounding of the point y, $\underline{\Lambda}(\sigma, y) = \Lambda(\tilde{\sigma}(y), y)$, $\quad \tilde{\sigma}_{ij}(y) = \int_\Omega \varphi(x,y)\sigma_{ij}(x)dx$ where $\tilde{\sigma}_{ij} \in C(\bar{\Omega})$ are components of an auxiliary non-local stress tensor, and $\varphi(x,y) \in C(\bar{\Omega}, L^2(\Omega))$ is a material characteristic, such as $\int_\Omega \varphi(x,y)dx = 1$. Then the strength condition for the whole body is $\underline{\Lambda}_\Omega(\sigma) := \sup_{y\in\Omega} \Lambda(\tilde{\sigma}, y) < 1$, where $\underline{\Lambda}_\Omega(\sigma) \in \mathbb{R}$ is the *body* normalised equivalent stress functional.

E x a m p l e 2. *(i) In the simplest case $\varphi(x,y) = \begin{cases} \frac{3}{4\pi d^3} & |x-y| < d \\ 0 & |x-y| \geq d \end{cases}$ for 3D, where d is a material constant, and $\tilde{\sigma}(y) = \frac{3}{4\pi d^3}\int\int\int_{|x-y|<d}\sigma(x)dx$. (ii) If $\varphi(x,y)$ is the Dirac-function, then $\tilde{\sigma}(y) = \sigma(y)$, and the non-local strength-condition coincides with the local one.*

3. Elements of homogenization technique

We use the following asymptotic expansion to the solution of (1)

$$u^{\varepsilon}(x) = u(x, \frac{x}{\varepsilon}) = u^0(x) + \varepsilon \left(N_q(\xi)\frac{\partial u^0(x)}{\partial x_q} + z(\xi)T(x) \right)\Bigg|_{\xi = \frac{x}{\varepsilon}} + O(\varepsilon^2), \quad x \in \Omega.$$

Here, $N_q = \{N_{pjq}\}_{3\times 3\times 3} \in H_{per[0]}(Y)$ and $z = \{z_i\}_{i=1,\ldots,3} \in H_{per[0]}(Y)$ are solutions to the auxilliary periodic weak problems of elasticity $\forall v_i \in H_{per[0]}(Y)$:

$$\int_Y a_{ihjk}(\xi)\frac{\partial(N_{pjq}(\xi)+\xi_p\delta_{jq})}{\partial\xi_k}\frac{\partial v_i(\xi)}{\partial\xi_h}d\xi=0,\quad \int_Y a_{ihjk}(\xi)\Big[\frac{\partial z_j(\xi)}{\partial\xi_k}-\alpha_{jk}(\xi)\Big]\frac{\partial v_i(\xi)}{\partial\xi_h}d\xi=0,\quad p,q=1,...,3.$$

The homogenized displacement and stress fields, $u_i^{(0)}\in H^1(\Omega)$, $\hat\sigma_{ij}\in L^2(\Omega)$ are a solution to the uniquely solvable homogenized problem coincident with (1) after replacement there the elastic constants $a_{ihjk}\in L^\infty_{per}(Y)$ and the linear expansion coefficients $\alpha_{ih}\in L^\infty_{per}(Y)$ by their homogenized counterparts:

$$\hat a_{ihjk}=\frac{1}{|Y|}\int_Y a_{ihqp}(\xi)\left[\delta_{jq}\delta_{kp}+\frac{\partial}{\partial\xi_p}N_{kqj}(\xi)\right]d\xi,\ \hat\alpha_{jk}=\frac{\hat a^{-1}_{jkih}}{|Y|}\int_Y a_{ihqp}(\xi)\left[\alpha_{pq}(\xi)-\frac{\partial z_p(\xi)}{\partial\xi_q}\right]d\xi.$$

Here $\hat a^{-1}_{pq\gamma\delta}$ is the homogenized compliance tensor, which is the inverse to the homogenized stiffness tensor $\hat a_{\gamma\delta\alpha\beta}$. Similar to [1] one can prove that $\sigma^\varepsilon\in L^2(\Omega)$ containes a subsequense, which two-scale converges to $\sigma^0\in L^2(\Omega\times Y)$ ($\xi=\frac{x}{\varepsilon}$), that is,

$$\lim_{\varepsilon\to0}\int_\Omega\varphi(x,\frac{x}{\varepsilon})\sigma^\varepsilon_{ij}(x)dx=\frac{1}{|Y|}\int_\Omega\int_Y\varphi(x,\xi)\sigma^0_{ij}(x,\xi)dxd\xi,\quad\forall\varphi\in L^2(\Omega,C_{per}(Y)),$$

$$\sigma^0_{ij}(x,\xi)=A_{ijkl}(\xi)[\hat\sigma_{kl}(x)-\hat\sigma'_{kl}(x)]+\sigma''_{ij}(x,\xi),\tag{2}$$

where $A_{ijkl}(\xi)=a_{ij\gamma\beta}(\xi)[\frac{\partial}{\partial\xi_\beta}N^q_{\gamma p}(\xi)+\delta_{\beta q}\delta_{\gamma p}]\hat a^{-1}_{pqkl}$ is the elastic stress concentration tensor [2], $\hat\sigma'_{kl}(x)=-T(x)\hat a_{klpq}(x)\hat\alpha_{pq}(x)$ is the macro-thermo-stress at constrained deformation, and $\sigma''_{ij}(x,\xi):=-T(x)a_{ijkl}(\xi)[\alpha_{kl}(\xi)-\frac{\partial}{\partial\xi_l}z_k(\xi)]$ is the micro-thermo-stress in the periodic medium at constrained deformation.

4. Homogenization of micro-strength

The functional $\underline{\Lambda}^\varepsilon(\sigma,y)$ depends generally both on the global coordinate of the considered point y, e.g., on its distance to the boundary of the body Ω, and on the position of the point y in the periodicity cell εY, that is on the material, to which the point belongs in the cell. Suppose first $\underline{\Lambda}^\varepsilon(\sigma^\varepsilon,y)=\Lambda(\sigma^\varepsilon(y),y,\frac{y}{\varepsilon})$

P r o p o s i t i o n 1 (homogenization of local micro-strength). *Let $\sigma^\varepsilon\in C(\bar\Omega)$ converges to a function $\sigma^0(y,\zeta)\in C(\bar\Omega,(C_{per}(Y))^{3\times3})$ uniformly w.r.t. y, i.e. $\lim_{\varepsilon\to0}\sup_{y\in\Omega}|\sigma^\varepsilon(y)-\sigma^0(y,\frac{y}{\varepsilon})|=0$ and $\Lambda(\sigma,y,\zeta)\in C(\Omega,C_{per}(Y,C_0(\mathbb{R}^{3\times3})))$. Then $\lim_{\varepsilon\to0}\underline{\Lambda}^\varepsilon_\Omega(\sigma^\varepsilon):=\lim_{\varepsilon\to0}\sup_{y\in\Omega}\Lambda\left(\sigma^\varepsilon(y),y,\frac{y}{\varepsilon}\right)\le\sup_{y\in\Omega}\sup_{\zeta\in\Omega}\Lambda\left(\sigma^0(y,\zeta),y,\zeta\right)$.*

Furthermore, if σ^0 is expressed by (2), then $\lim_{\varepsilon\to0}\underline{\Lambda}^\varepsilon_\Omega(\sigma^\varepsilon)\le\sup_{y\in\Omega}\underline{\hat\Lambda}(\hat\sigma-\hat\sigma',\sigma'';y)$, $\underline{\hat\Lambda}(\hat\sigma-\hat\sigma',\sigma'';y):=\sup_{\zeta\in\Omega}\Lambda(A_{ijkl}(\zeta)(\hat\sigma_{kl}(y)-\hat\sigma'_{kl}(y))+\sigma''_{ij}(y,\zeta),y,\zeta)$. Then the limit sufficient non-local macro-strength condition is $\sup_{y\in\Omega}\underline{\hat\Lambda}(\hat\sigma-\hat\sigma',\sigma'';y)<1$.

For $\varepsilon>0$, the last strength condition can be considered as approximate.

Let us consider the limit of $\underline{\Lambda}^\varepsilon_\Omega(\sigma^\varepsilon,y)$ as $\varepsilon\to0$ for non-local micro-strength condition. Suppose $\varphi^\varepsilon(x,y)=\varphi(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon})$, where $\varphi\in C(\bar\Omega,C_{per}(Y,L^2(\Omega,C_{per}(Y))))$,

$$\tilde\sigma^\varepsilon(y)=\int_\Omega\varphi(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon})\sigma^\varepsilon(x)dx,\quad\text{and}\quad\underline{\Lambda}^\varepsilon_\Omega(\sigma^\varepsilon):=\sup_{y\in\Omega}\Lambda\left(\tilde\sigma^\varepsilon(y,\frac{y}{\varepsilon}),y,\frac{y}{\varepsilon}\right)\tag{3}$$

P r o p o s i t i o n 2 (homogenization of non-local micro-strength). *Let $\sigma^\varepsilon\in L^2(\Omega)$ be a sequence of solutions to (1). Suppose the body microstrength condition is $\underline{\Lambda}^\varepsilon_\Omega(\sigma^\varepsilon)<1$, where $\underline{\Lambda}^\varepsilon_\Omega$ is given by (3). Then $\lim_{\varepsilon\to0}\underline{\Lambda}^\varepsilon_\Omega(\sigma^\varepsilon)\le\sup_{y\in\Omega}\underline{\hat\Lambda}(\hat\sigma-\hat\sigma',\tilde\sigma'';y)$, $\underline{\hat\Lambda}(\hat\sigma-\hat\sigma',\sigma'';y):=\sup_{\zeta\in\Omega}\Lambda(\int_\Omega\hat\varphi_ijkl(y,\zeta,x)(\hat\sigma_{kl}(x)-\hat\sigma'_{kl}(x))dx+\tilde\sigma''_{ij}(y,\zeta),y,\zeta)$, $\hat\varphi_{ijkl}(y,\zeta,x)=\frac{1}{|Y|}\int_Y\varphi(x,y,\xi,\zeta)A_{ijkl}(\xi)d\xi$, $\tilde\sigma''_{ij}(y,\zeta)=\frac{1}{|Y|}\int_\Omega\int_Y\varphi(x,y,\xi,\zeta)\sigma''_{ij}(x,\xi)dxd\xi$. Finally the limit sufficient non-local macro-strength condition is $\sup_{y\in\Omega}\underline{\hat\Lambda}(\hat\sigma-\hat\sigma',\sigma'';y)<1$.*

E x a m p l e 3. *In the particular case when the non-local weight function is independent of the cell characteristics, i.e. $\varphi^\varepsilon(x,y)=\varphi(x,y)$, we have $\hat\varphi_{ih\gamma\delta}(x,y,\zeta)=\varphi(x,y)$, $\tilde\sigma''_{ij}(y,\zeta)=\int_\Omega\varphi(x,y)\hat\sigma'_{ij}(x)dx$ and $\underline{\hat\Lambda}(\hat\sigma-\hat\sigma',\sigma'';y)=\sup_{\zeta\in Y}\Lambda\left(\int_\Omega\varphi(x,y)\hat\sigma(x)dx,y,\zeta\right)$, that is the cell stress concentration and micro-thermo-stress do not influence the composite strength for sufficiently small cells obeying the non-local strength condition.*

5. References

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