Proceedings of the Seventh International Conference on Biaxial/Multiaxial Fatigue & Fracture, DVM, Berlin, 2004, 409-414

LOCAL AND NON-LOCAL NORMALISED EQUIVALENT STRAIN FUNCTIONALS FOR CYCLIC FATIGUE

S.E. Mikhailov and I.V. Namestnikova
Dept. of Mathematics, Glasgow Caledonian Univ., Glasgow, G4 0BA, UK
s.mikhailov@gcal.ac.uk, i.namestnikova@gcal.ac.uk

ABSTRACT

The concept of Cyclic Normalised Equivalent Strain Functionals (CNESnFs) suitable for low-cyclic fatigue is developed in this paper. It reduces any fatigue strength and durability model to a unique form what facilitates the models comparison. Generalisation of some known fatigue durability models and introduction of new linear and nonlinear ones including those sensitive to the load sequence, are made. The functional concept allows to unite the strength and durability description under static, fatigue, creep and dynamic loading and their combinations. Examples of such new complex functional strength conditions are presented. The non-local versions of the CNESnFs applicable to strain fields sharply varying in space coordinates and to crack initiation and propagation analysis are also presented.

KEYWORDS

Durability; low-cyclic fatigue; multiaxial non-regularly oscillating loading; damage accumulation; local and non-local cyclic strength conditions; stress/strain concentration.

INTRODUCTION

There exist a number of cyclic fatigue theories accounting for damage accumulation under variable load intensity, multiaxiality and non-proportional loading. However their forms are often incomparable analytically and their refinements to include some additional observed fatigue effects are not evident. When one tries to develop a general approach to description of fatigue strength and durability for different classes of materials under complex multiaxial loading, one has to reduce all known fatigue theories to a unique comparable form and make sure the model refinements to include, e.g., overloading and sensitivity to the load sequence, are sufficiently simple. For a material under uniaxial or multiaxial non-regularly oscillating stress, such a theory was given in [6] by phenomenological strength conditions presented in terms of the Cyclic Normalised Equivalent Stress Functionals (CNESFs). Since the strain history (strain process) is more representative for low-cyclic fatigue, the concept of Cyclic Normalised Equivalent Strain Functionals (CNESnFs) will be developed in this paper.

To overcome the well-known shortcomings of the the traditional local approaches, such as inability to describe adequately the high stress concentration, short crack effects and other scale effects caused by the material pronounced micro-structure, non-local strength conditions and fracture criteria may be employed for time and history independent materials under monotone ("static") loading (see e.g. [8, 1, 2, 3, 4]). The non-local conditions use stresses or strains obtained on the assumption of solid macro-homogeneity by solving usual local boundary value problems (e.g. of elasticity). All micro-heterogeneity is then implicitly taken into account by a corresponding non-locality of the strength condition. Non-local condition parameters should be obtained from macro-tests, although they could in principle be estimated also by developing adequate micro-mechanical models where local strength conditions operate. Some modifications of the "static" non-local approaches to fatigue crack initiation stage were used in [9, 10, 11].

In this paper, we will describe the local and the non-local versions of the functional approach to fatigue, applicable to both the crack initiation and propagation.

LOCAL CYCLIC NORMALISED EQUIVALENT STRAIN FUNCTIONAL AND STRENGTH CONDITION

A general functional approach to the strength analysis of time dependent materials was described in [5]. It may still be applied to fatigue analysis using time as a natural parameter, however a special cyclic parameterisations seem to be more relevant for the cyclic fatigue. The corresponding cyclic approach, based on the *stress* history analysis was developed in [6, 7]. Here we present its counterpart based on the *strain* history, applicable also to low-cyclic fatigue.

Let $\{\varepsilon\} = \{\varepsilon^c(m)\}_{m=1,2,...}$ be a cyclic, generally non-periodic, multiaxial strain process independent on the space coordinates, where the tensor function $\varepsilon^c(m) = \{\varepsilon_{ij}(\tau); \tau_{m-1} \leq \tau \leq \tau_m\}$ is an $m-$th cycle and τ is time. Denoting by $n^*\{\varepsilon\}$ the durability (number of cycles to rupture) under a process $\{\varepsilon\}$, the notion of the generalised strain–durability Sn–N diagram, $\lambda \mapsto n^*\{\lambda\varepsilon\}$, (similar to the Coffin-Manson plot for plastic strain amplitude) may be introduced, where λ is a non-negative number by which the process $\{\varepsilon\}$ is multiplied. Then the cyclic functional safety factor, the cyclic normalised equivalent strain functional and the cyclic endurance normalised equivalent strain can be defined:

The cyclic strain functional safety factor $\underline{\lambda}^N(\{\varepsilon\};n)$ *is supremum of* $\lambda \geq 0$ *such that* $n^*\{\lambda''\epsilon\} > n$ *for any* $\lambda'' \in [0,\lambda]$*; if there is no such* λ*, we take* $\underline{\lambda}(\{\varepsilon\};n) = 0$*. The cyclic normalised equivalent strain functional, CNESnF, is defined as* $\underline{\Lambda}^N(\{\varepsilon\};n) := 1/\underline{\lambda}^N(\{\varepsilon\};n)$ *if* $\underline{\lambda}^N(\{\varepsilon\};n) \neq 0$*, and* $\underline{\Lambda}^N(\{\varepsilon\};n) := \infty$ *otherwise. The endurance normalised equivalent strain functional is* $\underline{\Lambda}^N_{th}(\{\varepsilon\}) = \underline{\Lambda}^N(\{\varepsilon\};\infty)$*.*

Note that the Cyclic Normalised Equivalent Strain Functional is a counterpart of the Cyclic Normalised Equivalent Stress Functional introduced in [6, 7] and coincides with it for linear elastic materials. As follows from the definition, $\underline{\Lambda}^N(\{\varepsilon\};n)$ is positively homogeneous in $\{\varepsilon\}$ and monotonously non-decreasing in n. The cyclic stable strength condition for a cycle n can be written in the form

$$\underline{\Lambda}^N(\{\varepsilon\};n) < 1.$$

The CNESnF $\underline{\Lambda}^N$ is a material characteristic which is not necessary connected with a geometrical, stiffness-related or abstract damage measure and can be identified from the cyclic strain durability tests under homogeneous strain process fields. Using the above definition, any cyclic strength condition written in terms of a damage measure can be expressed in terms of a corresponding $\underline{\Lambda}^N$, although not always analytically, and easily generalised to include e.g. instant overloading, creep or dynamic effects.

For *periodic* multiaxial fatigue processes (i.e. with the same strain loop ε^c_{ij} on all cycles), one can easily determine the CNESnFs $\underline{\Lambda}^N(\{\varepsilon^c\};n) = \underline{\Lambda}^{N*}(\varepsilon^c;n)$ from the corresponding Sn–N diagrams. Suppose, e.g., a material strength under a periodic multiaxial in-phase loading is described by a power Sn–N diagram (Basquin type relation) $n^*\{\varepsilon\} = (\|\varepsilon^c\|/\varepsilon^*(R))^{-b}$, where $\|\varepsilon^c\| = \sup_{\varepsilon\in\varepsilon^c}|\varepsilon|$; $|\varepsilon|$ is a matrix norm of the tensor function ε, e.g., $|\varepsilon| = \sqrt{\sum_{i,j=1}^3 \varepsilon_{ij}^2}$; $\varepsilon^*(R)$ and $b = b(R)$ are positive material characteristics depending generally on the cycle ε^c shape in the strain space, i.e. on the tensor counterpart $R_{ij} := \varepsilon^c_{ij}/\|\varepsilon\|$ of the asymmetry ratio but not on the cycle norm $\|\varepsilon^c\|$. Then we have the following CNESnF and fatigue strength condition,

$$\underline{\Lambda}^{N*}(\varepsilon^c;n) = \frac{\|\varepsilon^c\|}{\varepsilon^*(R)} n^{1/b} < 1. \tag{1}$$

The Sines type fatigue strength condition can be rewritten in terms of the associated CNESnF,

$$\underline{\Lambda}^{N*}(\varepsilon^c; n) = \frac{\sigma_{a,eq}}{\sigma^*_{-1}(n)} + 3\sigma_{h,m}\left[\frac{1}{\sigma^*_0(n)} - \frac{1}{\sigma^*_{-1}(n)}\right] < 1, \tag{2}$$

whereas the Crossland type fatigue strength condition can be rewritten as

$$\underline{\Lambda}^{N*}(\varepsilon^c; n) = \sigma_{eq,a}\left[\frac{2}{\sigma^*_{-1}(n)} - \frac{1}{\sigma^*_0(n)}\right] + 3\sigma_{h,max}\left[\frac{1}{\sigma^*_0(n)} - \frac{1}{\sigma^*_{-1}(n)}\right] < 1, \tag{3}$$

Here $\sigma_{a,eq} = \mu\varepsilon_{a,eq}$ is the von Mises form of the stress amplitude; $\sigma_{h,m} = K\varepsilon_{h,m} = K\varepsilon_{ii,m}/3$ is the mean hydrostatic stress, and $\sigma_{h,max} = K\varepsilon_{h,max} = K\varepsilon_{ii,max}/3$ is the maximum hydrostatic stress during the cycle; μ and K are the shear and volume expansion elastic moduli; $\sigma^*_{-1}(n)$ and $\sigma^*_0(n)$ are the (amplitude) S–N diagrams for the same material under uniaxial periodic loading with the asymmetry ratios $R = -1$ and $R = 0$, respectively.

Palmgren-Miner linear accumulation rule

For a Basquin type material under a multiaxial cyclic (non-periodic) in-phase *self–similar* loading or for b independent on R, the Palmgren-Miner linear accumulation rule leads to the following CNESnF and strain strength condition, c.f. [7],

$$\underline{\Lambda}^N_0(\{\varepsilon\}; n) = \left[\sum_{m=1}^{n}\left(\frac{\|\!|\varepsilon^c(m)|\!\|}{\varepsilon^*(R(m))}\right)^b\right]^{1/b} = \left[\int_0^n \left(\frac{\|\!|\varepsilon^c(m)|\!\|}{\varepsilon^*(R(m))}\right)^b dm\right]^{1/b} < 1, \tag{4}$$

where $\|\!|\varepsilon^c(m)|\!\|$ is a piece-wise constant function of the cycle number m, $\varepsilon^*(m) = \varepsilon^*(R(m))$ and $b = b(R)$ should be taken as positive material characteristics in the power Sn–N diagram for the corresponding multiaxial *periodic* process.

If one would like to take into account an influence of both the instantaneous overloads of material and creep durability, one can add a term with $\varepsilon_{eq}(\varepsilon)$ (e.g. von Mises, Tresca or other instantaneous equivalent strain) and a term $\underline{\Lambda}^T_0(\varepsilon; t') = \left[\frac{1}{\varepsilon^*_T}\int_0^{t'} |\varepsilon(\tau)|^{b_T} d\tau\right]^{1/b_T}$ connected with the linear Robinson rule of time-dependent damage accumulation and power time-durability diagram [5], and arrive at another, complex NESnF and local strength condition,

$$\underline{\Lambda}^{ITN}_0(\{\varepsilon\}; n(t)) = \sup_{0\le t'\le t}\left\{\frac{\varepsilon_{eq}(\varepsilon(t'))}{\varepsilon_r} + \underline{\Lambda}^T_0(\varepsilon; t') + \underline{\Lambda}^N_0(\{\varepsilon\}; n(t'))\right\} < 1 \tag{5}$$

where ε_r is a material strain strength under uniaxial monotone tensile loading; $n(t)$ is an integer-valued function of time t. The term $\underline{\Lambda}^T_0(\varepsilon; t')$ can be also considered as describing dynamic strength, c.f. [5].

Note that strength condition (5) lead to an accumulation rule sensitive to the loading sequence. One can create not only sums but also other homogeneous combinations of those three terms to get other possible simple forms of the NESnF describing instant, time-dependent and cycle-dependent effects on the durability.

Linear and non-linear accumulation rules for CNESnF partial increments

Let us consider another type of accumulation rules sensitive to loading sequence. In contrast to the Palmgren-Miner linear accumulation rule for partial life–times, we can write a linear (c.f. [6]) or non-linear accumulation rule for partial increments of CNESnF.

Let a CNESnF $\underline{\mathbf{\Lambda}}^{N*}(\varepsilon^c;n)$ for a *periodic* multiaxial fatigue process, i.e. with equal strain loop ε^c_{ij} on all cycles, be known, for example be given by one of expressions (1)-(3). Then a CNESnF and corresponding strength condition for a *non-periodic* oscillating process $\{\varepsilon\}$ can be taken in the following linear form

$$\begin{aligned}\underline{\mathbf{\Lambda}}^N(\{\varepsilon\};n) &= \max_{1\le n'\le n}\Bigg\{\underline{\mathbf{\Lambda}}^{N*}(\varepsilon^c(n');1)+\\ &\sum_{m=1}^{n'-1}\Big[\underline{\mathbf{\Lambda}}^{N*}(\varepsilon^c(m);n'-m+1)-\underline{\mathbf{\Lambda}}^{N*}(\varepsilon^c(m);n'-m)\Big]\Bigg\}<1. \qquad (6)\end{aligned}$$

Non-linear versions of the accumulation law can be also introduced. For example, one can consider the following one with the power-type nonlinearity,

$$\begin{aligned}\underline{\mathbf{\Lambda}}^N(\{\varepsilon\};n) &= \max_{1\le n'\le n}\Bigg\{\Big(\underline{\mathbf{\Lambda}}^{N*}(\varepsilon^c(n');1)\Big)^{\beta}+\\ &\sum_{m=1}^{n'-1}\Big[\Big(\underline{\mathbf{\Lambda}}^{N*}(\varepsilon^c(m);n'-m+1)\Big)^{\beta}-\Big(\underline{\mathbf{\Lambda}}^{N*}(\varepsilon^c(m);n'-m)\Big)^{\beta}\Big]\Bigg\}^{\frac{1}{\beta}}<1. \qquad (7)\end{aligned}$$

Here the constant $\beta > 0$ is a material parameter.

To obtain the fatigue strength condition for the Basquin, the Sines, or the Crossland type material under non-periodic cyclic loading, one has to substitute, respectively, expressions (1), (2) or (3) in (6) or (7). Similarly, one can use the linear (6) or non-linear (7) accumulation laws to extend the Dang Van, Kakuno, Mucha and other periodic fatigue strength conditions to the non-periodic cycling. One can check that (6) and (7) interpolate the Sn–N diagram, that is, degenerate into the corresponding strength conditions for a *periodic* processes if $\varepsilon^c(n) = \varepsilon^c$. Note that the strength conditions (6) and (7) are sensitive to the loading sequence in contrast to the Palmgren–Miner rule. They account for instant overloading and can be also combined with the creep/dynamic strength conditions similar to (5).

Local quasi-ductile and brittle fatigue strength conditions

The above fatigue strength conditions can be employed not only to the strain fields independent on space coordinates but also to ones moderately varying in the coordinates, leading to the so-called local cyclic quasi-ductile strength conditions at any material point y.

To describe cyclic *fracture*, i.e. crack initiation and propagation under cycling loading, one have to analyse the *brittle* strength, that is strength at a particular point y along a particular infinitesimal plane with a normal vector $\vec{\zeta}$ at that point. The local brittle cyclic strength condition *for a plane* $\vec{\zeta}$ *at a point* y can be taken in the form

$$\underline{\mathbf{\Lambda}}(\{\varepsilon^c(y)\};n,y,\vec{\zeta}) < 1. \qquad (8)$$

Here $\underline{\mathbf{\Lambda}}(\{\varepsilon^c(y)\};n,y,\vec{\zeta})$ is a local *brittle* cyclic normalised equivalent strain functional, which is defined similar to the above and is a material characteristics positively homogeneous in $\{\varepsilon^c_{ij}\}$ and non-decreasing in n. Particular forms of the brittle NESnFs can be obtained from (4)-(7), where the strain tensor loop sequence $\{\varepsilon^c_{ij}(m;y)\}_{m=1,2,...}$ is replaced by the strain vector loop sequence $\{\vec{\varepsilon}^c(m;y,\vec{\zeta})\}_{m=1,2,...}$ on the plane $\vec{\zeta}$.

There is no fracture in the body D if inequality (8) is satisfied on all planes $\vec{\zeta}$ at all points $y \in D$ but a crack nucleates or initiates on the plane $\vec{\zeta}^*(y^*)$ at the point y^* during the cycle $n^*(D)$, where and when the inequality violates. During the next cycles, the crack set Y^*

propagates and one can track it taking into account the strain field redistribution on each cycle due to the body shape change when a new portion of Y^* becomes a part of the body boundary. This allows to describe the growth of the CNESnF $\underline{\Lambda}^N(\{\varepsilon(D)\}; n, y_0^*, \vec{\zeta})$ on each plane at each point before and after the fracture initiation as well as the cyclic crack propagation through the damaged material points and planes.

Particularly, local strength condition (8) and CNESnF associated with (4) can be used for crack initiation as well as crack propagation analysis (for the CNESF it was shown in [7]). However, it works only for the power $b < 2$ in the Basquin relation and the crack start delay is not predicted. Due to the strain singularity at the crack tip, the CNESnFs like (6) with instant terms are not applicable to fatigue crack analysis unless their corresponding non-local counterparts are used.

NON-LOCAL CYCLIC NORMALISED EQUIVALENT STRAIN FUNCTIONAL AND STRENGTH CONDITION

To analyse strength and durability under oscillating in time and highly inhomogeneous strain fields and predict both the crack initiation and propagation as a united process, we merge in this section the above functional approach to the cyclic strength with the non-local approach of [3] and present a strain based counterpart of the corresponding stress-based non-local cyclic strength condition [6, 7].

In the non-local version of *brittle* fracture, we will suppose that *cyclic strength of an infinesimal plane $\vec{\zeta}$ at a point $y \in D$ does depend not only on the cyclic strain history at the point, $\{\varepsilon^c(m; y)\}_{m=1,2,...}$, but also on the stress history in its neighbourhood and generally, in the whole of the body, $\{\varepsilon\} = \{\varepsilon^c(m; x)\}_{m=1,2,...}, \quad x \in D$.*

Then we can repeat for a plane $\vec{\zeta}$ at a point y the reasonings of the previous section understanding under $\{\varepsilon\} = \{\varepsilon^c(m)\}_{m=1,2,...}$ the cyclic process field in D and arrive at the notions of non-local cyclic durability $n^{*\odot}(\{\varepsilon\}; y, \vec{\zeta})$, CNESnF $\underline{\Lambda}^{N\odot}(\{\varepsilon\}; n, y, \vec{\zeta})$ and the non-local cyclic strength condition *for a plane $\vec{\zeta}$ at a point $y \in D$*,

$$\underline{\Lambda}^{N\odot}(\{\varepsilon\}; n, y, \vec{\zeta}) < 1. \tag{9}$$

One can obtain the simplest examples of the non-local CNESnFs and strain strength conditions for a plane $\vec{\zeta}$ by replacing the local strain $\varepsilon_{ij}(\tau; y)$ by its non-local counterpart $\varepsilon_{ij}^{\odot}(\tau; y, \vec{\zeta})$ in the local CNESnFs, e.g. in $\underline{\Lambda}_0^N(\{\varepsilon(y)\}; n)$, $\underline{\Lambda}_0^{IN}(\{\varepsilon(y)\}; n(t))$ or $\underline{\Lambda}_0^{ITN}(\{\varepsilon(y)\}; n(t))$ above. Here $\varepsilon_{ij}^{\odot}(y, \vec{\zeta})$ is a weighted average of $\varepsilon_{ij}(x)$ along some neighbourhood (non-locality zone) $\Omega(y, \vec{\zeta})$ of y depending also on $\vec{\zeta}$. That is, one can take,

$$\underline{\Lambda}^{N\odot}(\{\varepsilon\}; n, y, \vec{\zeta}) = \underline{\Lambda}^N(\{\varepsilon^{\odot}(y, \vec{\zeta})\}; n), \quad \varepsilon_{ij}^{\odot}(\tau; y, \vec{\zeta}) = \int_{\Omega(y,\vec{\zeta})} w(x, y, \vec{\zeta})\varepsilon_{ij}(\tau; x)dx, \tag{10}$$

where $w(x, y, \vec{\zeta})$ as well as $\Omega(y, \vec{\zeta}) \subset D$ are characteristics of material and (generally) of the body D shape, such as $\int_{\Omega(y,\vec{\zeta})} w(x, y, \vec{\zeta})dx = 1$. Particularly, $\Omega(y, \vec{\zeta})$ can be taken as a disc of a diameter $2d$ (or a segment of a length $2d$ for a two-dimensional body D) in the plane $\vec{\zeta}$ with the center in y (or as intersection of the disc/segment with D for y near the boundary of D), where d is considered as a material parameter, and $w(x, y, \vec{\zeta}) = 1/|\Omega(y, \vec{\zeta})|$ where $|\Omega(y, \vec{\zeta})|$ is the area/length of $\Omega(y, \vec{\zeta})$.

As demonstrated in [7], the fatigue crack propagation numerical analysis using CNESnF is applicable for the Basquin type material with any power $b > 0$ of the Sn–N diagram and can predict the crack start delay and short–crack effect.

Note that one can similarly introduce the non-local cyclic *quasi-ductile* strength conditions and normalised equivalent strain functionals, if one discards above in this section the infinitesimal plane $\vec{\zeta}$ and considers $\Omega(y)$ in (10) as a 3D domain for the 3D body or, respectively, 2D domain for the 2D body.

CONCLUSION

Introduction of the notion of normalised equivalent strain functional and its use in the strength conditions allowed to unite the strength and durability description under static, fatigue, creep and dynamic loading and their combinations often appearing in practice.

Unlike some damage measures usually used for such purposes, the functionals are mechanically meaningful material characteristics, which can be experimentally determined and/or verified. This makes a refinement of the CNESnF approximations to reflect e.g. influence of the load sequence, more straightforward and justified than for damage measures.

The non-local versions of the CNESnFs allow to unite the cyclic strength, durability and fatigue crack propagation analysis including crack initiation and propagation through the damaged material. It makes also possible to analyse the fatigue strength and durability of structure elements with singular stress concentrators and describe the short–crack effects and crack start delay.

REFERENCES

[1] S. Bavaglia and S. E. Mikhailov. Application of a non-local failure criterion to a crack in heterogeneous media. In: A. Carpentery, C. A. Brebbia (eds), Damage and Fracture Mechanics: Computer aided assessment and control. Southampton-Boston: Computational Mechanics Publ., 1998, pp. 155–164.

[2] L. P. Isupov and S. E. Mikhailov. Comparative analysis of several non-local fracture criteria. Archive of Applied Mechanics 68 (1998) 597–612.

[3] S. E. Mikhailov. A functional approach to non–local strength conditions and fracture criteria. **I**. & **II**. Engng Fracture Mech. 52 (1995) 731–754.

[4] S. E. Mikhailov. Non-local strength conditions based on generalized δ_c cohesive models. ZAMM 40 (2000), S483–S486.

[5] S. E. Mikhailov. Theoretical backgrounds of durability analysis by normalized equivalent stress functionals. Mathematics and Mechanics of Solids 8 (2003) 105–142.

[6] S. E. Mikhailov and I. V. Namestnikova. Fatigue strength and durability analysis by normalised equivalent stress functionals. In: Proceedings of the 9th International Conference on the Mechanical Behaviour of Materials, ICM9, Geneva, 2003, 10p.

[7] S.E. Mikhailov and I.V. Namestnikova. Local and non-local approaches to fatigue crack initiation and propagation. In: A.B. Movchan (ed.) IUTAM Symposium on Asymptotics, Singularities and Homogenisation in Problems of Mechanics. Kluwer, 2003, pp. 285–294.

[8] H. J. Neuber. Theory of notch stresses: Principles for exact stress calculation. Ann Arbor, Michigan: J.W.Edwards, 1946.

[9] A. Seweryn and Z. Mróz. On the criterion of damage evolution for variable multiaxial stress states. Internat. J. Solids and Structures, 35 (1998) 1589–1616.

[10] D. Taylor. A mechanistic approach to critical-distance methods in notch fatigue. Fatigue and Fracture of Eng. Mater. and Struct. 24 (2001) 215–224.

[11] W. X. Yao, K. Xia, and Y. Gu. On the fatigue notch factor, k_f. Internat. J. of Fatigue 17 (1995) 245–251.

Contact address: s.mikhailov@gcal.ac.uk