Preprint of the paper published in: *Advances in Boundary Integral Methods — Proceedings of the 5th UK Conference on Boundary Integral Methods* (Edited by Ke Chen), University of Liverpool Publ., UK, ISBN 0 906370 39 6, 2005, 106-125.

Chapter 12

Analysis of extended boundary-domain integral and integro-differential equations of some variable-coefficient BVP

S.E. Mikhailov
Division of Mathematics
Glasgow Caledonian University, UK
Email: s.mikhailov@gcal.ac.uk

Abstract. *For a function from the Sobolev space $H^1(\Omega)$ definitions of non-unique external and unique internal co-normal derivatives are considered, which are related to possible extensions of a partial differential operator and its right hand side from the domain Ω, where they are prescribed, to the domain boundary, where they are not.*

The notions are then applied to formulation and analysis of direct boundary-domain integral and integro-differential equations (BDIEs and BDIDEs) based on a specially constructed parametrix and associated with the Dirichlet boundary value problems for the "Laplace" linear differential equation with a variable coefficient and a rather general right hand side.

The BDI(D)Es contain potential-type integral operators defined on the domain under consideration and acting on the unknown solution, as well as integral operators defined on the boundary and acting on the trace and/or co-normal derivative of the unknown solution or on an auxiliary function. Solvability, solution uniqueness, and equivalence of the BDIEs/BDIDEs/BDIDPs to the original BVP are investigated in appropriate Sobolev spaces.
Keywords. *Partial differential equation, variable coefficients, Sobolev spaces, external and internal co-normal derivatives, parametrix, integral equations, integro-differential equations, equivalence, invertibility.*

12.1 Introduction

Many applications in science and engineering can be modeled by boundary-value problems for equations with variable coefficients. Reduction of the BVPs with arbitrarily variable coefficients to boundary integral equations is usually not possible, since the fundamental solution necessary for such reduction is generally not available in an analytical form (except for some special dependence of the coefficients on coordinates). Using a parametrix (Levi function) as a substitute of a fundamental solution, it is possible however to reduce such a BVP to a boundary-domain integral equation (see e.g. [7, 6], [15, Sect. 18], [16, 17], where the Dirichlet, Neumann and

Robin problems for some PDEs were reduced to indirect BDIEs). However, many questions about their equivalence to the original BVP, solvability, solution uniqueness and invertibility of corresponding integral operator remained open.

In [1, 2, 12], the 3D mixed (Dirichlet-Neumann) boundary value problem (BVP) for the variable-coefficient "Laplace" equation with a square integrable right hand side was considered. Such equations appear e.g. in electrostatics, stationary heat transfer and other diffusion problems for inhomogeneous media. The BVP has been reduced to either segregated or united direct Boundary-Domain Integral or Integro-Differential Equations, BDI(D)Es, or Boundary-Domain Integro-Differential Problems, BDIDPs. Some of the BDI(D)Es/BDIDPs are associated with the BDIDE and BDIE formulated in [11]. Although several of the integral and integro-differential formulations for the mixed problem in [1, 2, 12] look like equations of the second kind, the spaces for the out-of-integral terms are different from the spaces for the right hand sides of the equations, thus the equations are of "almost" second kind. Note that genuinely second kind BDIEs and BDIDEs can be obtained for the pure Dirichlet BVP with a variable coefficient and square integrable right-hand side, as shown in [13].

While considering a second order partial differential equation for a function from the Sobolev space $H^1(\Omega)$, with a rather general right-hand side, a co-normal *external* derivative operator is usually defined with the help of the first Green identity, since the function derivatives do not generally exist on the boundary in the trace sense. However this definition is related to an extension of the PDE and its right hand side from the domain Ω, where they are prescribed, to the domain boundary, where they are not. Since the extension is non-unique, the co-normal derivative appears to be a non-unique operator, which is also non-linear in u unless a linear relation between u and the PDE right hand side extension is enforced. This creates some difficulties particularly in formulating the so-called united boundary-domain integro-differential equations.

To avoid this, we introduce in this paper a subspace of $H^1(\Omega)$, which is mapped by the PDE operator into the space $\widetilde{H}^{-1/2}(\Omega)$ for the right hand sides. This allows to define an *internal* co-normal derivative operator, which is unique, linear in u and coincides with the co-normal derivative in the trace sense if the latter does exist. The approach is applied to formulation and analysis of direct segregated and direct united BDIEs/BDIDE/BDIDP equivalent to the Dirichlet BVP for the "Laplace" PDE with a variable coefficient and right hand side from $\widetilde{H}^{-1}(\Omega)$ and $\widetilde{H}^{-1/2}(\Omega)$. Equivalence of the considered BDI(D)Es/ BDIDP to the original BVP is proved along with their solvability, solution uniqueness, and the operator invertibility in corresponding Sobolev-Slobodetski spaces. It is particularly shown that the Dirichlet problem can be reduced to genuine second-kind integral or integro-differential equations.

12.2 Co-normal derivatives and the boundary value problem

Let Ω be a bounded open three–dimensional region of $\mathbb{R}^3$. For simplicity, we assume that the boundary $\partial\Omega$ is a simply connected, closed, infinitely smooth surface. Let $a \in C^\infty(\overline{\Omega})$, $a(x) > 0$ for $x \in \overline{\Omega}$. Let also $\partial_{x_j} := \partial/\partial x_j$ $(j = 1, 2, 3)$, $\partial_x = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.

We consider the scalar elliptic differential equation, which for sufficiently smooth u has the following form,

$$Lu(x) := L(x, \partial_x)\, u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \Big(a(x)\, \frac{\partial u(x)}{\partial x_i} \Big) = f(x), \quad x \in \Omega, \tag{12.1}$$

where u is an unknown function and f is a given function in Ω.

In what follows $H^s(\Omega) = H^s_2(\Omega)$, $H^s(\partial\Omega) = H^s_2(\partial\Omega)$ are the Bessel potential spaces, where $s \in \mathbb{R}$ is an arbitrary real number (see, e.g., [8], [19]). We recall that H^s coincide with the Sobolev–Slobodetski spaces W^s_2 for any non-negative or integer s.

We denote by $\widetilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^3)$, $\widetilde{H}^s(\Omega) := \{g :\ g \in H^s(\mathbb{R}^3),\ \mathrm{supp}\, g \subset \overline{\Omega}\}$, while $H^s(\Omega)$ denotes the space of restriction on Ω of distributions from $H^s(\mathbb{R}^3)$, $H^s(\Omega) := \{r_\Omega g :\ g \in H^s(\mathbb{R}^3)\}$, where r_Ω denotes the restriction operator on Ω. We will also use notation $g|_\Omega := r_\Omega g$. We denote by $H^s_{\partial\Omega}$ the following subspace of $H^s(\mathbb{R}^3)$ (and $\widetilde{H}^s(\Omega)$), $H^s_{\partial\Omega} := \{g :\ g \in H^s(\mathbb{R}^3),\ \mathrm{supp}\, g \subset \partial\Omega\}$.

From the trace theorem (see e.g. [8, 19, 4, 9]) for $u \in H^1(\Omega)$, it follows that $u^+ := \tau^+ u \in H^{\frac{1}{2}}(\partial\Omega)$, where τ^+ is the trace operator on $\partial\Omega$ from Ω.

For $u \in H^2(\Omega)$ we can denote by T^+ the corresponding co–normal differentiation operator on $\partial\Omega$ in the sense of traces,

$$T^+(x, n^+(x), \partial_x)\, u(x) := \sum_{i=1}^{3} a(x)\, n_i^+(x) \left(\frac{\partial u(x)}{\partial x_i}\right)^+ = a(x) \left(\frac{\partial u(x)}{\partial n^+(x)}\right)^+,$$

where $n^+(x)$ is the outward (to Ω) unit normal vectors at the point $x \in \partial\Omega$.

Let us denote

$$\mathcal{E}(u, v) := \int_\Omega \sum_{i=1}^{3} a(x)\, \frac{\partial u(x)}{\partial x_i}\, \frac{\partial v(x)}{\partial x_i} dx.$$

Let $u \in H^1(\Omega)$. Then the Lu is understood as the following distribution,

$$\langle Lu, v\rangle_\Omega := -\mathcal{E}(u, v) \quad \forall v \in C^\infty_{comp}(\Omega). \tag{12.2}$$

The duality brackets $\langle\, g,\ \cdot\, \rangle_\Omega$ denote value of a linear functional (distribution) g, extending the usual L_2 inner product.

Since the set $C^\infty_{comp}(\Omega)$ is dense in $\widetilde{H}^1(\Omega)$, the above formula defines a bounded operator $L : H^1(\Omega) \to H^{-1}(\Omega) = [\widetilde{H}^1(\Omega)]^*$,

$$\langle Lu, v\rangle_\Omega := -\mathcal{E}(u, v) \quad \forall v \in \widetilde{H}^1(\Omega). \tag{12.3}$$

Let us consider also the following operator $\hat{L} : H^1(\Omega) \to \widetilde{H}^{-1}(\Omega) = [H^1(\Omega)]^*$,

$$\langle \hat{L}u, v\rangle_\Omega := -\mathcal{E}(u, v) \quad \forall v \in H^1(\Omega), \tag{12.4}$$

which is evidently bounded. For any $u \in H^1(\Omega)$, the functional $\hat{L}u$ belongs to $\widetilde{H}^{-1}(\Omega)$ and is an extension of the functional $Lu \in H^{-1}(\Omega)$ domain from $\widetilde{H}^1(\Omega)$ to $H^1(\Omega)$.

The extension is not unique, and any functional of the form

$$\hat{L}u + g, \quad g \in H^{-1}_{\partial\Omega} \tag{12.5}$$

provides another extension. On the other hand, any extension of Lu from $\widetilde{H}^1(\Omega)$ to $H^1(\Omega)$ has evidently form (12.5).

Let $u \in H^1(\Omega)$ and $Lu = f$ in Ω for some $f \in \widetilde{H}^{-1}(\Omega)$. Then one can correctly define *the generalised (or external) co–normal derivative* $\widetilde{T}^+(f, u) \in H^{-\frac{1}{2}}(\partial\Omega)$ with the help of Green's formula (c.f., for example, [3], [9, Lemma 4.3]),

$$\left\langle \widetilde{T}^+(f, u)\,,\, v^+ \right\rangle_{\partial\Omega} := \langle f, v\rangle_\Omega + \mathcal{E}(u, v) = \langle f - \hat{L}u, v\rangle_\Omega \quad \forall\, v \in H^1(\Omega). \tag{12.6}$$

Note that because of the involvement of f, the generalised co-normal derivative $\widetilde{T}^+(f, u)$ is generally *non-linear* in u. It becomes linear if a linear relation is imposed between u and f (including behaviour of the latter on the boundary $\partial\Omega$), thus fixing an extension of $f|_\Omega$ into $\widetilde{H}^{-1}(\Omega)$. For example, $f|_\Omega$ can be extended as $f = \hat{L}u$. Obviously, $\widetilde{T}^+(\hat{L}u, u)$=0.

Let us point out another case when the co-normal derivative operator becomes linear. Let a function $u \in H^1(\Omega)$ be such that $Lu = f|_\Omega$ in Ω, $f \in \widetilde{H}^t(\Omega)$, $-\frac{1}{2} \le t$. Then the distribution Lu can be extended to the functional $f \in \widetilde{H}^t(\Omega)$. Since $H^t_{\partial\Omega} = \{0\}$ for $t \ge -\frac{1}{2}$ (c.f. [9, L. 3.39]), the extension of Lu into $\widetilde{H}^s(\Omega)$ is unique, and we will call it *the canonical extension* and denote it by L^0u or still use the notation Lu if this will not lead to a confusion. For this case, we can introduce also *the canonical (or internal) co-normal derivative* $T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$,

$$\left\langle T^+u\,,\,v^+\right\rangle_{\partial\Omega} := \left\langle \widetilde{T}^+(L^0u, u)\,,\,v^+\right\rangle_{\partial\Omega} = \langle L^0u, v\rangle_\Omega \;+\; \mathcal{E}(u,v)$$
$$= \langle L^0u - \hat{L}u, v\rangle_\Omega \quad \forall\, v \in H^1(\Omega). \tag{12.7}$$

The canonical co-normal derivative is defined by the function u and operator L only and does not depend on the right hand side f (i.e. its behaviour on the boundary), unlike the generalised co-normal derivative defined in (12.6), and the operator T^+ is linear.
Note that the canonical co-normal derivative coincides with the classical co-normal derivative $T^+u = a\frac{\partial u}{\partial n}$ if the latter does exist in the trace sense.

Motivated by this, we define a subspace of $H^1(\Omega)$ for a linear operator L_* on Ω.

DEFINITION 1 *Let L_* be a linear operator on Ω and $t \ge -\frac{1}{2}$. We introduce the following subspace of $H^s(\Omega)$, $H^{s,t}(\Omega; L_*) := \{g :\; g \in H^s(\Omega),\; L_*g|_\Omega = f|_\Omega,\; f \in \widetilde{H}^t(\Omega)\}$ provided with the norm $\|g\|_{H^{s,t}(\Omega;L_*)} := \|g\|_{H^s(\Omega)} + \|L^0_*g\|_{\widetilde{H}^t(\Omega)}$, where $L^0_*g = f$ is the canonical extension of the distribution L_*g into $\widetilde{H}^t(\Omega)$.*

In this paper, we will particularly need the spaces $H^{s,-\frac{1}{2}}(\Omega, L_*)$ for L_* being either the operator L from (12.1) or the Laplace operator Δ. Since $Lu - \Delta u = \sum\limits_{i=1}^{3} \frac{\partial a}{\partial x_i}\frac{\partial u}{\partial x_i} \in H^{s-1}(\Omega)$ for $u \in H^s(\Omega)$, we have $H^{s,-\frac{1}{2}}(\Omega; L) = H^{s,-\frac{1}{2}}(\Omega; \Delta)$, $s \ge \frac{1}{2}$. Note that the spaces $H^{1,0}(\Omega; L_*)$ were used in [3, 2, 12].

If $u \in H^{1,-\frac{1}{2}}(\Omega; L)$, then evidently the canonical co-normal derivative $T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ is well defined.

Let $u \in H^{1,-\frac{1}{2}}(\Omega; L)$. Then definition (12.6) implies that the generalised co-normal derivative for any other extension $f \in \widetilde{H}^{-1}(\Omega)$ of the distribution Lu can be expressed as

$$\left\langle \widetilde{T}^+(f,u)\,,\,v^+\right\rangle_{\partial\Omega} = \left\langle T^+u\,,\,v^+\right\rangle_{\partial\Omega} + \langle f - L^0u, v\rangle_\Omega \quad \forall\, v \in H^1(\Omega). \tag{12.8}$$

Let $u \in H^1(\Omega)$ and $v \in H^{1,-\frac{1}{2}}(\Omega; L)$ be such that $Lv \in L_2(\Omega)$ in Ω. Swapping over the roles of u and v, we obtain the first Green identity defining T^+v,

$$\mathcal{E}(u,v) + \int_\Omega u(x)Lv(x)dx = \left\langle T^+v\,,\,u^+\right\rangle_{\partial\Omega}. \tag{12.9}$$

If, in addition, $Lu = f$ in Ω, where $f \in \widetilde{H}^{-1}(\Omega)$, then according to to definition of $\widetilde{T}^+(f,u)$, (12.6), the second Green identity can be written as

$$\langle f, v\rangle_\Omega \;-\int_\Omega u(x)Lv(x)dx = \left\langle \widetilde{T}^+(f,u)\,,\,v^+\right\rangle_{\partial\Omega} - \left\langle T^+v\,,\,u^+\right\rangle_{\partial\Omega}. \tag{12.10}$$

If $u \in H^{1,-\frac{1}{2}}(\Omega; L)$ and $v \in H^{1,-\frac{1}{2}}(\Omega; L)$ is such that $Lv \in L_2(\Omega)$ in Ω, then (12.10) becomes

$$\langle Lu, v\rangle_\Omega \;-\int_\Omega u(x)Lv(x)dx = \left\langle T^+u\,,\,v^+\right\rangle_{\partial\Omega} - \left\langle T^+v\,,\,u^+\right\rangle_{\partial\Omega}. \tag{12.11}$$

If, moreover, $Lu \in L_2(\Omega)$ in Ω, then we arrive at the familiar form of the second Green identity for the canonical extension and canonical co-normal derivatives

$$\int_\Omega [v(x)Lu(x) - u(x)Lv(x)]dx = \left\langle T^+u\,,\, v^+\right\rangle_{\partial\Omega} - \left\langle T^+v\,,\, u^+\right\rangle_{\partial\Omega}. \tag{12.12}$$

We will consider the following *Dirichlet boundary value problem.* Find a function $u \in H^1(\Omega)$ satisfying the conditions

$$L\,u = f \quad \text{in} \quad \Omega, \tag{12.13}$$

$$u^+ = \varphi_0 \quad \text{on} \quad \partial\Omega, \tag{12.14}$$

where $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, $f \in H^{-1}(\Omega)$ or $f \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$.

Equation (12.13) is understood in the distributional sense (12.2), and condition (12.14) in the trace sense.

Applying the first Green identity (12.9), (12.4) with $v = u$ as a solution of the homogeneous Dirichlet problem, i.e., with $f = 0$, $\varphi_0 = 0$, we have the following uniqueness theorem.

THEOREM 2 *BVP* (12.13)-(12.14) *with* $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ *and* $f \in H^{-1}(\Omega)$ *has at most one solution in* $H^1(\Omega)$.

12.3 Parametrix and potential type operators

We will say, a function $P(x, y)$ of two variables $x, y \in \Omega$ is a parametrix (the Levi function) for the operator $L(x, \partial_x)$ in $\mathbb{R}^3$ if (see, e.g., [7, 6, 15, 16, 17, 11])

$$L(x, \partial_x)\, P(x, y) = \delta(x - y) + R(x, y), \tag{12.15}$$

where $\delta(\cdot)$ is the Dirac distribution and $R(x, y)$ possesses a weak (integrable) singularity at $x = y$, i.e.,

$$R(x, y) = \mathcal{O}\left(|x - y|^{-\varkappa}\right) \quad \text{with} \quad \varkappa < 3. \tag{12.16}$$

It is easy to see that for the operator $L(x, \partial_x)$ given by the right-hand side in (12.1), the function

$$P(x, y) = \frac{-1}{4\pi\, a(y)\, |x - y|}\,, \quad x, y \in \mathbb{R}^3, \tag{12.17}$$

is a parametrix, the corresponding remainder function is

$$R(x, y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi\, a(y)\, |x - y|^3}\, \frac{\partial\, a(x)}{\partial x_i}\,, \quad x, y \in \mathbb{R}^3. \tag{12.18}$$

and satisfies estimate (12.16) with $\varkappa = 2$, due to the smoothness of the function $a(x)$.

Evidently, the parametrix $P(x, y)$ given by (12.17) is a fundamental solution to the operator $L(y, \partial_x) := a(y)\Delta(\partial_x)$ with "frozen" coefficient $a(x) = a(y)$, i.e., $L(y, \partial_x)\, P(x, y) = \delta(x - y)$.

For some scalar function g, let

$$Vg(y) := -\int_{\partial\Omega} P(x, y)\, g(x)\, dS_x, \quad y \notin \partial\Omega, \tag{12.19}$$

$$Wg(y) := -\int_{\partial\Omega} \left[\, T(x, n(x), \partial_x))\, P(x, y)\right]\, g(x)\, dS_x, \quad y \notin \partial\Omega, \tag{12.20}$$

be the single and the double layer surface potential operators.

The corresponding boundary integral (pseudodifferential) operators of direct surface values of the simple layer potential, $\mathcal{V}$, and of the double layer potential, $\mathcal{W}$, and the co-normal derivatives of the simple layer potential, $\mathcal{W}'$ and of the double layer potential, $\mathcal{L}^+$, for $y \in \partial\Omega$ are

$$\mathcal{V}\, g(y) := -\int\limits_{\partial\Omega} P(x,y)\, g(x)\, dS_x, \quad \mathcal{W}\, g(y) := -\int\limits_{\partial\Omega} \left[T_x^+\, P(x,y) \right]\, g(x)\, dS_x, \tag{12.21}$$

$$\mathcal{W}'\, g(y) := -\int\limits_{\partial\Omega} \left[T_y^+\, P(x,y) \right] g(x)\, dS_x, \quad \mathcal{L}^+ g(y) := T^+ W g(y). \tag{12.22}$$

The parametrix-based volume potential operator and the remainder potential operator, corresponding to parametrix (12.17) and to remainder (12.18) are

$$\mathcal{P}g(y) := \int\limits_{\Omega} P(x,y)\, g(x)\, dx, \quad \mathcal{R}g(y) := \int\limits_{\Omega} R(x,y)\, g(x)\, dx. \tag{12.23}$$

For $g_1 \in H^{-\frac{1}{2}}(\partial\Omega)$, and $g_2 \in H^{\frac{1}{2}}(\partial\Omega)$, there hold the jump relations on $\partial\Omega$

$$[Vg_1(y)]^+ = \mathcal{V}g_1(y), \quad [Wg_2(y)]^+ = -\frac{1}{2}\, g_2(y) + \mathcal{W}g_2(y), \tag{12.24}$$

$$[T(y,n(y),\partial_y)Vg_1(y)]^+ = \frac{1}{2}\, g_1(y) + \mathcal{W}'g_1(y), \quad y \in \partial\Omega. \tag{12.25}$$

The jump relations as well as mapping properties of potentials and operators (12.19)-(12.23) are well known for the case $a = const$. They were extended to the case of variable coefficient $a(x)$ in [1, 2], and in addition to (12.24)-(12.25) some of them are presented in the Appendix for convenience.

It is evident from definitions (12.16), (12.19), (12.20) that

$$\Delta(aVg) = 0, \quad \Delta(aWg) = 0 \text{ in } \Omega, \quad \forall g \in H^s(\partial\Omega) \quad \forall s \in \mathbb{R}. \tag{12.26}$$

Let us prove also that

$$\Delta(a\mathcal{P}g) = g \text{ in } \Omega, \quad \forall g \in \widetilde{H}^s(\Omega) \quad \forall s \in \mathbb{R}, \tag{12.27}$$

where the Laplace operator Δ is understood in the distributional sense. Indeed, property (12.27) holds true for $g \in C^\infty_{comp}(\Omega)$ since $a\mathcal{P}$ is the classical Newtonian volume potential operator. Taking into account that $C^\infty_{comp}(\Omega)$ is dense in $\widetilde{H}^s(\Omega)$ and the operators $\mathcal{P} : \widetilde{H}^s(\Omega) \to H^{s+2}(\Omega)$ and $\Delta : H^{s+2}(\Omega) \to H^s(\Omega)$ are continuous for any $s \in \mathbb{R}$, c.f. Theorem 12.8.2 in the Appendix, this implies (12.27) for $g \in \widetilde{H}^s(\Omega)$.

12.4 The third Green identities and integral relations

12.4.1 Generalised form

For $u \in H^1(\Omega)$ and $v(x) = P(x,y)$, where the parametrix $P(x,y)$ is given by (12.17), we obtain from (12.9), (12.4), (12.15) by the standard limiting procedures (cf. [15], [5, S. 3.8]) the following identity

$$u + \mathcal{R}u + Wu^+ = \mathcal{P}\hat{L}u \quad \text{in } \Omega, \tag{12.28}$$

where

$$\mathcal{P}\hat{L}u(y) := \langle \hat{L}u, P(\cdot,y) \rangle_\Omega = -\mathcal{E}(u, P(\cdot,y)) = -\int_\Omega \sum_{i=1}^{3} a(x)\, \frac{\partial u(x)}{\partial x_i}\, \frac{\partial P(x,y)}{\partial x_i}\, dx. \tag{12.29}$$

If $Lu = f|_\Omega$ in Ω, where $f \in \widetilde{H}^{-1}(\Omega)$, then recalling the definition of $\widetilde{T}^+(f,u)$, (12.6), we arrive at the *generalised third Green identity* in the following form,

$$\widetilde{G}(f,u) := u + \mathcal{R}u - V\widetilde{T}^+(f,u) + Wu^+ = \mathcal{P}f \qquad \text{in } \Omega, \tag{12.30}$$

where it was taken into account that

$$\langle \widetilde{T}^+(f,u), P^+(\cdot,y)\rangle_{\partial\Omega} = -V\widetilde{T}^+(f,u)(y), \quad \langle f, P(\cdot,y)\rangle_\Omega = \mathcal{P}f(y), \quad y{\in}\Omega.$$

For some functions f, Ψ, Φ, let us consider a more general "indirect" integral relation, associated with (12.30),

$$u(y) + \mathcal{R}u(y) - V\Psi(y) + W\Phi(y) \;=\; \mathcal{P}f(y) \quad y \in \Omega. \tag{12.31}$$

The following statement extends Lemma 4.1 from [1], where it was proved for $f \in L_2(\Omega)$.

LEMMA 3 *Let $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$, $f \in \widetilde{H}^{-1}(\Omega)$. Suppose a function $u \in H^1(\Omega)$ satisfies (12.31). Then*

$$Lu = f \quad \textit{in} \quad \Omega, \tag{12.32}$$

$$V(\Psi - \widetilde{T}^+(f,u)) - W(\Phi - u^+) = 0 \quad \textit{in} \quad \Omega, \tag{12.33}$$

$$u^+ + (\mathcal{R}u)^+ - \mathcal{V}\Psi u - \frac{1}{2}\Phi + \mathcal{W}\Phi = (\mathcal{P}f)^+ \quad \textit{on} \quad \partial\Omega, \tag{12.34}$$

$$\widetilde{T}^+(f,u) + T^+\mathcal{R}u - \frac{1}{2}\Psi - \mathcal{W}'\Psi + \mathcal{L}^+\Phi = \widetilde{T}^+(f + \mathcal{R}^0_* f, \mathcal{P}f) \quad \textit{on} \quad \partial\Omega, \tag{12.35}$$

where

$$\mathcal{R}^0_* f(y) = \begin{cases} \mathcal{R}_* f(y) & \textit{if} \quad y \in \Omega \\ 0 & \textit{if} \quad y \notin \Omega \end{cases}, \qquad \mathcal{R}_* f(y) := -\sum_{j=1}^{3} \partial_j[(\partial_j a)\mathcal{P}f](y). \tag{12.36}$$

Proof. Subtracting (12.31) from identity (12.28), we obtain

$$V\Psi(y) + W\Phi^*(y) = \mathcal{P}[\hat{L}u - f](y), \; y \in \Omega^+, \tag{12.37}$$

where $\Phi^* := u^+ - \Phi$.

Multiplying equality (12.37) by $a(y)$, applying the Laplace operator Δ and taking into account (12.26), (12.27), we get $f|_\Omega = (\hat{L}u)|_\Omega$. This means f is an extension of the distribution Lu in $\widetilde{H}^{-1}$, and u satisfies (12.32). Then (12.6) implies

$$\begin{aligned} \mathcal{P}[\hat{L}u - f](y) &= \langle \hat{L}u - f, P(\cdot,y)\rangle_\Omega \\ &= -\langle \widetilde{T}^+(f,u)\,,\, P(\cdot,y)\rangle_{\partial\Omega} = V\widetilde{T}^+(f,u), \quad y \in \Omega. \end{aligned} \tag{12.38}$$

Substituting (12.38) into (12.37) leads to (12.33). Equation (12.34) is implied by (12.24).

To prove (12.35), let us first remark that

$$L\mathcal{P}f = f + \mathcal{R}_* f \text{ in } \Omega, \tag{12.39}$$

which implies, due to (12.32), $L(\mathcal{P}f - u) = \mathcal{R}_* f$ in Ω, and $\mathcal{R}_* f \in L_2(\Omega)$ due to (12.92). Then $L(\mathcal{P}f - u)$ can be canonically extended to $L^0(\mathcal{P}f - u) = \mathcal{R}^0_* f \in \widetilde{H}^0(\Omega) \subset \widetilde{H}^{-1}(\Omega)$, where $\mathcal{R}^0_*$ is defined by (12.36). This implies that there exists a canonical co-normal derivative of $(\mathcal{P}f - u)$, which due to (12.7) is,

$$\langle T^+(\mathcal{P}f - u), v^+\rangle_{\partial\Omega} = \langle L^0(\mathcal{P}f - u) - \hat{L}\mathcal{P}f + \hat{L}u, v\rangle_\Omega = \langle \mathcal{R}^0_* f - \hat{L}\mathcal{P}f + \hat{L}u, v\rangle_\Omega =$$

$$\langle \mathcal{R}_*^0 f + f - f - \hat{L}\mathcal{P}f + \hat{L}u, v\rangle_\Omega = \langle \widetilde{L}\mathcal{P}f - \hat{L}\mathcal{P}f + \hat{L}u - f, v\rangle_\Omega =$$
$$\langle \widetilde{T}^+(\widetilde{L}\mathcal{P}f, \mathcal{P}f) - \widetilde{T}^+(f,u), v^+\rangle_{\partial\Omega} \quad \forall v \in H^1(\Omega), \quad (12.40)$$

where $\widetilde{L}\mathcal{P}f = f + \mathcal{R}_*^0 f \in \widetilde{H}^{-1}(\Omega)$ is the extension of $L\mathcal{P}f$ associated with (12.39).

From (12.31) we have $\mathcal{P}f - u = \mathcal{R}u - V\Psi + W\Phi$ in Ω. Substituting this in the left hand side of (12.40) and taking into account jump relation (12.25), we arrive at (12.35)

□

Lemma 3 and Green's identity (12.30) imply, the following

COROLLARY 4 *If $u \in H^1(\Omega)$ is such that $Lu = f$ in Ω, where $f \in \widetilde{H}^{-1}(\Omega)$, then*

$$\widetilde{\mathcal{G}}(f,u) := \frac{1}{2}u^+ + (\mathcal{R}u)^+ - \mathcal{V}\widetilde{T}^+(f,u) + \mathcal{W}u^+ = (\mathcal{P}f)^+ \quad on \quad \partial\Omega, \quad (12.41)$$

$$\widetilde{\mathcal{T}}(f,u) := \frac{1}{2}\widetilde{T}^+(f,u) + T^+\mathcal{R}u - \mathcal{W}'\widetilde{T}^+(f,u) + \mathcal{L}^+u^+$$
$$= \widetilde{T}^+(f + \mathcal{R}_*^0 f, \mathcal{P}f) \quad on \quad \partial\Omega. \quad (12.42)$$

The following statement is well known, see e.g. [1, L. 4.2] and references therein.

LEMMA 5

(i) Let $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$. If $V\Psi^*(y) = 0$, $y \in \Omega$, then $\Psi^* = 0$.

(ii) Let $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$. If $W\Phi^*(y) = 0$, $y \in \Omega$, then $\Phi^* = 0$.

THEOREM 6 *Let $f \in \widetilde{H}^{-1}(\Omega)$. A function $u \in H^1(\Omega)$ is a solution of PDE* (12.13) *in Ω if and only if it is a solution of BDIDE (12.30).*

Proof. If $u \in H^1(\Omega)$ solves PDE (12.13) in Ω, then it satisfies (12.30). On the other hand, if u solves BDIDE (12.30), then using Lemma 3 for $\Psi = \widetilde{T}^+(f,u)$, $\Phi = u^+$ completes the proof.

□

12.4.2 Canonical form

We specify here the results of Section 12.4.1 in the more narrow spaces $H^{1,-\frac{1}{2}}(\Omega; L)$, which we will especially need for the united integro-differential formulations.

Let $u \in H^{1,-\frac{1}{2}}(\Omega; L)$, then (12.28) and (12.7) imply

$$u(y) + \mathcal{R}u(y) - VT^+u(y) + Wu^+(y) = \mathcal{P}Lu(y), \quad y \in \Omega, \quad (12.43)$$

where Lu in (12.43) means the canonical extension $L^0 u$ of the distribution Lu into $\widetilde{H}^{-\frac{1}{2}}(\Omega)$.

If $u \in H^{1,-\frac{1}{2}}(\Omega; L)$ is a solution of equation (12.1) with $f \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$, then (12.43) gives

$$Lu = f \qquad \text{in } \Omega, \quad (12.44)$$

$$Gu := u + \mathcal{R}u - VT^+u + Wu^+ = \mathcal{P}f \qquad \text{in } \Omega, \quad (12.45)$$

$$\mathcal{G}u := \frac{1}{2}u^+ + (\mathcal{R}u)^+ - \mathcal{V}T^+u + \mathcal{W}u^+ = (\mathcal{P}f)^+ \qquad \text{on } \partial\Omega, \quad (12.46)$$

$$\mathcal{T}u := \frac{1}{2}T^+u + T^+\mathcal{R}u - \mathcal{W}'T^+u + \mathcal{L}^+u^+ = T^+\mathcal{P}f \qquad \text{on } \partial\Omega. \quad (12.47)$$

Equalities (12.46), (12.47) constitute a counterpart of Corollary 4 for $u \in H^{1,-\frac{1}{2}}(\Omega; L)$, $f = Lu$ if one takes into account that $\widetilde{T}^+(f,u) = T^+u$, $\widetilde{T}^+(f + \mathcal{R}_*^0 f, \mathcal{P}f) = T^+\mathcal{P}f$.

Let us consider the canonical case of indirect integral relation (12.31). Lemma 3 may be reformulated for this case as follows.

LEMMA 7 *Let $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$, $f \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$. Suppose a function $u \in H^1(\Omega)$ satisfies (12.31). Then $u \in H^{1,-\frac{1}{2}}(\Omega; L)$, it is a solution of PDE* (12.13) *in Ω and*

$$V(\Psi - T^+u) - W(\Phi - u^+) = 0 \quad \textit{in} \quad \Omega,$$
$$u^+ + (\mathcal{R}u)^+ - \mathcal{V}\Psi u - \frac{1}{2}\Phi + \mathcal{W}\Phi = (\mathcal{P}f)^+ \quad \textit{on} \quad \partial\Omega,$$
$$T^+u + T^+\mathcal{R}u - \frac{1}{2}\Psi - \mathcal{W}'\Psi + \mathcal{L}^+\Phi = T^+\mathcal{P}f \quad \textit{on} \quad \partial\Omega.$$

Proof. Equation (12.31) and mapping properties of the operators $\mathcal{R}$, $\mathcal{P}$, V and W (see the Appendix) imply $u \in H^{1,-\frac{1}{2}}(\Omega; L)$. The rest of the lemma claims follow from Lemma 3.

□

THEOREM 8 *Let $f \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$. A function $u \in H^{1,-\frac{1}{2}}(\Omega; L)$ is a solution of PDE* (12.13) *in Ω if and only if it is a solution of BDIDE* (12.45).

Proof. If $u \in H^{1,-\frac{1}{2}}(\Omega; L)$ solves PDE (12.13) in Ω, then it satisfies (12.45). On the other hand, if $u \in H^{1,-\frac{1}{2}}(\Omega; L)$ solves BDIDE (12.45), then using Lemma 7 for $\Psi = T^+u$, $\Phi = u^+$ completes the proof.

□

12.5 Segregated boundary-domain integral equations

Let us consider a segregated purely *integral* boundary-domain formulation for the Dirichlet problem, similar to the formulations introduced and analysed in [1, 2, 12] for the mixed problem with $u \in H^1(\Omega)$ and $u \in H^{1,0}(\Omega; \Delta)$ but $f \in L_2(\Omega)$. We will obtain here results for $u \in H^1(\Omega)$ with $f \in \widetilde{H}^{-1}(\Omega)$ and for $u \in H^{1,-\frac{1}{2}}(\Omega; L)$ with $f \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$.

12.5.1 Integral equation system $(G\mathcal{G})$

To reduce BVP (12.13)-(12.14) to a BDIE system in this section, we will use equation (12.30) in Ω and equation (12.41) on $\partial\Omega$, where the known function φ_0 is substituted for u^+ and an auxiliary unknown function $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$ for $\widetilde{T}^+(f, u)$. Then we arrive at the following system $(G\mathcal{G})$,

$$u + \mathcal{R}u - V\psi = F_0 \qquad \text{in } \Omega, \tag{12.48}$$
$$\mathcal{R}^+u - \mathcal{V}\psi = F_0^+ - \varphi_0 \qquad \text{on } \partial\Omega, \tag{12.49}$$

where

$$F_0 := \mathcal{P}f - W\varphi_0 \quad \text{in } \Omega. \tag{12.50}$$

Note that for $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, we have the inclusion $F_0 \in H^1(\Omega)$ if $f \in \widetilde{H}^{-1}(\Omega)$, and $F_0 \in H^{1,-\frac{1}{2}}(\Omega; L)$ if $f \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$, due to the mapping properties of the Newtonian (volume) and layer potentials, c.f. (12.85), (12.86), (12.90).

REMARK 9 *Let $f \in \widetilde{H}^{-1}(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Then $(F_0, F_0^+ - \varphi_0) = 0$ if and only if $(f,\ \varphi_0) = 0$. Indeed, the latter equality evidently implies the former. Inversely, let $(F_0, F_0^+ - \varphi_0) = 0$. Consequently $\varphi_0 = 0$. Taking in mind equation* (12.50)*, Lemma* 3 *with $u = F_0 = 0$, $\Phi = \varphi_0 = 0$, $\Psi = 0$ implies $f = 0$ in Ω (i.e. $f \in H^{-1}_{\partial\Omega}$) and $V(\widetilde{T}^+(f, 0)) = 0$ in Ω. Then Lemma 5(i) gives $\widetilde{T}^+(f, 0)) = 0$, which along with definition (12.6) means $f = 0$ in $\mathbb{R}^3$.*

Let us prove that BVP (12.13)–(12.14) in Ω is equivalent to the system of BDIEs (12.48)-(12.49).

THEOREM 10 *Let $f \in \widetilde{H}^{-1}(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$.*

(i) *If some $u \in H^1(\Omega)$ solves BVP* (12.13)–(12.14) *in Ω, then the solution is unique and the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, where*

$$\psi = \widetilde{T}^+(f, u) \quad on \quad \partial\Omega, \tag{12.51}$$

solves BDIE system (12.48)-(12.49).

(ii) *If a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system* (12.48)-(12.49)*, then the solution is unique, u solves BVP* (12.13)–(12.14)*, and ψ satisfies (12.51).*

Proof. Let $u \in H^1(\Omega)$ be a solution to BVP (12.13)–(12.14). It is unique due to Theorem 2. Setting ψ by (12.51) evidently implies $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$. Then it immediately follows from relations (12.30), (12.41) that the couple (u, ψ) solves system (12.48)-(12.49), which completes the proof of item (i).

Let now a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (12.48)-(12.49). Taking trace of equation (12.48) on $\partial\Omega$ and subtracting equation (12.49) from it, we obtain,

$$u^+(y) = \varphi_0(y), \quad y \in \partial\Omega, \tag{12.52}$$

i.e. u satisfies the Dirichlet condition (12.14).

Equation (12.48) and Lemma 3 with $\Psi = \psi$, $\Phi = \varphi_0$ imply that u is a solution of PDE (12.13) and $V\Psi^*(y) - W\Phi^*(y) = 0, \quad y \in \Omega$, where $\Psi^* = \psi - \widetilde{T}^+(f, u)$ and $\Phi^* = \varphi_0 - u^+$. Due to equation (12.52), $\Phi^* = 0$. Lemma 5(i) implies $\Psi^* = 0$, which completes the proof of conditions (12.51).

Uniqueness of the solution to BDIE system (12.48)-(12.49) follows from (12.51) along with Remark 9 and Theorem 2.

□

System (12.48)-(12.49) can be rewritten in the form

$$\mathcal{A}^{G\mathcal{G}}\mathcal{U} = \mathcal{F}^{G\mathcal{G}},$$

where $\mathcal{U}^\top := (u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$,

$$\mathcal{A}^{G\mathcal{G}} := \begin{bmatrix} I - \mathcal{R} & -V \\ \mathcal{R}^+ & -\mathcal{V} \end{bmatrix}, \quad \mathcal{F}^{G\mathcal{G}} := \begin{bmatrix} F_0 \\ F_0^+ - \varphi_0 \end{bmatrix}.$$

Due to the mapping properties of operators V, $\mathcal{V}$, W, $\mathcal{W}$, $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{R}^+$ (see [12, 2] and the Appendix), we have $\mathcal{F}^{G\mathcal{G}} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ if $f \in \widetilde{H}^{-1}(\Omega)$, and $\mathcal{F}^{G\mathcal{G}} \in H^{1,-\frac{1}{2}}(\Omega; L) \times H^{\frac{1}{2}}(\partial\Omega)$ if $f \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$, while the operators

$$\mathcal{A}^{G\mathcal{G}} \; : \; H^{1,-\frac{1}{2}}(\Omega; L) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,-\frac{1}{2}}(\Omega; L) \times H^{\frac{1}{2}}(\partial\Omega) \tag{12.53}$$

$$: \; H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{12.54}$$

are continuous. Due to Theorem 10 and the uniqueness Theorem 2, both operators (12.53), (12.54) are injective.

The proof of the following invertibility theorem below uses the scheme similar to [7, Th. 4.4].

THEOREM 11 *Operators* (12.53) *and* (12.54) *are continuous and continuously invertible.*

Proof. Let us consider the proof for the operator $\mathcal{A}^{G\mathcal{G}}$ given by (12.54) first. The continuity and injectivity is proved above. To prove the invertibility, let us consider the following operator

$$\mathcal{A}_0^{G\mathcal{G}} := \begin{bmatrix} I & -V \\ 0 & -\mathcal{V} \end{bmatrix}$$

As a result of compactness properties of the operators $\mathcal{R}$ and $\mathcal{R}^+$ (see Corollary 28), the operator $\mathcal{A}_0^{G\mathcal{G}}$ is a compact perturbation of the operator $\mathcal{A}^{G\mathcal{G}}$.

The operator $\mathcal{A}_0^{G\mathcal{G}}$ is an upper triangular matrix operator with the following scalar diagonal invertible operators

$$\begin{aligned} I &: H^1(\Omega) \to H^1(\Omega), \\ \mathcal{V} &: H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega), \end{aligned}$$

c.f. [4, Ch. XI, Part B, §2, T. 3] for $\mathcal{V}$. This implies that

$$\mathcal{A}_0^{G\mathcal{G}} : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is an invertible operator. Thus the operator $\mathcal{A}^{G\mathcal{G}}$ possesses the Fredholm property and its index is zero. The injectivity of the operator $\mathcal{A}^{G\mathcal{G}}$ already proved, completes the theorem proof for operator (12.54).

Let us now construct an inverse to operator (12.53). Let $(\mathcal{A}^{G\mathcal{G}})^{-1} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ be the operator inverse to (12.54). Thus, for any $H^{1,-\frac{1}{2}}(\Omega; L) \times H^{\frac{1}{2}}(\partial\Omega)$, the solution of the system $\mathcal{A}^{G\mathcal{G}}\mathcal{U} = \mathcal{F}^{G\mathcal{G}}$ in $H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is $\mathcal{U} = (\mathcal{A}^{G\mathcal{G}})^{-1}\mathcal{F}^{G\mathcal{G}}$. Taking into account that the operators $V : H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,0}(\Omega; \Delta)$ and $\mathcal{R} : H^1(\Omega) \to H^{2,-\frac{1}{2}}(\Omega; \Delta)$ are continuous due to Theorem 22 and Corollary 27, the first equation of this system then implies $u = \mathcal{U}_1 \in H^{1,-\frac{1}{2}}(\Omega; L)$ and the operator $(\mathcal{A}^{G\mathcal{G}})^{-1}$ is continuous also from $H^{1,-\frac{1}{2}}(\Omega; L) \times H^{\frac{1}{2}}(\partial\Omega)$ to $H^{1,-\frac{1}{2}}(\Omega; L) \times H^{-\frac{1}{2}}(\partial\Omega)$.

□

Original BVP (12.13) - (12.14) can be written in the form

$$A^D u = F^D,$$

where

$$A^D := \begin{bmatrix} L \\ \tau^+ \end{bmatrix}, \quad F^D = \begin{bmatrix} f \\ \varphi_0 \end{bmatrix}.$$

The operators

$$\begin{aligned} A^D &: H^{1,-\frac{1}{2}}(\Omega; L) \to \widetilde{H}^{-\frac{1}{2}}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) && (12.55) \\ &: H^1(\Omega) \to H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) && (12.56) \end{aligned}$$

are evidently continuous and due to the uniqueness theorem for the BVP are also injective.

COROLLARY 12 *Operators* (12.55) *and* (12.56) *are continuous and continuously invertible.*

Proof. The invertibility of the operators (12.53) and (12.54) and equivalence Theorem 10 immediately lead to the corollary claim for operator (12.55).

The claim for operator (12.56) will similarly follow if there exists a linear continuous extension operator $E : H^{-1}(\Omega) \to \widetilde{H}^{-1}(\Omega)$. Let $\tau^+ : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ be the bounded trace operator and $e : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ be a bounded extension operator, which do exist due to the trace theorem (although the latter operator is not unique). Then $u - e\tau^+u = (I - e\tau^+)u$ is a bounded projector from $H^1(\Omega)$ to $\widetilde{H}^1(\Omega)$. Thus any functional $g \in H^{-1}(\Omega)$ can be continuously mapped into the functional $\widetilde{g} \in \widetilde{H}^{-1}(\Omega)$ such that $\widetilde{g}u = g(I - e\tau^+)u$ for any $u \in H^1(\Omega)$, i.e., one can take $E = (I - e\tau^+)^*$, which finalise the proof.

□

Note that the Corollary statement for operator (12.56) is well known and can be obtained e.g. by the Lax-Milgram theorem.

12.5.2 Integral equation system $(G\mathcal{T})$

To obtain a segregated BDIE system of *the second kind*, we will use equation (12.30) in Ω and equation (12.42) on $\partial\Omega$, where again the known function φ_0 is substituted for u^+ and an auxiliary unknown function $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$ for $\widetilde{T}^+(f, u)$. Then we arrive at the following system $(G\mathcal{T})$,

$$u + \mathcal{R}u - V\psi = \mathcal{F}_1^{G\mathcal{T}} \quad \text{in} \quad \Omega, \tag{12.57}$$

$$\frac{1}{2}\psi + T^+\mathcal{R}u - \mathcal{W}'\psi = \mathcal{F}_2^{G\mathcal{T}} \quad \text{on} \quad \partial\Omega, \tag{12.58}$$

where

$$\mathcal{F}^{G\mathcal{T}} = \begin{bmatrix} \mathcal{P}f - W\varphi_0 \\ \widetilde{T}^+(f + \mathcal{R}_*^0 f, \mathcal{P}f) - \mathcal{L}^+\varphi_0 \end{bmatrix}. \tag{12.59}$$

Due to the mapping properties of the operators involved in (12.59) we have $\mathcal{F}^{G\mathcal{T}} \in H^1(\Omega^+) \times H^{-\frac{1}{2}}(\partial\Omega)$ if $f \in \widetilde{H}^{-1}(\Omega)$, and $\mathcal{F}^{G\mathcal{T}} \in H^{1,-\frac{1}{2}}(\Omega; L) \times H^{-\frac{1}{2}}(\partial\Omega)$ if $f \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$.

REMARK 13 *Let $f \in \widetilde{H}^{-1}(\Omega)$, $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\mathcal{F}^{G\mathcal{T}}$ be given by* (12.59). *Then $\mathcal{F}^{G\mathcal{T}} = 0$ if and only if $(f,\ \varphi_0) = 0$.*

Proof. The latter equality evidently implies the former. Inversely, let $\mathcal{F}^{G\mathcal{T}} = 0$. Taking in mind equation (12.59), Lemma 3 with $u = \mathcal{F}_1^{G\mathcal{T}} = 0$, $\Phi = \varphi_0$, $\Psi = 0$ gives,

$$f = 0 \quad \text{in} \quad \Omega, \tag{12.60}$$

$$-V(\widetilde{T}^+(f, 0)) - W(\varphi_0) = 0 \quad \text{in} \quad \Omega, \tag{12.61}$$

$$\widetilde{T}^+(f, 0) + \mathcal{L}^+\varphi_0 = \widetilde{T}^+(f + \mathcal{R}_*^0 f, \mathcal{P}f) \quad \text{on} \quad \partial\Omega. \tag{12.62}$$

Equation (12.60) means $f \in H^{-1}_{\partial\Omega}$. Equations $\mathcal{F}_2^{G\mathcal{T}} = 0$ and (12.62) imply $\widetilde{T}^+(f, 0) = 0$ on $\partial\Omega$, which along with definition (12.6) means $f = 0$ in $\mathbb{R}^3$. Then equation (12.61) and Lemma 5(ii) give $\varphi_0 = 0$.

□

Let us prove that BVP (12.13)–(12.14) is equivalent to system (12.57)–(12.58).

THEOREM 14 *Let $f \in \widetilde{H}^{-1}(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$.*

(i) *If some $u \in H^1(\Omega)$ solves BVP* (12.13)–(12.14) *in Ω, then the couple $(u, \psi)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, where*

$$\psi = \widetilde{T}^+(f, u) \quad on \quad \partial\Omega, \tag{12.63}$$

solves BDIE system (12.57)-(12.58).

(ii) *If a couple $(u, \psi)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system* (12.57)-(12.58), *then the solution is unique, u solves BVP* (12.13)–(12.14), *and ψ satisfies* (12.63).

Proof. Let $u \in H^1(\Omega)$ be a solution to BVP (12.13)–(12.14). Setting $\psi = \widetilde{T}^+(f, u)$ we evidently have $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$. Then it immediately follows from relations (12.30) and (12.42) that the couple (u, ψ) solves system (12.57)-(12.58) with the right hand side (12.59), which completes the proof of item (i).

Let now a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (12.57)-(12.58).

Lemma 3 for equation (12.57) implies that u is a solution of equation (12.1), and equations (12.33) and (12.35) hold for $\Psi = \psi$ and $\Phi = \varphi_0$. Subtracting (12.35) from equation (12.58) gives

$$\Psi^* := \psi - \widetilde{T}^+(f, u) = 0 \quad \text{on} \quad \partial\Omega, \tag{12.64}$$

that is equation (12.63) is proved.

Equations (12.33) and (12.64) give $W\Phi^*(y) = 0, \quad y \in \Omega$, where $\Phi^* = \varphi_0 - u^+$. Then Lemma 5(ii) implies $\Phi^* = 0$ on $\partial\Omega$. This means that u satisfies the Dirichlet condition (12.14).

Due to Remark 13, unique solvability of BDIE system (12.57)-(12.58) then follows from the unique solvability of BVP (12.13)–(12.14) and relation (12.63).

□

System (12.57)-(12.58)) can be rewritten in the form

$$\mathcal{A}^{G\mathcal{T}}\mathcal{U} = \mathcal{F}^{G\mathcal{T}},$$

where $\mathcal{U}^\top := (u, \psi)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and

$$\mathcal{A}^{G\mathcal{T}} := \begin{bmatrix} I + \mathcal{R} & -V \\ T^+\mathcal{R} & \dfrac{1}{2}I - \mathcal{W}' \end{bmatrix}. \tag{12.65}$$

Due to the mapping properties of the operators involved in (12.65), see [12, 2] and the Appendix, the operators

$$\begin{aligned} \mathcal{A}^{G\mathcal{T}} \quad &: \quad H^{1,-\frac{1}{2}}(\Omega; L) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^{1,-\frac{1}{2}}(\Omega; L) \times H^{-\frac{1}{2}}(\partial\Omega) &(12.66)\\ &: \quad H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) &(12.67)\end{aligned}$$

are continuous. Due to Theorem 14, and the uniqueness Theorem (2), both operators (12.66) and (12.67) are injective.

THEOREM 15 *Operators* (12.66) *and* (12.67) *are continuous and continuously invertible.*

Proof. The operator

$$\mathcal{A}_0^{G\mathcal{T}} := \begin{bmatrix} I & -V \\ 0 & \dfrac{1}{2}I \end{bmatrix}.$$

is a compact perturbation of both operators (12.66) and (12.67) due to compactness properties of the operators $\mathcal{R}$ and $\mathcal{W}$, see [1, 2, 12] and Corollary 28 from the Appendix. The invertibility of operators (12.66) and (12.67) then follows by the arguments similar to those in the proof of Theorem 11.

□

12.6 United boundary-domain integro-differential equations

In this section we consider the boundary-domain equations containing the canonical co-normal derivative operator T^+ of the internal field instead of introducing an auxiliary function ψ. For the operator T^+ to exist, we will work in the space $\widetilde{H}^{1,-\frac{1}{2}}(\Omega; L)$ for u.

12.6.1 United integro-differential problem (GD)

Let us supplement BDIDE (12.45) with the original Dirichlet boundary conditions and arrive at BDIDP (GD) constituted by equations (12.45), (12.14). The BDIDP is equivalent to the Dirichlet boundary value problem (12.13)–(12.14) in Ω, in the following sense.

THEOREM 16 *Let $f \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$, $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$. A function $u \in H^{1,-\frac{1}{2}}(\Omega; L)$ solves BVP* (12.13)–(12.14) *in Ω if and only if u solves BDIDP* (12.45), (12.14)*. Such solution does exist and is unique.*

Proof. A solution of BVP (12.13)–(12.14) does exist and is unique due to Corollary 12 and provides a solution to BDIDP (12.45), (12.14) due to Theorem 8. On the other hand, any solution of BDIDP (12.45), (12.14) satisfies also (12.13) due to the same Theorem 8.

□

BDIDP (12.45), (12.14) can be written in the form

$$\mathcal{A}^{GD} u = \mathcal{F}^{GD}, \tag{12.68}$$

where

$$\mathcal{A}^{GD} := \begin{bmatrix} I + \mathcal{R} - VT^+ + W\tau^+ \\ \tau^+ \end{bmatrix}, \quad \mathcal{F}^{GD} = \begin{bmatrix} \mathcal{P}f \\ \varphi_0 \end{bmatrix}.$$

Due to the mapping properties of operators V, W, $\mathcal{P}$ and $\mathcal{R}$ (see the Appendix), we have $\mathcal{F}^{GD} \in H^{1,-\frac{1}{2}}(\Omega; L) \times H^{\frac{1}{2}}(\partial\Omega)$, and the operator $\mathcal{A}^{GD} : H^{1,-\frac{1}{2}}(\Omega; L) \to H^{1,-\frac{1}{2}}(\Omega; L) \times H^{\frac{1}{2}}(\partial\Omega)$ is continuous. $\mathcal{A}^{GD}$ is also injective due to Theorem 16. Let us now characterise the range of the operator $\mathcal{A}^{GD}$.

THEOREM 17 *Let $\mathcal{F}^{GD}$ be a couple $(\mathcal{F}_1^{GD}, \mathcal{F}_2^{GD}) \in H^{1,-\frac{1}{2}}(\Omega; L) \times H^{\frac{1}{2}}(\partial\Omega)$. System* (12.68) *has a solution in $H^{1,-\frac{1}{2}}(\Omega; L)$ if and only if there exists $f_* \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$ such that*

$$\mathcal{F}_1^{GD} = \mathcal{P} f_* \ \textit{in}\ \Omega. \tag{12.69}$$

When the solution does exist, it is unique.

Proof. If condition (12.69) is satisfied, then, according to Theorem 16, there exists a unique solution of system (12.68).

On the other hand, if $u \in H^{1,-\frac{1}{2}}(\Omega; L)$ is a solution of system (12.68), then it satisfies the third Green identity (12.43). Comparing it with the first equation of system (12.68) implies representation (12.69) with $f_* = Lu$.

□

Let T_Δ^+, V_Δ and W_Δ denote the operators of co-normal derivative, simple layer potential and double layer potential associated with the Laplace operator, that is, for the coefficient $a = 1$.

REMARK 18 *Condition* (12.69) *for an $\mathcal{F}_1^{GD} \in H^{1,-\frac{1}{2}}(\Omega; L)$ is equivalent to the condition*

$$V_\Delta T_\Delta^+(a\mathcal{F}_1^{GD}) - W_\Delta(a\mathcal{F}_1^{GD})^+ = 0 \quad \textit{in}\ \Omega. \tag{12.70}$$

or, the same,

$$V\left[T^+\mathcal{F}_1^{GD} + \mathcal{F}_1^{GD+}\frac{\partial a}{\partial n^+}\right] - W(\mathcal{F}_1^{GD})^+ = 0 \quad \textit{in}\ \Omega. \tag{12.71}$$

Proof. Condition (12.69) can be rewritten as

$$a\mathcal{F}_1^{GD} = \mathcal{P}_\Delta f_* \text{ in } \Omega. \tag{12.72}$$

Third Green's identity (12.43) for $u = a\mathcal{F}_1^{GD}$ and for the potentials associated with the operator Δ gives

$$a\mathcal{F}_1^{GD} - V_\Delta T_\Delta^+(a\mathcal{F}_1^{GD}) + W(a\mathcal{F}_1^{GD})^+ = \mathcal{P}_\Delta \Delta(a\mathcal{F}_1^{GD}) \quad \text{in} \quad \Omega. \tag{12.73}$$

Thus (12.70) implies (12.72) with $f_* = \Delta(a\mathcal{F}_1^{GD})$.

On the other hand, if (12.72) is satisfied, then application of the Laplace operator to it gives $\Delta(a\mathcal{F}_1^{GD}) = f_*$, which substitution into (12.73) and comparison with (12.72) implies (12.70).

Condition (12.71) follows from (12.70) and the definitions of V and W.

□

To realise, how restrictive is condition (12.69) or, the same, conditions (12.70) and (12.71), let us prove the following statement.

LEMMA 19 *For any function $\mathcal{F}_1 \in H^{1,-\frac{1}{2}}(\Omega; L)$, there exists a unique couple $(f_*, \Phi_*) = \mathcal{C}_\Phi \mathcal{F}_1 \in \widetilde{H}^{-\frac{1}{2}}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ such that*

$$\mathcal{F}_1(y) \quad = \quad \mathcal{P} f_*(y) - W\Phi_*(y), \quad y \in \Omega, \tag{12.74}$$

and $\mathcal{C}_\Phi : H^{1,-\frac{1}{2}}(\Omega; L) \to \widetilde{H}^{-\frac{1}{2}}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is a linear bounded operator.

Proof. We adapt here the proof scheme from [2, Lemma 5.2].

Suppose first there exist some functions $f_*(y)$, $\Phi_*(y)$ satisfying (12.74) and find their expressions in terms of $\mathcal{F}_1(y)$. Taking into account definitions (12.17) and (12.19) for the volume and double layer potentials, ansatz (12.74) can be rewritten as

$$a(y)\mathcal{F}_1(y) = \mathcal{P}_\Delta f_*(y) - W_\Delta[a\Phi_*](y), \quad y \in \Omega. \tag{12.75}$$

Applying the Laplace operator to (12.75) we obtain that

$$f_* = \Delta(a\mathcal{F}_1) \qquad \text{in} \quad \Omega. \tag{12.76}$$

Then (12.75) can be rewritten as

$$W_\Delta[a\Phi_*](y) = Q(y), \quad y \in \Omega, \tag{12.77}$$

where

$$Q(y) := \mathcal{P}_\Delta[\Delta(a\mathcal{F}_1)](y) - a(y)\mathcal{F}_1(y), \quad y \in \Omega. \tag{12.78}$$

The trace of (12.77) on the boundary gives

$$\left[-\frac{1}{2}I + \mathcal{W}_\Delta\right][a\Phi_*](y) = Q^+(y), \quad y \in \partial\Omega. \tag{12.79}$$

Since $\left[-\frac{1}{2}I + \mathcal{W}_\Delta\right]$ is isomorphism in $H^{\frac{1}{2}}(\partial\Omega)$ (see e.g. [4, Ch. XI, Part B, §2, Remark 8]), and $a(y) \neq 0$, we obtain the following expression for Φ_*

$$\Phi_*(y) = \frac{1}{a(y)}\left[-\frac{1}{2}I + \mathcal{W}_\Delta\right]^{-1} Q^+(y), \quad y \in \partial\Omega, \tag{12.80}$$

Now we have to prove that $\Phi_*(y)$ and $f_*(y)$ given by (12.80) and (the canonical extension of) (12.76) do satisfy (12.74). First, (12.80) and (12.76) imply $\Phi_* \in H^{\frac{1}{2}}(\partial\Omega)$ and $f_* \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$. Then the potential $W_\Delta[a\Phi_*](y)$ with $\Phi_*(y)$ given by (12.80) is a harmonic function, and one can check that Q given by (12.78) is also harmonic. Since (12.79) implies that they coincide on the boundary, the two harmonic functions should coincide also in the domain, i.e. (12.77) holds true, which implies (12.74). Thus we constructed a bounded operator $\mathcal{C}_\Phi : H^{1,-\frac{1}{2}}(\Omega; L) \to \widetilde{H}^{-\frac{1}{2}}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ given by (12.76), (12.80), (12.78).

□

Lemma 19 implies that ansatz (12.69) does not cover the whole space $H^{1,-\frac{1}{2}}(\Omega; L)$.

12.6.2 United integro-differential equation (G)

In this section, we will get rid of the Dirichlet boundary condition to deal with only one integro-differential equation. Substituting the Dirichlet boundary condition (12.14) into Wu^+ in (12.45) leads to the following BDIE (G) for $u \in H^{1,-\frac{1}{2}}(\Omega; L)$,

$$\mathcal{A}^G u := u + \mathcal{R}u - VT^+u = \mathcal{F}^G \qquad \text{in } \Omega, \tag{12.81}$$

where

$$\mathcal{F}^G = F_0 = \mathcal{P}f - W\varphi_0. \tag{12.82}$$

Let us prove the equivalence of the BDIDE to the BVP (12.13)–(12.14).

THEOREM 20 *Let $f \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$, $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$. A function $u \in H^{1,-\frac{1}{2}}(\Omega; L)$ solves BVP* (12.13)–(12.14) *in Ω if and only if u solves BDIDE* (12.81) *with right-hand side* (12.82)*. Such solution does exist and is unique.*

Proof. Any solution of BVP (12.13)–(12.14) solves BDIDE (12.81) due to the third Green formula (12.45).

On the other hand, if u is a solution of BDIDE (12.81), then Lemma 7 implies that u satisfies equation (12.13) and $W(\varphi_0 - u^+) = 0$ in Ω. Lemma 5(ii) then implies that Dirichlet's boundary condition (12.14) is satisfied. The unique solvability of BDIDE (12.81) is implied by Corollary 12.

□

The mapping properties of operators V, W, $\mathcal{P}$ and $\mathcal{R}$ (see the Appendix) imply the membership $\mathcal{F}^G \in H^{1,-\frac{1}{2}}(\Omega; L)$ and continuity of the operator $\mathcal{A}^G$ in $H^{1,-\frac{1}{2}}(\Omega; L)$, while Theorem 20 implies its injectivity.

THEOREM 21 *The operator $\mathcal{A}^G$ is continuous and continuously invertible in $H^{1,-\frac{1}{2}}(\Omega; L)$.*

Proof. The continuity of $\mathcal{A}^G$ is already proved, and we have to prove existence of a bounded inverse operator $(\mathcal{A}^G)^{-1}$. Let us consider equation (12.81) with an arbitrary function $\mathcal{F}^G$ from $H^{1,-\frac{1}{2}}(\Omega; L)$. Due to Lemma 19, $\mathcal{F}^G$ can be presented as

$$\mathcal{F}^G(y) = \mathcal{P}f_*(y) - W\Phi_*(y) \quad y \in \Omega,$$

where $(f_*, \Phi_*) = \mathcal{C}_\Phi \mathcal{F}^G$ and $\mathcal{C}_\Phi$ is a bounded operator from $H^{1,-\frac{1}{2}}(\Omega; L)$ to $\widetilde{H}^{-\frac{1}{2}}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Then Theorem 20 and Corollary 12 imply that equation (12.81) has a unique solution $u = (A^D)^{-1}(f_*, \Phi_*)^\top$, where $(A^D)^{-1}$ is a bounded operator from $\widetilde{H}^{-\frac{1}{2}}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ to $H^{1,-\frac{1}{2}}(\Omega; L)$.

□

12.7 Concluding remarks

After introduction of external and internal co-normal derivative operators, the Dirichlet problem for a variable–coefficient PDE with a right-hand side function from $H^{-1}(\Omega)$ or from $\widetilde{H}^{-\frac{1}{2}}(\Omega)$, and with the Dirichlet data from $H^{\frac{1}{2}}(\partial\Omega)$, was considered in the paper. It was shown that the BVP can be equivalently reduced to two direct segregated boundary-domain integral equation systems, one of them of the second kind. On the other hand, the BVP can be equivalently reduced to a united boundary-domain integro-differential problem, or to a united boundary-domain integro-differential equation of the second kind. It was shown that the operators associated with the left-hand sides of all the four systems/problems/equations, are continuous and continuously invertible in the corresponding Sobolev-Slobodetski spaces.

A further analysis of spectral properties of the two second kind equations obtained in the paper is needed to decide whether the resolvent theory and the Neumann series method (c.f. [10, 18] and references therein) are efficient for solving the equations.

This study can serve as a step forward approaching BDIDEs/BDIDPs based on the *localised* parametrices, leading after discretization to sparsely populated systems of linear algebraic equations, attractive for computations, c.f. [11]. This can be then extended to analysis of localised BDIDEs/BDIDPs of nonlinear problems, c.f. [14].

12.8 Appendix – Properties of potential operators

12.8.1 Surface potentials

The mapping and jump properties of the potentials of type (12.19)-(12.20) and the corresponding boundary integral and pseudodifferential operators in the Hölder ($C^{k+\alpha}$), Bessel potential (H_p^s) and Besov ($B_{p,q}^s$) spaces are well studied nowadays for the constant coefficient, $a = const$, (see, e.g., a list of references in [1]). Some of the properties were extended in [1, 2] to the case of variable positive coefficient $a \in C^\infty(\mathbb{R})$. A selection of those results is provided here without proofs, while proofs are given only for new statements involving spaces $H^{s,-\frac{1}{2}}(\Omega; L)$.

THEOREM 22 *The following operators are continuous,*

$$\begin{aligned} V \ &: \ H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\Omega), \quad s \in \mathbb{R}, & (12.83)\\ &: \ H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s,-\frac{1}{2}}(\Omega; L), \quad s > \frac{1}{2}; & (12.84)\\ W \ &: \ H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega), \quad s \in \mathbb{R}, & (12.85)\\ &: \ H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s,-\frac{1}{2}}(\Omega; L), \quad s > \frac{1}{2}. & (12.86)\end{aligned}$$

Proof.

$$V\Psi(y) = \frac{1}{a(y)} V_\Delta \Psi(y), \qquad V_\Delta \Psi(y) := \int\limits_S P_\Delta(x,y)\,\Psi(x)\,dx \tag{12.87}$$

$$W\Phi(y) = \frac{1}{a(y)} W_\Delta[a\Phi](y), \qquad W_\Delta[a\Phi](y) := \int\limits_S \frac{\partial P_\Delta(x,y)}{\partial n(x)}\,a(x)\Phi(x)\,dx, \tag{12.88}$$

where $P_\Delta(x,y) := -(4\pi)^{-1}\,|x-y|^{-1}$ is the fundamental solution to the Laplace equation.

This is well known that the operators

$$V_\Delta : H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\Omega), \qquad W_\Delta : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \tag{12.89}$$

are continuous for any $s \in \mathbb{R}$ (see e.g. the above references). Since $a(x) \neq 0$ and $a \in C^\infty(\mathbb{R})$, equalities (12.87), (12.88) imply the similar properties, (12.83), (12.85), for the operators V and W.

On the other hand,

$$[\Delta V\Psi](y) = \left[\Delta \frac{1}{a(y)}\right] V_\Delta\Psi(y) + \sum_{i=1}^{3} \frac{\partial}{\partial y_i}\left[\frac{1}{a(y)}\right] \frac{\partial V_\Delta \Psi(y)}{\partial y_i},$$

$$[\Delta W\Phi](y) = \left[\Delta \frac{1}{a(y)}\right] V_\Delta[a\Phi](y) + \sum_{i=1}^{3} \frac{\partial}{\partial y_i}\left[\frac{1}{a(y)}\right] \frac{\partial W_\Delta[a\Phi](y)}{\partial y_i},$$

since $\Delta V_\Delta \Psi(y) = \Delta W_\Delta[a\Phi](y) = 0$ for $y \in \Omega$.

Due to the continuity of operators (12.89) this implies the operators $\Delta V : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-1}(\Omega)$ and $\Delta W : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-1}(\Omega)$ are continuous for $s \in \mathbb{R}$. Since $H^{s-1}(\Omega) \subset \widetilde{H}^{-\frac{1}{2}}(\Omega)$ for $s > \frac{1}{2}$, this implies (12.84), (12.86) and completes the theorem.

□

THEOREM 23 *Let $s \in \mathbb{R}$. The following pseudodifferential operators are continuous*

$$\begin{aligned} \mathcal{V} &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega), \\ \mathcal{W},\ \mathcal{W}' &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega), \\ \mathcal{L}^+ &: H^s(\partial\Omega) \to H^{s-1}(\partial\Omega). \end{aligned}$$

THEOREM 24 *Let $s \in \mathbb{R}$. The following operators are compact,*

$$\begin{aligned} r_{S_2}\, \mathcal{V} &: H^s(\partial\Omega) \to H^s(\partial\Omega), \\ r_{S_2}\, \mathcal{W} &: H^s(\partial\Omega) \to H^s(\partial\Omega), \\ r_{S_2}\, \mathcal{W}' &: H^s(\partial\Omega) \to H^s(\partial\Omega). \end{aligned}$$

THEOREM 25 *The operator $\mathcal{V} : H^{s-1}(\partial\Omega) \to H^s(\partial\Omega)$ is continuously invertible for all $s \in \mathbb{R}$.*

12.8.2 Volume potentials

The following theorem about mapping properties was proved in [1].

THEOREM 26 *Let Ω be a bounded open three–dimensional region of $\mathbb{R}^3$ with a simply connected, closed, infinitely smooth boundary. The following operators are continuous*

$$\begin{aligned}
\mathcal{P} &: \widetilde{H}^s(\Omega) \to H^{s+2}(\Omega), \quad s \in \mathbb{R}, && (12.90)\\
&: H^s(\Omega) \to H^{s+2}(\Omega), \quad s > -\frac{1}{2}; && (12.91)\\
\mathcal{R} &: \widetilde{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R}, && (12.92)\\
&: H^s(\Omega) \to H^{s+1}(\Omega), \quad s > -\frac{1}{2}; && (12.93)\\
\mathcal{P}^+ &: \widetilde{H}^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{3}{2}, && (12.94)\\
&: H^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{1}{2}; && (12.95)\\
\mathcal{R}^+ &: \widetilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, && (12.96)\\
&: H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}; && (12.97)\\
T^+\mathcal{P} &: \widetilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, && (12.98)\\
&: H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}; && (12.99)\\
T^+\mathcal{R} &: \widetilde{H}^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2}, && (12.100)\\
&: H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2}. && (12.101)
\end{aligned}$$

COROLLARY 27 *The following operators are continuous,*

$$\mathcal{P} \ : \ \widetilde{H}^s(\Omega) \to H^{s+2,-\frac{1}{2}}(\Omega; L), \quad s \geq -\frac{1}{2}, \tag{12.102}$$

$$: \ H^s(\Omega) \to H^{s+2,-\frac{1}{2}}(\Omega; L), \quad s > -\frac{1}{2}; \tag{12.103}$$

$$\mathcal{R} \ : \ H^s(\Omega) \to H^{s+1,-\frac{1}{2}}(\Omega; L), \quad s > \frac{1}{2}. \tag{12.104}$$

Proof. Continuity of operators (12.90), (12.91) and (12.93) imply continuity of operator (12.102) for $s > -\frac{1}{2}$, as well as (12.103) and (12.104). Let us prove (12.102) for $s = -\frac{1}{2}$. For $g \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$, we have, $\mathcal{P}\, g \in H^{\frac{3}{2}}(\Omega)$ due to (12.90), and

$$\Delta \mathcal{P}\, g = \Delta \left[\frac{1}{a}\, \mathcal{P}_\Delta\, g\right] =$$

$$\frac{1}{a} g + 2 \sum_{j=1}^{3} \partial_j \left[\frac{1}{a}\right]\, \partial_j \left[\mathcal{P}_\Delta\, g\right] + \left[\Delta \frac{1}{a}\right]\, \mathcal{P}_\Delta\, g \qquad \text{in} \quad \mathbb{R}^3, \tag{12.105}$$

where $\mathcal{P}_\Delta := \mathcal{P}|_{a=1}$, and we taken into account that $\Delta \mathcal{P}_\Delta\, g = g$. Since $a \in C^\infty(\bar{\Omega})$, $a > 0$, the first term in (12.105) belongs to $\widetilde{H}^{-\frac{1}{2}}(\Omega)$, while the sum of the second and the third term belongs to $H^{\frac{1}{2}}(\Omega)$ and can be extended by zero to $\widetilde{H}^0(\Omega) \subset \widetilde{H}^{-\frac{1}{2}}(\Omega)$, which completes the proof. □

COROLLARY 28 *The operators*

$$\mathcal{R} \ : \ H^s(\Omega) \to H^s(\Omega), \quad s > -\frac{1}{2}, \tag{12.106}$$

$$: \ H^s(\Omega) \to H^{s,-\frac{1}{2}}(\Omega; L), \quad s > \frac{1}{2}, \tag{12.107}$$

$$\mathcal{R}^+ \ : \ H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2}, \tag{12.108}$$

$$T^+\mathcal{R} \ : \ H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad s > \frac{1}{2}, \tag{12.109}$$

are compact for any infinitely smooth boundary curve $\partial\Omega$.

Proof. Compactness of the operators (12.106), (12.108) and (12.109) follows from (12.93), (12.97), and (12.101), respectively, and the Rellich compact embedding theorem. Then (12.106) and (12.93) imply (12.107). □

References for Chapter 12

[1] O. Chkadua, S. E. Mikhailov, D. Natroshvili (2004), Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient. Submitted for publication, preprint available from http://www.gcal.ac.uk/cms/global/contactmaps/staff/sergey/CMS-MAT-PP-2004-1.pdf

[2] O. Chkadua, S. E. Mikhailov, D. Natroshvili (2004), Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient. Submitted for publication, preprint available from http://www.gcal.ac.uk/cms/global/contactmaps/staff/sergey/CMS-MAT–2004-12.pdf

[3] M. Costabel (1988) Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal.* **19**:613–626.

[4] R. Dautray and J.L. Lions (1990), *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. 4, Integral Equations and Numerical Methods, Springer.

[5] G. Hellwig (1977), *Partial Differential Equations*, B.G. Teubner.

[6] D. Hilbert (1912), *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen.* Teubner.

[7] E. E. Levi (1909), I problemi dei valori al contorno per le equazioni lineari totalmente ellittiche alle derivate parziali. *Mem. Soc. Ital. dei Sc. XL.* **16**:1-112.

[8] J.-L. Lions, E. Magenes (1972), *Non-Homogeneous Boundary Value Problems and Applications*, Vol. 1. Springer.

[9] W. McLean (2000), *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press.

[10] S. E. Mikhailov (1983), On an integral equation of some boundary value problems for harmonic functions in plane multiply connected domains with nonregular boundary. *Matematicheskii Sbornik.* **121**:533–544. (Engl. Translation: *Mathematics of the USSR. - Sbornik.* 1984; **49**:525–536.)

[11] S. E. Mikhailov (2002), Localized boundary–domain integral formulation for problems with variable coefficients. *Int. J. Engineering Analysis with Boundary Elements.* **26**:681–690.

[12] S. E. Mikhailov (2004), Analysis of united boundary-domain integro-differential and integral equations for a mixed BVP with variable coefficient. Submitted for publication, preprint available from http://www.gcal.ac.uk/cms/contact/staff/sergey/CMS-MAT-2004-11.pdf

[13] S. E. Mikhailov (2005), Analysis of boundary-domain integral and integro-differential equations for a Dirichlet problem with variable coefficient, to appear in: *Integral Methods in Science and Engineering: Analytic and Numerical Techniques*, (Edited by C.Constanda), Birkhäuser, 16p.

[14] S. E. Mikhailov (2005), Localized direct boundary-domain integro-differential formulations for scalar nonlinear problems with variable coefficients. *J. Engineering Math.*, **51**, 283-302.

[15] C. Miranda (1970), *Partial Differential Equations of Elliptic Type*, 2-nd ed. Springer.

[16] A. Pomp (1998), Levi functions for linear elliptic systems with variable coefficients including shell equations, Computational Mechanics; **22**:93–99.

[17] A. Pomp (1998), *The boundary-domain integral methid for elliptic systems. With applications in shells.* Lecture Notes in Mathematics. Vol. 1683, Springer.

[18] O. Steinbach, W. L. Wendland (2001) On C. Neumann's method for second–order elliptic systems in domains with non–smooth boundaries. *J. Math. Anal. Appl.* **262**:733–748.

[19] H. Triebel (1978) *Interpolation Theory, Function Spaces, Differential Operators.* North–Holland.