ON CONTRACTING HYPERPLANE ELEMENTS FROM A 3-CONNECTED MATROID

RHIANNON HALL

Abstract. Let $\tilde{K}_{3,n}$, $n \geq 3$, be the simple graph obtained from $K_{3,n}$ by adding three edges to a vertex part of size three. We prove that if H is a hyperplane of a 3-connected matroid M and $M \not\cong M^*(\tilde{K}_{3,n})$, then there is an element x in H such that the simple matroid associated with M/x is 3-connected.

1. Introduction.

Much work has been done recently on chain-type theorems and splitter-type theorems for 3-connected matroids. In trying to solve major problems, it is believed that we will need many of these "tools" in order to make further progress. Our paper contributes to this recent work by establishing the existence of removable elements in particular structures of matroids. Now, 3-connectivity plays a major role in such work since many difficulties arise when working with matroids having 2-separations. However, considering only 3-connected matroids does not restrict the power of our results. We rarely lose generality by considering only 3-connected matroids, since all matroids can be constructed from sums and 2-sums of 3-connected matroids.

This paper is part of a current body of work in which we are given a 3-connected matroid M with sets of elements X and Y such that we wish to contract some element of X or delete some element of Y while maintaining 3-connectivity (or "almost" 3-connectivity). What structure might M have if for all $x \in X$ and for all $y \in Y$, $\mathrm{si}(M/x)$ and $\mathrm{co}(M\backslash y)$ (the simplification of M/x and cosimplification of $M\backslash y$) are not 3-connected? Oxley et. al. [7] considered this problem in the case where X is a basis and Y is its corresponding cobasis, in fact they also proved a stronger result, a splitter-type theorem. Hall

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Date: August 2007.

1991 *Mathematics Subject Classification.* 05B35.

Key words and phrases. matroid, 3-connected, hyperplane, minor, chain theorem.

This research was supported by a Nuffield Foundation Award for Newly Appointed Lecturers in Science, Engineering and Mathematics.

and Mayhew [4] considered the problem where X is a cocircuit and also proved a splitter-type theorem. This paper focuses on the case where X is a hyperplane, a problem suggested by Whittle in a private communication. In other words, which 3-connected matroids M have the property that they contain a hyperplane H, such that for all $h \in H$, $\mathrm{si}(M/h)$ is not 3-connected? Let $\tilde{K}_{3,n}$, $n \geq 3$, be the simple graph obtained from $K_{3,n}$ by the addition of three edges to a vertex part of size three, see Figure 1 for a depiction of $\tilde{K}_{3,n}$. We will show that the set of all 3-connected matroids having the hyperplane contraction property just stated, is the family $\mathcal{P}^*$, which we define to be the family of all cographic matroids $M^*(\tilde{K}_{3,n})$, $n \geq 3$. Note that if we define $\mathcal{P}$ to be the family of all matroids whose dual is a member of $\mathcal{P}^*$, then $\mathcal{P}$ is equal to the family of all matroids M, such that $E(M) = \{c_1, c_2, c_3, t_{11}, t_{12}, t_{13}, t_{21}, t_{22}, t_{23}, \ldots, t_{n1}, t_{n2}, t_{n3}\}$, $n \geq 3$, where $\{c_1, c_2, c_3\}$ is a triangle, $\{t_{i1}, t_{i2}, t_{i3}\}$ is a triad, and where $M|\{c_1, c_2, c_3, t_{i1}, t_{i2}, t_{i3}\} \cong K_4$ for all $1 \leq i \leq n$. This equivalence can be seen by observing that these matroids are all graphic, as none of them have a minor isomorphic to $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$ or $M^*(K_{3,3})$ (the five excluded minors for graphic matroids found by Tutte [8], see also Oxley [5, Theorem 6.6.5]). Diagrams of graphic and geometric representations of the matroids of $\mathcal{P}$ are given in Figure 1.

Stated formally, the main theorem of this paper is as follows.

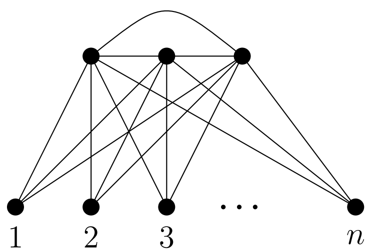
Theorem 1.1. *Let M be a 3-connected matroid. Then M has a hyperplane H such that for all $h \in H$, $\mathrm{si}(M/h)$ is not 3-connected if and only if $M \cong M^*(\tilde{K}_{3,n})$ for some $n \geq 3$.*

Note that within the matroid $M = M^*(\tilde{K}_{3,n})$, the hyperplane H in question, consists of the elements that correspond to the edges of the original $K_{3,n}$ graph. Furthermore, if $h \in H$ then $\mathrm{si}(M/h)$ is not 3-connected due to the existence of a single series pair, and $\mathrm{co}(\mathrm{si}(M/h))$ is 3-connected with $\mathrm{co}(\mathrm{si}(M/h)) \cong M^*(\tilde{K}_{3,n-1})$.

In proving Theorem 1.1, we also prove a lemma that will be of independent interest, as it may be applicable to various situations where we have a set of elements from which we wish to contract some member and keep a particular 3-connected minor. This lemma is:

Theorem 1.2. *Let (X_1, x, X_2) be a vertical 3-partition of a 3-connected matroid M. Then there exists $y \in X_i$, $i = 1, 2$, such that $\mathrm{si}(M/y)$ is 3-connected.*

The paper is structured as follows. Section 2 provides some preliminary lemmas on matroid connectivity. In Section 3, we prove Theorem 1.2 as well as some lemmas specific to our hyperplane problem.



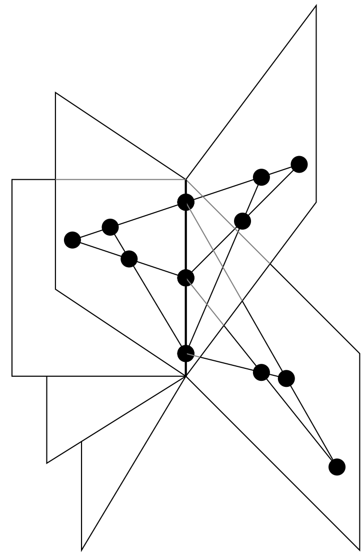


FIGURE 1. A graphic representation and a geometric representation of matroid from the class $\mathcal{P}$, the class of matroids having a cohyperplane H such that for all $x \in H$, $\mathrm{co}(M \backslash x)$ is not 3-connected. Within the graph, H consists of all of the edges of the $K_{3,n}$ subgraph. Within the geometric representation, H consists of all elements that are not in the three-point line that is common to all copies of K_4.

Section 4 completes the proof of Theorem 1.1. All terminology is taken from Oxley [5], with the exception that $\mathrm{si}(M)$ and $\mathrm{co}(M)$ denote the simplification and cosimplification of M respectively.

2. Preliminaries.

This section will provide definitions and results, mostly on connectivity, that are useful tools when applied to problems in matroid structure theory. We begin the section with some definitions on matroid connectivity. Let M be a matroid on the groundset $E(M)$. The function defined on all subsets of $E(M)$, given by $\lambda(A) = \mathrm{r}(A) + \mathrm{r}(E(M) - A) - \mathrm{r}(M)$, is known as the *connectivity function* of M. We say that a subset $A \subseteq E(M)$ is *k-separating* or a *k-separator of* M if $\lambda(A) \leq k - 1$, and we say that a partition $(A, E(M) - A)$ is a *k-separation* of M if $\lambda(A) \leq k - 1$ and $|A|, |E(M) - A| \geq k$. A k-separator or k-separation is *exact* if $\lambda(A) = k - 1$. A matroid M is said to be *k-connected* if M has no k'-separation for any $k' < k$. We define a *k-partition* of M to be a partition $(A_1, A_2, \ldots, A_n)$ of $E(M)$ in which A_i is k-separating for all

$1 \leq i \leq n$, and an *exact k-partition* is where A_i is exactly k-separating for all $1 \leq i \leq n$.

It is well known that the connectivity function of a matroid is submodular, that is for all $X, Y \subseteq E(M)$, we have $\lambda(X \cap Y) + \lambda(X \cup Y) \leq \lambda(X) + \lambda(Y)$. From this the following result of Geelen and Whittle [2] is elementary and will be used repeatedly. It is commonly referred to as "uncrossing".

Lemma 2.1. *Let M be a 3-connected matroid with 3-separating sets X and Y. Then*

- *if $|X \cap Y| \geq 2$ then $X \cup Y$ is a 3-separator of M;*
- *if $|E(M) - (X \cup Y)| \geq 2$ then $X \cap Y$ is a 3-separator of M;*
- *if $|X \cap Y| = 1$ then $X \cup Y$ is a 4-separator of M.*

The following is straight forward and can be easily proved by considering the dual matroid.

Lemma 2.2. *Let X be a series class of a 2-connected matroid M with $y \in \mathrm{cl}(X) - X$. Then $X \cup \{y\}$ is a circuit of M.*

We now define segments, cosegments and fans. These structures have appeared often in the literature due to their high quantities of triangles and triads. A *segment* of a 3-connected matroid is a set of elements in which every three-element subset is a triangle. A *cosegment* of a 3-connected matroid is a set of elements in which every three element subset is a triad. Segments and cosegments are known in some literature as lines and colines respectively. A *fan* of a 3-connected matroid is an ordered set of elements $\{f_1, f_2, \ldots, f_n\}$ in which $\{f_i, f_{i+1}, f_{i+2}\}$ is a triangle or a triad for all $1 \leq i \leq n-2$, where if $\{f_i, f_{i+1}, f_{i+2}\}$ is a triangle then $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triad, and if $\{f_i, f_{i+1}, f_{i+2}\}$ is a triad then $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triangle.

The next three lemmas appear in [3], and are useful when considering elements of a matroid that can be moved from one side of a k-separation to the other.

Lemma 2.3. *Let M be a matroid and let $(X, Y, \{z\})$ be a partition of $E(M)$. Then $z \in \mathrm{cl}^*(Y)$ if and only if $z \notin \mathrm{cl}(X)$.*

Lemma 2.4. *Let $(X, Y, \{z\})$ be a partition of $E(M)$ for some matroid M. If $\lambda(X) = \lambda(Y)$ then either $z \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$ or $z \in \mathrm{cl}^*(X) \cap \mathrm{cl}^*(Y)$.*

Lemma 2.5. *Let M be a 3-connected matroid with a 3-separating set A. For some $n \geq 3$, let $\{x_1, x_2, \ldots, x_n\} \subseteq E(M) - A$. If $x_i \in \mathrm{cl}(A)$ for all $1 \leq i \leq n$, then $\{x_1, x_2, \ldots, x_n\}$ is a segment of M. Dually, if $x_i \in \mathrm{cl}^*(A)$ for all $1 \leq i \leq n$, then $\{x_1, x_2, \ldots, x_n\}$ is a cosegment of M.*

The following is well known and straight forward.

Lemma 2.6. *Let M be a 3-connected matroid with a cosegment D such that $|D| \geq 4$. Then for all $d \in D$, M/d is 3-connected.*

Proof. Suppose this is false, and let (A, B) be a 2-separation of M/d. We may assume without loss of generality that $|A \cap D| \geq 2$, and hence $d \in \mathrm{cl}^*_M(A)$. Then $(A \cup \{d\}, B)$ is a 2-separation of M, a contradiction. □

The next two results are well known, see for example [7]. They provide us with information on why a matroid might lose some level of connectivity upon the contraction of an element. The second of these lemmas focuses on the case where we contract an element from a 3-connected matroid and not only do we lose 3-connectivity in the resultant matroid, but we lose 3-connectivity in its simplification as well.

Lemma 2.7. *Let M be a k-connected matroid of size $|E(M)| \geq 2k-1$, with an element z such that M/z is not k-connected. Then M/z is $(k-1)$-connected with at least one $(k-1)$-separation. Moreover, if (X, Y) is a $(k-1)$-separation of M/z then $(X, Y, \{z\})$ is a partition of $E(M)$ such that $\lambda_M(X) = \lambda_M(Y) = k$, $z \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$, and $|X|, |Y| \geq k-1$.*

Lemma 2.8. *Let M be a 3-connected matroid with an element z such that $\mathrm{si}(M/z)$ is not 3-connected. Then M has a partition $(X, Y, \{z\})$ such that $\lambda(X) = \lambda(Y) = 2$, $z \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$ and $\mathrm{r}(X), \mathrm{r}(Y) \geq 3$.*

We refer to a partition $(X, Y, \{z\})$ in which $\lambda(X) = \lambda(Y) = k-1$, $z \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$, and $\mathrm{r}(X), \mathrm{r}(Y) \geq k$ as a *vertical k-partition.* Note how this differs slightly from a *vertical k-separation*, which is defined in many papers as a k-separation (A, B) in which $\mathrm{r}(A), \mathrm{r}(B) \geq k$, see for example [7]. The following is a useful tool when considering vertical k-partitions.

Lemma 2.9. *Let M be a k-connected matroid with a vertical k-partition $(X, Y, \{z\})$. Then $(X - \mathrm{cl}(Y), \mathrm{cl}(Y) - \{z\}, \{z\})$ is also a vertical k-partition of M.*

Proof. Suppose that $x \in X \cap \mathrm{cl}(Y)$. Then $\lambda(X - \{x\}) \in \{k-2, k-1\}$. If $\lambda(X - \{x\}) = k-2$ then $(X - \{x\}, Y \cup \{x, z\})$ is a $(k-1)$-separation of M, contradicting that M is k-connected. Hence $\lambda(X - \{x\}) = k-1$ which implies that $\mathrm{r}(X - \{x\}) = \mathrm{r}(X)$ and $z \in \mathrm{cl}(X - \{x\})$. It follows that $(X - \{x\}, Y \cup \{x\}, z)$ is a vertical k-partition of M. Continuing this process, we see that $(X - \mathrm{cl}(Y), \mathrm{cl}(Y) - \{z\}, \{z\})$ is a vertical k-partition of M. □

We now state the version of Bixby's Theorem [1] that is most natural for our requirements in this paper.

Theorem 2.10 (Bixby's theorem)**.** *Let M be a 3-connected matroid and let $x \in E(M)$. Then either $M\backslash x$ is 3-connected up to series pairs or M/x is 3-connected up to parallel pairs.*

We now discuss the important concept of local connectivity. The *local connectivity function* of a matroid M is defined on pairs of subsets of $E(M)$ as $\sqcap(A, B) = \mathrm{r}(A) + \mathrm{r}(B) - \mathrm{r}(A \cup B)$. Note that we do not require A and B to be disjoint. It is helpful to think of local connectivity as the connectivity between A and B in the matroid $M|(A \cup B)$. A good introduction to the local connectivity function can be found in Oxley, Semple, & Whittle [6]. The following two results on local connectivity appear in [6].

Lemma 2.11. *Let (X, Y, Z) be an exact 3-partition of the 3-connected matroid M. Then $\sqcap(X, Y) = \sqcap(X, Z) = \sqcap(Y, Z)$.*

Lemma 2.12. *Let X and Y be subsets of $E(M)$, with $X' \subseteq X$ and $Y' \subseteq Y$. Then $\sqcap(X', Y') \leq \sqcap(X, Y)$.*

At this point, we use Lemma 2.12 to prove the following.

Lemma 2.13. *Let X and Y be disjoint subsets of $E(M)$, with $X' \subseteq X$, and such that $\sqcap(X', Y) = \sqcap(X, Y)$. If there exists $y \in Y$ such that $y \in \mathrm{cl}(X)$, then $y \in \mathrm{cl}(X')$.*

Proof. Firstly, since $y \in \mathrm{cl}(X)$, we have

$$\begin{aligned}
\sqcap(X \cup \{y\}, Y) &= \mathrm{r}(X \cup \{y\}) + \mathrm{r}(Y) - \mathrm{r}(X \cup \{y\} \cup Y) \\
&= \mathrm{r}(X) + \mathrm{r}(Y) - \mathrm{r}(X \cup Y) \\
&= \sqcap(X, Y) \\
&= \sqcap(X', Y).
\end{aligned}$$

By Lemma 2.12, we have $\sqcap(X' \cup \{y\}, Y) \leq \sqcap(X \cup \{y\}, Y) = \sqcap(X', Y)$. Lemma 2.12 also gives $\sqcap(X', Y) \leq \sqcap(X' \cup \{y\}, Y)$, therefore we may deduce that $\sqcap(X' \cup \{y\}, Y) = \sqcap(X', Y)$. It follows that $\mathrm{r}(X' \cup \{y\}) + \mathrm{r}(Y) - \mathrm{r}(X' \cup \{y\} \cup Y) = \mathrm{r}(X') + \mathrm{r}(Y) - \mathrm{r}(X' \cup Y)$, and by cancelling terms, we obtain $\mathrm{r}(X' \cup \{y\}) = \mathrm{r}(X')$. The result now follows. $\square$

3. Some useful lemmas.

The purpose of this section is to prove some "larger" lemmas including Theorem 1.2. Most of the lemmas of this section are specific to our problem, however Theorem 1.2 and Lemma 3.2 may be applicable to a number of far more general settings.

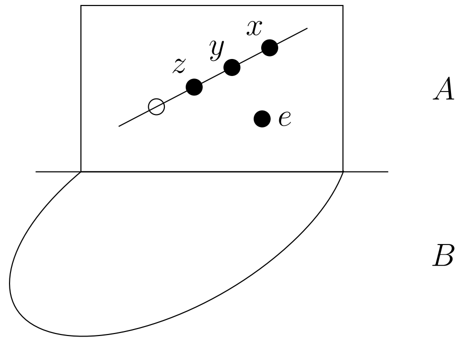


FIGURE 2. A 3-separation (A, B), such that $\mathrm{r}(A) = 3$ and $A - \{e\} - \mathrm{cl}(B)$ is a segment with at least three elements.

We begin by generating a lower bound on the size of a matroid that has a hyperplane from which contraction of any element creates a vertical 2-separation.

Lemma 3.1. *Let M be a 3-connected matroid with a hyperplane H, such that for all $h \in H$, $\mathrm{si}(M/h)$ is not 3-connected. Then $|E(M)| \geq 7$.*

Proof. Let M be such a matroid. By Lemma 2.8, M has a vertical 3-partition $(X, Y, \{z\})$. Since $\mathrm{r}(X), \mathrm{r}(Y) \geq 3$, we have $|X|, |Y| \geq 3$. Hence $|E(M)| \geq 3 + 3 + 1 = 7$. $\square$

We next consider what happens if our matroid has a specific type of 3-separator.

Lemma 3.2. *Let M be a 3-connected matroid with a 3-separation (A, B) such that $\mathrm{r}(A) = 3$ and there exists e with $\mathrm{r}(A - \{e\}) = 2$ and $|A - \{e\} - \mathrm{cl}(B)| \geq 3$. See Figure 2 for a geometrical representation of (A, B). Suppose that $\mathrm{si}(M/a)$ is not 3-connected for all $a \in A - \{e\}$. Then M has a cosegment D such that $e \in D$, $|D - \{e\}| \geq |A - \{e\} - \mathrm{cl}(B)|$, and $A \cup D$ is 3-separating in M.*

Proof. Firstly, we may assume that $\mathrm{cl}(A - \{e\}) \subseteq A$ (we will use this assumption in the proof of Sublemma 3.2.4). Now, $M \backslash e$ has the vertical 2-separation $(A - e, B)$, so $e \in \mathrm{cl}^*(A - \{e\}) \cap \mathrm{cl}^*(B)$. Since $\mathrm{si}(M/a)$ is not 3-connected for any $a \in A - \{e\}$, upon the contraction of any member of $A - \{e\}$, we will obtain a vertical 2-separation, which corresponds to a vertical 3-partition of the original matroid M. Let $x, y \in A - \{e\} - \mathrm{cl}(B)$, and consider their vertical 3-partitions. Let (X_1, x, X_2) be a vertical 3-partition. Then, since $x \in \mathrm{cl}(X_i)$, $i = 1, 2$, and $x \notin \mathrm{cl}(B)$, we see that $X_i \cap (A - \mathrm{cl}(B)) \neq \emptyset$. Assume without loss of generality that $e \in X_2$. Then $\mathrm{r}(A \cap X_1) \leq 2$, and as $\mathrm{r}(X_1) \geq 3$ we have $X_1 \cap B \neq \emptyset$.

Sublemma 3.2.1. *$X_2 \cap A = \{e\}$ and $A \not\subseteq \mathrm{cl}(X_i)$, $i = 1, 2$.*

Proof. Suppose that $A \subseteq \mathrm{cl}(X_2)$. It follows that if $|X_1 - A| \geq 2$ then $X_1 - A$ is 2-separating in M, and if $|X_1 - A| = 1$ then $X_1 - A$ is separating in M, and both possibilities contradict the 3-connectivity of M. We conclude that $A \not\subseteq \mathrm{cl}(X_2)$.

It now follows that since $e \in X_2$ and $x \in \mathrm{cl}(X_2)$, no member of $A - \{e, x\}$ can be in X_2, otherwise A would be contained in $\mathrm{cl}(X_2)$. Thus $X_2 \cap A = \{e\}$.

Now suppose that $e \in \mathrm{cl}(X_1)$. Then as $X_2 \cap A = \{e\}$, we have $e \notin \mathrm{cl}(X_2 - \{e\})$, implying that $X_2 - \{e\}$ is a 2-separator of M of size at least two, a contradiction. Thus $e \notin \mathrm{cl}(X_1)$. Therefore $A \not\subseteq \mathrm{cl}(X_1)$ as required. □

Sublemma 3.2.2. *$\sqcap(X_2 - A, A) = \sqcap(X_2 - A, \{e, x\}) = 1$ and $\sqcap(X_1 - A, A) = \sqcap(X_1 - A, A - \{e\}) = \begin{cases} 0 & \text{if } |X_1 - A| = 1; \\ 1 & \text{if } |X_1 - A| \geq 2, \end{cases}$.*

Proof. Since $X_i - \mathrm{cl}(B) \neq \emptyset$ and $X_i - A \subseteq B$, with $\sqcap(B, A) = 2$ and $A \not\subseteq \mathrm{cl}(X_i)$, we see that $\sqcap(X_i - A, A) \leq 1$, $i = 1, 2$. It is easily seen that since $x \in \mathrm{cl}(X_2)$ with $X_2 \cap A = \{e\}$, we must have $\sqcap(X_2 - A, \{e, x\}) = \sqcap(X_2 - A, A) = 1$.

Now, if $|X_1 - A| \geq 2$ then $\lambda(X_1 - A) \geq 2$. Since $\lambda(X_1 \cup \{x\}) = 2$ and $\mathrm{r}(X_2 \cup A) = \mathrm{r}(X_2) + 1$, we must have $\sqcap(X_1 - A, A - \{e\}) = 1$, otherwise we would have $\lambda(X_1 - A) = 1$. Now, by Lemma 2.11, $\sqcap(X_1 - A, A) = 1$.

If $|X_1 - A| = 1$, then since $\mathrm{r}(X_1) \geq 3$ and $A \not\subseteq \mathrm{cl}(X_1)$, we must have $X_1 - A \subseteq (B - \mathrm{cl}(A))$, thus $\sqcap(X_1 - A, A - \{e\}) = \sqcap(X_1 - A, A) = 0$. □

Sublemma 3.2.3. *$X_1 - A$ and $X_2 - A$ are 3-separators of M.*

Proof. This follows immediately from the fact that $\mathrm{r}(X_i \cup A) = \mathrm{r}(X_i) + 1$ and $\mathrm{r}(X_i - A) < \mathrm{r}(X_i)$, $i = 1, 2$. □

Let (Y_1, y, Y_2) be a vertical 3-partition of M, where $e \in Y_2$. By symmetry, the same conditions as described in Sublemmas 3.2.1–3.2.3 for (X_1, x, X_2) also hold for (Y_1, y, Y_2). Let $X_i' = X_i \cap B$ and $Y_i' = Y_i \cap B$, $i = 1, 2$.

Sublemma 3.2.4. *Either $|X_1'| = 1$ or $|Y_1'| = 1$.*

Proof. Suppose that $|X_1'| \geq 2$ and $|Y_1'| \geq 2$ (note that $|X_2'| \geq 2$ and $|Y_2'| \geq 2$ since $X_2 \cap A = Y_2 \cap A = \{e\}$). Then $\sqcap(\{x, y\}, X_1') = 1$,

$\sqcap(\{x,y\},Y_1') = 1$, $\sqcap(\{e,x\},X_2') = 1$ and $\sqcap(\{e,y\},Y_2') = 1$ from Sublemma 3.2.2. We consider how these sets can intersect. It is evident that X_2' contains some element that is not a member of Y_2' since $x \notin \mathrm{cl}(Y_2' \cup \{e\})$ and $x \in \mathrm{cl}(X_2' \cup \{e\})$. Similarly, Y_2' contains some element that is not a member of X_2'. Furthermore, since $e \in \mathrm{cl}(X_2' \cup \{x\})$ but $e \notin \mathrm{cl}(Y_1' \cup \{x\}) = \mathrm{cl}(Y_1)$, we see that X_2' contains some element not in Y_1'. Finally, as $x \in \mathrm{cl}(X_1' \cup \{y\})$ and $x \notin \mathrm{cl}(Y_2' \cup \{y\})$, there must exist some element of X_1' that is not in Y_2'. The result of this is that each of the sets $X_1' \cap Y_1'$, $X_1' \cap Y_2'$, $X_2' \cap Y_1'$, and $X_2' \cap Y_2'$ are nonempty. Now, since $e \in X_2 \cap Y_2$ and $X_2' \cap Y_2' \neq \emptyset$, it follows that $|X_2 \cap Y_2| \geq 2$, so by uncrossing $X_2 \cup Y_2$ is 3-separating in M.

Now observe that no member of $A - \{e\}$ is in $X_2 \cup Y_2$, but every member of $A - \{e\}$ is in $\mathrm{cl}(X_2 \cup Y_2)$ since $\{x,y\} \subseteq \mathrm{cl}(X_2 \cup Y_2)$. Therefore $\mathrm{r}(X_2 \cup Y_2 \cup A) = \mathrm{r}(X_2 \cup Y_2)$. However, $E(M) - (X_2 \cup Y_2 \cup A) = X_1' \cap Y_1'$. Suppose that $|X_1' \cap Y_1'| \geq 2$ then $\mathrm{r}(X_1' \cap Y_1') \leq \mathrm{r}((X_1 \cap Y_1) \cup \{x,y\}) - 1$ implying that $\lambda(X_1' \cap Y_1') \leq 1$ (because $\lambda(X_2 \cup Y_2 \cup A) \leq \lambda(X_2 \cup Y_2) \leq 2$), contradicting the connectivity of M. Therefore $|X_1' \cap Y_1'| = 1$, but since $\mathrm{cl}(A - \{e\}) \subseteq A$, we have $\mathrm{r}(X_1' \cap Y_1') \leq \mathrm{r}(X_1 \cap Y_1) \cup \{x,y\}) - 2$ implying that $\lambda(X_1' \cap Y_1') = 0$, another contradiction to the connectivity of M. This contradiction shows that it is not possible to have $|X_1'| \geq 2$ and $|Y_1'| \geq 2$. □

We may now assume by Sublemma 3.2.4 and by symmetry that $|X_1'| = 1$. Recall that $|X_2'|, |Y_2'| \geq 2$ because $|X_2|, |Y_2| \geq 3$. First note that since $\mathrm{r}(X_1) \geq 3$ and $e \notin \mathrm{cl}(X_1)$, it follows that the element x' of X_1' is not in $\mathrm{cl}(A)$. Then $x' \in \mathrm{cl}^*(A - \{e\}) \cap \mathrm{cl}^*(B \cup \{e\} - \{x'\})$ by Lemma 2.4. Now, since $y \in \mathrm{cl}(Y_2' \cup \{e\})$ and $y \notin \mathrm{cl}(X_2' \cup \{e\})$, it follows that Y_2' must contain x', and hence $Y_1' \subseteq X_2'$.

Sublemma 3.2.5. *$|Y_1'| = 1$.*

Proof. Suppose that $|Y_1'| \geq 2$. Then $\sqcap(Y_1', A - \{e\}) = 1$ by Sublemma 3.2.2, which implies that $A - \{e,x\} \subseteq \mathrm{cl}(Y_1' \cup \{x\})$. However $y \notin \mathrm{cl}(X_2' \cup \{x\}) \supseteq \mathrm{cl}(Y_1' \cup \{x\})$, a contradiction. This implies that $|Y_1'| = 1$. □

A similar argument to the one above shows that the element y' of Y_1' is a member of $\mathrm{cl}^*(A - \{e\}) \cap \mathrm{cl}^*(B \cup \{e\} - \{y'\})$. Also, since $y' \in Y_1'$, we have $x' \neq y'$.

We may apply symmetric arguments to any pair of elements $x, y \in A - \{e\} - \mathrm{cl}(B)$, where (X_1, x, X_2) and (Y_1, y, Y_2) are vertical 3-partitions of M such that $e \in X_2$ and $e \in Y_2$. These arguments show that $|X_1 - A| = |Y_1 - A| = 1$, and if $x' \in X_1 - A$ and $y' \in Y_1 - A$, then $x' \neq y'$ and $x', y' \in \mathrm{cl}^*(A - \{e\})$. Now, let $D = \mathrm{cl}^*(A - \{e\}) - (A - \{e\})$.

Then D contains e and $\{x', y'\}$ for all distinct $x, y \in A - \{e\} - \mathrm{cl}(B)$. Furthermore, $|D - \{e\}| \geq |A - \{e\} - \mathrm{cl}(B)|$ since $x' \neq y'$ for all distinct $x, y \in A - \{e\} - \mathrm{cl}(B)$. We also see that $A \cup D$ is 3-separating by construction, and that D is a cosegment of M by Lemma 2.5. □

We may now apply Lemma 3.2 to our problem in the following corollary.

Corollary 3.3. *Let M be a 3-connected matroid with a 3-separation (A, B) such that $\mathrm{r}(A) = 3$ and there exists e with $\mathrm{r}(A - \{e\}) = 2$ and $|A - \{e\} - \mathrm{cl}(B)| \geq 3$. Again, refer to Figure 2 for a geometrical representation of (A, B). Suppose that M has a hyperplane H that contains $A - \{e\}$. Then there exists $h \in H$ such that $\mathrm{si}(M/h)$ is 3-connected.*

Proof. Suppose we have a matroid satisfying such conditions, and suppose that for all $a \in A - \{e\}$, $\mathrm{si}(M/a)$ is not 3-connected. Then by Lemma 3.2, e is a member of a cosegment D of size at least four, such that $A \cup D$ is 3-separating in M. In order for H to have a rank of $\mathrm{r}(M) - 1$, H must intersect D. Let $h \in H \cap D$. Then by Lemma 2.6, M/h is 3-connected. □

The following lemma allows us to choose vertical 3-partitions that have a certain type of "minimality" on one of the large sides of the partition.

Lemma 3.4. *Let M be a 3-connected matroid with a set of elements J, such that for all $j \in J$, $\mathrm{si}(M/j)$ is not 3-connected. Suppose $x \in J$ and (X_1, x, X_2) is a vertical 3-partition such that for all $y \in (X_1 \cup \{x\}) \cap J$, whenever (Y_1, y, Y_2) is a vertical 3-partition with $Y_1 \subseteq X_1$ then $Y_1 \cap J \neq \emptyset$. Then there exists $z \in (X_1 \cup \{x\}) \cap J$ with a vertical 3-partition (Z_1, z, Z_2) such that*
- *$Z_1 \subseteq X_1$ and $Z_1 \cap J \neq \emptyset$, and*
- *$Z_2 \cup \{z\}$ is closed, and*
- *for all $j \in Z_1 \cap J$, whenever (J_1, j, J_2) is a vertical 3-partition, then $J_1 \cap X_2 \neq \emptyset$ and $J_2 \cap X_2 \neq \emptyset$.*

Proof. In order to construct such a partition (Z_1, z, Z_2), we begin by checking the vertical 3-partition (A_1, j_1, B_1), where $A_1 = X_1 - \mathrm{cl}(X_2)$, $j_1 = x$, and $B_1 = \mathrm{cl}(X_2) - \{x\}$. Clearly, $A_1 \subseteq X_1$, $A_1 \cap J \neq \emptyset$ (by the conditions set out in the statement of the lemma), and $B_1 \cup \{j_1\}$ is closed. Then either we have constructed the desired vertical 3-partition, or there is some $j_2 \in A_1 \cap J$ such that there exists a vertical 3-partition (A_2, j_2, B_2) with $A_2 \subseteq A_1$ and $B_2 \cup \{j_2\}$ is closed. Since $B_1 \cup \{j_1\}$ is closed, and $j_2 \in A_1 \cap \mathrm{cl}(B_2)$, it follows that $\mathrm{r}(A_2) < \mathrm{r}(A_1)$.

	Y_1	y	Y_2
X_1	$\neq \emptyset$	●	$\neq \emptyset$
x	●		
X_2	$\neq \emptyset$		$\neq \emptyset$

FIGURE 3. A Venn diagram showing the 3-partitions (X_1, x, X_2) and (Y_1, y, Y_2).

Also $A_2 \cap J \neq \emptyset$, by the conditions set out in the statement of the lemma. We may repeat this process, each time choosing $j_i \in A_i \cap J$, until we produce the desired vertical 3-partition (A_k, h_k, B_k). We will eventually achieve this since $\mathrm{r}(A_i) < \mathrm{r}(A_{i-1})$ always. □

We now prove Theorem 1.2, which tells us that when we have a 3-connected matroid with a vertical 3-partition, then we can always find some element on either of the large sides of the partition, whose contraction keeps us 3-connected up to parallel classes. We restate Theorem 1.2 here for ease of reading.

Theorem 3.5. *Let (X_1, x, X_2) be a vertical 3-partition of a 3-connected matroid M. Then there exists $y \in X_i$, $i = 1, 2$, such that $\mathrm{si}(M/y)$ is 3-connected.*

Proof. Suppose that the lemma is false, and suppose that (X_1, x, X_2) is a vertical 3-partition of M, such that for all $y \in X_1$, $\mathrm{si}(M/y)$ is not 3-connected. Then we may assume by the construction detailed in the proof of Lemma 3.4, that $X_2 \cup \{x\}$ is closed and for all $y \in X_1$, whenever (Y_1, y, Y_2) is a vertical 3-partition, then $Y_1 \cap X_2 \neq \emptyset$ and $Y_2 \cap X_2 \neq \emptyset$.

Let $y \in X_1$ and (Y_1, y, Y_2) be a vertical 3-partition of M with $x \in Y_1$. Consider the Venn diagram for $E(M)$ of Figure 3. By construction, $Y_1 \cap X_2 \neq \emptyset$ and $Y_2 \cap X_2 \neq \emptyset$. Also, since $X_2 \cup \{x\}$ is closed, we see that $y \notin \mathrm{cl}(X_2 \cup \{x\})$. However $y \in \mathrm{cl}(Y_1)$ and $y \in \mathrm{cl}(Y_2)$, meaning that $Y_1 \cap X_1 \neq \emptyset$ and $Y_2 \cap X_1 \neq \emptyset$. We now consider the connectivity of these sets. We know that $X_2 \cup \{x\}$ and Y_1 are 3-separating in M and intersect in at least two elements, so by uncrossing, $(X_1 \cap Y_2) \cup \{y\}$ is 3-separating in M. By a similar argument, $X_1 \cap Y_2$ is also 3-separating.

Sublemma 3.5.1. $\mathrm{r}((X_1 \cap Y_2) \cup \{y\}) = 2.$

Proof. Suppose that $\mathrm{r}((X_1 \cap Y_2) \cup \{y\}) \geq 3$. Then $\lambda((X_1 \cap Y_2) \cup \{y\}) = \lambda(X_1 \cap Y_2) = 2$. Also $y \in \mathrm{cl}(Y_1)$, hence by Lemma 2.4, $y \in \mathrm{cl}(X_1 \cap Y_2)$, so that $\mathrm{r}(X_1 \cap Y_2) \geq 3$. It follows that $(X_2 \cup Y_1, y, X_1 \cap Y_2)$ is a vertical 3-partition of M with $(X_1 \cap Y_2) \cap X_2 = \emptyset$, a contradiction to our construction of (X_1, x, X_2). □

Sublemma 3.5.2. *If* $(X_1 \cap Y_1) \cup \{x, y\}$ *is* 3*-separating in* M *then* $\mathrm{r}((X_1 \cap Y_1) \cup \{x, y\}) = 2$.

Proof. Suppose $(X_1 \cap Y_1) \cup \{x, y\}$ is 3-separating. Then since $x \in \mathrm{cl}(X_2)$ and $y \in \mathrm{cl}(Y_2)$, it follows that each of $X_1 \cap Y_1$, $(X_1 \cap Y_1) \cup \{x\}$, $(X_1 \cap Y_1) \cup \{y\}$ and $(X_1 \cap Y_1) \cup \{x, y\}$ is 3-separating. By a similar argument to Sublemma 3.5.1, we see that $\mathrm{r}((X_1 \cap Y_1) \cup \{x, y\}) = 2$. □

Sublemma 3.5.3. $(X_1 \cap Y_1) \cup \{x, y\}$ *is not* 3*-separating in* M*.*

Proof. Suppose $(X_1 \cap Y_1) \cup \{x, y\}$ is 3-separating in M. Then by Sublemmas 3.5.1 and 3.5.2, $\mathrm{r}((X_1 \cap Y_2) \cup \{y\}) = 2$ and $\mathrm{r}((X_1 \cap Y_1) \cup \{x, y\}) = 2$, hence $\mathrm{r}(X_1) = 3$. Now suppose that $|X_1 \cap Y_2| \geq 2$. Then $y \in \mathrm{cl}(X_1 \cap Y_2)$ by Lemma 2.4. We now choose $z \in X_1 \cap Y_1$ and consider a vertical 3-partition (Z_1, z, Z_2). We may assume by symmetry that $Z_1 \cap ((X_1 \cap Y_1) \cup \{x, y\}) \neq \emptyset$ and that $Z_1 \cup \{z\}$ is closed by Lemma 2.9. Then since $z \in \mathrm{cl}(Z_1)$, it follows that $(X_1 \cap Y_1) \cup \{x, y\} \subseteq \mathrm{cl}(Z_1)$, therefore $(X_1 \cap Y_1) \cup \{x, y\} \subseteq Z_1 \cup \{z\}$. Observe that $Z_2 \cap X_1 \neq \emptyset$ because $z \in \mathrm{cl}(Z_2)$ and $z \notin \mathrm{cl}(X_2 \cup \{x\})$, and as a result, $Z_2 \cap (X_1 \cap Y_2) \neq \emptyset$. Furthermore, as $Z_1 \cup \{z\}$ is closed and $y \in Z_1$, it follows that $Z_1 \cap (X_1 \cap Y_2) = \emptyset$, otherwise we would have the contradiction that $(X_1 \cap Y_2) \subseteq \mathrm{cl}(Z_1)$. This means that $(X_1 \cap Y_2) \subseteq Z_2$, resulting in $\{y, z\} \subseteq \mathrm{cl}(Z_2)$. Furthermore, this implies that $(X_1 \cap Y_1) \cup \{y\} \subseteq \mathrm{cl}(Z_2)$, because $(X_1 \cap Y_1) \cup \{y\} \subseteq \mathrm{cl}(\{y, z\})$. By Lemma 2.9, we may now construct the vertical 3-partition $(Z_1 - \mathrm{cl}(Z_2), z, \mathrm{cl}(Z_2) - \{z\})$ which has $X_1 \subseteq \mathrm{cl}(Z_2)$. This is a contradiction since that would mean that $z \in \mathrm{cl}(Z_1 - \mathrm{cl}(Z_2))$, which is impossible as $Z_1 - \mathrm{cl}(Z_2) \subseteq X_2 \cup \{x\}$. This contradiction shows that if $(X_1 \cap Y_1) \cup \{x, y\}$ is 3-separating, then we cannot have $|X_1 \cap Y_2| \geq 2$, and we see that $|X_1 \cap Y_2| = 1$.

Letting $w \in X_1 \cap Y_2$, it follows easily that $((X_1 \cap Y_1) \cup \{x, y\}, X_2)$ is a vertical 2-separation of $M \backslash w$. By Bixbys Theorem 2.10, it follows that $\mathrm{si}(M/w)$ is 3-connected, contradicting our original assumption that for all $e \in X_1$, $\mathrm{si}(M/e)$ is not 3-connected. The result follows. □

Consider the size of $X_2 \cap Y_2$. If $|X_2 \cap Y_2| \geq 2$, then by uncrossing, $(X_1 \cap Y_1) \cup \{x, y\}$ is 3-separating, contradicting Sublemma 3.5.3. Hence, it must be the case that $|X_2 \cap Y_2| = 1$ and $\lambda((X_1 \cap Y_1) \cup \{x, y\}) = 3$ (by uncrossing, we must have $\lambda((X_1 \cap Y_1) \cup \{x, y\}) \leq 3$).

Proceeding from here, it is helpful to continue to refer to the Venn diagram of Figure 3 to gain intuition. We have $|Y_2| \geq 3$ and $|Y_2 \cap X_2| =$

1, hence $|X_1 \cap Y_2| \geq 2$ meaning that $(X_1 \cap Y_2) \cup \{y\}$ is a segment of size at least three. It is clear also that since $|X_2 \cap Y_2| = 1$, $\mathrm{r}(X_1 \cap Y_2) = 2$, and $\mathrm{r}(Y_2) \geq 3$, it follows that $Y_1 \cup \{y\}$ is closed and $\mathrm{r}(Y_2) = 3$. Evidently, no member of $X_1 \cap Y_1$ can extend $(X_1 \cap Y_2) \cup \{y\}$ to a larger segment because $Y_1 \cup \{y\}$ is closed. Hence $(X_1 \cap Y_2) \cup \{y\}$ is a maximal segment contained in X_1.

Sublemma 3.5.4. $y \in \mathrm{cl}((X_1 \cap Y_1) \cup \{x\})$.

Proof. Suppose that $y \notin \mathrm{cl}((X_1 \cap Y_1) \cup \{x\})$. Then since $y \in \mathrm{cl}(Y_2)$, we have $\lambda((X_1 \cap Y_1) \cup \{x\}) = \lambda((X_1 \cap Y_1) \cup \{x, y\}) - 1 = 2$.

We now see that $((X_1 \cap Y_1) \cup \{x\}, X_2, (X_1 \cap Y_2) \cup \{y\})$ is an exact 3-partition of M with $\sqcap((X_1 \cap Y_1) \cup \{x\}, X_2) \geq 1$ since $x \in \mathrm{cl}(X_2)$. By Lemma 2.11, $\sqcap((X_1 \cap Y_1) \cup \{x\}, (X_1 \cap Y_2) \cup \{y\}) \geq 1$. We also see that since $(X_1 \cap Y_2) \cup \{y\}$ is a segment and $X_1 \cap Y_2 \not\subseteq \mathrm{cl}(Y_1)$, we have $\sqcap((X_1 \cap Y_2) \cup \{y\}, Y_1) = 1$. By Lemma 2.11, $\sqcap((X_1 \cap Y_1) \cup \{x\}, (X_1 \cap Y_2) \cup \{y\}) = 1$, and by Lemma 2.13, we have $y \in \mathrm{cl}((X_1 \cap Y_1) \cup \{x\})$ because $y \in \mathrm{cl}(Y_1)$. This contradicts our initial assumption, and we conclude that $y \in \mathrm{cl}((X_1 \cap Y_1) \cup \{x\})$. □

Let $s \in X_1 \cap Y_2$, and consider a vertical 3-partition (S_1, s, S_2) with $x \in S_1$. By the symmetry of the situation, (S_1, s, S_2) shares many of the same properties as (Y_1, y, Y_2), for example $|X_2 \cap S_2| = 1$, $(X_1 \cap S_2) \cup \{s\}$ is a maximal segment contained in X_1, also $s \in \mathrm{cl}((X_1 \cap S_1) \cup \{x\})$ and $S_1 \cup \{s\}$ is closed with $\mathrm{r}(S_2) = 3$. Consider the members of the segment $(X_1 \cap S_2) \cup \{s\}$. Since $s \in \mathrm{cl}(X_1 \cap S_2)$ and $s \notin \mathrm{cl}((X_1 \cap Y_1) \cup \{y\})$ (recall that $Y_1 \cup \{y\}$ is closed), there must be some member s of $X_1 \cap Y_2$ that is contained in $X_1 \cap S_2$. Now, as $\{s, s\}$ is a subset of $(X_1 \cap Y_2) \cup \{y\}$ and $(X_1 \cap S_2) \cup \{s\}$, both of which are maximal segments contained in X_1, it follows that $(X_1 \cap Y_2) \cup \{y\} = (X_1 \cap S_2)\{s\}$. This implies that $X_1 \cap Y_1 = X_1 \cap S_1$, and we see that $\{y, s\} \subseteq \mathrm{cl}((X_1 \cap Y_1) \cup \{x\})$, a contradiction as we have already established that $Y_1 \cup \{y\}$ is closed and $s \notin Y_1 \cup \{y\}$.

We conclude from this final contradiction that our original assumption, that for all $e \in X_1$, $\mathrm{si}(M/e)$ is not 3-connected, must be false. The result now follows by a symmetric argument on X_2. □

The result of Theorem 1.2 can now be put to use on our problem, and we obtain the following corollary.

Corollary 3.6. *Let M be a 3-connected matroid with a hyperplane H, such that for all $h \in H$, $\mathrm{si}(M/h)$ is not 3-connected. Let (X_1, x, X_2) be a vertical 3-partition of M with $x \in H$, and let C be the cocircuit whose complement is H. Then*

(1) $X_i \cap C \neq \emptyset$ *for* $i = 1, 2$, *and*
(2) $X_i \cap H \neq \emptyset$ *for* $i = 1, 2$.

Proof. We have that (X_1, x, X_2) is a vertical 3-partition of M with $x \in H$. Then by Theorem 1.2, there exists $y \in X_i$, $i = 1, 2$, such that $\mathrm{si}(M/y)$ is 3-connected. Since $\mathrm{si}(M/h)$ is not 3-connected for all $h \in H$, we see that $X_i \cap C \neq \emptyset$, $i = 1, 2$. This proves (1).

Now suppose that $X_1 \cap H = \emptyset$, so that $X_1 \subseteq C$ and $H \subseteq X_2 \cup \{x\}$. Since (X_1, x, X_2) is a vertical 3-partition, $\mathrm{r}(X_2 \cup \{x\}) < \mathrm{r}(M)$ meaning that $X_2 \cup \{x\}$ is contained in some hyperplane H' of M. Thus $H \subseteq X_2 \cup \{x\} \subseteq H'$, implying that $H = X_2 \cup \{x\}$, contradicting part (1) above which states that $X_2 \cap C \neq \emptyset$. The result now follows by a symmetric argument on $X_2 \cap H$. □

The following lemma will be used in the proof of Lemma 3.8, which is an important part of the proof of our main theorem.

Lemma 3.7. *Let* M *be a* 3*-connected matroid, and let* X *and* Y *be disjoint subsets of* $E(M)$*, where* X *is a cosegment. If for some* $x \in X$, $\sqcap(X - \{x\}, Y) \geq 1$*, then* X *is a maximal member of the class of all cosegments of* M *that do not intersect* Y.

Proof. Suppose this is false, and that for some $e \in E(M) - (X \cup Y)$, $X \cup \{e\}$ is a cosegment of M. Then since e and x are distinct members of $X \cup \{e\}$, the remaining members of $X \cup \{e\}$ become coloops in the matroid $M \backslash \{e, x\}$. It follows that $\sqcap(X - \{x\}, E(M) - (X \cup \{e\})) = 0$, implying that $\sqcap(X - \{x\}, Y) = 0$ by Lemma 2.12. The result follows by contradiction. □

For the next lemma, we define *covertical* k*-partitions* and *covertical* k*-separations* of a matroid to be vertical k-partitions and k-separations of the dual matroid respectively. For this lemma, we consider the dual of our problem, namely that our 3-connected matroid has a cohyperplane from which deletion of any element leaves the matroid with a covertical 2-separation. Here, we consider only the case where the complement of the cohyperplane is a triangle.

Lemma 3.8. *Let* M *be a* 3*-connected matroid with a cohyperplane* H*, such that for all* $h \in H$, $\mathrm{co}(M \backslash h)$ *is not* 3*-connected, and let* C *be the circuit whose complement is* H*. Suppose that* C *is a triangle of* M*. Then* M *is a member of the family* $\mathcal{P}$ *of matroids defined in Section 1.*

Proof. Firstly note that in M^*, H is a hyperplane such that for all $h \in H$, $\mathrm{si}(M/h)$ is not 3-connected. We may assume by Lemma 3.1 that $|E(M)| \geq 7$. Let $C = \{c_1, c_2, c_3\}$, and let $x \in H$ with (X_1, x, X_2)

a covertical 3-partition of M. Then by Corollary 3.6, $C \cap X_i \neq \emptyset$, $i = 1, 2$. We may assume without loss of generality that $C \cap X_1 = \{c_1\}$, giving $c_1 \in \mathrm{cl}(X_2)$ (because $\{c_1, c_2, c_3\}$ is a triangle). This implies that $(X_1 - \{c_1\}, X_2 \cup \{c_1\})$ is a 2-separation of $M \backslash x$, however it is not a covertical 2-separation since $(X_1 - \{c_1\}) \cap C = \emptyset$ which would contradict Corollary 3.6. Since $(X_1 - \{c_1\}, X_2 \cup \{c_1\})$ is not covertical, $X_1 - \{c_1\}$ is a series class of $M \backslash x$. It follows that $X_1 - \{c_1\} \cup \{x\}$ is a cosegment of M. We also see that $c_1 \in \mathrm{cl}_M(X_1 - \{c_1\})$, because otherwise $X_1 - \{c_1\}$ would be a separator of $M \backslash x$, a contradiction to the connectivity of M. Thus by Lemma 2.2, X_1 is a circuit of $M \backslash x$, and hence a circuit of M.

Now, let $y \in X_1 - \{c_1\}$ and let (Y_1, y, Y_2) be a covertical 3-partition of M, where $Y_1 \cap C = \{c_i\}$. Again, $Y_1 - \{c_i\} \cup \{y\}$ is a cosegment of M. Let $\{y, z, w\}$ be a triad of this cosegment that contains y. By orthogonality, $\{z, w\} \cap (X_1 - \{c_1\}) \neq \emptyset$ since X_1 is a circuit containing y, and $c_1 \notin \{z, w\}$. It now follows that $\{y, z, w\}$ intersects a triad of $X_1 - \{c_1\} \cup \{x\}$ in at least two members, so that $X_1 - \{c_1\} \cup \{x, y, z, w\}$ is a cosegment of M. Now, observe that $X_1 - \{c_1\} \cup \{x\}$ is a maximal cosegment of $E(M) - C$ by Lemma 3.7, because $\sqcap(X_1 - \{c_1\}, C) = 1$. It now follows that $X_1 - \{c_1\} \cup \{x, y, z, w\} = X_1 - \{c_1\} \cup \{x\}$, and we deduce that $Y_1 - \{c_i\} \subseteq X_1 \cup \{x\}$. A symmetric argument now shows that $Y_1 - \{c_i\} \cup \{y\} = X_1 - \{c_1\} \cup \{x\}$. Now, observe that since X_1 is a circuit of M, $y \in \mathrm{cl}(X_1 - \{y\})$, but also $y \notin \mathrm{cl}(Y_1)$, implying that $c_i \neq c_1$. Thus we have without loss of generality that $c_i = c_2$, that is $\{c_2\} = Y_1 \cap C$.

We now consider $z \in X_1 - \{c_1, y\}$ and a covertical 3-partition (Z_1, z, Z_2) of M, with $|Z_1 \cap C| = 1$. Then by the symmetry of the argument above, $c_3 \in Z_1$ and $Z_1 - \{c_3\} \cup \{z\} = X_1 - \{c_1\} \cup \{x\}$ is a cosegment of M. Now suppose that there is another member $w \in X_1 - \{c_1, y, z\}$. Then if (W_1, w, W_2) were a covertical 3-partition of M with $|W_1 \cap C| = 1$, a symmetrical argument tells us that $c_1, c_2, c_3 \notin W_1$, a contradiction. We conclude that no such w exists, and that $X_1 = \{c_1, y, z\}$. The result of this is that $\{x, y, z\}$ is a maximal cosegment of M, and since X_1, Y_1 and Z_1 are circuits, we have $M|\{x, y, z, c_1, c_2, c_3\} \cong K_4$.

We may apply the argument above to any member $h \in H$, to show that h is contained in a triad T, and that $M|(T \cup C) \cong K_4$. We conclude that $M \in \mathcal{P}$. □

4. Proof of main theorem.

In this section, we complete the proof of the main theorem of the paper, Theorem 1.1.

Proof of Theorem 1.1. It is easily seen that all members of $\mathcal{P}^*$ have a hyperplane as described in Theorem 1.1 by letting H be the set of elements $\{t_{11}, t_{12}, t_{13}, \ldots, t_{n1}, t_{n2}, t_{n3}\}$.

We must now show that if M has a hyperplane H with the contraction property stated in Theorem 1.1, then $M \in \mathcal{P}^*$. Let C be the cocircuit whose complement is H. Let $x \in H$ and let (X_1, x, X_2) be a vertical 3-partition of M such that $X_2 \cup \{x\}$ is closed, and for all $y \in X_1 \cap H$, whenever (Y_1, y, Y_2) is a vertical 3-partition of M, then $Y_1 \cap X_2 \neq \emptyset$ and $Y_2 \cap X_2 \neq \emptyset$ (we know that such a 3-partition exists by Lemma 3.4 and Corollary 3.6). Corollary 3.6 now tells us that $X_i \cap C \neq \emptyset$ and $X_i \cap H \neq \emptyset$, for $i = 1, 2$.

Let $y \in X_1 \cap H$ and (Y_1, y, Y_2) be a vertical 3-partition of M with $x \in Y_1$. Then as in the proof of Theorem 3.5, each of $X_1 \cap Y_1$, $X_1 \cap Y_2$, $X_2 \cap Y_1$ and $X_2 \cap Y_2$ is nonempty. Now, observe by uncrossing that $X_1 \cap Y_2$ and $(X_1 \cap Y_2) \cup \{y\}$ are 3-separators of M. If $|X_1 \cap Y_2| \geq 2$ then $\mathrm{r}((X_1 \cap Y_2) \cup \{y\}) = 2$ by a proof similar to that of Sublemma 3.5.1.

Sublemma 4.0.1. *If $|X_2 \cap Y_2| \geq 2$, then $(X_1 \cap Y_1) \cup \{x, y\}$ is a segment contained in H, and $X_1 \cap Y_2 \subseteq C$.*

Proof. Suppose that $|X_2 \cap Y_2| \geq 2$. Then by uncrossing, $(X_1 \cap Y_1) \cup \{x, y\}$, $(X_1 \cap Y_1) \cup \{y\}$, $(X_1 \cap Y_1) \cup \{x\}$ and $X_1 \cap Y_1$ are 3-separators of M. Since $y \in \mathrm{cl}(Y_2)$, we have $y \in \mathrm{cl}((X_1 \cap Y_1) \cup \{x\})$ by Lemma 2.4, and a similar argument gives $x \in \mathrm{cl}((X_1 \cap Y_1) \cup \{y\})$. Also, if $|X_1 \cap Y_1| \geq 2$ then $x, y \in \mathrm{cl}(X_1 \cap Y_1)$. We see that $\mathrm{r}((X_1 \cap Y_1) \cup \{x, y\}) = 2$ by a similar proof to that of Sublemma 3.5.1. Now, since $x, y \in H$, and H is closed, we see that $X_1 \cap Y_1 \subseteq H$. Furthermore, $(X_1 \cap Y_2) \cap C \neq \emptyset$ because we have $X_1 \cap C \neq \emptyset$ and $((X_1 \cap Y_1) \cup \{y\}) \cap C = \emptyset$. We now see that $X_1 \cap Y_2 \subseteq C$, because H is closed, $y \in H$, and $\mathrm{r}((X_1 \cap Y_2) \cup \{y\}) = 2$. $\square$

Sublemma 4.0.2. *If $|X_2 \cap Y_2| \geq 2$ then $|X_1 \cap Y_1| = 1$.*

Proof. Suppose that $|X_2 \cap Y_2| \geq 2$ and $|X_1 \cap Y_1| \geq 2$. Then by Corollary 3.3, we must have $|X_1 \cap Y_2| \geq 2$ as well. Choose $z \in X_1 \cap Y_1$ and consider a vertical 3-partition (Z_1, z, Z_2) of M. We may assume without loss of generality that $x \in Z_1$. Combine this with the fact that $z \in \mathrm{cl}(Z_1)$, to obtain $(X_1 \cap Y_1) \cup \{x, y\} \subseteq \mathrm{cl}(Z_1)$. Hence we may assume by Lemma 2.9 that $(X_1 \cap Y_1) \cup \{x, y\} \subseteq Z_1$. Now, $z \in \mathrm{cl}(Z_2)$ and $z \notin \mathrm{cl}(X_2)$ so we see that $(X_1 \cap Y_2) \cap Z_2 \neq \emptyset$. Suppose that $X_1 \cap Y_2 \not\subseteq Z_2$ and that $w \in (X_1 \cap Y_2) \cap Z_1$. Then $\mathrm{r}(Z_1 \cap X_1) = 3$, which implies that $X_1 \subseteq \mathrm{cl}(Z_1)$. Then by Lemma 2.9, $(\mathrm{cl}(Z_1) - \{z\}, z, Z_2 - \mathrm{cl}(Z_1))$ is a vertical 3-partition of M with $Z_2 - \mathrm{cl}(Z_1) \subseteq X_2$, contradicting the fact that $z \notin \mathrm{cl}(X_2)$. We conclude that $X_1 \cap Y_2 \subseteq Z_2$. Now, since $y \in \mathrm{cl}(X_1 \cap Y_2)$, we see that $\{y, z\} \subseteq \mathrm{cl}(Z_2)$ implying that

$(X_1 \cap Y_1) \cup \{x, y\} \subseteq \mathrm{cl}(Z_2)$. By Lemma 2.9, we may construct the new 3-partition $(Z_1 - \mathrm{cl}(Z_2), z, \mathrm{cl}(Z_2) - \{z\})$ in which $Z_1 - \mathrm{cl}(Z_2) \subseteq X_2$, contradicting that $z \notin \mathrm{cl}(X_2)$. We may conclude from this that it is not possible to have $|X_1 \cap Y_1| \geq 2$, and the result follows. □

Sublemma 4.0.3. *If $|X_2 \cap Y_2| \geq 2$, then M is a member of the class $\mathcal{P}^*$.*

Proof. Suppose that $|X_2 \cap Y_2| \geq 2$. Then by Sublemma 4.0.2, $|X_1 \cap Y_1| = 1$. Corollary 3.3 tells us that $|X_1 \cap Y_2| = 1$. Thus X_1 is a triad of M. Let $z \in X_1 \cap Y_1$, and let $a \in X_1 \cap Y_2$. Then $z \in H$ as H is closed, and by Corollary 3.6, $a \in C$. We also see by uncrossing that $(X_2 \cap Y_1) \cup \{x\}$ is a 3-separator of M, and hence a 2-separator of $M \backslash z$, which by Bixby's Theorem 2.10 implies that $|X_2 \cap Y_1| = 1$. We thus have Y_1 a triad of M. Let $b \in X_2 \cap Y_1$. Then by Corollary 3.6, $Y_1 \cap C \neq \emptyset$ implying that $b \in C$.

Consider a vertical 3-partition (Z_1, z, Z_2) of M, and assume without loss of generality that $x \in Z_1$. Then since $z \in \mathrm{cl}(Z_1)$, we have $y \in \mathrm{cl}(Z_1)$, so we may assume by Lemma 2.9, that $y \in Z_1$.

Now, the triads $\{a, y, z\}$ and $\{b, x, z\}$ must both intersect Z_2 in order for $z \in \mathrm{cl}(Z_2)$, thus $a, b \in Z_2$. We have $\lambda(\{x, y, z\}) = 2$, and $\mathrm{r}(X_2 \cap Y_2) = \mathrm{r}((X_2 \cap Y_2) \cup \{a, b\}) - 2$, thus $\sqcap(\{x, y, z\}, X_2 \cap Y_2) = 0$, giving $\mathrm{r}(Z_1) = \mathrm{r}(\{x, y\}) + \mathrm{r}(Z_1 - \{x, y\}) = \mathrm{r}(Z_1 - \{x, y\}) + 2$. Now, since $z \in \mathrm{cl}(Z_2)$, $\mathrm{r}(Z_2 \cup \{x, y\}) \leq \mathrm{r}(Z_2) + 1$. It follows that $\lambda(Z_1 - \{x, y\}) \leq 1$ which implies that $|Z_1 - \{x, y\}| = 1$. Then Z_1 is a triad of M, $Z_1 \cup \{z\}$ is a fan of M, and letting $c \in Z_1 - \{x, y\}$, $M \backslash c$ has the vertical 2-separation $(\{x, y, z\}, Z_2)$, thus $c \in C$. We may now deduce that $a, b, c \in \mathrm{cl}^*(\{x, y, z\})$, and by Lemma 2.5, $\{a, b, c\}$ is a triad of M contained in C. As C is a cocircuit, we have $C = \{a, b, c\}$. We also see by the list of triads and triangles in $\{x, y, z, a, b, c\}$, that $M^*|\{x, y, z, a, b, c\} \cong K_4$. We may now apply Lemma 3.8 in order to obtain the result that M^* is a member of $\mathcal{P}$. □

Having considered the case where $|X_2 \cap Y_2| \geq 2$, we must now look at the case where $|X_2 \cap Y_2| = 1$. Firstly, $|Y_2| \geq 3$ giving $|X_1 \cap Y_2| \geq 2$, hence $(X_1 \cap Y_2) \cup \{y\}$ is a segment of size at least three. Let $e \in X_2 \cap Y_2$. Then $((X_1 \cap Y_2) \cup \{y\}, Y_1)$ is a vertical 2-separation of $M \backslash e$, implying that $e \in C$ by Bixby's Theorem 2.10. Note that $X_1 \cap Y_2$ contains an element of H, by Corollary 3.6 applied to (Y_1, y, Y_2). Since H is closed and $y \in H$, it follows that $(X_1 \cap Y_2) \cup \{y\} \subseteq H$. We may now apply Corollary 3.3 to the 3-separation $(Y_1, Y_2 \cup \{y\})$ to obtain $|X_1 \cap Y_2| = 2$.

Sublemma 4.0.4. $y \in \mathrm{cl}((X_1 \cap Y_1) \cup \{x\})$.

Proof. This is identical to the proof of Sublemma 3.5.4. □

Let $z \in X_1 \cap Y_2$ and consider a vertical 3-partition (Z_1, z, Z_2) of M with $x \in Z_1$. We know by Sublemma 4.0.3 that if $|Z_2 \cap X_2| \geq 2$, then M is in the class $\mathcal{P}^*$, so we may assume that $|Z_2 \cap X_2| = 1$, and by symmetry, we see that Z_2 is a triad of M with $(X_1 \cap Z_2) \cup \{z\}$ a triangle contained in H.

Let w be the third member of the triangle $(X_1 \cap Y_2) \cup \{y\}$. Then since $Z_1 \cup \{z\}$ is closed, either $\{y, w\} \subseteq Z_1$ or $\{y, w\} \subseteq Z_2$. If $\{y, w\} \subseteq Z_1$ then $(X_1 \cap Z_2) \cup \{z\}$ is a triangle with $X_1 \cap Z_2 \subseteq X_1 \cap Y_1$, but this is not possible because z is not in $\mathrm{cl}(Y_1)$. Therefore, we must have $\{y, w\} \subseteq Z_2$, and by the sizes of Y_2 and Z_2, we have $\{y, w\} = X_1 \cap Z_2$, which implies that $X_1 \cap Y_1 = X_1 \cap Z_1$. However, $y \in \mathrm{cl}((X_1 \cap Y_1) \cup \{x\})$ and hence $y \in \mathrm{cl}(Z_1)$, contradicting that $Z_1 \cup \{z\}$ is closed. This contradiction completes the analysis of the case where $|X_2 \cap Y_2| = 1$, and the result of Theorem 1.1 now follows. $\square$

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Brunel University, Uxbridge UB8 3PH, United Kingdom
E-mail address: `rhiannon.hall@brunel.ac.uk`