THE STRUCTURE OF 3-CONNECTED MATROIDS OF PATH WIDTH THREE

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Abstract. A 3-connected matroid M is sequential or has path width 3 if its ground set $E(M)$ has a sequential ordering, that is, an ordering $(e_1, e_2, \ldots, e_n)$ such that $(\{e_1, e_2, \ldots, e_k\}, \{e_{k+1}, e_{k+2}, \ldots, e_n\})$ is a 3-separation for all k in $\{3, 4, \ldots, n-3\}$. In this paper, we consider the possible sequential orderings that such a matroid can have. In particular, we prove that M essentially has two fixed ends, each of which is a maximal segment, a maximal cosegment, or a maximal fan. We also identify the possible structures in M that account for different sequential orderings of $E(M)$. These results rely on an earlier paper of the authors that describes the structure of equivalent non-sequential 3-separations in a 3-connected matroid. Those results are extended here to describe the structure of equivalent sequential 3-separations.

1. Introduction

The matroid terminology used here will follow Oxley [3]. Let M be a matroid. When M is 2-connected, Cunningham and Edmonds [1] gave a tree decomposition of M that displays all of its 2-separations. Now suppose that M is 3-connected. Oxley, Semple, and Whittle [5] showed that there is a corresponding tree decomposition of M that displays all non-sequential 3-separations of M up to a certain natural equivalence. Both this equivalence and the definition of a non-sequential 3-separation are based on the notion of full closure in M. For a set Y, if Y equals its closure in both M and M^*, we say that Y is *fully closed* in M. The *full closure*, $\mathrm{fcl}(Y)$, of Y is the intersection of all fully closed sets containing Y. It is obtained by beginning with Y and alternately applying the closure operators of M and M^* until no new elements can be added. If (X, Y) is a 3-separation of M, then (X, Y) is *sequential* if $\mathrm{fcl}(X)$ or $\mathrm{fcl}(Y)$ is $E(M)$. Two 3-separations (Y_1, Y_2) and (Z_1, Z_2) of M are *equivalent* if $\{\mathrm{fcl}(Y_1), \mathrm{fcl}(Y_2)\} = \{\mathrm{fcl}(Z_1), \mathrm{fcl}(Z_2)\}$.

While the introduction of this notion of equivalence is an essential tool in proving the main result of [5], this equivalence ignores some of the finer structure of the matroid. Hall, Oxley, and Semple [2] made a detailed examination of this equivalence and described precisely what substructures in the matroid result in two non-sequential 3-separations being equivalent. The assumption that the 3-separations are non-sequential is helpful in that it gives two fixed ends for the 3-separations in an equivalence class $\mathcal{K}$. More precisely, if (A_1, B_1) is in $\mathcal{K}$,

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then $(A_1 - \mathrm{fcl}(B_1), \mathrm{fcl}(B_1))$ and $(\mathrm{fcl}(A_1), B_1 - \mathrm{fcl}(A_1))$ are also in $\mathcal{K}$. Letting $A = A_1 - \mathrm{fcl}(B_1)$ and $B = B_1 - \mathrm{fcl}(A_1)$, we have, for every 3-separation (A_2, B_2) in $\mathcal{K}$, that $\{A_2 - \mathrm{fcl}(B_2), B_2 - \mathrm{fcl}(A_2)\} = \{A, B\}$. Thus we can view A and B as the fixed ends of the members of the equivalence class $\mathcal{K}$. Moreover, we can associate with $\mathcal{K}$ a sequence $(A, x_1, x_2, \ldots, x_n, B)$ where $E(M) = A \cup \{x_1, x_2, \ldots, x_n\} \cup B$ and, for all i in $\{0, 1, \ldots, n\}$, the partition $(A \cup \{x_1, x_2, \ldots, x_i\}, \{x_{i+1}, x_{i+2}, \ldots, x_n\} \cup B)$ is a 3-separation. In [2], we described what reorderings of $(x_1, x_2, \ldots, x_n)$ produce another such sequence and specified what kinds of substructures of M result in these reorderings.

In this paper, we consider the behaviour of sequential 3-separations in M. In particular, our aim is to associate fixed ends with such a 3-separation so that we can use the results of [2]. Since we want this paper to include a description of sequential matroids that is as self-contained as possible, we shall state here a number of results from [2]. Let (A_1, B_1) be a sequential 3-separation. We call (A_1, B_1) *bisequential* if both $\mathrm{fcl}(A_1)$ and $\mathrm{fcl}(B_1)$ equal $E(M)$, and *unisequential* otherwise. In the latter case, suppose that $\mathrm{fcl}(A_1) = E(M)$. Then (A_1, B_1) is equivalent to $(A_1 - \mathrm{fcl}(B_1), \mathrm{fcl}(B_1))$ and, if $A = A_1 - \mathrm{fcl}(B_1)$, then, for every member (A_2, B_2) of the equivalence class $\mathcal{K}$ containing (A_1, B_1), we have $\{A_2 - \mathrm{fcl}(B_2), B_2 - \mathrm{fcl}(A_2)\} = \{A, \emptyset\}$. This gives us A as one fixed end for every member of $\mathcal{K}$. Our first task in treating the members of such an equivalence class $\mathcal{K}$ is to show that we can associate a second fixed end with the members of $\mathcal{K}$. Let S be a subset of $E(M)$ with $|S| \geq 3$. Then S is a *segment* if every 3-element subset of S is a triangle, and S is a *cosegment* if every 3-element subset of S is a triad. We call S a *fan* if there is an ordering $(s_1, s_2, \ldots, s_n)$ of the elements of S such that, for all $i \in \{1, 2, \ldots, n-2\}$,

(i) $\{s_i, s_{i+1}, s_{i+2}\}$ is either a triangle or a triad, and
(ii) when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triad, and when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triangle.

This ordering $(s_1, s_2, \ldots, s_n)$ is called a *fan ordering* of S. When $n \geq 4$, the elements s_1 and s_n are the only elements of the fan that are not in both a triangle and a triad contained in S. We call these elements the *ends* of the fan S. The remaining elements of S are the *internal elements* of the fan. We denote the set of such elements by $I(S)$.

The second fixed end that one can associate with an equivalence class of unisequential 3-separations is obtained using the following result.

Theorem 1.1. *Let M be a 3-connected matroid with a 3-sequence $(X, x_1, x_2, \ldots, x_n)$ where $|X|$ and $n-1$ are both at least two and $E(M) - X$ is fully closed. Then $E(M) - X$ contains either*

(i) *a subset W such that, for every 3-sequence $(X, y_1, y_2, \ldots, y_n)$, the set W is the unique maximal subset of $E(M) - X$ that contains $\{y_{n-2}, y_{n-1}, y_n\}$ and is a segment, a cosegment, or a fan; or*
(ii) *a triangle or triad T such that, for every 3-sequence $(X, y_1, y_2, \ldots, y_n)$, the set $\{y_{n-3}, y_{n-2}, y_{n-1}, y_n\}$ contains T. Moreover, T is the set of internal elements of exactly three 5-element maximal fans, F_1, F_2, and F_3 of M, at*

least two of these fans contain $\{y_{n-2}, y_{n-1}, y_n\}$, and there is no other maximal subset of $E(M) - X$ that contains $\{y_{n-2}, y_{n-1}, y_n\}$ and is a segment, a cosegment, or a fan.

If (A_1, B_1) is a bisequential 3-separation of M, then it is straightforward to show that M itself is sequential. Then, when we describe all members of the equivalence class $\mathcal{K}$ containing (A_1, B_1), we are also describing all possible sequential orderings of M. Two elementary examples of sequential matroids are $U_{2,n}$ and $U_{n-2,n}$. In each case, every possible ordering of the ground set is a sequential ordering of the matroid. All wheels and whirls are also sequential and, as we shall see in Lemma 6.12, all possible sequential orderings of such matroids are also easily described. We shall prove in Proposition 4.7 that these examples are the only matroids having a cyclic ordering on their ground sets so that any linear ordering produced by cutting this cyclic ordering is a sequential ordering. Thus such sequential matroids can be viewed as having no fixed end. We shall show that every other sequential matroid has two distinct fixed ends. More precisely, we shall prove the following.

Theorem 1.2. *Let M be a sequential matroid of rank and corank at least three and assume that M is neither a wheel nor a whirl. Let $(e_1, e_2, \dots, e_n)$ and $(g_1, g_2, \dots, g_n)$ be sequential orderings of $E(M)$. Then $E(M)$ has distinct subsets U and V each of which is a maximal segment, a maximal cosegment, or a maximal fan such that, for some $(h_1, h_2, \dots, h_n)$ in $\{(g_1, g_2, \dots, g_n), (g_n, g_{n-1}, \dots, g_1)\}$, both $\{e_1, e_2, e_3\} \cup \{h_1, h_2, h_3\} \subseteq U$ and $\{e_{n-2}, e_{n-1}, e_n\} \cup \{h_{n-2}, h_{n-1}, h_n\} \subseteq V$.*

The last theorem may mislead the reader into thinking that the sets U and V are the same for all pairs of sequential orderings of M. But, when U or V is a maximal fan, this notion needs some refinement. Let M be a sequential matroid that has rank and corank at least three and is not a wheel or a whirl. Let $(a_1, a_2, \dots, a_n)$ be a sequential ordering Σ of M. Clearly $\{a_1, a_2, a_3\}$ is a triangle or a triad. If this set is in no larger segment, cosegment, or fan of M, we let $L(\Sigma) = \{a_1, a_2, a_3\}$ and call $L(\Sigma)$ a *triangle end* or a *triad end* of M. If $\{a_1, a_2, a_3\}$ is contained in a segment or cosegment of size at least four, let $L(\Sigma)$ be the maximal such segment or cosegment and call $L(\Sigma)$ a *segment end* or a *cosegment end* of M. Finally, if $\{a_1, a_2, a_3\}$ is contained in a fan of size at least four, then take a maximal such fan F. By Theorem 3.4 below, although F need not be unique, every choice of F has the same set of internal elements and we let $L(\Sigma) = I(F)$ and call $L(\Sigma)$ a *fan end* of Σ. We define $R(\Sigma)$ analogously. The *type* of $L(\Sigma)$ or $R(\Sigma)$ is its designation as a triangle end, a triad end, a segment end, a cosegment end, or a fan end

Theorem 1.3. *Let M be a sequential matroid that has rank and corank at least three and is not a wheel or a whirl. Then there are distinct subsets $L(M)$ and $R(M)$ of $E(M)$ such that $\{L(M), R(M)\} = \{L(\Sigma), R(\Sigma)\}$ for all sequential orderings Σ of $E(M)$.*

We call $L(M)$ and $R(M)$ the *left* and *right ends*, respectively, of M. In view of the last result, once the sets $L(M)$ and $R(M)$ have been specified, if Σ is a sequential ordering of M and $\overleftarrow{\Sigma}$ is its reversal, we *normalize* Σ by replacing it by the member Σ_0 of $\{\Sigma, \overleftarrow{\Sigma}\}$ for which $L(\Sigma_0) = L(M)$ and $R(\Sigma_0) = R(M)$.

The labelling of $L(M)$ and $R(M)$ may suggest that these sets occur at the ends of every sequential ordering of M. This oversimplifies the situation somewhat although not dramatically so. The next result gives a precise description of what can happen at the left end. Evidently, the corresponding result also holds for the right end.

Theorem 1.4. *Let M be a sequential matroid that has rank and corank at least three and is not a wheel or a whirl. Let Σ and Σ' be normalized sequential orderings of M. Then $L(\Sigma)$ and $L(\Sigma')$ have the same type. Moreover,*

(i) *if $L(\Sigma)$ is a triangle or a triad end of M, then the first three elements of Σ' are in $L(\Sigma)$;*
(ii) *if $L(\Sigma)$ is a segment or cosegment end of M, then the first $|L(\Sigma)|-1$ elements of Σ' are in $L(\Sigma)$; and*
(iii) *if $L(\Sigma)$ is a fan end of M, then either the first $|L(\Sigma)|$ elements of Σ' are in $L(\Sigma)$, or there is a maximal fan F of M having $L(\Sigma)$ as its set of internal elements such that the first $|L(\Sigma)|+1$ elements of Σ' include $L(\Sigma)$ and are contained in F.*

The last result guarantees that the types of $L(M)$ and $R(M)$ are well-defined. We conclude this section by briefly describing the organization of the paper. In the next section, we introduce some basic terminology and present the results from [2] that are needed to appreciate the main results of this paper. That section concludes with Theorem 2.3 which specifies how to determine all sequential orderings of a 3-connected matroid. Section 3 contains some elementary preliminaries. In Section 4, we prove several attractive results about possible reorderings of sequential matroids. These results will be basic tools in the material that follows. In Sections 5 and 6, we investigate how segments and fans can occur at the ends of sequential orderings, and, in Section 7, we complete the proofs of Theorems 1.1–1.4.

2. Overview

The primary concern in this paper will be with 3-connected matroids. Indeed, from now on, unless otherwise specified, all matroids considered will be 3-connected.

Let M be a matroid. A partition (X,Y) of $E(M)$ into possibly empty sets is 3-*separating* if $r(X)+r(Y)-r(M)\le 2$. When equality holds here, (X,Y) is called *exactly* 3-*separating*; and (X,Y) is a 3-*separation* when $|X|,|Y|\ge 3$. In [2], the term "3-sequence" was used for an ordered partition $(A,x_1,x_2,\dots,x_n,B)$ of the ground set $E(M)$ of M such that $|A|,|B|\ge 2$ and $A\cup\{x_1,x_2,\dots,x_i\}$ is exactly 3-separating for all i in $\{0,1,\dots,n\}$. In this paper, we shall call such a sequence an (A,B) 3-*sequence* and we shall extend the use of the term 3-*sequence* to an ordered partition $(E_1,E_2,\dots,E_n)$ of $E(M)$ into sets such that, for all i in $\{0,1,\dots,n\}$, the set $\bigcup_{j=1}^{i}E_j$ is 3-separating with this set being exactly 3-separating if both $|\bigcup_{j=1}^{i}E_j|$ and $|\bigcup_{j=i+1}^{n}E_j|$ exceed one. If, for some m in $\{1,2,\dots,n\}$, there is an ordering $\overrightarrow{E_m}$ of E_m, say $\overrightarrow{E_m}=(x_1,x_2,\dots,x_k)$, such that $(E_1,E_2,\dots,E_{m-1},\{x_1\},\{x_2\},\dots,\{x_k\},E_{m+1},\dots,E_n)$ is a 3-sequence, then we also write this 3-sequence as $(E_1, E_2, \ldots, E_{m-1}, x_1, x_2, \ldots, x_k, E_{m+1}, \ldots, E_n)$ or $(E_1, E_2, \ldots, E_{m-1}, \overrightarrow{E_m}, E_{m+1}, \ldots, E_n)$. Usually the sets $E_1, E_2, \ldots, E_n$ in a 3-sequence will be non-empty but there is one case, in particular, when we want to allow the last set to be empty. A 3-sequence $(e_1, e_2, \ldots, e_t, Z)$ is *fully expanded* if $\mathrm{fcl}(\{e_1, e_2, \ldots, e_t\}) = \{e_1, e_2, \ldots, e_t\}$. We call Z the *tail* of this 3-sequence. If Z is empty, then Z is *trivial*. Observe that if Z is non-trivial, then $|Z| \geq 4$. Let (A, X, B) be a 3-sequence. If x and y are elements of X and there are 3-sequences $(A, \overrightarrow{X_1}, B)$ and $(A, \overrightarrow{X_2}, B)$ such that x precedes y in $\overrightarrow{X_1}$, and y precedes x in $\overrightarrow{X_2}$, then x is in the *jump-set* J_y of y, and y is in J_x.

If (Y, Z) is an exactly 3-separating partition of $E(M)$, then this partition is *sequential* if Y or Z has an ordering, $\overrightarrow{Y}$ or $\overrightarrow{Z}$, such that $(\overrightarrow{Y}, Z)$ or $(Y, \overrightarrow{Z})$ is a 3-sequence. The matroid M itself is *sequential* if there is an ordering $e_1, e_2, \ldots, e_t$ of $E(M)$ such that $(e_1, e_2, \ldots, e_t)$ is a 3-sequence. Note that we can view such a sequential ordering as a fully expanded 3-sequence with trivial tail. When we refer to a sequential ordering of M of the form (U, V), we mean that there are orderings $\overrightarrow{U}$ and $\overrightarrow{V}$ of U and V such that $(\overrightarrow{U}, \overrightarrow{V})$ is a sequential ordering of M. A sequential matroid M is also said to have *path width three* because there is a path P with vertex set $E(M)$ such that the partition of $E(M)$ induced by each edge of P is a 3-separating partition of $E(M)$.

The next result is elementary. Its proof appears, for example, in [2, Lemma 4.1].

Lemma 2.1. *Let $(A, x_1, x_2, \ldots, x_n, B)$ be a 3-sequence of a 3-connected matroid. Then, for each i, exactly one of the following holds:*

(i) $x_i \in \mathrm{cl}(A \cup \{x_1, \ldots, x_{i-1}\}) \cap \mathrm{cl}(\{x_{i+1}, \ldots, x_n\} \cup B)$; *or*
(ii) $x_i \in \mathrm{cl}^*(A \cup \{x_1, \ldots, x_{i-1}\}) \cap \mathrm{cl}^*(\{x_{i+1}, \ldots, x_n\} \cup B)$.

We call x_i a *guts* or *coguts* element of the 3-sequence $(A, x_1, x_2, \ldots, x_n, B)$ depending on whether (i) or (ii) of the last lemma holds or, equivalently, on whether x_i is in the closure or coclosure of $A \cup \{x_1, x_2, \ldots, x_{i-1}\}$. It was shown in [2, Lemma 4.6] that this labelling is robust in that if x_i is a guts element of some (A, B) 3-sequence, then it is a guts element of all (A, B) 3-sequences.

Next we introduce the matroid structures that arise within X in an (A, B) 3-sequence $(A, \overrightarrow{X}, B)$. Evidently X can contain segments, cosegments, and fans. Such structures having more than three elements fall unambiguously under one of these three descriptions. It was shown in [2] that a triangle contained in X must consist of either three guts elements or two guts elements and a coguts element. In the former case, the triangle will be viewed as a segment, in the latter case as a fan. Likewise, a triad in X that consists of three coguts elements will be viewed as a cosegment, while a triad consisting of two coguts elements and a guts element will be viewed as a fan. For a 2-element subset S of X, if there is an (A, B) 3-sequence in which the elements of S are consecutive, then S is a *degenerate segment* if it consists of two guts elements, a *degenerate cosegment* if it consists of two coguts elements, and a *degenerate fan* if it consists of a guts and a coguts element that are not in each other's jump-sets. It was shown in [2, Theorem 6.9] that, whenever a fan F occurs in an (A,B) 3-sequence, its elements occur in a fan ordering and this ordering is the same for all (A,B) 3-sequences. Thus the ends of the fan are its first and last elements.

A *maximal segment* in a 3-sequence (A,X,B) is a segment Y in X so that there is no segment Z contained in X that properly contains Y. When x is a guts element in X that is not contained in any segment with two or more elements, it will be convenient to view $\{x\}$ as a maximal segment. *Maximal cosegments* are defined dually.

The main result of [2] is that once four special substructures are eliminated from a 3-sequence $(A,\overrightarrow{X},B)$, the set X can be partitioned into maximal segments and maximal cosegments. Moreover, there is a canonical ordering on these maximal segments and maximal cosegments that induces an ordering $\overrightarrow{X_0}$ on X such that $(A,\overrightarrow{X},B)$ is a 3-sequence if and only if $\overrightarrow{X}$ is obtained from $\overrightarrow{X_0}$ by arbitrarily permuting the elements within each maximal segment and each maximal cosegment or, at each interface between a maximal segment and a maximal cosegment, by interchanging a guts element and a coguts element. Before giving a formal statement of this theorem, we shall identify the special substructures that occur within an (A,B) 3-sequence. Precisely the same structures arise within sequential matroids with only a predictable variation occurring at the ends of such matroids.

Let $(A,\overrightarrow{X},B)$ be a 3-sequence in a matroid M. Let F_1 and F_2 be disjoint fans contained in X such that each is maximal with this property. We call $F_1 \cup F_2$ a *clock* if there is a partition (A',B') of $E(M)-(F_1 \cup F_2)$ such that A' and B' contain A and B, respectively, and each of $A', A' \cup F_1, A' \cup F_2$, and $A' \cup F_1 \cup F_2$ is 3-separating. Observe that we allow F_1 or F_2 to be a degenerate fan consisting of one guts and one coguts element. When both F_1 and F_2 are degenerate fans, we call $F_1 \cup F_2$ a *degenerate clock*.

In a 3-sequence (A,X,B), if Y is a clock, a maximal segment, or a maximal cosegment, then the elements of Y can be made consecutive in some (A,B) 3-sequence. Moreover, when this is done, the sets L_Y and R_Y of elements of X that occur to the left and right of Y are uniquely determined. Thus such a set Y forms a barrier in the 3-sequence in that no element e of $X-Y$ has all of Y in its jump-set.

There is a natural ordering on the set of non-degenerate clocks in X. Since a clock is the union of two fans, the clock contains the first and last elements of these fans. All other clock elements are called *internal*. A degenerate clock is *even* if its first elements are both guts elements or are both coguts elements, and is *odd* otherwise. The ordering on non-degenerate clocks referred to above can be extended to include odd degenerate clocks. However, degenerate clocks that are even can interlock in a structure we call a non-degenerate crocodile. Such a structure is built from a maximal segment and a maximal cosegment with the property that at least two elements in the segment have distinct cosegment elements in their jump-sets.

Let $(A,\overrightarrow{X},B)$ be a 3-sequence in a matroid M. Let S and S^* be a maximal segment and a maximal cosegment in X such that $|S \cup S^*| \geq 5$. We call $S \cup S^*$

a *crocodile* if, for some (C, D) in $\{(A, B), (B, A)\}$, there is a partition (C', D') of $E(M) - (S \cup S^*)$ such that

(i) C' and D' contain C and D, respectively;
(ii) each of $C', C' \cup S$, and $C' \cup S \cup S^*$ is 3-separating; and
(iii) for some $k \geq 2$, there are k-element subsets $S_w = \{s_1, s_2, \ldots, s_k\}$ and $S^*_w = \{s^*_1, s^*_2, \ldots, s^*_k\}$ of S and S^* such that
  (a) $C' \cup (S - \{s_i\}) \cup \{s^*_i\}$ is 3-separating for all i in $\{1, 2, \ldots, k\}$; and
  (b) if $s \in S - S_w$, there is no s^* in S^* such that $C' \cup (S - \{s\}) \cup \{s^*\}$ is 3-separating.

The crocodile $S \cup S^*$ is *degenerate* if $|S| = 2$ or $|S^*| = 2$. It was shown in [2, Lemma 8.2] that if $S \cup S^*$ is a non-degenerate crocodile in X, then, for all distinct i and j in $\{1, 2, \ldots, k\}$, there is an even degenerate clock with fans (s_i, s^*_j) and (s_j, s^*_i).

While segments, cosegments, and fans are well-known substructures of matroids, crocodiles are less so although the matroids that give rise to crocodiles have appeared previously in the literature [4] and have a natural and attractive structure. For each $k \geq 3$, take a basis $\{b_1, b_2, \ldots, b_k\}$ of $PG(k-1, \mathbb{R})$ and a line L that is freely placed relative to this basis. By modularity, for each i, the hyperplane of $PG(k-1, \mathbb{R})$ that is spanned by $\{b_1, b_2, \ldots, b_k\} - \{b_i\}$ meets L. Let a_i be the point of intersection. We shall denote by Θ_k the restriction of $PG(k-1, \mathbb{R})$ to $\{b_1, b_2, \ldots, b_k, a_1, a_2, \ldots, a_k\}$. The reader can easily check that Θ_3 is isomorphic to $M(K_4)$. We extend the definition above to include the case $k = 2$. In that case, we begin with two independent points b_1 and b_2 and we add a_1 in parallel with b_2, and a_2 in parallel with b_1. As noted in [4, Lemma 2.1], for all $k \geq 2$, the matroid Θ_k is isomorphic to its dual under the map that interchanges a_i and b_i for all i. In [2, Theorem 8.3], it was shown that every occurrence of a crocodile in an (A, B) 3-sequence corresponds to the presence of a minor isomorphic to Θ_k.

The phenomenon of an element being able to jump a non-trivial sequence of guts and coguts elements, which occurs in a clock, also occurs in a more general context. This leads us to the following generalization of the idea of a fan.

Let $(A, \overrightarrow{X}, B)$ be a 3-sequence and z be a guts element of X. For a subset F of $X - \{z\}$, we call $F \cup \{z\}$ a *pointed flan* or *p-flan* if there is an ordered partition $(\{z\}, F_1, F_2, \ldots, F_m)$ of $F \cup \{z\}$ with $m \geq 3$ such that the following hold:

(i) for all $i \in \{1, 2, \ldots, m\}$, either F_i consists of a single coguts element or $F_i \cup \{z\}$ is a maximal segment;
(ii) if $i \in \{1, 2, \ldots, m-1\}$, then F_i contains a coguts element if and only if F_{i+1} does not;
(iii) if $i \in \{1, 2, \ldots, m-2\}$ and F_i is a singleton coguts set, then $F_i \cup F_{i+1} \cup F_{i+2}$ is a cocircuit; and
(iv) if $i \in \{1, 2, \ldots, m-2\}$ and F_i is a set of guts elements, then $F_i \cup F_{i+1} \cup F_{i+2} \cup \{z\}$ has rank three.

We call z the *tip* of the p-flan $F \cup \{z\}$. Dually, $F \cup \{z\}$ is a *p-coflan* of M with cotip z if it is a p-flan of M^* with tip z. Note that, in the definition of a p-flan, if F_i contains exactly one guts element, then $F_i \cup \{z\}$ is a degenerate segment. A p-flan $(\{z\}, F_1, F_2, \dots, F_m)$ in a 3-sequence (A, X, B) is *maximal* if there is no p-flan $F \cup \{z\}$ such that F properly contains $F_1 \cup F_2 \cup \cdots \cup F_m$.

Three decomposition results for a 3-sequence $(A, \overrightarrow{X}, B)$ are given in [2]. The first of these, Theorem 10.2, begins with a non-degenerate or odd degenerate clock C in X and breaks $(A, \overrightarrow{X}, B)$ into two 3-sequences $(A', \overrightarrow{Y}, B)$ and $(A, \overrightarrow{Z}, B')$. Recalling that L_C is the set of elements of $X - C$ that occur to the left of C in an (A, B) 3-sequence having the elements of C consecutive, we have that A' is the union of A with L_C and all of C except the last elements of its fans, and B' is defined symmetrically. The key point about this decomposition is that every pair of orderings $\overrightarrow{Y_1}$ and $\overrightarrow{Z_1}$ of Y and Z such that $(A', \overrightarrow{Y_1}, B)$ and $(A, \overrightarrow{Z_1}, B')$ are 3-sequences can be combined to produce a 3-sequence $(A, \overrightarrow{X_1}, B)$, and every (A, B) 3-sequence arises in this way.

The second decomposition result of [2], Theorem 10.3, has the same flavour as the first. It breaks up an (A, B) 3-sequence having an even degenerate clock. This decomposition theorem is applicable to (A, B) 3-sequences with non-degenerate crocodiles. It also applies to degenerate crocodiles that occur within 3-sequences with no non-degenerate clocks.

The final decomposition result of [2], Theorem 10.7, treats (A, B) 3-sequences having no clocks. In such a 3-sequence (A, X, B), when the elements of a p-flan $z \cup F$ are consecutive, the sets L_F and R_F of elements of X occurring to the left and right of $z \cup F$ in X are determined. This enables us to obtain the third decomposition theorem, which has a similar form to the first two.

In view of the three decomposition theorems, it is natural to exclude clocks, p-flans, and p-coflans from the 3-sequences we are considering. In such a 3-sequence, every guts element is in a unique maximal segment and every coguts element is in a unique maximal cosegment. Moreover, for such a 3-sequence, we have the following theorem, the main result of [2]. A slightly more explicit statement of this result appears as [2, Theorem 10.13].

Theorem 2.2. *Let $(A, \overrightarrow{X}, B)$ be a 3-sequence that contains no clocks, no p-flans, and no p-coflans and suppose that $|X| \geq 3$. Let $T_1, T_2, \dots, T_n$ be the collection of maximal segments and maximal cosegments in X. Then there is a unique ordering on these sets such that $(A, T_1, T_2, \dots, T_n, B)$ is a 3-sequence. Moreover, every (A, B) 3-sequence can be obtained from this one by the following two steps:*

(i) *arbitrarily reorder the elements of each T_i; and*
(ii) *look among these reorderings at when the last element of T_i is in the jump-set of the first element of T_{i+1}. These swap pairs are disjoint and, for each i and $i+1$ in $\{1, 2, \dots, n\}$, there is at most one such pair. Pick some subset of these swap pairs and swap each element with its partner.*

Next we describe how we shall associate with a sequential matroid M a new sequential matroid M_{LR} and two disjoint subsets $A(M)$ and $B(M)$ of $E(M_{LR})$ such that knowledge of all $(A(M), B(M))$ 3-sequences in M_{LR} will enable us to determine all sequential orderings of M. The matroid M_{LR} is the same as M except that, in order to help describe what happens at the ends, when $L(M)$ or $R(M)$ is a segment of M or M^* of size at least four, we shall freely add three points to this segment. Assume that M is sequential having rank and corank at least three and that M is not a wheel or a whirl. If $L(M)$ is a triangle or triad end or a fan end, let $A(M) = L(M)$ and $M_L = M$. If L is a segment end, then adjoin three elements l_1, l_2, and l_3 freely to the flat $L(M)$ of M to give M_L and let $A(M) = \{l_1, l_2, l_3\}$. If L is a cosegment end, then, in M^*, adjoin three elements l_1, l_2, and l_3 freely to the flat $L(M)$ of M^* to give M_L^* and let $A(M) = \{l_1, l_2, l_3\}$. If $R(M)$ is a triangle or triad end or a fan end, let $B(M) = R(M)$ and $M_{LR} = M_L$. If R is a segment or cosegment end, adjoin three elements r_1, r_2, and r_3 freely to the flat $R(M)$ of M_L or M_L^* to give M_{LR} or M_{LR}^*, respectively, and let $B(M) = \{r_1, r_2, r_3\}$.

Let $\overrightarrow{F}$ be an ordering $(f_1, f_2, \dots, f_m)$ of a fan F in a matroid M. If $\{f_1, f_2, \dots, f_k\}$ is 3-separating for all $k \in \{3, 4, \dots, m\}$, then $\overrightarrow{F}$ is a *left-end ordering* of F. If $\{f_{k-2}, f_{k-1}, \dots, f_m\}$ is 3-separating for all $k \in \{3, 4, \dots, m\}$, then $\overrightarrow{F}$ is a *right-end ordering* of F. We call $\overrightarrow{F}$ an *end ordering* of F if it is a left-end ordering or a right-end ordering of F.

We now describe how to move from $(A(M), B(M))$ 3-sequences in M_{LR} to sequential orderings of M. Let $(A(M), \overrightarrow{X}, B(M))$ be an $(A(M), B(M))$ 3-sequence of M_{LR}. Let $D(M) \in \{A(M), B(M)\}$. If $|D(M)| \leq 3$, let $\overrightarrow{D(M)}$ be an arbitrary ordering of $D(M)$. If $|D(M)| \geq 4$, then $D(M) = I(F)$ for some fan F of M and we let $\overrightarrow{D(M)}$ be a left-end or right-end ordering of the fan $I(F)$ depending on whether $D(M)$ is $A(M)$ or $B(M)$, respectively. We shall show that $(\overrightarrow{A(M)}, \overrightarrow{X}, \overrightarrow{B(M)})$ is a sequential ordering Σ_{LR} of M_{LR}. We shall say that it has been obtained from the 3-sequence $(A(M), \overrightarrow{X}, B(M))$ by *ordering ends*. Let Σ be obtained from Σ_{LR} by deleting the elements not in $E(M)$. This deletes the first three elements of Σ_{LR} when $A(M) = \{l_1, l_2, l_3\}$ or the last three elements of Σ_{LR} when $B(M) = \{r_1, r_2, r_3\}$. We refer to this process as *pruning surplus elements*. Now let $\Sigma = (e_1, e_2, \dots, e_n)$. Suppose that $A(M) = I(F)$ for some fan F and suppose that $\{e_1, e_2, \dots, e_k\} \subseteq F$ but $e_{k+1} \notin F$. Then Σ' is obtained from Σ by replacing $(e_1, e_2, \dots, e_k)$ by a left-end ordering of this fan. We call this move a *fan shuffle*. Finally, if $B(M) = I(F')$ for some fan F', then we can do a fan shuffle at the right end of Σ' to get a sequential ordering Σ'' of M. We shall show that every sequential ordering of M arises in this way.

Theorem 2.3. *Let M be a sequential matroid that has rank and corank at least three and is not a wheel or a whirl. Then the collection of sequential orderings of M coincides with the collection of orderings of $E(M)$ that can be obtained from $(A(M), B(M))$ 3-sequences of M_{LR} by ordering ends, pruning surplus elements, and fan shuffles.*

If $r(M)$ or $r(M^*)$ is 2, then every ordering of $E(M)$ is sequential. We show in Lemma 6.12 that if M is a wheel or a whirl with ground set $\{e_1, e_2, \dots, e_n\}$, then

$(e_1, e_2, \ldots, e_n)$ is a sequential ordering of M if and only if, for all k in $\{3, 4, \ldots, n-3\}$, the sequences $(e_1, e_2, \ldots, e_k)$ and $(e_{k+1}, e_{k+2}, \ldots, e_n)$ are, respectively, a left-end ordering and a right-end ordering of a fan contained in $E(M)$.

Now suppose that (A_1, B_1) is a unisequential 3-separation of a 3-connected matroid M with $\mathrm{fcl}(A_1) = E(M)$ and $\mathrm{fcl}(B_1) = E(M) - A_1$. The results proved later in the paper will guarantee that the construction described above can be applied to M to give a matroid M_R and sets $A(M)$ and $B(M)$ such that $A(M) = A_1$ and knowledge of all $(A(M), B(M))$ 3-sequences of M_R will determine all 3-separations that are equivalent to (A_1, B_1). The formal statement of this result is given in Theorem 7.2 at the end of the paper.

3. Preliminaries

The property of a matroid M that a circuit and a cocircuit cannot intersect in exactly one element is referred to as *orthogonality*. This property is equivalent to the assertion that if $(X, \{z\}, Y)$ is a partition of $E(M)$ into possibly empty sets, then $z \in \mathrm{cl}(X)$ if and only if $z \notin \mathrm{cl}^*(Y)$. We shall write $e \in \mathrm{cl}^{(*)}(Z)$ to mean that $e \in \mathrm{cl}(Z)$ or $e \in \mathrm{cl}^*(Z)$. The next three lemmas will be used frequently throughout the paper often without explicit reference.

Lemma 3.1. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.*

Lemma 3.2. *Let $(A, x_1, x_2, \ldots, x_n, B)$ be a 3-sequence of a 3-connected matroid M, and let $i < j$. If $x_j \in \mathrm{cl}(A \cup \{x_1, \ldots, x_i\})$ or $x_j \in \mathrm{cl}^*(A \cup \{x_1, x_2, \ldots, x_i\})$, then*

$$(A, x_1, \ldots, x_i, x_j, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, x_n, B)$$

is also a 3-sequence of M.

Lemma 3.3. *Let $(e_1, e_2, \ldots, e_n, Z)$ be a fully expanded 3-sequence in a 3-connected matroid M that is neither a wheel nor a whirl and has rank and corank at least three. If S is a maximal segment, a maximal cosegment, or a maximal fan containing $\{e_1, e_2, e_3\}$, then $S \subseteq E(M) - Z$ and $S \neq E(M)$.*

As noted earlier, fans play a prominent role in this paper. We end this short section by introducing some more terminology and some basic properties of fans. Let $(f_1, f_2, \ldots, f_m)$ be a fan ordering of a fan F. If $m \geq 4$ and g is an end of F, then g is a *triangle end* if g is in a triangle of F, and g is a *triad end* if g is in a triad of F. The fan F is of one of three types: *type*-1 if F is a triangle or both ends of F are triangle ends; *type*-2 if F is a triad or both ends of F are triad ends; and *type*-3 when one end of F is a triangle end and the other end is a triad end.

An element e of a 3-connected matroid is *essential* if neither $M\backslash e$ nor M/e is 3-connected. The following result is proved in [6].

Theorem 3.4. *Let M be a 3-connected matroid that is neither a wheel nor a whirl. Suppose that e is an essential element of M. Then e is in a maximal fan. Moreover, this maximal fan is unique unless one of the following holds:*

(i) *every maximal fan containing e consists of a single triangle and any two such triangles meet in $\{e\}$;*
(ii) *every maximal fan containing e consists of a single triad and any two such triads meet in $\{e\}$; or*
(iii) *e is in exactly three maximal fans, where*
  (a) *these three maximal fans are of the same type, each has five elements, and together they contain a total of six elements, and*
  (b) *depending on whether these fans are of type-1 or type-2, the restriction or contraction, respectively, of M to this set of six elements is isomorphic to $M(K_4)$.*

The routine proofs of the next two lemmas are omitted.

Lemma 3.5. *Let M be a 3-connected matroid with $|E(M)| \geq 4$ and let F be a fan of M. If $|F| \geq |E(M)| - 1$, then M is a wheel or a whirl.*

Lemma 3.6. *Let M be a 3-connected matroid that is neither a wheel nor a whirl. Let F be a fan in M that has $(f_1, f_2, \dots, f_k)$ as a fan ordering. Then the only triangles and triads of M contained in F are the sets $\{f_i, f_{i+1}, f_{i+2}\}$ for $1 \leq i \leq k-2$. Moreover, when $k \geq 5$, the only other fan ordering of F is $(f_k, f_{k-1}, \dots, f_1)$; and, when $k = 4$, there are these two fan orderings and two others, (f_1, f_3, f_2, f_4) and its reversal.*

Lemma 3.7. *In a 3-connected matroid M with at least five elements, let F be a fan and S be a segment or a cosegment. If $|F|, |S| \geq 4$, then $|F \cap S| \leq 2$.*

Proof. Assume that $|F \cap S| \geq 3$ and suppose without loss of generality that S is a segment. Then some element of $F \cap S$ is certainly in a triad T^* contained in F. As M has at least five elements, T^* is not a triangle. By orthogonality, if T^* meets a triangle, it contains exactly two elements of that triangle. Thus $|T^* \cap S| = 2$. But since $|S| \geq 4$, there is certainly a triangle contained in S that has exactly one common element with T^*; a contradiction to orthogonality. □

4. Properties of Sequential Matroids and 3-Sequences

In this section, we shall establish some general properties of 3-sequences. We also prove that segments, cosegments, wheels, and whirls are the only 3-connected matroids having a cyclic ordering in which every consecutive subsequence of elements is 3-separating. Recall that we are assuming, unless otherwise specified, that all matroids being considered are 3-connected.

An immediate consequence of the first lemma is that every 3-separation of a 3-connected sequential matroid is either bisequential or unisequential. If M is a matroid and Σ is a 3-sequence $(e_1, e_2, \dots, e_n, Z)$ of $E(M)$, then one can obtain another 3-sequence with tail Z by arbitrarily permuting $\{e_1, e_2, e_3\}$ or by arbitrarily permuting $\{e_{n-2}, e_{n-1}, e_n\}$ when Z is trivial. Such a move is called an *end shuffle* on Σ.

Lemma 4.1. *Let M be a sequential matroid. If (A, B) is a 3-separation of M, then either A or B is sequential. In particular, if there is sequential ordering $(e_1, e_2, \ldots, e_n)$ of M in which $e_1, e_2 \in A$, then B is sequential.*

Proof. Let Σ be a sequential ordering $(e_1, e_2, \ldots, e_n)$ of M. Without loss of generality, we may assume that two of e_1, e_2, and e_3 are in A. Furthermore, by performing an end shuffle on Σ, we may assume that $e_1, e_2 \in A$. Since Σ is a sequential ordering of M, it follows that $e_i \in \mathrm{cl}^{(*)}(\{e_1, e_2, \ldots, e_{i-1}\})$ for all $i \in \{3, 4, \ldots, n\}$. Therefore, for all such i, we have $e_i \in \mathrm{cl}^{(*)}(A \cup \{e_3, e_4, \ldots, e_{i-1}\})$. This implies that $\mathrm{fcl}(A) = E(M)$ and so B is sequential. □

The class of sequential matroids is well-behaved. Indeed, as the following elementary result shows, every 3-connected minor of a sequential matroid is sequential.

Lemma 4.2. *Let M be a sequential matroid and let $e \in E(M)$. If $M \backslash e$ is 3-connected, then $M \backslash e$ is sequential. Moreover, if $(e_1, e_2, \ldots, e_n)$ is a sequential ordering for M, then $(e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n)$ is a sequential ordering for $M \backslash e_i$.*

Proof. If (X, Y) is a 3-separation of M and $e_i \in X$, then $(X - e_i, Y)$ is a 3-separating partition of $E(M \backslash e_i)$. Moreover, if $|X - e_i| \geq 2$, then, since $M \backslash e_i$ is 3-connected, $(X - e_i, Y)$ is a 3-separation of this matroid. □

The next lemma is particularly useful. We shall apply it in the two subsequent corollaries and also in the proof of Theorem 1.1.

Lemma 4.3. *In a matroid M, let (A, B) be an exactly 3-separating partition of $E(M)$. If there is an ordering $(b_1, b_2, \ldots, b_n)$ of B such that both $(A, b_1, b_2, \ldots, b_n)$ and $(A, b_n, b_{n-1}, \ldots, b_1)$ are 3-sequences, then B is a segment, a cosegment, or a fan.*

Proof. If $i \in \{1, 2, \ldots, n-2\}$, then both $\{b_1, b_2, \ldots, b_{i+2}\}$ and $\{b_i, b_{i+1}, \ldots, b_n\}$ are 3-separating and their union avoids A. Thus, by Lemma 3.1, their intersection $\{b_i, b_{i+1}, b_{i+2}\}$ is 3-separating. Hence, by Lemma [2, Lemma 3.2], B is a segment, a cosegment, or a fan. □

Corollary 4.4. *Let M be a matroid and (A, U, V) be a partition of its ground set into sets with at least two elements. Suppose that $(A, \overrightarrow{U_1}, \overrightarrow{V_1})$ and $(A, \overrightarrow{V_2}, \overrightarrow{U_2})$ are 3-sequences for some orderings $\overrightarrow{U_1}$ and $\overrightarrow{U_2}$ of U, and $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ of V. Then $U \cup V$ is a segment, a cosegment, or a fan.*

Proof. Let $\overrightarrow{U_1} = (u_1, u_2, \ldots, u_m)$ and $\overrightarrow{V_2} = (v_1, v_2, \ldots, v_n)$. Then, because $A \cup \{u_1, u_2, \ldots, u_k\}$ is 3-separating for all k in $\{0, 1, \ldots, m\}$, it follows, by beginning with the 3-sequence $(A, \overrightarrow{V_2}, \overrightarrow{U_2})$, that $(A, \overrightarrow{U_1}, \overrightarrow{V_2})$ is a 3-sequence. By symmetry, so too is $(A, \overrightarrow{V_2}, \overrightarrow{U_1})$. From $(A, \overrightarrow{U_1}, \overrightarrow{V_2})$, since $|V| \geq 2$, we see that $V \cup \{u_m, u_{m-1}, \ldots, u_k\}$

is 3-separating for all k in $\{1, 2, \dots, m+1\}$. Thus, by applying Lemma 3.1(i) to this 3-separating set and $A \cup V$, we deduce that $(A, v_1, v_2, \dots, v_n, u_m, u_{m-1}, \dots, u_1)$ is a 3-sequence. By symmetry, so is $(A, u_1, u_2, \dots, u_m, v_n, v_{n-1}, \dots, v_1)$. Hence, by Lemma 4.3, $U \cup V$ is a segment, a cosegment, or a fan. □

Corollary 4.5. *Let M be a sequential matroid, and let (A, B) be a 3-separation of M. If A is not sequential, then B is a segment, a cosegment, or a fan.*

Proof. Let Σ be a sequential ordering $(e_1, e_2, \dots, e_n)$ of M. Then, by performing end shuffles on Σ, we may assume that $\{e_1, e_2\}$ is contained in A or B, and that $\{e_{n-1}, e_n\}$ is contained in A or B. Since A is not sequential, it follows by Lemma 4.1 that B is sequential and $\{e_1, e_2, e_{n-1}, e_n\} \subseteq A$. Let $(b_1, b_2, \dots, b_k)$ be the ordering induced on B by Σ. Since $e_1, e_2 \in A$, it follows that $(A, b_1, b_2, \dots, b_k)$ is a 3-sequence of M. Now $(e_n, e_{n-1}, \dots, e_2, e_1)$ is also a sequential ordering of M and $e_n, e_{n-1} \in A$. Thus $(A, b_k, b_{k-1}, \dots, b_1)$ is also a 3-sequence of M. The corollary now follows immediately from Lemma 4.3. □

The next result is useful in the proof of Theorem 1.1, which is given in the last section. For an ordered set $\overrightarrow{Y}$ and an ordinary set Z, we shall denote the ordering induced on the set $Y - Z$ by $\overrightarrow{Y} - Z$.

Lemma 4.6. *Suppose that $(X, y_1, y_2, \dots, y_n)$ and $(X, z_1, z_2, \dots, z_n)$ are 3-sequences in a matroid M where $|X|$ and $n-1$ are both at least two. Then M has a segment, a cosegment, or a fan that is contained in $E(M) - X$ and contains $\{y_{n-2}, y_{n-1}, y_n\}$ and $\{z_{n-2}, z_{n-1}, z_n\}$.*

Proof. We argue by induction on n. The result is immediate if $n = 3$. Assume it true for $n < m$ and let $n = m \geq 4$. Write $\overrightarrow{Y}$ and $\overrightarrow{Z}$ for $(y_1, y_2, \dots, y_n)$ and $(z_1, z_2, \dots, z_n)$, respectively. We can relabel y_{n-2}, y_{n-1}, y_n so that, in $\overrightarrow{Z}$, they occur in the the order y_n, y_{n-1}, y_{n-2}. Let $z_{k+1} = y_n$. Now, for all $i \leq k$, we have $\{z_1, z_2, \dots, z_i\} \cap \{y_{n-1}, y_n\} = \emptyset$, so the union of the two 3-separating sets $X \cup \{z_1, z_2, \dots, z_i\}$ and $X \cup \{y_1, y_2, \dots, y_{n-2}\}$ avoids $\{y_{n-1}, y_n\}$. Thus, by Lemma 3.1, their intersection is 3-separating. It follows that $(X, z_1, z_2, \dots, z_k, \overrightarrow{Y} - \{z_1, z_2, \dots, z_k\}, y_{n-1}, y_n)$ is a 3-sequence. Moreover, $y_{n-2} \notin \{z_1, z_2, \dots, z_k\}$. Hence $(X \cup \{z_1, z_2, \dots, z_k\}, z_{k+1}, z_{k+2}, \dots, z_n)$ and $(X \cup \{z_1, z_2, \dots, z_k\}, \overrightarrow{Y} - \{z_1, z_2, \dots, z_k\}, y_{n-1}, y_n)$ are both 3-sequences. As the first 3-sequence ends with z_{n-2}, z_{n-1}, z_n and the second with y_{n-2}, y_{n-1}, y_n, the lemma follows by induction unless $k + 1 = 1$. Thus we may assume that $z_1 = y_n$.

Suppose that $n = 4$. Then (X, y_4, z_2, z_3, z_4) and (X, y_1, y_2, y_3, y_4) are both 3-sequences. By performing an end shuffle in the first of these 3-sequences, we deduce that (X, y_4, y_3, y_2, y_1) is a 3-sequence. The lemma now follows by Lemma 4.3. We conclude that we may assume that $n \geq 5$.

Suppose that $z_2 = y_j$ where $j \neq n-1$. Then $j \neq n-2$. Moreover, $X \cup \{y_n, z_2\}$ and $X \cup \{y_1, y_2, \dots, y_{j-1}, z_2\}$ are 3-separating and their union avoids $\{y_{n-2}, y_{n-1}\}$, so $X \cup z_2$ is 3-separating. Thus both $(X, z_2, y_n, \overrightarrow{Z} - \{z_2, y_n\})$ and $(X, z_2, \overrightarrow{Y} - \{z_2\})$ are 3-sequences. Therefore so too are $(X \cup z_2, y_n, \overrightarrow{Z} - \{z_2, y_n\})$ and $(X \cup z_2, \overrightarrow{Y} - \{z_2\})$.

Since $n \geq 5$, these orderings end in z_{n-2}, z_{n-1}, z_n and y_{n-2}, y_{n-1}, y_n, respectively. In that case, the result follows by induction.

We may now assume that $z_2 = y_{n-1}$. Then $\overrightarrow{Z} = (y_n, y_{n-1}, z_3, z_4, \ldots, z_n)$. Since $\{y_n, y_{n-1}, \ldots, y_{n-t}\}$ is 3-separating for all t in $\{2, 3, \ldots, n-1\}$, it follows that $(X, y_n, y_{n-1}, y_{n-2}, \ldots, y_1)$ is a 3-sequence. We conclude by Lemma 4.3 that $\{y_1, y_2, \ldots, y_n\}$ is a segment, a cosegment, or a fan thereby completing the proof of the lemma. □

Proposition 4.7. *Let M be a 3-connected matroid with at least three elements. Then there is a cyclic ordering $x_1, x_2, \ldots, x_n$ of $E(M)$ such that, for all i and j in $\{1, 2, \ldots, n\}$, the set $\{x_i, x_{i+1}, \ldots, x_{i+j-1}\}$ is 3-separating if and only if M is a segment, a cosegment, a wheel, or a whirl.*

Proof. It is clear that if M is one of the four specified types of matroids, then the required cyclic ordering exists. Now assume that such a cyclic ordering exists. We may suppose that $|E(M)| \geq 6$ since all smaller 3-connected matroids with at least three elements are segments or cosegments. Because $\{x_i, x_{i+1}, x_{i+2}\}$ is 3-separating for all i in $\{1, 2, \ldots, n\}$, each such set is a triangle or a triad. If both $\{x_i, x_{i+1}, x_{i+2}\}$ and $\{x_{i+1}, x_{i+2}, x_{i+3}\}$ are triangles for some i, then, by orthogonality, $\{x_{i+2}, x_{i+3}, x_{i+4}\}$ is also a triangle. Continuing in this way, we deduce that M is a segment. Hence we may assume that, for each i, exactly one of $\{x_i, x_{i+1}, x_{i+2}\}$ and $\{x_{i+1}, x_{i+2}, x_{i+3}\}$ is a triangle and exactly one is a triad. Thus every element of M is in both a triangle and a triad. Thus, by Tutte's Wheels-and-Whirls Theorem [7], M is a wheel or a whirl. □

Corollary 4.8. *Let M be a sequential matroid, and suppose that $(x_1, x_2, \ldots, x_n)$ is a sequential ordering of M such that $\{x_1, x_2\} \cup \{x_{n-1}, x_n\}$ is a segment. Then M is a segment.*

Proof. Let X be the set $\{x_i, x_{i+1}, \ldots, x_j\}$ where $1 \leq i < j \leq n$. We shall show that X is 3-separating. This is certainly true if $i = 1$ or $j = n$. Hence we may assume that $1 < i < j < n$. Now $\{x_1, x_2, \ldots, x_{i-1}\}$ is 3-separating as is $\{x_{j+1}, x_{j+2}, \ldots, x_n\}$. If $j \leq n-2$, then $\{x_{j+1}, x_{j+2}, \ldots, x_n\}$ spans $\{x_1, x_2\}$. Hence $\{x_{j+1}, x_{j+2}, \ldots, x_n, x_1\}$ and $\{x_{j+1}, x_{j+2}, \ldots, x_n, x_1, x_2\}$ are 3-separating. So, by Lemma 3.1, $\{x_{j+1}, x_{j+2}, \ldots, x_n, x_1, x_2, \ldots, x_{i-1}\}$ is 3-separating. Thus, so is its complement, which equals X. Thus we may assume that $j = n-1$ and, by symmetry, that $i = 2$. Then $|E(M) - X| = 2$, so X is 3-separating. We conclude that, in general, X is 3-separating and so is its complement. Thus the hypotheses of Proposition 4.7 are satisfied and we conclude that M is a segment. □

5. Sequential Orderings

Evidently, to describe all fully expanded 3-sequences of M with tail Z, it suffices to do so to within end shuffles. The next lemma shows, for example, that if a triangle or a triad T occurs at one end of some sequential ordering of a sequential matroid M, then, in any other sequential ordering of M, the set T must belong to a segment, cosegment, or fan that occurs very close to an end.

Lemma 5.1. *Let Σ be a fully expanded 3-sequence $(x_1, x_2, \dots, x_n, Z)$ in a matroid M. Let $X_k = \{x_1, x_2, \dots, x_k\}$ for some k with $3 \le k \le |E| - 3$. Let Σ' be another fully expanded 3-sequence of M with tail Z and let $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ be the ordering induced on X_k by Σ'. Suppose that $\{x_1, x_2, x_3\} = \{x_{i_s}, x_{i_t}, x_{i_u}\}$ where $i_s < i_t < i_u$. Let L be the set of elements of $E - X_k$ that occur to the left of x_{i_t} in Σ' and, if Z is trivial, let R be the set of elements of $E - X_k$ that occur to the right of x_{i_t} in Σ'. Then the following hold.*

(i) *When Z is non-trivial, if $|L| \ge 2$, then $X_k \cup L$ is a segment, a cosegment, or a fan.*
(ii) *When Z is trivial, if $|L| \ge 2$ and $|R| \ge 2$, then $X_k \cup L$ or $X_k \cup R$ is a segment, a cosegment, or a fan.*

Proof. We know that x_{i_t} is the second element of $\{x_1, x_2, x_3\}$ in Σ'. From Σ', we can obtain another fully expanded 3-sequence of M by moving x_{i_s} and x_{i_u} so that they immediately precede and immediately succeed x_{i_t}. Then we move $x_4, x_5, \dots, x_k$ within this fully expanded 3-sequence so that the elements of X_k form a consecutive subsequence of a fully expanded 3-sequence Σ'' of M. Let P be the set of elements of $E - X_k$ that precede x_{i_t} in Σ''. If Z is non-trivial, let $Q = E - (P \cup X_k)$ and, if Z is trivial, let Q be the set of elements of $E - X_k$ that succeed x_{i_t} in Σ''. Then, by the construction of Σ'', we have $P = L$ and, when Z is trivial, $Q = R$. Thus, when Z is non-trivial, Σ'' is a 3-sequence of the form $(\overrightarrow{P'}, \overrightarrow{X'_k}, Q)$, where $\overrightarrow{P'}$ and $\overrightarrow{X'_k}$ are orderings of P and X_k. Moreover, when Z is trivial, Q has an ordering $\overrightarrow{Q'}$ such that $\Sigma'' = (\overrightarrow{P'}, \overrightarrow{X'_k}, \overrightarrow{Q'})$.

Now suppose that Z is trivial and consider Σ. At least two of x_{n-2}, x_{n-1}, and x_n are in P or at least two such elements are in Q. An end shuffle in Σ ensures that its last two elements are both in Q or are both in P. In the latter case, we interchange the labels on P and Q and then replace Σ'' by its reversal $\overleftarrow{\Sigma''}$, which has the form $(\overrightarrow{P''}, \overrightarrow{X''_k}, \overrightarrow{Q''})$. Thus, when Z is trivial, we may assume that the last two elements of Σ are in Q and that M has a 3-sequence of the form $(\overrightarrow{P}, \overrightarrow{X_k}, Q)$.

Now, when Z is trivial or non-trivial, let $x_{j_1}, x_{j_2}, \dots, x_{j_p}$ be the ordering induced on P by Σ. Then $X_k \cup P$ is 3-separating as is the initial subsequence of Σ that ends with x_{j_v} for all v in $\{1, 2, \dots, p\}$. As the union of these two 3-separating sets avoids Z when Z is non-trivial and avoids $\{x_{n-1}, x_n\}$ when Z is trivial, the intersection of these two 3-separating sets is 3-separating; that is, $X_k \cup \{x_{j_1}, x_{j_2}, \dots, x_{j_v}\}$ is 3-separating. Hence we can move the elements of P in Σ to get a 3-sequence of the form $(\overrightarrow{X'''_k}, \overrightarrow{P'''}, Q)$, where $\overrightarrow{P'''}$ and $\overrightarrow{X'''_k}$ are orderings of P and X_k, respectively. Since we also have a 3-sequence of the form $(\overrightarrow{P}, \overrightarrow{X_k}, Q)$, Corollary 4.4 implies that $X_k \cup P$ is a segment, a cosegment, or a fan. Since P is L when Z is non-trivial, and P is L or R when Z is trivial, the lemma follows. □

Now, suppose we have a sequential ordering $(x_1, x_2, \dots, x_n)$ of a 3-connected matroid M. Then, up to duality, $\{x_1, x_2, x_3\}$ is a triangle. Moreover, one of the following three possibilities occurs:

(i) $\{x_1, x_2, x_3, x_4\}$ contains another triangle, so $\{x_1, x_2, x_3, x_4\}$ is a segment;
(ii) $\{x_1, x_2, x_3, x_4\}$ contains a triad, so $\{x_1, x_2, x_3, x_4\}$ is a fan; or
(iii) $\{x_1, x_2, x_3, x_4\}$ is a cocircuit.

We shall analyze these cases separately with a view to showing that a sequential matroid has only two possible ends.

Lemma 5.2. *Let S be a maximal segment in a matroid M with $|S| = m \geq 4$, and let $(e_1, e_2, \ldots, e_n, Z)$ be a fully expanded 3-sequence Σ of M. If $|S \cap \{e_1, e_2, e_3\}| \geq 2$, then the first $m-1$ elements of Σ are in S.*

Proof. The result is immediate if $r(M) = 2$ so assume that $r(M) > 2$. The set $\{e_1, e_2, e_3\}$ is a triangle or a triad. Since $|S| \geq 4$, by orthogonality, $\{e_1, e_2, e_3\}$ must be a triangle. Because this triangle contains at least two elements of S, it must be a subset of S. Suppose that $\{e_1, e_2, \ldots, e_k\} \subseteq S$, but $e_{k+1} \notin S$. Then $e_{k+1} \notin \mathrm{cl}(\{e_1, e_2, \ldots, e_k\})$ so $e_{k+1} \in \mathrm{cl}^*(\{e_1, e_2, \ldots, e_k\})$. Hence e_{k+1} is contained in a cocircuit that is contained in $\{e_1, e_2, \ldots, e_{k+1}\}$. By orthogonality, at most one member of S avoids this cocircuit. Thus $k \geq m - 1$. □

Lemma 5.3. *Let S be a maximal segment in a matroid M with $|S| = m \geq 4$. Suppose that M has a fully expanded 3-sequence with tail Z such that the first three elements of this 3-sequence are in S. Let Σ be an arbitrary fully expanded 3-sequence of M with tail Z. Then*

(i) *Z is non-trivial and the first three elements of Σ are in S; or*
(ii) *Z is trivial and either the first or last three elements of Σ are in S.*

Proof. Let Σ be $(e_1, e_2, \ldots, e_n, Z)$ and let $(s_1, s_2, \ldots, s_m)$ be the ordering induced on S by Σ. We shall prove (ii), omitting the similar proof of (i). Suppose that Z is trivial and assume that neither $\{e_1, e_2, e_3\}$ nor $\{e_{n-2}, e_{n-1}, e_n\}$ is contained in S. If $|S \cap \{e_1, e_2, e_3\}| \geq 2$ or $|S \cap \{e_{n-2}, e_{n-1}, e_n\}| \geq 2$, then the result follows by Lemma 5.2. Thus $|(E(M) - S) \cap \{e_1, e_2, e_3\}| \geq 2$ and $|(E(M) - S) \cap \{e_{n-2}, e_{n-1}, e_n\}| \geq 2$. Hence, if L_2 and R_2 are the sets of elements of $E(M) - S$ occurring to the left and right of s_2 in Σ, then $|L_2| \geq 2$ and $|R_2| \geq 2$. Thus, by Lemma 5.1, $S \cup L_2$ or $S \cup R_2$ is a segment, a cosegment, or a fan. Because S is a maximal segment and $|S| \geq 4$, this gives a contradiction. □

The condition that the size of S is at least 4 in the statement of Lemma 5.3 cannot be weakened. To see this, consider a maximal fan F of size 4 whose full closure is F. If (f_1, f_2, f_3, f_4) is a fan ordering of F, then there is a fully expanded 3-sequence of M starting with f_1, f_2, and f_3 as well as one starting with f_2, f_3, and f_4.

6. Fans

Theorem 1.1 focuses attention on segments, cosegments, and fans occurring at an end of a 3-sequence. In this section, we treat the fan case, which is the most difficult.

Lemma 6.1. *Let Σ be a fully expanded* 3*-sequence* $(e_1, e_2, \dots, e_n, Z)$ *of a matroid* M *and suppose that* $\{e_1, e_2, \dots, e_k\}$ *is a fan* F*. Assume that* $|E(M)| \geq 7$ *and that* M *is neither a wheel nor a whirl. Suppose that* F *is not a maximal fan contained in* $E(M) - Z$ *and that* $F \cup e_{k+i}$ *is* 3*-separating for some* $i \geq 2$*. Then either*

(i) e_{k+i} *is unique,* $F \cup e_{k+1}$ *or* $F \cup e_{k+i}$ *is a fan, and*
  (a) $F \cup \{e_{k+1}, e_{k+i}\}$ *is a fan having ends* e_{k+1} *and* e_{k+i}*; or*
  (b) e_{k+1} *and* e_{k+i} *both belong to the line* $\mathrm{cl}(F) \cap \mathrm{cl}(E - F)$*; or*
  (c) e_{k+1} *and* e_{k+i} *both belong to the line* $\mathrm{cl}^*(F) \cap \mathrm{cl}^*(E - F)$ *of* M^**;*
or
(ii) $F \cup e_{k+j}$ *is* 3*-separating for some* $j \geq 2$ *with* $j \neq i$ *and* $\{e_{k+1}, e_{k+i}, e_{k+j}\}$ *is contained in* $\mathrm{cl}(F) \cap \mathrm{cl}(E - F)$ *or* $\mathrm{cl}^*(F) \cap \mathrm{cl}^*(E - F)$*.*

Proof. By Lemma 3.5, $|E - F| \geq 2$. Assume that $|E - F| \in \{2, 3\}$. Then Z is trivial and $\{e_{n-2}, e_{n-1}, e_n\}$ is, without loss of generality, a triangle but not a triad of M. Since M is 3-connected, F spans $E(M)$ and it follows that (i)(b) or (ii) holds. We may now assume that $|E - F| \geq 4$. Since we can move e_{k+i} in the fully expanded 3-sequence Σ so that it immediately follows e_{k+1}, we may assume that $i = 2$. Since F is not maximal, $F \cup e_{k+t}$ is a fan for some $t \geq 1$.

Assume first that $F \cup e_{k+j}$ is 3-separating for some $j \geq 3$. Then we may suppose that $j = 3$. If $|E-F| \geq 5$, then we have that $(F, e_{k+1}, e_{k+2}, e_{k+3}, \{e_{k+4}, \dots, e_n\} \cup Z)$ is a 3-sequence in which e_{k+1}, e_{k+2}, and e_{k+3} are mutually jumping. Thus, by [2, Lemma 6.4], $\{e_{k+1}, e_{k+2}, e_{k+3}\}$ is a triangle or a triad of M. Hence $\{e_{k+1}, e_{k+2}, e_{k+3}\}$ is contained in $\mathrm{cl}(F) \cap \mathrm{cl}(E-F)$ or $\mathrm{cl}^*(F) \cap \mathrm{cl}^*(E-F)$, respectively, and so (ii) holds. Assume $|E - F| = 4$. Then Z is trivial, otherwise $4 \leq |Z| < |E - F|$. Since all of $F \cup \{e_{k+1}\}, F \cup \{e_{k+2}\}$, and $F \cup \{e_{k+3}\}$ are 3-separating, each of $\{e_{k+4}, e_{k+3}, e_{k+2}\}$, $\{e_{k+4}, e_{k+3}, e_{k+1}\}$, and $\{e_{k+4}, e_{k+2}, e_{k+1}\}$ is a triangle or a triad of M. By duality, we may assume that two of these sets are triangles. Hence so is the third. It follows that $\{e_{k+1}, e_{k+2}, e_{k+3}\}$ is contained in $\mathrm{cl}(F) \cap \mathrm{cl}(E - F)$ and again (ii) holds.

We may now assume that e_{k+i} is unique and that, as before, $i = 2$. Then, since $F \cup e_{k+t}$ is a fan and hence is 3-separating, $t \in \{1, 2\}$. By interchanging e_{k+1} and e_{k+2} if necessary, we may assume that $t = 1$. Each of e_{k+1} and e_{k+2} is in exactly one of $\mathrm{cl}(F)$ and $\mathrm{cl}^*(F)$. If both e_{k+1} and e_{k+2} are in $\mathrm{cl}(F)$ or both are in $\mathrm{cl}^*(F)$, then (i)(b) or (i)(c) holds. Thus we may assume that $e_{k+1} \in \mathrm{cl}(F)$ and $e_{k+2} \in \mathrm{cl}^*(F)$. It is not difficult to see that we can order $(e_1, e_2, \dots, e_k)$ as $(f_1, f_2, \dots, f_k)$ such that $(f_1, f_2, \dots, f_k, e_{k+1}, e_{k+2}, \dots, e_n, Z)$ is a fully expanded 3-sequence and $(f_1, f_2, \dots, f_k, e_{k+1})$ is a fan ordering of $F \cup e_{k+1}$. As $e_{k+1} \in \mathrm{cl}(F)$, we have that $\{f_{k-1}, f_k, e_{k+1}\}$ is a triangle of M.

Now $\{f_1, f_2, f_3\}$ is a triangle or a triad. Assume the latter occurs. Then $f_1 \in \mathrm{cl}^*(F - f_1)$. We know that $e_{k+2} \in \mathrm{cl}^*(F)$. Suppose first that $e_{k+2} \in \mathrm{cl}^*(F - f_1)$. Then $(F - f_1, f_1, e_{k+1}, e_{k+2}, \{e_{k+3}, \dots, e_n\} \cup Z)$ is an (A, B) 3-sequence in which f_1, e_{k+1}, e_{k+2} are mutually jumping, so, by [2, Lemma 6.4], $\{f_1, e_{k+1}, e_{k+2}\}$ is a triangle or a triad. In each case, we get a contradiction to orthogonality using either the triad $\{f_1, f_2, f_3\}$ or the triangle $\{f_{k-1}, f_k, e_{k+1}\}$. Hence we may assume that $e_{k+2} \notin \mathrm{cl}^*(F - f_1)$. Thus every cocircuit of M contained in $F \cup e_{k+2}$ that contains e_{k+2} must also contain f_1. Moreover, such a cocircuit C^* exists. But $\{f_1, f_2, f_3\}$ is a triad of M. By strong circuit elimination, M has a cocircuit D^* that contains e_{k+2} and is contained in $(\{f_1, f_2, f_3\} \cup C^*) - f_1$; a contradiction.

We may now assume that $\{f_1, f_2, f_3\}$ is a triangle of M. Then it follows, since $\{f_{k-1}, f_k, e_{k+1}\}$ is also a triangle, that k is even. Since $e_{k+2} \in \text{cl}^*(F)$, we deduce that $e_{k+2} \in \text{cl}^*(\{f_2, f_4, \dots, f_k\} \cup \{f_1\})$. Thus there is a cocircuit C^* of M such that $e_{k+2} \in C^*$ and $C^* - e_{k+2} \subseteq \{f_2, f_4, \dots, f_k\} \cup \{f_1\}$. The triangles $\{f_3, f_4, f_5\}, \{f_5, f_6, f_7\}, \dots, \{f_{k-1}, f_k, e_{k+1}\}$ of M imply, by orthogonality, that none of $f_4, f_6, \dots, f_k$ is in C^*. Hence $C^* = \{f_1, f_2, e_{k+2}\}$. It follows that M contains a fan with an ordering of $(e_{k+2}, f_1, f_2, \dots, f_k, e_{k+1})$, so (i)(a) holds. □

Lemma 6.2. *Let Σ be a fully expanded 3-sequence with tail Z in a matroid M that is not a wheel or a whirl. Suppose that F is a fan contained in $E(M) - Z$ such that the ordering induced on F by Σ is $(f_1, f_2, \dots, f_m)$. If $\{f_1, f_2, \dots, f_k\}$ is 3-separating for all $k \in \{3, 4, \dots, m\}$, then $\{f_1, f_2, \dots, f_k\}$ is a fan for all such k.*

Proof. Since M is not a wheel or a whirl, $|E(M) - F| \geq 2$. The result is immediate if $m \leq 4$. Hence we may assume that $m \geq 5$. We argue by induction on k noting that it holds when $k = 3$. Assume it holds for $k < t$ and let $k = t \geq 4$. We may assume that $t < m$. Since the only triangles and triads of M contained in F consist of consecutive triples of elements in a fan ordering of F, we deduce that there is a fan ordering $\overrightarrow{F}$ of F such that $\{f_1, f_2, \dots, f_{t-1}\}$ consists of a consecutive subsequence of $\overrightarrow{F}$. In fact, since $|F| \geq 5$, the fan ordering $\overrightarrow{F}$ is unique up to reversal by Lemma 3.6. Let $F' = \{f_1, f_2, \dots, f_{t-1}\}$. Now $F' \cup f_t$ is 3-separating. If this set is a fan, then we have what we want. Thus assume that $F' \cup f_t$ is not a fan. Since F' is a fan contained in the fan F, the set $F' \cup f_s$ is a fan for some s in $\{t+1, t+2, \dots, m\}$.

Suppose first that $F' \cup f_u$ is a fan for some u in $\{t+1, t+2, \dots, m\} - s$. Then $F' \cup \{f_s, f_u\}$ is a fan having f_s and f_u as its ends. Now $F' \cup f_p$ is 3-separating for all $p \in \{s, t, u\}$. Thus, by Lemma 6.1 and duality, we may assume that $\{f_s, f_t, f_u\} \subseteq \text{cl}(F') \cap \text{cl}(E - F')$. Hence $\{f_s, f_t, f_u\}$ is a triangle. Since $F' \cup \{f_s, f_u\}$ is contained in F and has ends f_s and f_u, the fan F cannot have $\{f_s, f_t, f_u\}$ as a triangle; a contradiction. Hence we may assume that $F' \cup f_s$ is the unique fan contained in F that contains F' and has size $|F'| + 1$. This means that the elements of F' must occur consecutively all at the left-hand end or all at the right-hand end of $\overrightarrow{F}$. We may assume the former.

Next assume that $t = m - 1$. Then $s = m$ and $\overrightarrow{F} = (F', f_s, f_t)$. Without loss of generality, $f_s \in \text{cl}(F')$. Then $f_t \in \text{cl}^*(F' \cup f_s) \cap \text{cl}^*(E - F)$. But, by Lemma 6.1, $f_t \in \text{cl}(F')$. Thus we have a contradiction to orthogonality.

We may now assume that $t < m - 1$. Because the elements of F' form an initial subsequence of F, the set $F - F'$ is also a fan and so is 3-separating. In addition, $F' \cup \{f_s, f_t, f_{t+i}\}$ is 3-separating where $i = 1$ unless $s = t + 1$, in which case, $i = 2$. As the union of the last two 3-separating sets avoids $E - F$, their intersection, $\{f_s, f_t, f_{t+i}\}$, is 3-separating. Using duality, we may assume that $\{f_s, f_t, f_{t+i}\}$ is a triangle. As each of F', $F' \cup f_s$, and $F' \cup f_t$ is 3-separating, it follows by orthogonality that $f_s, f_t \in \mathrm{cl}(F')$. Hence the triangle $\{f_s, f_t, f_{t+i}\}$ is contained in $\mathrm{cl}(F')$. But this means that the fan F' spans three elements of $F - F'$, which does not occur in a fan. □

The next result is an immediate consequence of the last lemma.

Corollary 6.3. *Let Σ be a fully expanded 3-sequence $(e_1, e_2, \dots, e_n, Z)$ of a matroid M and suppose that $\{e_1, e_2, \dots, e_m\}$ is a fan of M. Then $\{e_1, e_2, \dots, e_k\}$ is a fan for all $k \in \{3, 4, \dots, m\}$.*

Lemma 6.4. *Let M be a matroid that is neither a wheel nor a whirl, and suppose that an ordering $\overrightarrow{F}$ of a fan F is both a left-end ordering and a right-end ordering of F. Then $\overrightarrow{F}$ is a fan ordering of F.*

Proof. Let $\overrightarrow{F} = (f_1, f_2, \dots, f_m)$. Since M is neither a wheel nor a whirl, $|E(M) - F| \geq 2$. Now, for all $k \in \{3, 4, \dots, m\}$, both $\{f_1, f_2, \dots, f_k\}$ and $\{f_{k-2}, f_{k-1}, \dots, f_m\}$ are 3-separating and their union avoids $E(M) - F$. Hence their intersection, $\{f_{k-2}, f_{k-1}, f_k\}$, is 3-separating and so is a triangle or a triad. It follows that $\overrightarrow{F}$ is a fan ordering of F. □

Lemma 6.5. *Let F be a maximal fan in a sequential matroid M that is not a wheel or a whirl and let Σ be a sequential ordering $(e_1, e_2, \dots, e_n)$ of M. Then either $\{e_1, e_2, e_3\} \not\subseteq F$ or $\{e_{n-2}, e_{n-1}, e_n\} \not\subseteq F$.*

Proof. Assume that both $\{e_1, e_2, e_3\} \subseteq F$ and $\{e_{n-2}, e_{n-1}, e_n\} \subseteq F$. In any fan ordering of F, the elements of $\{e_1, e_2, e_3\}$ are consecutive as are the elements of $\{e_{n-2}, e_{n-1}, e_n\}$. Clearly there is a fan ordering $\overrightarrow{F}$ of F in which $\{e_1, e_2, e_3\}$ precedes $\{e_{n-2}, e_{n-1}, e_n\}$. In $\overrightarrow{F}$, let L be the set of elements of F that precede the first member of $\{e_{n-2}, e_{n-1}, e_n\}$ and let R be $F - L$. Evidently, both L and R are fans. Now within Σ, we can move all the elements of L to the left end of the sequence and all the elements of R to the right end. This gives a sequential ordering of the form $(L, E(M) - F, R)$. Because $L \cup R$ is a fan, there is also a sequential ordering of the form $(L, R, E(M) - F)$. By comparing these two sequential orderings, we deduce from Corollary 4.4 that $(E(M) - F) \cup R$ is a segment, a cosegment, or a fan. Because this set has at least three common elements with F, Lemma 3.7 implies that it must be a fan. But $(E(M) - F) \cup R$ is contained in a maximal fan F', which is distinct from F. Since $|F| \geq 6$ and the elements of R are in both F and F', at least one of the elements of R is an essential element of M that is in two distinct maximal fans at least one of which has more than five elements. This contradicts Theorem 3.4. □

Lemma 6.6. *Let Σ be a fully expanded 3-sequence $(e_1, e_2, \dots, e_n, Z)$ in a matroid M that is not a wheel or a whirl. Let F be a maximal fan of M and suppose that $|F| \geq 4$. If $|F \cap \{e_1, e_2, e_3\}| \geq 2$, then one of the following holds:*

(i) *$|F \cap \{e_1, e_2, e_3\}| = 3$; or*
(ii) *$|F \cap \{e_1, e_2, e_3\}| = 2$ and, for some N in $\{M, M^*\}$,*
  (a) *N is spanned by $F - \{e_1, e_2, e_3\}$ and has ground set consisting of the union of F and a segment that is spanned by $\{e_1, e_2, e_3\}$; or*

(b) *$\{e_1, e_2, e_3\}$ is a triangle of N, the fan F has five elements and a unique triad of N, the elements of $F \cap \{e_1, e_2, e_3\}$ are in $I(F)$, and $N|(F \cup \{e_1, e_2, e_3\}) \cong M(K_4)$.*

Proof. Assume that $|F \cap \{e_1, e_2, e_3\}| = 2$. Then, by permuting e_1, e_2, and e_3, we may assume that $(e_1, e_2, e_3) = (x, f_1, f_2)$ where $x \notin F$ and $f_1, f_2 \in F$. Without loss of generality, we may assume that $\{x, f_1, f_2\}$ is a triangle.

Suppose first that f_1 or f_2 is in a triad T^* of F. Then, by orthogonality with the triangle $\{x, f_1, f_2\}$, we deduce that T^* contains both f_1 and f_2. It follows that T^* is the unique triad of F meeting $\{f_1, f_2\}$. Now $\{f_1, f_2\}$ is not contained in a triangle T of F otherwise $T \cup x$ is a segment having three common elements with F, contradicting Lemma 3.7. Suppose f_1 and f_2 are consecutive in some fan ordering of F. If neither f_1 nor f_2 is an end of the ordering, then $\{f_1, f_2\}$ is contained in a triangle of F; a contradiction. If one of f_1 and f_2 is an end of the ordering, then $F \cup x$ is a fan; a contradiction. We conclude that no fan ordering of F has f_1 and f_2 consecutive. It follows easily that $|F| \neq 4$, so $|F| \geq 5$. To avoid f_1 or f_2 being in more than one triad of F, we must have that $|F| = 5$ and F has a fan ordering $(g_1, f_1, g_2, f_2, g_3)$ where $\{g_1, f_1, g_2\}$ and $\{g_2, f_2, g_3\}$ are triangles and $\{f_1, g_2, f_2\}$ is a triad. Hence f_1 and f_2 are internal elements of F. Since $\text{cl}(F) \cap \text{cl}(E - F)$ has rank at most 2 and contains $\{g_1, g_3, x\}$, we deduce that the last set is a triangle, so $N|(F \cup \{e_1, e_2, e_3\}) \cong M(K_4)$ and (ii)(b) holds.

We may now assume that neither f_1 nor f_2 is in a triad of F. Hence F has f_1 and f_2 as its ends, F has triangles at both ends, and $|F| \geq 5$. Let S be the segment with which Σ begins and let e_s be the first element of Σ that is not in S. Then $e_s \in \text{cl}^*(\{e_1, e_2, \ldots, e_{s-1}\})$. Thus there is a cocircuit C^* of M that contains e_s and is contained in $\{e_1, e_2, \ldots, e_s\}$. By orthogonality, C^* must contain all but at most one element of $\{e_1, e_2, \ldots, e_{s-1}\}$. In particular, C^* contains f_1 or f_2. But each of these elements is contained in a triangle contained in F. Hence, by orthogonality, e_s must also be in one of these triangles, so $e_s \in F$. This means that $\{e_1, e_2, \ldots, e_s\}$ contains two elements of one of the end triangles of F, so it follows that we can move the elements of $F - \{f_1, f_2, e_s\}$ in Σ so that they immediately follow e_s. Hence, for some ordering $\overrightarrow{F}$ of F, we have a 3-sequence of the form

$$(e_1, e_2, \ldots, e_{s-1}, \overrightarrow{F} - \{f_1, f_2\}, E - (F \cup \{e_1, e_2, \ldots, e_{s-1}\})).$$

Since $F - \{f_1, f_2\}$ is a fan that spans the segment $\{e_1, e_2, \ldots, e_{s-1}\}$, it follows that there is an ordering $\overrightarrow{F'}$ of F such that M also has a 3-sequence of the form

$$(\overrightarrow{F'} - \{f_1, f_2\}, e_1, e_2, \ldots, e_{s-1}, E - (F \cup \{e_1, e_2, \ldots, e_{s-1}\})).$$

Thus, provided $|E - (F \cup \{e_1, e_2, \ldots, e_{s-1}\})| \geq 2$, we can apply Corollary 4.4 to obtain the contradiction that $F \cup \{e_1, e_2, \ldots, e_{s-1}\}$ is a segment, a cosegment, or a fan. We deduce that $|E - (F \cup \{e_1, e_2, \ldots, e_{s-1}\})| \leq 1$, so Z must be trivial. Suppose that $E - (F \cup \{e_1, e_2, \ldots, e_{s-1}\}) = \{z\}$. Then, arguing as for x using $(e_1, e_2, \ldots, e_{s-1}, \overrightarrow{F} - \{f_1, f_2\}, z)$, we deduce that there are two elements f_{k-1} and f_k of $F - \{f_1, f_2\}$ such that $\{z, f_k, f_{k-1}\}$ is a triangle or a triad. Let $(f_1, h_1, h_2, \ldots, h_{k-2}, f_2)$ be a fan ordering of F. Then, by [6, Lemma 3.4], none of $h_2, h_3, \ldots, h_{k-3}$ is in any triangles or triads of M other than those in F. Thus

$\{z, f_k, f_{k-1}\} = \{z, h_1, h_{k-2}\}$. As $\{f_1, h_1, h_2\}$ is a triangle, orthogonality implies that $\{z, h_1, h_{k-2}\}$ is not a triad and so is a triangle. As $\{h_1, h_2, h_3\}$ is a triad, orthogonality implies that $k-2 = 3$, so $k = 5$ and $|F| = 5$. We deduce that $r(N) = 3$ and (ii)(a) holds. □

Lemma 6.7. *Let Σ be a fully expanded 3-sequence $(e_1, e_2, \dots, e_n, Z)$ in a matroid M that is not a wheel or a whirl. Let F be a maximal fan of M and suppose that $|F| \geq 4$ and $|F \cap \{e_1, e_2, e_3\}| \geq 2$. Let $(g_1, g_2, \dots, g_n, Z)$ be another fully expanded 3-sequence Σ' of M. If $\{e_1, e_2, e_3\}$ is a triangle of M, then either*

(i) *M is the union of F and the segment spanned by $\{e_1, e_2, e_3\}$; or*
(ii) (a) *Z is non-trivial and $|F \cap \{g_1, g_2, g_3\}| \geq 2$, or*
(b) *Z is trivial, and $|F \cap \{g_1, g_2, g_3\}| \geq 2$ or $|F \cap \{g_{n-2}, g_{n-1}, g_n\}| \geq 2$.*

Proof. First suppose that Z is trivial. Assume that $|F \cap \{g_1, g_2, g_3\}| \leq 1$ and $|F \cap \{g_{n-2}, g_{n-1}, g_n\}| \leq 1$. Then, by end shuffles, we may assume that F avoids both $\{g_1, g_2\}$ and $\{g_{n-1}, g_n\}$. Let $\overrightarrow{F}$ be the ordering $(f_1, f_2, \dots, f_m)$ induced on F by Σ'. Then, by considering the intersection of F with $\{g_1, g_2, \dots, g_k\}$ for all k in $\{3, 4, \dots, n-2\}$, we deduce that $\overrightarrow{F}$ is a left-end ordering of F. By symmetry, $\overrightarrow{F}$ is a right-end ordering of F. By Lemma 6.4, $\overrightarrow{F}$ is a fan ordering of F, and we can move $f_1, f_3, f_4, \dots, f_m$ within Σ' to get $\overrightarrow{F}$ as a consecutive subsequence of a sequential ordering Σ'' of M in which the sets X and Y of elements of $E - F$ that precede and succeed f_2 are the same as the corresponding sets in Σ'. Hence M has a non-sequential ordering of the form $(X, \overrightarrow{F}, Y)$.

Now consider Σ and assume that (i) does not hold. Since $|F \cap \{e_1, e_2, e_3\}| \geq 2$, by Lemma 6.6, either $|F \cap \{e_1, e_2, e_3\}| = 3$, or $|F \cap \{e_1, e_2, e_3\}| = 2$ and $M|(F \cup \{e_1, e_2, e_3\}) \cong M(K_4)$. In the latter case, two of the three elements of the unique triad of F are in $\{e_1, e_2, e_3\}$ and the third element is e_4. Hence $\{e_1, e_2, e_3, e_4\}$ is a fan of M and a fan shuffle on these four elements enables us to assume that, in general, $|F \cap \{e_1, e_2, e_3\}| = 3$ in Σ.

The structure of Σ enables us to move the elements of F within Σ to obtain a sequential ordering of the form $(F, X \cup Y)$. Now, by permuting the last three elements of this sequential ordering, we may assume that the last two elements of this sequential ordering are both in X or are both in Y. By symmetry, we may assume the latter. Hence, we have a sequential ordering Σ''' of the form $(F, (X \cup Y) - \{y_1, y_2\}, y_1, y_2)$ where y_1 and y_2 are in Y. Then by taking the intersection of the 3-separating set $F \cup X$ with initial subsequences of Σ''', we can, by using Lemma 3.1, obtain a sequential ordering of the form $(F, X, Y - \{y_1, y_2\}, y_1, y_2)$, that is, of the form (F, X, Y). Since we also have a sequential ordering of the form (X, F, Y), Corollary 4.4 implies that $X \cup F$ is a segment, a cosegment, or a fan. This contradicts Lemma 3.7 or the fact that F is a maximal fan. Thus the lemma holds in the case Z is trivial. The proof for the case when Z is non-trivial is similar and is omitted. □

Evidently the dual of the last result holds when $\{e_1, e_2, e_3\}$ is a triad.

Lemma 6.8. *Let Σ be a fully expanded 3-sequence $(e_1, e_2, \ldots, e_n, Z)$ in a matroid M that is not a wheel or a whirl. Assume that there is a maximal fan F of M with $|F| \geq 4$ and $F \supseteq \{e_1, e_2, e_3\}$. Assume that neither M nor M^* is spanned by F and that, when $|F| = 5$, the only triangles and triads of M meeting $I(F)$ are contained in F. Let Σ' be an arbitrary fully expanded 3-sequence $(g_1, g_2, \ldots, g_n, Z)$ for M. Then*

(i) *when Z is non-trivial, $|F \cap \{g_1, g_2, g_3\}| = 3$ and, when Z is trivial, $|F \cap \{g_1, g_2, g_3\}| = 3$ or $|F \cap \{e_{n-2}, e_{n-1}, e_n\}| = 3$.*

Moreover, one of the following holds in Σ'.

(ii) *If F has type-1 or type-2, then either*
  (a) *Z is non-trivial and all of the elements in $I(F)$ precede every element in $E(M) - F$, or*
  (b) *Z is trivial and all of the elements in $I(F)$ either precede every element in $E(M) - F$ or succeed every element in $E(M) - F$.*

(iii) *If F has type-3, then either*
  (a) *Z is non-trivial and all of the elements in $I(F)$ together with one of the ends of F precede every element in $E(M) - F$, or*
  (b) *Z is trivial and all of the elements in $I(F)$ together with one of the ends of F either precede every element in $E(M) - F$ or succeed every element in $E(M) - F$.*

Proof. By using Lemma 6.7, we deduce that $|F \cap \{g_1, g_2, g_3\}| \geq 2$ when Z is non-trivial, and $|F \cap \{g_1, g_2, g_3\}| \geq 2$ or $|F \cap \{e_{n-2}, e_{n-1}, e_n\}| \geq 2$ when Z is trivial. Then, by using Lemma 6.6 and the hypotheses of the current lemma, we obtain (i). Now, by possibly replacing Σ' by its reversal when Z is trivial, we may assume that, in general, $\{g_1, g_2, g_3\} \subseteq F$. Then, when Z is trivial, by Lemmas 6.5 and 6.6, we may assume that, after an end shuffle, the last two elements of Σ' avoid F.

Now Σ' has at least two elements at its right end that are not in F, possibly including members of Z. Thus, by repeated applications of Lemma 3.1, we deduce that the ordering $(f_1, f_2, \ldots, f_m)$ induced on F by Σ' is a left-end ordering of F. Therefore, if $|F| = 4$, then the lemma holds. Thus we may assume that $|F| \geq 5$, so a fan ordering, $\overrightarrow{F}$, of F is unique up to reversal.

Let g_i be the first element of Σ' that is not in F. Since $(f_1, f_2, \ldots, f_m)$ is a left-end ordering of F, the elements of $\{g_1, g_2, \ldots, g_{i-1}\}$ are consecutive in the fan ordering $\overrightarrow{F}$. Suppose that some element f_k of $I(F)$ succeeds g_i in Σ'. Then f_k and one of its neighbours in $\overrightarrow{F}$, say f_j, occur to the right of g_i in Σ'. Since $\mathrm{fcl}(\{f_k, f_j\}) \supseteq F$, we can shift all of the elements of F in Σ' so that they occur to the right of g_i. This gives a fully expanded 3-sequence Σ'' with tail Z having g_i as its first element and, when Z is trivial, neither of its last two elements in F. Since (i) fails for Σ'', we have a contradiction which establishes that all the elements of $I(F)$ precede the first element of $E(M) - F$ in Σ'. Hence (ii) holds.

To complete the proof of (iii), suppose that both ends f and f' of F succeed g_i in Σ' where $g_i \in E(M) - F$. Then, by Lemma 6.1, $\{g_i, f, f'\}$ is a subset of $\mathrm{cl}(F - \{f, f'\}) \cap \mathrm{cl}(E - (F - \{f, f'\}))$ or of $\mathrm{cl}^*(F - \{f, f'\}) \cap \mathrm{cl}^*(E - (F - \{f, f'\}))$ and so is a triangle or a triad. Then either f and f' are both triangle ends of F, or f and f' are both triad ends of F. Hence F does not have type-3 and (iii) follows. □

The next two lemmas address situations that were excluded by the hypotheses of the last lemma.

Lemma 6.9. *Let M be a matroid of rank at least three that is neither a wheel nor a whirl. Assume that $E(M)$ is the union of a maximal fan F and a maximal segment S with $|S| \geq 3$. Let Σ be an ordering $(e_1, e_2, \ldots, e_n)$ of the ground set of M. Then $F - S$ is a fan, as is $(F - S) \cup s$ for each element s of $F \cap S$. Moreover:*

- (i) *If M is not a copy of $M(K_4)$ with extra points added to one of its 3-point lines, then, up to end shuffles, Σ is a sequential ordering of M if and only if Σ or its reversal is one of the following:*
  - (a) *$(\overrightarrow{S}, \overrightarrow{F - S})$, where $\overrightarrow{S}$ is an arbitrary ordering of S and $\overrightarrow{F}$ is an arbitrary right-end ordering of the fan $F - S$; or*
  - (b) *$(\overrightarrow{S - s}, \overrightarrow{(F - S) \cup s})$, where $s \in F \cap S$ and $\overrightarrow{S - s}$ and $\overrightarrow{(F - S) \cup s}$ are an arbitrary ordering of $S - s$ and an arbitrary right-end ordering of the fan $(F - S) \cup s$.*
- (ii) *If M is a copy of $M(K_4)$ to which extra points have been added on one of the 3-point lines to give the segment S, then, up to end shuffles, Σ is a sequential ordering of M if and only if Σ or its reversal is one of the following:*
  - (a) *$(\overrightarrow{S}, a, b, c)$, where $\{a, b, c\}$ is the unique triad of M and $\overrightarrow{S}$ is an arbitrary ordering of S; or*
  - (b) *$(\overrightarrow{S - s}, z, u, v, w)$, where $s \in S \cap \{u, v, w\}$, where $\{u, v, w\}$ is one of the three lines of M having exactly three points, and $\overrightarrow{S - s}$ is an arbitrary ordering of $S - s$.*

Proof. Since $(S, F - S)$ is a 3-separation of M, it follows that $F - S$ spans M. By Lemma 3.5, $|S - F| \geq 2$. Thus M has a sequential ordering Σ' of the form $(\overrightarrow{S - F}, \overrightarrow{F})$ where $\overrightarrow{S - F}$ and $\overrightarrow{F}$ are an arbitrary ordering of $S - F$ and an arbitrary right-end ordering of F. If $|F \cap S| \neq \emptyset$, we can move the elements of $F \cap S$ in Σ' to get another sequential ordering that begins with the elements of S and ends with the elements of $F - S$. Then, by Corollary 6.3, $F - S$ is a fan as is $(F - S) \cup s$ for each element s of $F \cap S$. By arguing as above, it is straightforward to check that, in cases (i) and (ii), all of the orderings specified under (a) and (b) are sequential. Now assume that Σ is a sequential ordering of M. Because we have a sequential ordering of M that begins with the elements of S, it follows, using Lemma 5.1, that by replacing Σ by its reversal if necessary, we may assume that $|S \cap \{e_1, e_2, e_3\}| \geq 2$. Then, by an end shuffle, we may suppose that $\{e_1, e_2\} \subseteq S$.

Assume that $\{e_1, e_2, \ldots, e_k\} \subseteq S$ and that $e_{k+1} \notin S$. Then $k \geq 2$ and $e_{k+1} \in \mathrm{cl}^*(\{e_1, e_2, \ldots, e_k\})$. Thus $e_{k+1} \in C^* \subseteq \{e_1, e_2, \ldots, e_{k+1}\}$ for some cocircuit C^*. By orthogonality, $|C^* \cap S| \geq |S| - 1$. Thus either

(a) $S = \{e_1, e_2, \ldots, e_k\}$; or
(b) $S - \{e_1, e_2, \ldots, e_k\} = \{s\}$.

In case (a), Σ consists of an arbitrary ordering of S followed by a right-end ordering of $F - S$. Now consider case (b). Then e_{k+1} is a coloop of $M \backslash \{e_1, e_2, \ldots, e_k\}$. Thus e_{k+1} is in no triangles of the fan $F - S$. Hence $F - S$ has a fan ordering of the form $(e_{k+1}, g_1, g_2, \ldots, g_m)$. Now s is spanned by $F - S$. Since e_{k+1} is a coloop of $M \backslash \{e_1, e_2, \ldots, e_k\}$, it follows that s is spanned by G where G is $\{g_1, g_3, \ldots, g_m\}$ when m is odd, and G is $\{g_1, g_3, \ldots, g_{m-1}, g_m\}$ when m is even. Hence $s \cup G$ contains a circuit C. By orthogonality with the triads of $F - S$, we deduce that m is even and $C = \{g_{m-1}, g_m, s\}$. Thus $(F - S) \cup s$ is a fan. This fan is contained in a maximal fan F'.

Suppose first that $F' \neq F$. Then, by Theorem 3.4, $|F| = |F'| = 5$ and $M|(F \cup F') \cong M(K_4)$. In that case, $r(M) = 3$ and $|F \cap S| = 2$. Since M is not a wheel or a whirl, we see that M is obtained from a copy of $M(K_4)$ by adding extra points to one of the 3-point lines of this matroid. Let the resulting line be S and $F - S = \{a, b, c\}$. Then, up to end shuffles and reversals, the possible sequential orderings of M are of two types: $(\overrightarrow{S}, a, b, c)$ for an arbitrary ordering $\overrightarrow{S}$ of S; and $(\overrightarrow{S} - s, z, u, v, w)$ where $s \in S \cap \{u, v, w\}$ and $\{u, v, w\}$ is one of the three lines of M having exactly three points.

We may now assume that $F' = F$. So $s \in F \cap S$. We know that $(F - S) \cup s$ is a fan, and it follows that Σ consists of an arbitrary ordering of $S - s$ followed by a right-end ordering of the fan $(F - S) \cup s$. □

Lemma 6.10. *Let Σ be a fully expanded 3-sequence $(e_1, e_2, \ldots, e_n, Z)$ in a matroid M that is neither a wheel nor a whirl. Assume that there is a maximal fan F of M with $|F| = 5$ and that M has a triangle T that meets $I(F)$ but is not contained in F. Then either*

(i) *Z is non-trivial and all of the elements in $I(F)$ precede every element in $E(M) - F$; or*
(ii) *Z is trivial and all of the elements in $I(F)$ either precede every element in $E(M) - F$ or succeed every element in $E(M) - F$.*

Proof. By the assumptions of the lemma, F is a type-1 fan. Let $\{a, b, c\}$ be the unique triad of F. Then $I(F) = \{a, b, c\}$ and, by Theorem 3.4, $M|(T \cup F) \cong M(K_4)$. Let $\{u, v, w\}$, $\{a, b, u\}$, $\{b, c, v\}$, and $\{a, c, w\}$ be the triangles of this restriction. By Lemma 3.7, $\{a, b, c\}$ is a maximal cosegment and each of $\{a, b, u\}$, $\{b, c, v\}$, and $\{a, c, w\}$ is a maximal segment of M.

First we show:

6.11. *When Z is non-trivial, $|\{a,b,c\} \cap \{e_1,e_2,e_3\}| \geq 2$; and, when Z is trivial, $|\{a,b,c\} \cap \{e_1,e_2,e_3\}| \geq 2$ or $|\{a,b,c\} \cap \{e_{n-2},e_{n-1},e_n\}| \geq 2$.*

Let Z be trivial and assume that $|\{a,b,c\} \cap \{e_1,e_2,e_3\}| \leq 1$ and $|\{a,b,c\} \cap \{e_{n-2},e_{n-1},e_n\}| \leq 1$. In Σ, suppose that the ordering induced on $\{a,b,c\}$ is (a,b,c). Now move a and c in Σ to make $a,b,$ and c consecutive, leaving Σ otherwise unchanged. Let the resulting 3-sequence be (X,a,b,c,Y). By assumption, $|X| \geq 2$ and $|Y| \geq 2$. Without loss of generality, we may assume that Y contains at least two elements of $\{u,v,w\}$. Then $a \in \text{cl}(Y \cup \{b,c\})$, so $a \in \text{cl}(X)$. But this contradicts orthogonality with the triad $\{a,b,c\}$. Thus (6.11) holds when Z is trivial, and a similar argument establishes it when Z is non-trivial.

By (6.11), we may assume that $\{e_1,e_2,e_3\} = \{x,a,b\}$ so $\{x,a,b\}$ is either a triad or a triangle. In the first case, since $\{a,b,c\}$ is a maximal cosegment, $x = c$ and the lemma holds. Thus we may assume that $\{x,a,b\}$ is a triangle. By orthogonality, this implies $x = u$.

Consider e_4. We may assume this is different from c otherwise the lemma holds. Since $\{u,a,b\}$ is a triangle that is contained in no larger segment, $e_4 \in \text{cl}^*(\{u,a,b\})$. Therefore $\{u,a,b,e_4\}$ contains a cocircuit C^* containing e_4. Now $u \in C^*$ otherwise $\{a,b,c\}$ is contained in a larger cosegment. By orthogonality between C^* and the triangle $\{u,v,w\}$, we deduce that $e_4 \in \{v,w\}$. Since $M \not\cong M(K_4)$, it follows that $|C^*| \neq 3$, so $C^* = \{u,a,b,e_4\}$. But this contradicts orthogonality with either $\{a,c,w\}$ or $\{b,c,v\}$. □

Lemma 6.12. *Let M be a wheel or a whirl. Then Σ is a sequential ordering $(e_1,e_2,\ldots,e_n)$ of M if and only if, for all $k \in \{3,4,\ldots,n-3\}$, both $\{e_1,e_2,\ldots,e_k\}$ and $\{e_{k+1},e_{k+2},\ldots,e_n\}$ are fans.*

Proof. If, for all $k \in \{3,4,\ldots,n-3\}$, both $\{e_1,e_2,\ldots,e_k\}$ and $\{e_{k+1},e_{k+2},\ldots,e_n\}$ are fans, then Σ is a sequential ordering of M. Now assume that Σ is a sequential ordering of M. If $|E(M)| \leq 6$, then it is routine to check using Lemma 6.9(ii) that the lemma holds. Therefore assume that $|E(M)| \geq 8$. Since M is a wheel or a whirl, there is a cyclic ordering $\Psi = (x_1,x_2,\ldots,x_n)$ of the elements of M so that, for all i, the set $T_i = \{x_i,x_{i+1},x_{i+2}\}$ is either a triangle or a triad, and T_i is a triangle if and only if T_{i+1} is a triad, where the subscripts are read modulo n. Moreover, the sets T_i are the only triangles and triads of M.

It suffices to show that the elements in $\{e_1,e_2,\ldots,e_k\}$ are consecutive in Ψ for all $k \in \{3,4,\ldots,n-3\}$. We do this by induction on k noting that it is clear for $k = 3$. Now let $\{e_1,e_2,\ldots,e_{k-1}\} = A$ where $k-1 \geq 3$ and assume that the elements in A are consecutive in Ψ and that the immediate neighbours of this consecutive subsequence of Ψ are x_i and x_j. Suppose that $e_k \notin \{x_i,x_j\}$ and let $B = E(M) - (A \cup \{x_i,x_j,e_k\})$. Since M has sequential orderings in which each of x_i,x_j, and e_k immediately succeeds A, if $|B| = 1$, then two of the 3-element subsets of $\{x_i,x_j,e_k\} \cup B$ are triangles or two are triads, so $\{x_i,x_j,e_k\} \cup B$ is a 4-element segment or cosegment contained in $E(M)$, a contradiction; and if $|B| \geq 2$, we have an (A,B) 3-sequence in which x_i,x_j, and e_k are mutually jumping so, by

[2, Lemma 6.4], $\{x_i, x_j, e_k\}$ is a triangle or a triad of M, contradicting the fact that these elements are not consecutive in Ψ. □

Let M be a 3-connected matroid, and suppose that it has a fully expanded 3-sequence with tail Z in which the set T of its first three elements is not contained in a 4-element segment, cosegment, or fan. The next lemma shows that T is a fixed end for M.

Lemma 6.13. *Let $(e_1, e_2, \dots, e_n, Z)$ be a fully expanded 3-sequence of a matroid M. Suppose that no 4-element set containing $\{e_1, e_2, e_3\}$ is a segment, cosegment, or fan. If $(g_1, g_2, \dots, g_n, Z)$ is a fully expanded 3-sequence of M, then $\{e_1, e_2, e_3\}$ is $\{g_1, g_2, g_3\}$ when Z is non-trivial, and $\{e_1, e_2, e_3\}$ is $\{g_1, g_2, g_3\}$ or $\{g_{n-2}, g_{n-1}, g_n\}$ when Z is trivial.*

Proof. By Lemma 5.1, at least two elements of $\{e_1, e_2, e_3\}$ are in $\{g_1, g_2, g_3\}$ when Z is non-trivial, or at least two such elements are in $\{g_1, g_2, g_3\}$ or $\{g_{n-2}, g_{n-1}, g_n\}$ when Z is trivial. Now $\{g_1, g_2, g_3\}$ is a triangle or a triad, and, when Z is trivial, so is $\{g_{n-2}, g_{n-1}, g_n\}$. The lemma follows because no segment, cosegment, or fan containing $\{e_1, e_2, e_3\}$ has more than three elements. □

7. Proofs of the Main Results

In this section, we prove the four theorems from the introduction together with Theorem 2.3.

Proof of Theorem 1.1. By duality, we may assume that $\{x_{n-2}, x_{n-1}, x_n\}$ is a triangle. If $E(M)-X$ has a maximal segment K that properly contains $\{x_{n-2}, x_{n-1}, x_n\}$, then, by orthogonality and Lemma 3.7, K is the unique maximal subset of $E(M)-X$ that contains $\{x_{n-2}, x_{n-1}, x_n\}$ and is a segment, a cosegment, or a fan. It follows by Lemma 4.6 that, for every 3-sequence $(X, y_1, y_2, \dots, y_n)$, the set $\{y_{n-2}, y_{n-1}, y_n\}$ is contained in K, so (i) holds. We may now assume that $\{x_{n-2}, x_{n-1}, x_n\}$ is a maximal segment. If it is also a maximal fan, then, by Lemma 4.6 again, (i) holds. Thus we may assume that $\{x_{n-2}, x_{n-1}, x_n\}$ is properly contained in a maximal fan F. Now suppose that M has a 3-sequence $(X, y_1, y_2, \dots, y_n)$ such that $\{y_{n-2}, y_{n-1}, y_n\}$ is not contained in F. Then, by Lemma 4.6, $\{x_{n-2}, x_{n-1}, x_n\} \cup \{y_{n-2}, y_{n-1}, y_n\}$ is contained in a maximal fan F'. By Theorem 3.4, $|F| = |F'| = 5$ and, for some $N \in \{M, M^*\}$, we have $N|(F \cup F') \cong M(K_4)$. Moreover, $F \cup F'$ contains a third 5-element maximal fan F'', all of F, F', and F'' have the same set T of internal elements, and $\{y_{n-2}, y_{n-1}, y_n\}$ is contained in at least two of these fans.

Now let $(X, z_1, z_2, \dots, z_n)$ be another 3-sequence of M. Then, by Lemma 4.6, $\{x_{n-2}, x_{n-1}, x_n\} \cup \{z_{n-2}, z_{n-1}, z_n\}$ is contained in a maximal fan F_0. By Theorem 3.4, $F_0 \in \{F, F', F''\}$. Since every triangle or triad of one of these fans is in at least two such fans, we deduce that $\{z_{n-2}, z_{n-1}, z_n\}$ is contained in at least two of F, F', and F''. Moreover, by Theorem 3.4 and Lemma 3.7, $\{z_{n-2}, z_{n-1}, z_n\}$ is contained in no other subset of $E(M) - X$ that is maximal being a segment, a cosegment, or a fan.

Finally, by applying Lemma 6.10 to F_0, we deduce that $I(F_0)$ succeeds every element of $E(M) - F_0$ in $(X, z_1, z_2, \dots, z_n)$. It now follows by Lemma 6.2 that $T \subseteq \{z_{n-3}, z_{n-2}, z_{n-1}, z_n\}$. □

Proofs of Theorems 1.2 and 1.3. Let the sequential orderings $(e_1, e_2, \dots, e_n)$ and $(g_1, g_2, \dots, g_n)$ of M be denoted by Σ_e and Σ_g. In the argument that follows, we shall frequently replace sequential orderings by their reversals, often without explicitly mentioning this.

Suppose first that M is a copy of $M(K_4)$ to which extra points have been added on one of the 3-point lines to give a segment S. Let $\{a, b, c\}$ be the unique triad of M. Then $\{a, b, c\}$ is the set of internal elements of every maximal fan of M with more than three elements. Lemma 6.9(ii) lists all possible sequential orderings of M. From that result, we deduce that $\{L(M), R(M)\} = \{S, \{a, b, c\}\}$ and that, after possibly replacing Σ_e or Σ_g by its reversal, $\{e_1, e_2, e_3\} \cup \{g_1, g_2, g_3\} \subseteq S$ and $\{e_{n-2}, e_{n-1}, e_n\} \cup \{g_{n-2}, g_{n-1}, g_n\}$ is contained in a maximal fan of M that has $\{a, b, c\}$ as its set of internal elements.

Suppose next that M is the union of a maximal fan F and a maximal segment S but that M is not a copy of $M(K_4)$ with extra points added to one of its 3-point lines. If $|S| \geq 4$ or if $|S| \geq 3$ and $|F \cap S| = 0$, then, by Lemma 6.9(i), $\{L(M), R(M)\} = \{S, I(F)\}$. Moreover, $\{e_1, e_2, e_3\} \cup \{g_1, g_2, g_3\} \subseteq S$ and $\{e_{n-2}, e_{n-1}, e_n\} \cup \{g_{n-2}, g_{n-1}, g_n\} \subseteq F$. Now suppose that $|S| = 3$ and $|F \cap S| > 0$. Since M is not a wheel or a whirl, $|F \cap S| \neq 2$, so $|F \cap S| = 1$. Let $F \cap S = \{x_1\}$. Then x_1 is an end of the fan F. Let x_m be the other end of F. Then F has type-3 and $(S - x_1) \cup x_m$ is a triad of M. Thus $S \cup x_m$ is a 4-element fan F' of M. In this case, $\{L(M), R(M)\} = \{I(F'), I(F)\}$ and $\{e_1, e_2, e_3\} \cup \{g_1, g_2, g_3\} \subseteq F'$ and $\{e_{n-2}, e_{n-1}, e_n\} \cup \{g_{n-2}, g_{n-1}, g_n\} \subseteq F$.

We may now assume that neither M nor M^* is the union of a maximal fan and a maximal segment. By duality, we may also assume that $\{e_1, e_2, e_3\}$ is a triangle.

Next we prove the following.

7.1. *By replacing Σ_g by its reversal if necessary,*

(i) $\{e_1, e_2, e_3\} = \{g_1, g_2, g_3\}$ *and this set is contained in no segment or fan with more than three elements; or*
(ii) $\{e_1, e_2, e_3\} \cup \{g_1, g_2, g_3\}$ *is contained in a maximal segment or a maximal fan with at least four elements.*

Suppose first that M has no 4-element segment and no 4-element fan containing $\{e_1, e_2, e_3\}$. Then it follows immediately by Lemma 6.13 that (i) holds. Suppose next that M has a set S properly containing $\{e_1, e_2, e_3\}$ such that S is a segment or a fan. Then, by Lemma 3.7, either

(a) every such set is a segment; or
(b) every such set is a fan.

Assume that (a) holds and let S be a maximal segment containing $\{e_1, e_2, e_3\}$. Then S is unique. Moreover, by Lemma 5.3 and Corollary 4.8, exactly one of $\{g_1, g_2, g_3\}$ and $\{g_{n-2}, g_{n-1}, g_n\}$ is contained in S. By reversing Σ_g if necessary, we may assume that $\{g_1, g_2, g_3\} \subseteq S$. Now suppose that (b) holds and let S be a maximal fan containing $\{e_1, e_2, e_3\}$. Then, by Lemma 6.7, we may assume that $|S \cap \{g_1, g_2, g_3\}| \geq 2$. Then, by the earlier assumptions and Lemma 6.6, either $S \supseteq \{g_1, g_2, g_3\}$ and (7.1) holds; or $|S \cap \{g_1, g_2, g_3\}| = 2$ and, for some $N \in \{M, M^*\}$, the fan S has five elements and contains a unique triad T^* of N, the set $\{g_1, g_2, g_3\}$ is a triangle of N, the elements of $S \cap \{g_1, g_2, g_3\}$ are in $I(S)$, and $N|(S \cup \{g_1, g_2, g_3\}) \cong M(K_4)$. In the latter case, one easily checks that $\{e_1, e_2, e_3\} \cup \{g_1, g_2, g_3\}$ is contained in a 5-element maximal fan of N. This completes the proof of (7.1).

If we apply (7.1) to the other end of Σ_e, then we will get information about either $\{e_{n-2}, e_{n-1}, e_n\} \cup \{g_1, g_2, g_3\}$ or $\{e_{n-2}, e_{n-1}, e_n\} \cup \{g_{n-2}, g_{n-1}, g_n\}$. Next we eliminate the first of these possibilities. If $\{e_{n-2}, e_{n-1}, e_n\}$ equals $\{g_1, g_2, g_3\}$ and is not contained in any segment, cosegment, or fan with more than three elements, then we have a contradiction to (7.1). If $\{e_{n-2}, e_{n-1}, e_n\} \cup \{g_1, g_2, g_3\}$ is contained in a segment, cosegment, or fan with more than three elements, then, by (7.1), $\{e_1, e_2, e_3\} \cup \{g_1, g_2, g_3\}$ is contained in a maximal segment or a maximal fan with at least four elements. In the first case, it follows by Corollary 4.8 that $r(M) = 2$; a contradiction. In the second case, we get a contradiction by Lemma 6.5.

It now follows by (7.1) that

(iii) $\{e_{n-2}, e_{n-1}, e_n\} = \{g_{n-2}, g_{n-1}, g_n\}$ and this set is contained in no segment, cosegment, or fan with more than three elements; or
(iv) $\{e_{n-2}, e_{n-1}, e_n\} \cup \{g_{n-2}, g_{n-1}, g_n\}$ is contained in a maximal segment, cosegment, or fan with at least four elements.

From (7.1)(i) and (ii), and from (iii) and (iv), we deduce that each of $\{e_1, e_2, e_3\} \cup \{g_1, g_2, g_3\}$ and $\{e_{n-2}, e_{n-1}, e_n\} \cup \{g_{n-2}, g_{n-1}, g_n\}$ is contained in a maximal segment, a maximal cosegment, or a maximal fan. The fact that these segments, cosegments, or fans are distinct follows by Corollary 4.8 and Lemma 6.5. Thus Theorem 1.2 is proved.

To complete the proof of Theorem 1.3, we observe that if $L(\Sigma_e)$ is a triangle end, a triad end, a segment end, or a cosegment end, then, by (7.1), $L(\Sigma_e) = L(\Sigma_g)$. Finally, suppose that $L(\Sigma_e)$ is a fan end. Then $L(\Sigma_g)$ is also a fan end. Since, by Theorem 3.4, all maximal fans containing $\{e_1, e_2, e_3\}$ have the same sets of internal elements, we deduce that $L(\Sigma_e) = L(\Sigma_g)$. By a similar argument, we have $R(\Sigma_e) = R(\Sigma_g)$. The fact that $L(\Sigma_e)$ and $R(\Sigma_e)$ are distinct follows by Corollary 4.8 and Lemma 6.5. □

Proof of Theorem 1.4. The fact that $L(\Sigma)$ and $L(\Sigma')$ have the same type is immediate from Theorem 1.3. Moreover, parts (i) and (ii) follow by Lemmas 6.13 and 5.2. Finally, suppose that $L(\Sigma)$ is a fan end. Then part (iii) follows by Lemmas 6.8, 6.9, 6.10, and 6.5. □

Proof of Theorem 2.3. Consider the construction of M_{LR}. We know that M is sequential. Suppose that $L(M)$ is a segment end of M or M^*, say M, having at least four elements. Let Σ be a normalized sequential ordering of M. Then, by Theorem 1.4, Σ begins with $|L(M)|-1$ elements of $L(M)$. Evidently adjoining an arbitrary ordering of $\{l_1, l_2, l_3\}$ to the beginning of Σ gives a sequential ordering Σ_L of M_L. If $L(M)$ is a triangle end, a triad end, or a fan end of M, then $M_L = M$ and we let $\Sigma_L = \Sigma$. Next we show that $R(M_L) = R(M)$. This is certainly true if $M_L = M$. If $M_L \neq M$, then $L(M_L) = L(M) \cup \{l_1, l_2, l_3\}$. In a sequential ordering Σ' of M_L that begins with $|L(M_L)|-1$ elements of $L(M_L)$, all of l_1, l_2, and l_3 precede every element of $E(M)-L(M)$. To see this, suppose that Σ' has an element of $E(M)-L(M)$ preceding some l_i. Let e be the first such element. Then $e \in \mathrm{cl}^*_{M_L}(L(M_L)-l_i)$. Thus M_L has a cocircuit containing e that is contained in $(L(M_L)-l_i)\cup e$. By orthogonality, this cocircuit is $(L(M_L)-l_i)\cup e$. Hence M_L has a hyperplane meeting $L(M_L)$ in $\{l_i\}$. This contradicts the fact that l_i was freely placed on $L(M)$.

We deduce that when $M_L \neq M$, the sequential orderings of M_L are exactly the orderings of $E(M_L)$ that can be obtained by arbitrarily inserting l_1, l_2, and l_3 into a normalized sequential ordering of M so that each precedes every element of $E(M)-L(M)$. It follows from this that $R(M_L) = R(M)$. Then the argument used above to show that M_L is sequential establishes that M_{LR} is also sequential.

Now let $(A(M), \overrightarrow{X}, B(M))$ be an $(A(M), B(M))$ 3-sequence of M_{LR} and let Σ_{LR} be obtained from this 3-sequence by ordering ends. Then Σ_{LR} is a sequential ordering of M_{LR}. Moreover, by Lemma 4.2, pruning surplus elements from Σ_{LR} gives a sequential ordering Σ of M. Clearly by performing fan shuffles at the ends of Σ, we maintain a sequential ordering of M.

It remains to show that if Σ' is an arbitrary sequential ordering $(e_1, e_2, \dots, e_n)$ of M, then Σ' can be obtained using the above procedure. If neither $L(M)$ nor $R(M)$ is a fan end, then the definitions of $A(M)$ and $B(M)$ ensure that the construction described produces an $(A(M), B(M))$ 3-sequence. Moreover, Σ' can be obtained from this $(A(M), B(M))$ 3-sequence by ordering ends and pruning surplus elements.

Now suppose that $L(M)$ is a fan end. Then $\{e_1, e_2, e_3\}$ is contained in a maximal fan F of M, and F has $L(M)$ as its set of internal elements. Then, by Theorem 1.4, either the first $|L(M)|$ elements of Σ' all lie in $L(M)$, or the first $|L(M)|+1$ elements of Σ' are contained in F and include $L(M)$. In either case, a fan shuffle produces a sequential ordering of M whose first $|L(M)|$ elements consist of a left-end ordering of the fan $I(F)$.

To complete the proof of the theorem, we need only analyze what happens at the right end of Σ'. This depends on the type of $R(M)$ and a similar argument to the above establishes the required result. □

The final result of the paper specifies how to determine all 3-separations that are equivalent to a unisequential 3-separation of a 3-connected matroid. The proof, which is similar to that given above, is omitted.

Theorem 7.2. *Let (A_1, B_1) be a 3-separation of a 3-connected matroid M such that $\text{fcl}(B_1) = E(M)$ and $\text{fcl}(A_1) = E(M) - B_1$. Let $(e_1, e_2, \dots, e_n, B_1)$ be a 3-sequence and suppose $\{e_1, e_2, e_3\}$ is a triangle. Let $B(M) = B_1$. If $\{e_1, e_2, e_3\}$ is contained in no segment or fan with more than three elements, let $A(M) = \{e_1, e_2, e_3\}$, and $M^+ = M$. If $\{e_1, e_2, e_3\}$ is contained in a segment with at least four elements, adjoin l_1, l_2, and l_3 freely to the line spanned by $\{e_1, e_2, e_3\}$ to give M^+ and let $A(M) = \{l_1, l_2, l_3\}$. If $\{e_1, e_2, e_3\}$ is contained in a maximal fan F with at least four elements, let $A(M) = I(F)$, and $M^+ = M$. Then every fully expanded 3-sequence of M with tail B_1 can be obtained from some $(A(M), B(M))$ 3-sequence of M^+ by*

(i) *ordering $A(M)$ arbitrarily if $|A(M)| \leq 3$, and with a left-end ordering if $|A(M)| \geq 4$;*
(ii) *pruning surplus elements if $A(M) \cap E(M) = \emptyset$; and*
(iii) *a fan shuffle when $A(M) = I(F)$.*

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