A CHAIN THEOREM FOR 4–CONNECTED MATROIDS

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Abstract. A matroid M is said to be *k–connected up to separators of size l* if whenever A is $(k-1)$–separating in M, then either $|A| \leq l$ or $|E(M) - A| \leq l$. We use $\mathrm{si}(M)$ and $\mathrm{co}(M)$ to denote the simplification and cosimplification of the matroid M. We prove that if a 3–connected matroid M is 4–connected up to separators of size 5, then there is an element x of M such that either $\mathrm{co}(M\backslash x)$ or $\mathrm{si}(M/x)$ is 3–connected and 4–connected up to separators of size 5, and has a cardinality of $|E(M)| - 1$ or $|E(M)| - 2$.

1. Introduction

We begin by recalling Tutte's definition of matroid connectivity [8]. Let M be a matroid with ground set E. The *connectivity function* of M is given by $\lambda_M(A) = \mathrm{r}(A) + \mathrm{r}(E - A) - \mathrm{r}(M) + 1$ where A is a subset of E. A subset A of E is *k–separating* if $\lambda_M(A) \leq k$. Thus, a partition (A, B) of E is a *k–separation* of M if A is k–separating and $|A|, |B| \geq k$. We say that M is *k–connected* if M has no k'–separation where $k' < k$.

Historically, the focus of much attention in matroid theory has been on 3–connected matroids. One reason for this is that 3–connected matroids possess significant structure in that a number of the degeneracies caused by low connectivity are ironed out in the 3–connected case. A second crucial reason is that there exist satisfactory chain theorems such as Tutte's Wheels and Whirls Theorem and Seymour's Splitter Theorem that enable strong inductive arguments to be made in the class of 3–connected matroids.

However, over recent years evidence has accumulated that 3–connectivity is not enough for substantial progress in matroid representation theory and that higher connectivity is needed. On the other hand it is also clear that strict 4–connectivity is too strong a notion to be really useful. This notion excludes highly structured objects such as matroids of complete graphs. Moreover, it does not appear possible to find a reasonable analogue for chain theorems such as the Wheels and Whirls Theorem. Given this, it is natural to look for weakenings of 4–connectivity. To be useful, such a weakening should allow natural structures such as matroids of complete graphs and it should also be possible to prove reasonable chain theorems. One such weakening is the notion of sequential 4–connectivity introduced by Geelen

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and Whittle [3]. With this notion it is possible to prove an analogue of the Wheels and Whirls Theorem.

Theorem 1.1. *Let M be a sequentially 4–connected matroid. If M is not a wheel or a whirl, then there exists an element $e \in E(M)$ such that either $M\backslash e$ or M/e is sequentially 4–connected.*

Sequential 4–connectivity is certainly a natural notion. However, if (A, B) is a 3–separation in a sequentially 4–connected matroid, then, while one of A or B is forced to have a certain simple structure, no bound can be placed on the sizes of A or B, that is, we may have arbitrarily large 3–separations. In this paper we consider an alternative weakening of 4–connectivity. A matroid M is *k–connected up to separators of size l* if whenever A is $(k-1)$–separating in M, then either $|A| \leq l$ or $|E(M) - A| \leq l$. Here, rather than focusing on the *structure* of 3–separators, we focus solely on their *size*. The main theorem of this paper proves

Theorem 1.2. *If a 3–connected matroid M is 4–connected up to 3–separators of size 5 then there is an element $x \in E(M)$ such that $\mathrm{co}(M\backslash x)$ or $\mathrm{si}(M/x)$ is 3–connected and 4–connected up to 3–separators of size 5, with a cardinality of $|E(M)| - 1$ or $|E(M)| - 2$.*

The paper is structured as follows. In Section 2 we deal with the case where the matroid M is 4–connected. Section 3 deals with the internally 4–connected case. We prove Theorem 3.1 which is stronger than we need for proving Theorem 1.2 however it is of independent interest, for example it is used in bounding the size of excluded minors for the matroids of branch–width 3 [4]. Unfortunately the proof of Theorem 3.1 is rather cumbersome as it involves case analysis. In Section 4, we deal with the case where the matroid is 4–connected up to separators of size 4, and in Section 5 we complete the proof of Theorem 1.2. Section 5 begins with a relatively straightforward proof for the matroids with more than 15 elements, however we require case analysis when we look at the matroids smaller than this.

We assume that the reader is familiar with matroid theory as set forth in Oxley [5]. Also notation follows Oxley with the following exceptions. We use $\mathrm{si}(M)$ and $\mathrm{co}(M)$ for the simplification and cosimplification of the matroid M. We let $\mathrm{cl}^{(*)}(X)$ denote $\mathrm{cl}(X) \cup \mathrm{cl}^*(X)$.

Finally we note a lemma [3, Proposition 3.2] that will be used frequently.

Lemma 1.3. *Let λ_M be the connectivity function of a matroid M, and let A and B be subsets of the groundset of M. If A and B are 3–separating and $\lambda_M(A \cap B) \geq 3$, then $\lambda_M(A \cup B) \leq 3$.*

2. The 4–connected Case

In this section we deal with the case where the matroid is 4–connected. The following lemma is [2, Lemma 5.2].

Lemma 2.1. *Let x be an element of a matroid M, and let A and B be subsets of $E(M) - \{x\}$. Then*

$$\lambda_{M\backslash x}(A) + \lambda_{M/x}(B) \geq \lambda_M(A \cap B) + \lambda_M(A \cup B \cup \{x\}) - 1.$$

The next lemma is a straightforward consequence of Lemma 2.1. It appears well known but does not seem to appear in the literature.

Lemma 2.2. *Let M be k–connected up to separators of size l. Then, for all $x \in E(M)$, either $M\backslash x$ or M/x is k–connected up to separators of size $2l$.*

Proof. Let $x \in E(M)$. Suppose that $M\backslash x$ is not k–connected up to separators of size $2l$, so that there is a $(k-1)$–separation (A_1, A_2) of $M\backslash x$ where $|A_1|, |A_2| \geq 2l + 1$. Consider M/x. Let (B_1, B_2) be a $(k-1)$–separation of M/x. Then from Lemma 2.1, $\lambda_{M\backslash x}(A_1) + \lambda_{M/x}(B_1) \geq \lambda_M(A_1 \cap B_1) + \lambda_M(A_1 \cup B_1 \cup \{x\}) - 1$ so that $\lambda_M(A_1 \cap B_1) + \lambda_M(A_2 \cap B_2) \leq 2k - 1$, and it follows that either $\lambda_M(A_1 \cap B_1) \leq k - 1$ or $\lambda_M(A_2 \cap B_2) \leq k - 1$. But if $A_1 \cap B_1$ or $A_2 \cap B_2$ is $(k-1)$–separating in M, then $|A_1 \cap B_1| \leq l$ or $|A_2 \cap B_2| \leq l$ respectively. By the same argument as above, we see that $|A_1 \cap B_2| \leq l$ or $|A_2 \cap B_1| \leq l$. We can assume without loss of generality that $|A_1 \cap B_1| \leq l$. It is not possible to have $|A_1 \cap B_2| \leq l$ because $|A_1| \geq 2l + 1$, so we must have $|A_2 \cap B_1| \leq l$ and as a result $|B_1| \leq 2l$. From this we see that M/x is k–connected up to separators of size $2l$. □

An immediate corollary of Lemma 2.2 is

Corollary 2.3. *Let x be an element of the 4–connected matroid M. Then $M\backslash x$ or M/x is 4–connected up to 3–separators of size 4.*

3. The Internally 4–connected Case

Recall that a matroid is internally 4–connected if it is 3–connected and 4–connected up to separators of size 3. It is easily seen that if e is an element of a triangle in an internally 4–connected matroid M with at least eight elements, then $M\backslash e$ is 3–connected.

The object of this section is to prove the following theorem.

Theorem 3.1. *Let M be an internally 4–connected matroid, and let $\{a, b, c\}$ be a triangle of M. Then at least one of the following hold.*

(1) *At least one of $M\backslash a$, $M\backslash b$ and $M\backslash c$ is 4–connected up to 3–separators of size 4.*
(2) *At least two of $M\backslash a$, $M\backslash b$ and $M\backslash c$ are 4–connected up to 3–separators of size 5.*

Before proving Theorem 3.1, we establish some preliminary lemmas. We begin with a definition. Let (X_1, X_2) and (Y_1, Y_2) be k–separations of a matroid M. Then (X_1, X_2) is *meatier* than (Y_1, Y_2) if $\min\{|X_1|, |X_2|\} > \min\{|Y_1|, |Y_2|\}$. A *meaty* 3–separation (X_1, X_2) of a matroid M is one where $|X_1| \geq 5$ and $|X_2| \geq 5$.

Lemma 3.2 ([3, Lemma 6.1.1.]). *Let M be an internally 4–connected matroid. Let $\{a,b,c\}$ be a triangle of M, and let (X,Y) be a meaty 3–separation of $M\backslash a$. Then $b \in X$, $c \in Y$, $b \in \mathrm{cl}(X-\{b\})$ and $c \in \mathrm{cl}(Y-\{c\})$.*

Proof. M is internally 4–connected and $|X|, |Y| \geq 5$, so $a \notin \mathrm{cl}(X)$ and $a \notin \mathrm{cl}(Y)$. However $a \in \mathrm{cl}(\{b,c\})$ so without loss of generality, we must have $b \in X$ and $c \in Y$.

Now suppose that $b \notin \mathrm{cl}(X-\{b\})$, then $(X-\{b\}, Y\cup\{b\})$ is a 3–separation of $M\backslash a$. But $a \in \mathrm{cl}(Y\cup\{b\})$ so $(X-\{b\}, Y\cup\{a,b\})$ is a 3–separation of M where $|X-\{b\}| \geq 4$ and $|Y\cup\{a,b\}| \geq 7$, contradicting the fact that M is internally 4–connected. As a result, we see that $b \in \mathrm{cl}(X-\{b\})$, and similarly $c \in \mathrm{cl}(Y-\{c\})$. □

In what follows, M is an internally 4–connected matroid, $\{a,b,c\}$ is a triangle of M, and (A_b, A_c), (B_a, B_c) and (C_a, C_b) are meaty 3–separations of $M\backslash a$, $M\backslash b$ and $M\backslash c$ respectively, where $b \in A_b$, $c \in A_c$, $a \in B_a$, $c \in B_c$, $a \in C_a$, and $b \in C_b$. We use the following lemma of [3] to prove the lemma which follows it, which also appears in [3].

Lemma 3.3 ([3, Lemma 6.1.4.]). *If $A_b \cap B_c$ (respectively $A_c \cap B_c$ or $A_c \cap B_a$) is k–separating in $M\backslash a,b$, then $A_b \cap B_c$ (respectively $A_c \cap B_c$ or $A_c \cap B_a$) is k–separating in M.*

Proof. We have $a \in \mathrm{cl}(B_a-\{a\})$ and $b \in \mathrm{cl}(\{a,c\})$. Therefore, if $A_b \cap B_c$ is k–separating in $M\backslash a,b$ then $A_b \cap B_c$ is k–separating in M. Similarly, if $A_c \cap B_a$ is k–separating in $M\backslash a,b$, then $A_c \cap B_a$ is k–separating in M. Moreover, since $a \in \mathrm{cl}(B_a-\{a\})$ and $b \in \mathrm{cl}(A_b-\{b\})$, we see that if $A_c \cap B_c$ is k–separating in $M\backslash a,b$ then $A_c \cap B_c$ is k–separating in M. □

Lemma 3.4 ([3, Lemma 6.1.5.]).

(i) *If $|A_b \cap B_c| \geq 2$, then $A_c \cap B_a$ is 3–separating in M and $|A_c \cap B_a| \leq 3$.*
(ii) *If $|A_c \cap B_a| \geq 2$, then $A_b \cap B_c$ is 3–separating in M and $|A_b \cap B_c| \leq 3$.*
(iii) *If $\lambda_{M\backslash a,b}(A_b \cap B_a) \geq 3$, then $A_c \cap B_c$ is 3–separating in M and $|A_c \cap B_c| \leq 3$.*
(iv) *If $\lambda_{M\backslash a,b}(A_b \cap B_a) = 2$, then $A_b \cap B_a$ is 3–separating in M and $|A_b \cap B_a| \leq 3$.*

Proof.

(i) If $|A_b \cap B_c| \geq 2$, then $A_b \cap B_c$ cannot be 2–separating in $M\backslash a,b$ because then it would be 2–separating in M, by Lemma 3.3. So $A_b \cap B_c$ must be 3–separating in $M\backslash a,b$. Now, from Lemma 1.3, $A_c \cap B_a$ is 3–separating in $M\backslash a,b$, and by Lemma 3.3, $A_c \cap B_a$ is 3–separating in M. Now, M is internally 4–connected, so $|A_c \cap B_a| \leq 3$.
(ii) This argument follows from (i) and from the symmetry of $\{a,b,c\}$.

(iii) If $\lambda_{M\backslash a,b}(A_b \cap B_a) \geq 3$, then from Lemma 1.3, $A_c \cap B_c$ is 3–separating in $M\backslash a, b$, and from Lemma 3.3, it is 3–separating in M. Now, M is internally 4–connected, so $|A_c \cap B_c| \leq 3$.

(iv) If $\lambda_{M\backslash a,b}(A_b \cap B_a) = 2$, then since $M\backslash a$ is 3–connected, $A_b \cap B_a$ is 3–separating in $M\backslash a$. Now, $a \in \mathrm{cl}(\{b, c\})$, so $A_b \cap B_a$ is 3–separating in M, and M is internally 4–connected so $|A_b \cap B_a| \leq 3$.

□

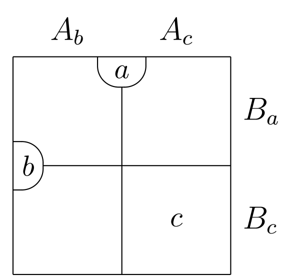
Lemma 3.5. *If $|A_b \cap B_a| = 3$ and $\lambda_{M\backslash a,b}(A_b \cap B_a) = 2$, then $M\backslash c$ is 4–connected up to 3–separators of size 5.*

Proof. By Lemma 3.4, $A_b \cap B_a$ is 3–separating in M, so it is a triangle or triad of M. Let $A_b \cap B_a = \{x_1, x_2, x_3\}$. Now, $\lambda_{M\backslash a,b}(A_b \cap B_a) = 2$, but $M\backslash a$ and $M\backslash b$ are 3–connected so $\lambda_{M\backslash a}(A_b \cap B_a) = 3$ and $\lambda_{M\backslash b}(A_b \cap B_a) = 3$. As a result, $a, b \notin \mathrm{cl}(A_c \cup B_c)$ and hence $a \in \mathrm{cl}^*_M(\{b, x_1, x_2, x_3\})$ and $b \in \mathrm{cl}^*_M(\{a, x_1, x_2, x_3\})$. Now consider the 3–separation (C_a, C_b) of $M\backslash c$. We need to show that $|C_a| \leq 5$ or $|C_b| \leq 5$. By symmetry, there are two cases to check. In the first case $\{x_1, x_2, x_3\} \subseteq C_a$, and in the second case $x_1, x_2 \in C_a$ and $x_3 \in C_b$.

We begin with the first case where $\{x_1, x_2, x_3\} \subseteq C_a$. Since $b \in \mathrm{cl}^*_M(\{x_1, x_2, x_3, a\})$, we know that $(C_a \cup \{b\}, C_b - \{b\})$ is a 3–separation of $M\backslash c$, and therefore $(C_a \cup \{b, c\}, C_b - \{b\})$ is a 3–separation of M. Now, M is internally 4–connected so $|C_b - \{b\}| \leq 3$ and thus, $|C_b| \leq 4$

Now consider the second case where $x_1, x_2 \in C_a$ and $x_3 \in C_b$. Since $\{x_1, x_2, x_3\}$ is a triangle or triad of M, $x_3 \in \mathrm{cl}^{(*)}(\{x_1, x_2\})$, hence $(C_a \cup \{x_3\}, C_b - \{x_3\})$ is a 3–separation of $M\backslash c$. Now, $b \in \mathrm{cl}^*_M(\{a, x_1, x_2, x_3\})$, so $(C_a \cup \{x_3, b\}, C_b - \{x_3, b\})$ is a 3–separation of $M\backslash c$, and as a result $(C_a \cup \{x_3, b, c\}, C_b - \{x_3, b\})$ is a 3–separation of M. But M is internally 4–connected, so $|C_b - \{x_3, b\}| \leq 3$ and therefore $|C_b| \leq 5$. This shows that $M\backslash c$ is 4–connected up to 3–separators of size 5. □

Having proved these preliminary lemmas, we will now start bounding the size of the 3–separators in the matroids $M\backslash a$, $M\backslash b$, and $M\backslash c$. In the following, we assume that (A_b, A_c), (B_a, B_c) and (C_a, C_b) are the meatiest 3–separators of $M\backslash a$, $M\backslash b$, and $M\backslash c$ respectively. Also in what follows, we make frequent use of Venn diagrams. The diagram below illustrates the 3–separations (A_b, A_c) and (B_a, B_c), and may assist the reader in following the proof of Lemma 3.6.



Lemma 3.6. *Let (A_b, A_c), (B_a, B_c) and (C_a, C_b) be the meatiest 3–separations of $M\backslash a$, $M\backslash b$, and $M\backslash c$ respectively. If $|A_b \cap B_a| \le 1$, $|A_b \cap B_c| \le 1$, $|A_c \cap B_a| \le 1$, or $|A_c \cap B_c| \le 2$, then Theorem 3.1 is satisfied.*

Proof. The above Venn diagram may assist in following the proof. First suppose that $|A_b \cap B_a| \le 1$. It is easily seen that if $|A_b \cap B_c| \le 3$ and $|A_c \cap B_a| \le 3$, then $|A_b| \le 5$ and $|B_a| \le 5$, as required. If $|A_c \cap B_a| > 3$, then $|A_b \cap B_c| \le 1$ by Lemma 3.4, so $|A_b| \le 3$ as required. The argument is symmetric if $|A_b \cap B_c| > 3$.

Secondly, we suppose that $|A_c \cap B_c| \le 2$. If $|A_b \cap B_c| \le 3$ and $|A_c \cap B_a| \le 3$, then $|A_c| \le 5$ and $|B_c| \le 5$ as required. If $|A_c \cap B_a| > 3$ then $|A_b \cap B_c| \le 1$ by Lemma 3.4, so $|B_c| \le 3$ as required. The argument is symmetric if $|A_b \cap B_c| > 3$.

Now suppose that $|A_b \cap B_c| \le 1$. If $|A_c \cap B_c| \le 3$, then $|B_c| \le 4$ as required. If $|A_c \cap B_c| > 3$, then by Lemma 3.4, $\lambda_{M\backslash a,b}(A_b \cap B_a) < 3$ and $|A_b \cap B_a| \le 3$. Firstly, if $|A_b \cap B_a| \le 2$ then $|A_b| \le 4$. Secondly, if $|A_b \cap B_a| = 3$ then $|A_b| \le 5$, and by Lemma 3.5 $M\backslash c$ is 4–connected up to 3–separators of size 5.

Finally, the case where $|A_c \cap B_a| \le 1$ is symmetric to the case where $|A_b \cap B_c| \le 1$. □

Given 3–separations (A_b, A_c), (B_a, B_c) and (C_a, C_b), we use the following notation to simplify the statements of Lemmas 3.7—3.10.

$$\begin{aligned} \lambda_1 &:= |A_b \cap B_c \cap C_a| & \lambda_2 &:= |A_c \cap B_a \cap C_b| \\ \nu_{1a} &:= |A_b \cap B_a \cap C_a| & \nu_{2a} &:= |A_c \cap B_a \cap C_a| \\ \nu_{1b} &:= |A_b \cap B_c \cap C_b| & \nu_{2b} &:= |A_b \cap B_a \cap C_b| \\ \nu_{1c} &:= |A_c \cap B_c \cap C_a| & \nu_{2c} &:= |A_c \cap B_c \cap C_b| \end{aligned}$$

It is easily seen that the eight sets listed above along with $\{a, b, c\}$ partition the elements of the matroid M.

Lemma 3.7. *Let (A_b, A_c), (B_a, B_c) and (C_a, C_b) be meaty 3–separations of the matroids $M\backslash a$, $M\backslash b$ and $M\backslash c$ respectively, such that $A_b \cap B_c$, $A_c \cap B_a$, $A_b \cap C_a$, $A_c \cap C_b$, $B_a \cap C_b$ and $B_c \cap C_a$ all have at least two elements. If $\lambda_1 \ge 2$, then $\lambda_1 + \nu_{1a} + \nu_{1b} + \nu_{1c} \le 3$. And similarly, if $\lambda_2 \ge 2$ then $\lambda_2 + \nu_{2a} + \nu_{2b} + \nu_{2c} \le 3$.*

Proof. Assume that $\lambda_1 = |A_b \cap B_c \cap C_a| \ge 2$. We know from Lemma 3.4 that each of $A_b \cap B_c$, $A_c \cap B_a$, $A_b \cap C_a$, $A_c \cap C_b$, $B_a \cap C_b$ and $B_c \cap C_a$ is 3–separating in M and has at most three elements. Suppose first that two of $A_b \cap B_c$, $B_c \cap C_a$ and $A_b \cap C_a$ have three elements, and that $|A_b \cap B_c \cap C_a| = 2$. We may assume by symmetry that $|A_b \cap B_c| = |B_c \cap C_a| = 3$. Then $A_b \cap C_a$ and $B_c \cap C_a$ are 3–separating subsets of M whose intersection has two elements. But then Lemma 1.3 tells us that their union forms a four-element 3–separator of M, contradicting that M is internally 4–connected. Thus, we see that if two of $A_b \cap B_c$, $B_c \cap C_a$ and $A_b \cap C_a$ have three elements, then $|A_b \cap B_c \cap C_a| = 3$.

Now suppose that two of $A_b \cap B_c$, $B_c \cap C_a$ and $A_b \cap C_a$ have three elements, and assume by symmetry that they are $A_b \cap B_c$ and $B_c \cap C_a$. Then $|A_b \cap B_c \cap C_a| = 3$, and hence $A_b \cap C_a = A_b \cap B_c \cap C_a$, since $|A_b \cap C_a| \leq 3$. But then $A_b \cap C_a \subseteq B_c$, thus $A_b \cap B_a \cap C_a = \emptyset$ and $\nu_{1a} = 0$. Similarly we see that $\nu_{1b} = \nu_{1c} = 0$, and hence $\lambda_1 + \nu_{1a} + \nu_{1b} + \nu_{1c} = 3$.

Just as with the paragraph above, if $|A_b \cap B_c| = |B_c \cap C_a| = |A_b \cap C_a| = |A_b \cap B_c \cap C_a| = 2$, then $\nu_{1a} = \nu_{1b} = \nu_{1c} = 0$. And hence $\lambda_1 + \nu_{1a} + \nu_{1b} + \nu_{1c} = 2$.

Now suppose that $|A_b \cap B_c \cap C_a| = 2$, $|A_b \cap B_c| = 3$, $|B_c \cap C_a| = 2$ and $|A_b \cap C_a| = 2$. Then as with the paragraph above, $\nu_{1a} = \nu_{1c} = 0$. Also $|A_b \cap B_c| = 3$ and $|A_b \cap B_c \cap C_a| = 2$, so two elements of $A_b \cap B_c$ are contained in C_a, while the other is in C_b. Therefore, $\nu_{1b} = |A_b \cap B_c \cap C_b| = 1$, and $\lambda_1 + \nu_{1a} + \nu_{1b} + \nu_{1c} = 3$.

By symmetry we know that if $\lambda_1 = 2$ and $|B_c \cap C_a| = 3$, then $\nu_{1c} = 1$ and $\nu_{1a} = \nu_{1b} = 0$. And we know that if $\lambda_1 = 2$ and $|A_b \cap C_a| = 3$, then $\nu_{1a} = 1$ and $\nu_{1b} = \nu_{1c} = 0$. Therefore, if $\lambda_1 \geq 2$, then $\lambda_1 + \nu_{1a} + \nu_{1b} + \nu_{1c} \leq 3$.

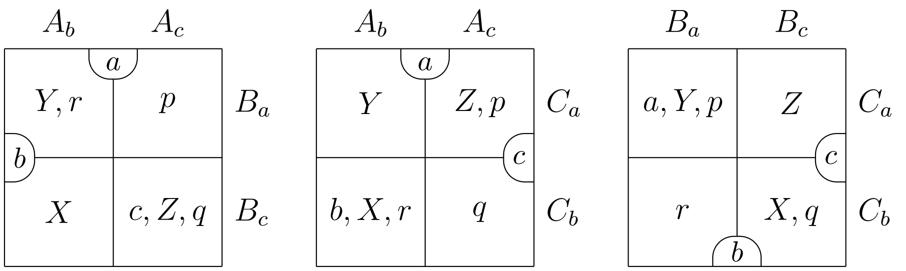
Again, we may apply symmetry to the situation above to obtain the result that if $\lambda_2 \geq 2$, then $\lambda_2 + \nu_{2a} + \nu_{2b} + \nu_{2c} \leq 3$. □

Let M be an internally 4–connected matroid with a triangle $\{a, b, c\}$. And let $M \backslash a$, $M \backslash b$ and $M \backslash c$ have meaty 3–separations (A_b, A_c), (B_a, B_c) and (C_a, C_b) respectively, such that each of $A_b \cap B_c$, $A_c \cap B_a$, $A_b \cap C_a$, $A_c \cap C_b$, $B_a \cap C_b$ and $B_c \cap C_a$ have at least two elements. Then $|E(M)| = 3 + \lambda_1 + \lambda_2 + \nu_{1a} + \nu_{2a} + \nu_{1b} + \nu_{2b} + \nu_{1c} + \nu_{2c}$ so by Lemma 3.7, if $\lambda_1 \geq 2$ and $\lambda_2 \geq 2$ then $|E(M)| \leq 9$. This means that we may assume that $\lambda_2 \leq 1$ in proving Theorem 3.1. Our proof will be divided into the following three cases.

(1) $\lambda_1 \leq 1$. This is the topic of Lemma 3.8.
(2) $\lambda_1 = 2$ and $\nu_{1a} = 1$. This is the topic of Lemma 3.9.
(3) $\lambda_1 \geq 2$ and $\nu_{1a} = \nu_{1b} = \nu_{1c} = 0$. This is the topic of Lemma 3.10.

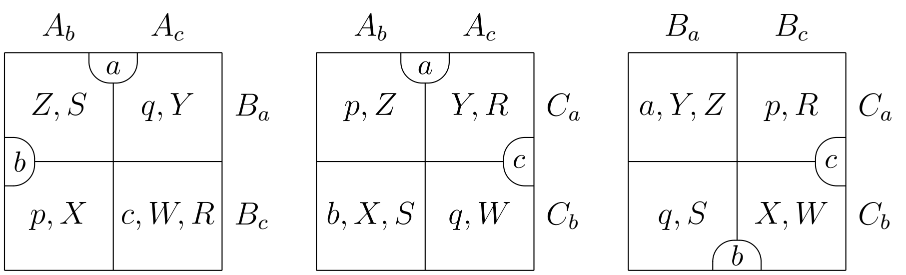
Lemma 3.8. *Let M be an internally 4–connected matroid with a triangle $\{a, b, c\}$. And let (A_b, A_c), (B_a, B_c) and (C_a, C_b) be the meatiest 3–separations of $M \backslash a$, $M \backslash b$ and $M \backslash c$ respectively, such that each of $A_b \cap B_c$, $A_c \cap B_a$, $A_b \cap C_a$, $A_c \cap C_b$, $B_a \cap C_b$ and $B_c \cap C_a$ have at least two elements. If $\lambda_1 \leq 1$ and $\lambda_2 \leq 1$, then Theorem 3.1 is satisfied.*

Proof. First suppose that $\lambda_1 = |A_b \cap B_c \cap C_a| = 0$ and $\lambda_2 = |A_c \cap B_a \cap C_b| \leq 1$. Let $X := A_b \cap B_c$, $Y := A_b \cap C_a$ and $Z := B_c \cap C_a$, then since $|A_b \cap B_c \cap C_a| = 0$, basic set theory tells us that $X \subseteq A_b \cap C_b$, $X \subseteq B_c \cap C_b$, $Y \subseteq B_a \cap C_a$, $Y \subseteq A_b \cap B_a$, $Z \subseteq A_c \cap C_a$ and $Z \subseteq A_c \cap B_c$. These are illustrated on the following Venn diagrams which may assist the reader.



Now, $\lambda_2 = |A_c \cap B_a \cap C_b| \leq 1$, while $A_c \cap B_a$, $A_c \cap C_b$ and $B_a \cap C_b$ have at least two elements each, therefore $|A_c \cap B_a \cap C_a| \geq 1$, $|A_c \cap B_c \cap C_b| \geq 1$ and $|A_b \cap B_a \cap C_b| \geq 1$. Let $p \in A_c \cap B_a \cap C_a$, $q \in A_c \cap B_c \cap C_b$ and $r \in A_b \cap B_a \cap C_b$. These elements are shown above in the Venn diagrams. Now, since $|X| \geq 2$, $|Y| \geq 2$ and $|Z| \geq 2$, we see that $|A_c \cap B_c| \geq 4$, $|A_b \cap C_b| \geq 4$ and $|B_a \cap C_a| \geq 4$. But Lemma 3.4 then tells us that $|A_b \cap B_a| = 3$ and $\lambda_{M\backslash a,b}(A_b \cap B_a) = 2$; $|A_c \cap C_a| = 3$ and $\lambda_{M\backslash a,c}(A_c \cap C_a) = 2$; and $|B_c \cap C_b| = 3$ and $\lambda_{M\backslash b,c}(B_c \cap C_b) = 2$. And it follows from Lemma 3.5 that $M\backslash a$, $M\backslash b$ and $M\backslash c$ are all 4–connected up to 3–separators of size 5.

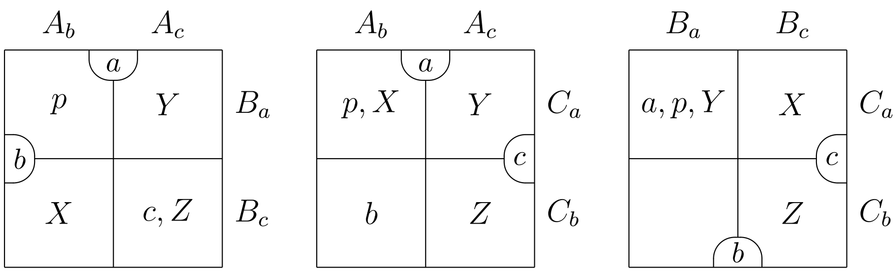
By symmetry, we obtain the same result if $\lambda_2 = 0$ and $\lambda_1 \leq 1$. Hence, we may now assume that $\lambda_1 = |A_b \cap B_c \cap C_a| = 1$ and $\lambda_2 = |A_c \cap B_a \cap C_b| = 1$. Let $\{p\} := A_b \cap B_c \cap C_a$ and $\{q\} := A_c \cap B_a \cap C_b$, $X := A_b \cap B_c - \{p\}$, $Y := A_c \cap B_a - \{q\}$, $Z := A_b \cap C_a - \{p\}$, $W := A_c \cap C_b - \{q\}$, $R := B_c \cap C_a - \{p\}$ and $S := B_a \cap C_b - \{q\}$. It is easily seen by Lemma 3.4 that each of X, Y, Z, W, R and S have either one or two elements. The following Venn diagrams are obtained by basic set theory. And basic set theory tells us that $E(M) = \{a, b, c, p, q\} \cup X \cup Y \cup Z \cup W \cup R \cup S$.



It is easily seen that if each of X, Y, Z, W, R and S has just one element, then $M\backslash a$, $M\backslash b$ and $M\backslash c$ are all 4–connected up to 3–separators of size 5, because M has only eleven elements. Hence we may assume that one of X, Y, Z, W, R and S has two elements, and we may assume by symmetry that it is Y. By Lemma 3.4, we see that $\{q\} \cup Y$ is a triangle or triad of M. Now, suppose that $\lambda_{M\backslash a,c}(A_c \cap C_a) = 2$, then $A_c \cap C_a \cup \{q\}$ is a 3–separator of M with more than three elements, contradicting that M is internally 4–connected. Thus we know that $\lambda_{M\backslash a,c}(A_c \cap C_a) \geq 3$ and by Lemma 3.4, $A_b \cap C_b$ is 3–separating in M, and hence $|X| = |S| = 1$. Next suppose that $|W| = 2$. Then $\{q\} \cup W$ would be a triangle or triad of M. But $|B_a \cap C_a| \geq 4$ so Lemma 3.4 tells us that $B_c \cap C_b$ is a 3–separator of M, and hence $B_c \cap C_b \cup \{q\}$ is a four-element 3–separator of M. This contradiction tells us that $|W| = 1$, and hence $|C_b| = 5$. A similar argument tells us that not both of R and Z may have two elements, thus either $|A_b| = 5$ or $|B_c| = 5$. And since (A_b, A_c), (B_a, B_c) and (C_a, C_b) are the meatiest 3–separations of $M\backslash a$, $M\backslash b$ and $M\backslash c$ respectively, Theorem 3.1 is satisfied. □

Lemma 3.9. *Let M be an internally 4–connected matroid with a triangle $\{a, b, c\}$. And let (A_b, A_c), (B_a, B_c) and (C_a, C_b) be the meatiest 3–separations of $M\backslash a$, $M\backslash b$ and $M\backslash c$ respectively, such that each of $A_b \cap B_c$, $A_c \cap B_a$, $A_b \cap C_a$, $A_c \cap C_b$, $B_a \cap C_b$ and $B_c \cap C_a$ have at least two elements. If $\lambda_1 = 2$, $\lambda_2 \leq 1$ and $\nu_{1a} = 1$, then Theorem 3.1 is satisfied.*

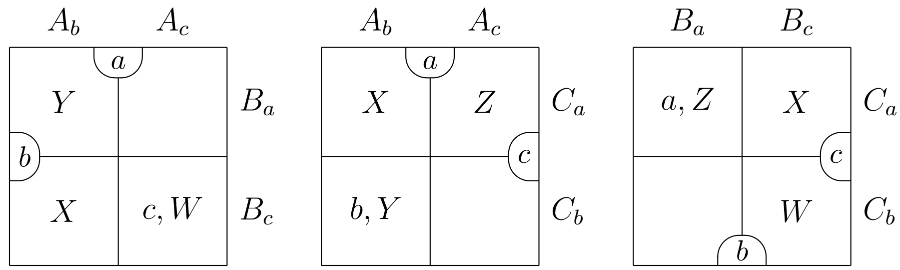
Proof. Since $\lambda_1 = |A_b \cap B_c \cap C_a| = 2$ and $\nu_{1a} = |A_b \cap B_a \cap C_a| = 1$, we see from the proof of Lemma 3.7 that $\nu_{1b} = |A_b \cap B_c \cap C_b| = 0$, $\nu_{1c} = |A_c \cap B_c \cap C_a| = 0$, $|A_b \cap C_a| = 3$, $|A_b \cap B_c| = 2$ and $|B_c \cap C_a| = 2$. Let $\{p\} := A_b \cap B_a \cap C_a$ and let $X := A_b \cap B_c \cap C_a$. These are shown in the Venn diagrams below. Let $Y := A_c \cap C_a$, then since $|A_c \cap B_c \cap C_a| = 0$, $Y \subseteq B_a$. And let $Z := B_c \cap C_b$, then since $|A_b \cap B_c \cap C_b| = 0$, $Z \subseteq A_c$. These are also shown in the Venn diagrams below.



Now, if $|Y| \geq 2$ and $|Z| \geq 2$, then $|B_a \cap C_a| \geq 4$ and Lemma 3.4 tells us that $\lambda_{M\backslash b,c}(B_c \cap C_b) = 2$. But then $M\backslash b$ and $M\backslash c$ are 3–connected so $b \in \mathrm{cl}^*_M(Z \cup \{c\})$, thus $b \in \mathrm{cl}^*_M(A_c)$ contradicting Lemma 3.2 which states that $b \in \mathrm{cl}_M(A_b - \{b\})$. As a consequence, we see that either $|Y| = |A_c \cap C_a| \leq 1$ or $|Z| = |B_c \cap C_b| \leq 1$. And it follows from Lemma 3.6, that Theorem 3.1 is satisfied. □

Lemma 3.10. *Let M be an internally 4–connected matroid with a triangle $\{a, b, c\}$. And let (A_b, A_c), (B_a, B_c) and (C_a, C_b) be the meatiest 3–separations of $M\backslash a$, $M\backslash b$ and $M\backslash c$ respectively, such that each of $A_b \cap B_c$, $A_c \cap B_a$, $A_b \cap C_a$, $A_c \cap C_b$, $B_a \cap C_b$ and $B_c \cap C_a$ have at least two elements. If $\lambda_1 \geq 2$, $\lambda_2 \leq 1$ and $\nu_{1a} = \nu_{1b} = \nu_{1c} = 0$, then Theorem 3.1 is satisfied.*

Proof. We see from the proof of Lemma 3.7, that since $\lambda_1 \geq 2$ and $\nu_{1a} = \nu_{1b} = \nu_{1c} = 0$, $A_b \cap B_c \cap C_a = A_b \cap B_c = A_b \cap C_a = B_c \cap C_a$. Let $X := A_b \cap B_c \cap C_a$. Now, since $\nu_{1a} = |A_b \cap B_a \cap C_a| = 0$, $A_b \cap B_a \subseteq C_b$, and since $\nu_{1b} = |A_b \cap B_c \cap C_b| = 0$, $A_b \cap C_b - \{b\} \subseteq B_a$, so by simple set theory, $A_b \cap B_a = A_b \cap C_b - \{b\}$. Let $Y := A_b \cap B_a$. Also, since $\nu_{1c} = |A_c \cap B_c \cap C_a| = 0$, $A_c \cap C_a \subseteq B_a$, and since $\nu_{1a} = |A_b \cap B_a \cap C_a| = 0$, $B_a \cap C_a - \{a\} \subseteq A_c$, and thus $A_c \cap C_a = B_a \cap C_a - \{a\}$. Let $Z := A_c \cap C_a$. By a similar argument we see that $B_c \cap C_b = A_c \cap B_c - \{c\}$. Let $W := B_c \cap C_b$. These are all shown below on the Venn diagrams.



Suppose that $A_b \cap B_a$ and $B_c \cap C_b$ have at least two elements each. If $A_b \cap B_a$ is 2–separating in $M\backslash a, b$, then since $M\backslash b$ is 3–connected, $a \in \mathrm{cl}^*_{M\backslash b}(Y)$. Hence $a \in \mathrm{cl}^*_M(Y \cup \{b\})$. But $Y \cup \{b\} \subseteq C_b$ so $a \notin \mathrm{cl}(C_a - \{a\})$ contradicting Lemma 3.2. As a result, we see that $\lambda_{M\backslash a,b}(A_b \cap B_a) \geq 3$, and Lemma 3.4 then tells us that $\{c\} \cup W$ is 3–separating in M. Now, M is internally 4–connected so $\{c\} \cup W$ is either a triangle or a triad. And $c \in \mathrm{cl}_M(\{a, b\})$, so $c \notin \mathrm{cl}^*_M(W)$, thus $\{c\} \cup W$ is a triangle of M. But $W \subseteq C_b$, which means that $c \in \mathrm{cl}_M(C_b)$ contradicting the fact that $(C_a, C_b \cup \{c\})$ is not a 3–separation of M. It follows that either $|A_b \cap B_a| \leq 1$ or $|B_c \cap C_b| \leq 1$, and Lemma 3.6 then tells us that Theorem 3.1 holds. □

Proof of Theorem 3.1. Let M be an internally 4–connected matroid with a triangle $\{a, b, c\}$. And let (A_b, A_c), (B_a, B_c) and (C_a, C_b) be the meatiest 3–separations of $M\backslash a$, $M\backslash b$ and $M\backslash c$ respectively. By Lemma 3.6, if any of $A_b \cap B_c$, $A_c \cap B_a$, $A_b \cap C_a$, $A_c \cap C_b$, $B_a \cap C_b$ and $B_c \cap C_a$ has less than two elements, then Theorem 3.1 holds. Hence we now assume that each of $A_b \cap B_c$, $A_c \cap B_a$, $A_b \cap C_a$, $A_c \cap C_b$, $B_a \cap C_b$ and $B_c \cap C_a$ has at least two elements. By Lemma 3.7, we know that if $\lambda_1 \geq 2$ and $\lambda_2 \geq 2$, then $|E(M)| \leq 9$ so Theorem 3.1 holds. Also from the proof of Lemma 3.7 and the symmetry of the situation, we may now assume that one of the following holds, (1) $\lambda_1 \leq 1$ and $\lambda_2 \leq 1$; or (2) $\lambda_2 \leq 1$, $\lambda_1 = 2$ and $\nu_{1a} = 1$; or (3) $\lambda_2 \leq 1$, $\lambda_1 \geq 2$ and $\nu_{1a} = \nu_{1b} = \nu_{1c} = 0$ In case (1), Lemma 3.8 tells us that Theorem 3.1 holds; in case (2), Lemma 3.9 tells us that Theorem 3.1 holds; and in case (3), Lemma 3.10 tells us that Theorem 3.1 holds. □

4. Separators of Size 4

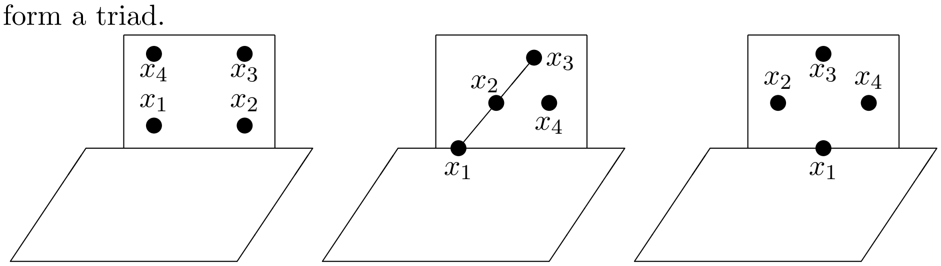
In this section we deal with the case where the matroid is 4–connected up to separators of size 4. A *segment* in a matroid M is a subset A of $E(M)$ with the property that every 3–element subset of A is a triangle. A *cosegment* is a subset of $E(M)$ that is a segment in the dual matroid M^*.

Lemma 4.1. *If a 3–connected matroid M is 4–connected up to 3–separators of size k and contains a 4–element segment or cosegment, then there is an* *element $x \in E(M)$ such that $M\backslash x$ or M/x is 3–connected and 4–connected up to 3–separators of size k.*

Proof. Suppose M contains a 4–element segment. Let x be an element of the segment. Then it is easily checked that $M\backslash x$ is 3–connected. Let (X, Y) be a 3–separation of $M\backslash x$. We can assume that X contains two elements of the segment, so $x \in \mathrm{cl}(X)$. Then $(X \cup \{x\}, Y)$ is a 3–separation of M, so $|X \cup \{x\}| \leq k$ or $|Y| \leq k$ as required. The case where M has a 4–element cosegment follows by duality. □

Theorem 4.2. *Let M be a 3–connected matroid with more than nine elements. If M is 4–connected up to 3–separators of size 4 and contains a 3–separator of size 4, then there is an element $x \in E(M)$ such that $M\backslash x$ or M/x is 3–connected and 4–connected up to 3–separators of size 5.*

Proof. By Lemma 4.1, we can assume that M does not have a 4–element segment or cosegment, so by duality M contains one of the three following structures. The first structure is a quad. It is a 4–element circuit–cocircuit. The second structure is a 4–element fan. The elements $\{x_1, x_2, x_3\}$ form a triangle, while the elements $\{x_2, x_3, x_4\}$ form a triad. The third structure is a type–4 3–separator. It is a 4–element circuit where the elements $\{x_2, x_3, x_4\}$ form a triad.



It is easily checked that $M\backslash x_1$ is 3–connected for each of these structures. Let $T := \{x_2, x_3, x_4\}$. In each case, T is a triad in $M\backslash x_1$ and $x_1 \in \mathrm{cl}(T)$. Now let (X, Y) be a 3–separation of $M\backslash x_1$ with $|X \cap T| \geq 2$. Then, since T is a triad in $M\backslash x_1$, $(X \cup T, Y - T)$ is a 3–separation in $M\backslash x_1$, and, since $x_1 \in \mathrm{cl}(T)$, $(X \cup T \cup \{x_1\}, Y - T)$ is a 3–separation of M. Thus $|X| \leq 2$ or $|Y| \leq 5$, as required. □

5. Proof of Main Theorem

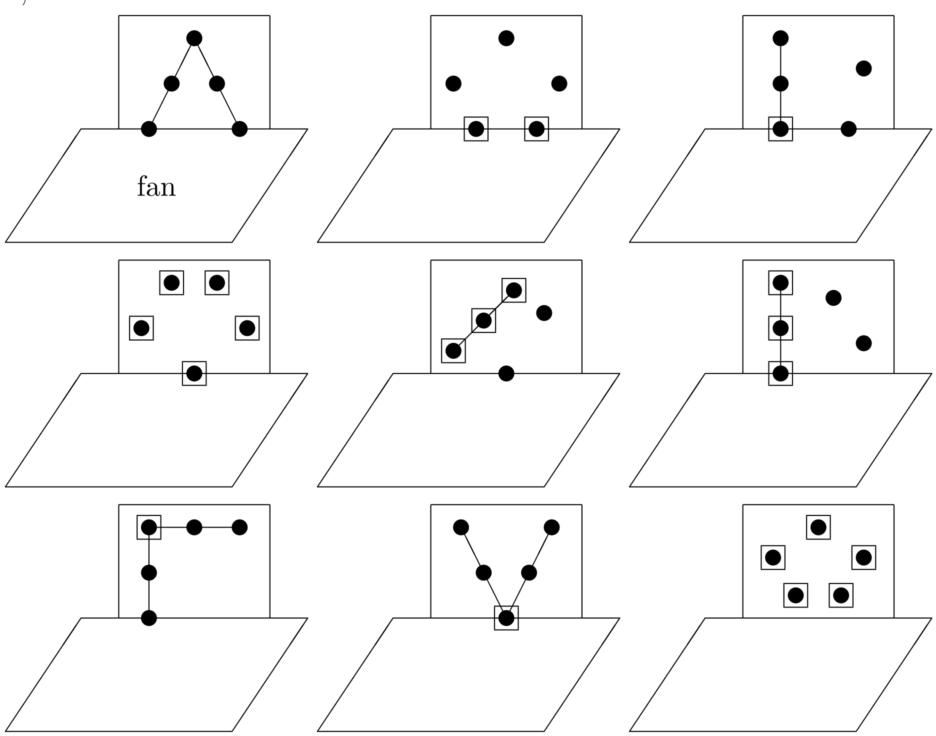
In this section we prove Theorem 1.2. The following theorem of Tutte is from [8].

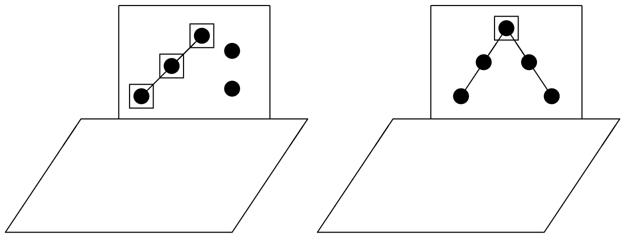
Theorem 5.1 (Wheels and Whirls Theorem). *If M is a 3–connected matroid that is neither a wheel nor a whirl, then M has an element x such that either $M\backslash x$ or M/x is 3–connected.*

If the 3–connected matroid M has at most 12 elements then by Theorem 5.1, for some $x \in E(M)$, either $\mathrm{co}(M\backslash x)$ or $\mathrm{si}(M/x)$ is 3–connected with cardinality $|E(M)| - 1$ or $|E(M)| - 2$. Furthermore, since this minor can have at most eleven elements, it is automatically 4–connected up to 3–separators of size 5. Now, if M is 4–connected, internally 4–connected, or 4–connected up to separators of size 4, then by Corollary 2.3, Theorem 3.1, and Theorem 4.2, there is an element $x \in E(M)$ such that $M\backslash x$ or M/x is 3–connected and 4–connected up to 3–separators of size 5. As a result, from here on we are interested in matroids that have at least 13 elements and have a 5–element 3–separator, and by Lemma 4.1 we can assume that they don't contain a 4–element segment or cosegment. It is easily checked that such a 3–separator A has rank 3 or rank 4. Using the equation $\mathrm{r}^*(X) = |X| - \mathrm{r}(M) + \mathrm{r}(E - X)$ we see that $\mathrm{r}(A) = 3$ if and only if $\mathrm{r}^*(A) = 4$. So by duality we can assume that $\mathrm{r}(A) = 3$.

Lemma 5.2. *Let M be a 3–connected matroid that is 4–connected up to 3–separators of size 5, and has a cardinality of at least 16. If A is a 5–element 3–separator, then there is some $x \in A$ such that $\mathrm{co}(M\backslash x)$ or $\mathrm{si}(M/x)$ is 3–connected and 4–connected up to 3–separators of size 5, and has a cardinality of $|E(M)| - 1$ or $|E(M)| - 2$.*

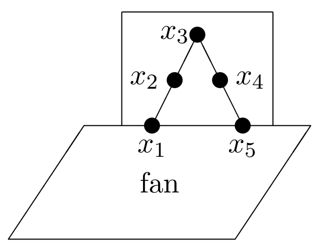
Proof. From the paragraph above, we may assume that the 5–element 3–separator, A, has a rank of 3. Then there are eleven possible structures for A, shown below.





For each 3–separator *except* for the fan, let x be one of the elements with a box around it. Then it is easily checked that (i) $M\backslash x$ is 3–connected, and (ii) $A - x$ does not contain a triangle. Suppose that $M\backslash x$ has a 3–separation (X, Y) where $|X|, |Y| \geq 6$. Then $x \notin \mathrm{cl}(X)$ and $x \notin \mathrm{cl}(Y)$ since M is 4–connected up to 3–separators of size 5. But $\mathrm{r}(A) = 3$ and $A - \{x\}$ has no triangle so $|A \cap X| = |A \cap Y| = 2$. Also $M\backslash x$ is 3–connected so $\lambda_{M\backslash x}(A \cap X) = \lambda_{M\backslash x}(A \cap Y) = 3$. It follows that $X \cap (E(M) - A)$ and $Y \cap (E(M) - A)$ are 3–separators of $M\backslash x$, and since $x \in \mathrm{cl}(A - \{x\})$, they are 3–separators of M. But M is 4–connected up to 3–separators of size 5 so $|X \cap (E(M) - A)| \leq 5$ and $|Y \cap (E(M) - A)| \leq 5$, hence $|E(M)| \leq 15$. This contradiction shows that $M\backslash x$ is 3–connected and 4–connected up to 3–separators of size 5.

Now consider the fan with elements labelled $x_1, \ldots, x_5$ as shown below. It follows from results in [6] that $M\backslash x_1$, $M\backslash x_5$, and $\mathrm{co}(M\backslash x_3)$ are 3–connected, with $|\mathrm{co}(M\backslash x_3)| = |E(M)| - 2$.



Suppose that $M\backslash x_1$ has a 3–separation (X, Y) where $|X|, |Y| \geq 6$. Then $x_1 \notin \mathrm{cl}(X)$ and $x_1 \notin \mathrm{cl}(Y)$. $M\backslash x_1$ is 3–connected so if $|X \cap A| = |Y \cap A| = 2$ then we use the previous argument to show $|E(M)| \leq 15$. So we can assume that $x_2 \in X$ and $\{x_3, x_4, x_5\} \subseteq Y$. Now, $x_2 \in \mathrm{cl}^*_{M\backslash x_1}(Y)$ and $x_1 \in \mathrm{cl}_M(Y \cup \{x_2\})$ so $X - \{x_2\}$ is a 3–separator of M, so $|X - \{x_2\}| \leq 5$. But $|X| \geq 6$ so $|X| = 6$.

Now consider $M\backslash x_3$. Suppose that $M\backslash x_3$ has a 3–separation (C, D) where $|C|, |D| \geq 6$. Then $x_3 \notin \mathrm{cl}(C)$ and $x_3 \notin \mathrm{cl}(D)$ so $|A \cap C| = |A \cap D| = 2$. If $\lambda_{M\backslash x_3}(A \cap C) = \lambda_{M\backslash x_3}(A \cap D) = 3$, then we can use the previous argument to show $|E(M)| \leq 15$. So we can assume that $\lambda_{M\backslash x_3}(A \cap D) = 2$ so that $A \cap C = \{x_1, x_5\}$ and $A \cap D = \{x_2, x_4\}$. Now, $\mathrm{r}(C \cup \{x_2, x_4\}) = \mathrm{r}(C) + 1$ and $\mathrm{r}(D - \{x_2, x_4\}) \leq \mathrm{r}(D) - 1$ so $D - \{x_2, x_4\}$ is a 3–separator of $M\backslash x_3$. And $x_3 \in \mathrm{cl}(C \cup \{x_2, x_4\})$ hence $D - \{x_2, x_4\}$ is a 3–separator of M, and

$|D - \{x_2, x_4\}| \leq 5$ so $|D| \leq 7$. If $|D| = 6$ then consider $\text{co}(M\backslash x_3)$ and let (C, D') be the resulting 3–separation of $\text{co}(M\backslash x_3)$. Then $|D'| = 5$ so $\text{co}(M\backslash x_3)$ is 4–connected up to 3–separators of size 5. As a result, we can assume that $|D| = 7$.

5.2.1. *$x_2 \in \text{cl}(D - \{x_2\})$ and $x_4 \in \text{cl}(D - \{x_4\})$.*

Proof. Suppose that $x_2 \notin \text{cl}(D - \{x_2\})$, then $(C \cup \{x_2\}, D - \{x_2\})$ is a 3–separation of $M\backslash x_3$. So $(C \cup \{x_2, x_3\}, D - \{x_2\})$ is a 3–separation of M. But $x_4 \in \text{cl}(C \cup \{x_2, x_3\})$ and $x_4 \in \text{cl}^*(C \cup \{x_2, x_3\})$ so $D - \{x_2, x_4\}$ is a 2–separator of M. This is a contradiction as M is 3–connected, so we see that $x_2 \in \text{cl}(D - \{x_2\})$ and similarly $x_4 \in \text{cl}(D - \{x_4\})$. □

Now we compare the 3–separators X and D. Let $X' = X \cap (E(M) - A)$, $Y' = Y \cap (E(M) - A)$, $C' = C \cap (E(M) - A)$ and $D' = D \cap (E(M) - A)$. Then $|X'|, |D'| = 5$, and they are both 3–separators of M. There are four cases to consider. They are (1) $X' = D'$, (2) $2 \leq |X' \cap D'| \leq 4$, (3) $|X' \cap D'| = 1$, and (4) $|X' \cap D'| = 0$.

(1) If $X' = D'$ then $Y' = C'$. But since $x_1 \notin \text{cl}(Y)$, we have $x_1 \notin \text{cl}(C - \{x_1\})$ so $(D \cup \{x_1\}, C - \{x_1\})$ is a 3–separation of $M\backslash x_3$, and hence $(D \cup \{x_1, x_3\}, C - \{x_1\})$ is a 3–separation of M. This implies that $|C - \{x_1\}| \leq 5$ so $|E(M)| \leq 14$.

(2) If $2 \leq |X' \cap D'| \leq 4$ then $X' \cup D'$ is a 3–separator of M. But $6 \leq |X' \cup D'| \leq 8$ so $|E - (X' \cup D')| \leq 5$ hence $|E(M)| \leq 13$.

(3) If $|X' \cap D'| = 1$ and $X' \cap D' = \{e\}$ then either (i) $e \in \text{cl}(X' - \{e\})$ and $e \in \text{cl}(D' - \{e\})$, or (ii) $e \in cl^*_M(X' - \{e\})$ and $e \in \text{cl}^*_M(D' - \{e\})$. So $(X - \{e\}, Y \cup \{e\})$ is a 3–separation of $M\backslash x_1$ with $D - \{x_2\} \subseteq Y \cup \{e\}$. But since $x_2 \in \text{cl}(D - \{x_2\})$, we have $x_2 \in \text{cl}(Y \cup \{e\})$. And $x_1 \in \text{cl}(Y \cup \{e, x_2\})$ so $x_1 \in \text{cl}(Y \cup \{e\})$. Therefore $X - \{e\}$ is a 3–separator of M. But $x_2 \in \text{cl}(Y \cup \{e, x_1\})$ and $x_2 \in \text{cl}^*(Y \cup \{e, x_1\})$ so $X - \{e, x_2\}$ is a 2–separator of M. This is a contradiction as M is 3–connected.

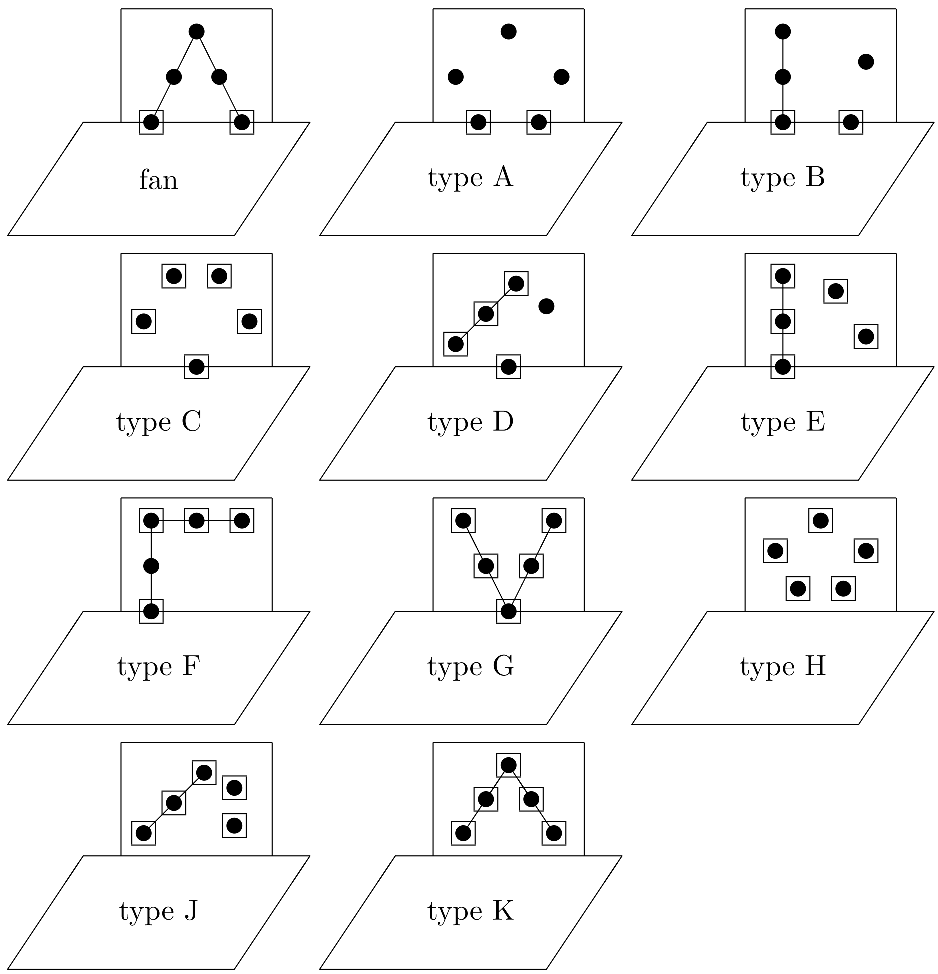
(4) If $|X' \cap D'| = 0$ then $D - \{x_2\} \subseteq Y$. But $x_2 \in \text{cl}(D - \{x_2\})$ so $x_2 \in \text{cl}(Y)$. And $x_1 \in \text{cl}(Y \cup \{x_2\})$ so $x_1 \in \text{cl}(Y)$. This is a contradiction since $x_1 \notin \text{cl}(Y)$.

As a result of the contradictions above, we see that if $|E(M)| \geq 16$, then $M\backslash x_1$ or $\text{co}(M\backslash x_3)$ is 3–connected and 4–connected up to 3–separators of size 5, with $|\text{co}(M\backslash x_3)| = |E(M)| - 2$. □

Now we know that Theorem 1.2 holds for matroids with more than 15 elements. The following argument for matroids with at most 15 elements is just a finite case check.

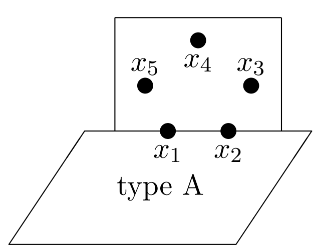
Proof of Theorem 1.2. By the previous lemmas, it suffices to prove that if $13 \leq |E(M)| \leq 15$ and M has a 5–element 3–separator, A, then there is an element $x \in A$ such that $\text{co}(M\backslash x)$ or $\text{si}(M/x)$ is 3–connected and 4–connected up to 3–separators of size 5, with a cardinality of $|E(M)| - 1$ or

$|E(M)| - 2$. It is easily checked that for each of the 3–separators below, if x is one of the elements with a box around it, then $M\backslash x$ is 3–connected (provided M has more than eleven elements).



5.2.2. *Theorem 1.2 is satisfied if M has a type–A 3–separator.*

Proof. We label the elements of the type–A 3–separator $x_1, \ldots, x_5$ as shown below.



Suppose $M\backslash x_1$ has a 3–separation (X, Y) with $|X|, |Y| \geq 6$, then $x_1 \notin \mathrm{cl}(X)$ and $x_1 \notin \mathrm{cl}(Y)$ so without loss of generality $x_4, x_5 \in X$ and $x_2, x_3 \in Y$. Let

$X' = X - \{x_4, x_5\}$ and let $Y' = Y - \{x_2, x_3\}$. As with Lemma 5.2, X' and Y' are 3–separating in M with $4 \leq |X'| \leq 5$, and since $x_3 \in \mathrm{cl}^*(X)$, $Y' \cup \{x_2\}$ is 3–separating in $M \backslash x_1$. Now, $x_1 \in \mathrm{cl}(X \cup \{x_3\})$ so $Y' \cup \{x_2\}$ is 3–separating in M and $|Y'| = 4$.

Now suppose that $M \backslash x_2$ has a 3–separation (B, C) where $|B|, |C| \geq 6$. Then $x_2 \notin \mathrm{cl}(B)$ and $x_2 \notin \mathrm{cl}(C)$ so without loss of generality $x_1, x_i \in B$ and $x_j, x_k \in C$ where $\{x_i, x_j, x_k\} = \{x_3, x_4, x_5\}$. Let $B' = B - \{x_1, x_i\}$ and let $C' = C - \{x_j, x_k\}$. As above, C' and B' are 3–separating in M with $4 \leq |C'| \leq 5$ and $|B'| = 4$. Also since $x_1 \in \mathrm{cl}(E(M) - B')$, we have $x_1 \in \mathrm{cl}(B')$. Now, since $x_1 \notin \mathrm{cl}(X')$ and $x_1 \notin \mathrm{cl}(Y')$, $B' \nsubseteq X'$ and $B' \nsubseteq Y'$ so $B' \cap X' \neq \emptyset$ and $B' \cap Y' \neq \emptyset$. We now compare the 3–separators B', X' and Y'. There are two possible cases.

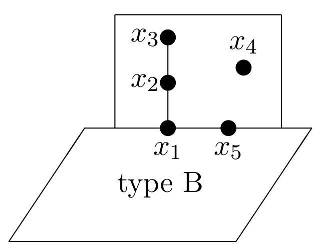
(1) If $|B' \cap Y'| \geq 2$ then $B' \cup Y'$ is 3–separating in M, and since $x_1 \in \mathrm{cl}(B')$, $B' \cup Y' \cup \{x_1\}$ is 3–separating in M. But $6 \leq |B' \cup Y' \cup \{x_1\}| \leq 7$ and $|E(M)| \geq 13$ contradicting that M is 4–connected up to 3–separators of size 5.

(2) If $|B' \cap X'| \geq 2$ then $B' \cup X'$ is 3–separating in M, and as above, $B' \cup X' \cup \{x_1\}$ is 3–separating in M. But either $6 \leq |B' \cup X'| \leq 7$ or $6 \leq |B' \cup X' \cup \{x_1\}| \leq 7$ contradicting that $|E(M)| \geq 13$ and that M is 4–connected up to 3–separators of size 5.

These contradictions show that either $M \backslash x_1$ or $M \backslash x_2$ is 4–connected up to 3–separators of size 5. □

5.2.3. *Theorem 1.2 is satisfied if M has a type–B 3–separator.*

Proof. We label the elements of the type–B 3–separator $x_1, \ldots, x_5$ as shown below.



Suppose that $M \backslash x_1$ has a 3–separation (X, Y) where $|X|, |Y| \geq 6$. Then $x_1 \notin \mathrm{cl}(X)$ and $x_1 \notin \mathrm{cl}(Y)$ so without loss of generality $x_2, x_5 \in X$ and $x_3, x_4 \in Y$. Let $X' = X - \{x_2, x_5\}$ and let $Y' = Y - \{x_3, x_4\}$. Then as with 5.2.2, X', $X' \cup \{x_5\}$, and Y' are 3–separators of M with $4 \leq |Y'| \leq 5$, and $|X'| = 4$ and $x_5 \in (X')$. As with Lemma 5.2, $x_3 \in \mathrm{cl}(Y' \cup \{x_4\})$ and $x_4 \in \mathrm{cl}(Y' \cup \{x_3\})$.

Suppose that $M \backslash x_5$ has a 3–separation (B, C) where $|B|, |C| \geq 6$. Then $x_5 \notin \mathrm{cl}(B)$ and $x_5 \notin \mathrm{cl}(C)$. If $|A \cap B| = |A \cap C| = 2$, then we obtain a similar contradiction to the one in 5.2.2, so we can assume that $x_4 \in B$ and $\{x_1, x_2, x_3\} \subseteq C$. Let $B' = B - \{x_4\}$ and let $C' = C - \{x_1, x_2, x_3\}$. Then since $x_4 \in \mathrm{cl}^*(C)$ and $x_5 \in \mathrm{cl}(C \cup \{x_4\})$, we see that B' is a 3–separator of M and $|B'| = 5$. Now, since $x_5 \in \mathrm{cl}(X')$, we have $X' \nsubseteq B'$ and $X' \nsubseteq C'$ so $X' \cap B' \neq \emptyset$ and $X' \cap C' \neq \emptyset$. We now compare the 3–separators B' and X'. There are two cases to consider.

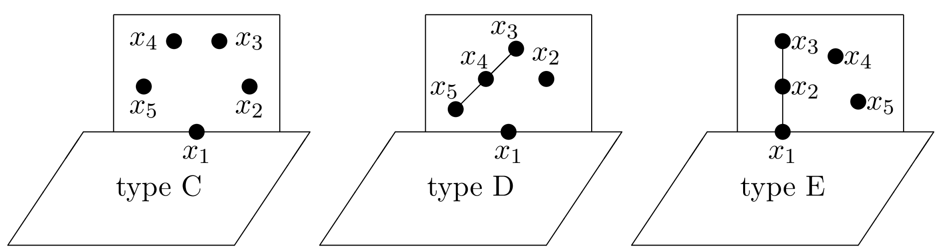
(1) If $|B' \cap X'| \geq 2$ then $B' \cup X'$ is 3–separating in M. But $6 \leq |B' \cup X'| \leq 7$ contradicting that $|E(M)| \geq 13$ and that M is 4–connected up to 3–separators of size 5.

(2) If $|B' \cap X'| = 1$ and $|B' \cap Y'| = 4$ then $B' \cup Y'$ is 3–separating in M. If $|B' \cup Y'| = 6$ then we have the same contradiction as above. If $|B' \cup Y'| = 5$ then $Y' \subset B'$ and $C' \subset X'$, and since $x_3 \in \mathrm{cl}(Y' \cup \{x_4\})$, we have $x_3 \in \mathrm{cl}(B)$. Also $x_2 \in \mathrm{cl}^*(B \cup \{x_3\})$ and $x_5 \in \mathrm{cl}(B \cup \{x_2, x_3\})$ so $C' \cup \{x_1\}$ is 3–separating in M. Now, $x_1 \in \mathrm{cl}(A - \{x_1\})$ so $x_1 \in \mathrm{cl}(C')$. But $C' \subset X'$, hence $x_1 \in \mathrm{cl}(X')$. This is a contradiction since $x_1 \notin \mathrm{cl}(X)$.

As a result, we see that $M\backslash x_1$ or $M\backslash x_5$ is 4–connected up to 3–separators of size 5. □

5.2.4. *Theorem 1.2 is satisfied if M has a type–C, type–D, or type–E 3–separator.*

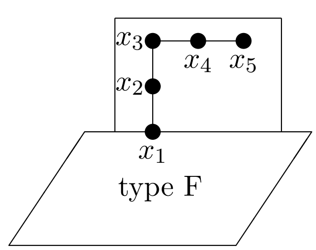
Proof. We label the elements $x_1, \ldots, x_5$ as shown below.



If M has a type–C 3–separator then we look at $M\backslash x_1$ and $M\backslash x_2$ and construct a similar argument to 5.2.2. If M has a type–D 3–separator then we look at $M\backslash x_3$ and $M\backslash x_4$ and construct a similar argument to 5.2.2. If M has a type–E 3–separator then we look at $M\backslash x_1$ and $M\backslash x_2$ and construct a similar argument to 5.2.2. □

5.2.5. *Theorem 1.2 is satisfied if M has a type–F 3–separator.*

Proof. We label the elements $x_1, \ldots, x_5$ as shown below.



Suppose that $M\backslash x_3$ has a 3–separation (X,Y) where $|X|,|Y| \geq 6$. Then $x_3 \notin \mathrm{cl}(X)$ and $x_3 \notin \mathrm{cl}(Y)$ so without loss of generality $x_1, x_4 \in X$ and $x_2, x_5 \in Y$. Let $X' = X - \{x_1, x_4\}$ and let $Y' = Y - \{x_2, x_5\}$. As with 5.2.2, Y', X', and $X' \cup \{x_1\}$ are 3–separating in M with $4 \leq |Y'| \leq 5$, $|X'| = 4$ and $x_1 \in \mathrm{cl}(X')$.

Suppose that $M\backslash x_1$ has a 3–separation (B,C) where $|B|,|C| \geq 6$. Then $x_1 \notin \mathrm{cl}(B)$ and $x_1 \notin \mathrm{cl}(C)$. If $|A \cap B| = |A \cap C| = 2$ then we obtain a similar contradiction to the one in 5.2.2, so we can assume that $\{x_3, x_4, x_5\} \subseteq B$ and $x_2 \in C$. Let $B' = B - \{x_3, x_4, x_5\}$ and let $C' = C - \{x_2\}$. As with 5.2.3, C' is a 3–separator of M and $|C'| = 5$. Since $x_1 \in \mathrm{cl}(X')$, $X' \nsubseteq B'$ and $X' \nsubseteq C'$ so $X' \cap B' \neq \emptyset$ and $X' \cap C' \neq \emptyset$. We now compare the 3–separators X' and C'. There are two possible cases.

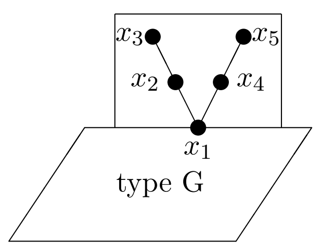
(1) If $|X' \cap C'| \geq 2$, or if $|X' \cap C'| = 1$ and $|Y'| = 5$, then we obtain a similar contradiction to the one in 5.2.2.

(2) If $|X' \cap C'| = 1$ and $|Y'| = 4$, then $Y' \subset C'$ and $B' \subset X'$. Let $\{e\} = C' - Y'$ (or equivalently $\{e\} = X' - B'$), then since C' and Y' are both 3–separators of M, we have $e \in \mathrm{cl}^{(*)}(Y')$. But $X' \cup \{x_1\}$ is a 3–separator of M with $e \in \mathrm{cl}^{(*)}(E(M) - (X' \cup \{x_1\}))$, so $B' \cup \{x_1\}$ is a 3–separator of M. But $x_1 \in \mathrm{cl}(A - \{x_1\})$ so $x_1 \in \mathrm{cl}(B')$ contradicting that $x_1 \notin \mathrm{cl}(B)$.

These contradictions show that either $M\backslash x_1$ or $M\backslash x_3$ is 4–connected up to 3–separators of size 5. □

5.2.6. *Theorem 1.2 is satisfied if M has a type–G 3–separator.*

Proof. We label the elements $x_1, \ldots, x_5$ as shown below.



Suppose that $M\backslash x_1$ has a 3–separation (X,Y) where $|X|,|Y| \geq 6$. Then $x_1 \notin \mathrm{cl}(X)$ and $x_1 \notin \mathrm{cl}(Y)$ so without loss of generality $x_2, x_5 \in X$ and $x_3, x_4 \in Y$. Let $X' = X - \{x_2, x_5\}$ and let $Y' = Y - \{x_3, x_4\}$. Then as with 5.2.2, X' and Y' are 3–separators of M with $4 \leq |X'|, |Y'| \leq 5$. As with Lemma 5.2, $x_3 \in \mathrm{cl}(Y' \cup \{x_4\})$ and $x_2 \in \mathrm{cl}(X' \cup \{x_5\})$.

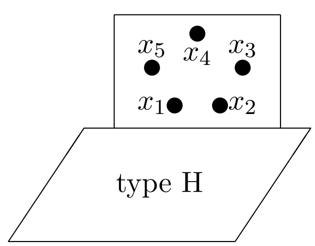
Suppose that $M\backslash x_3$ has a 3–separation (B,C) where $|B|,|C| \geq 6$. Then $x_3 \notin \mathrm{cl}(B)$ and $x_3 \notin \mathrm{cl}(C)$. If $|A \cap B| = |A \cap C| = 2$ then we obtain a similar contradiction to the one in 5.2.2, so we can assume that $\{x_1, x_4, x_5\} \subseteq C$ and $x_2 \in B$. Let $B' = B - \{x_2\}$ and let $C' = C - \{x_1, x_4, x_5\}$. Then as with 5.2.3, B' is 3–separating in M and $|B'| = 5$. We now compare the 3–separators X', Y', and B'. There are three cases to check.

(1) If $X' \nsubseteq B'$ and $Y' \nsubseteq B'$ then we obtain a similar contradiction to the one in 5.2.2.
(2) If $X' \subseteq B'$ then $C' \subseteq Y'$. If $C' = Y'$ then $x_3 \in \mathrm{cl}(C' \cup \{x_4\})$ contradicting that $x_3 \notin \mathrm{cl}(C)$. So we see that $C' \subsetneq Y'$ and $X' \subsetneq B'$. Let $\{e\} = B' - X'$ (or equivalently $\{e\} = Y' - C'$). Since X' and B' are both 3–separators of M, we have $e \in \mathrm{cl}^{(*)}(X')$. And since Y' is a 3–separator with $e \in \mathrm{cl}^{(*)}(E(M) - Y')$, we see that C' is also a 3–separator with $e \in \mathrm{cl}^{(*)}(C')$. But then $(Y' \cup \{x_1, x_4, x_5\}, X' \cup \{x_2\})$ is a 3–separation of $M \backslash x_3$ with $x_3 \in \mathrm{cl}(Y' \cup \{x_4\})$. So $X' \cup \{x_2\}$ is a 3–separator of M. But $x_2 \in \mathrm{cl}(A - \{x_2\})$ and $x_2 \in \mathrm{cl}^*(A - \{x_2\})$ so X' is a 2–separator of M contradicting the fact that M is 3–connected.
(3) If $Y' \subseteq B'$ then $C' \subseteq X'$. If $C' = X'$ then $x_2 \in \mathrm{cl}(C' \cup \{x_5\})$ so $x_2 \in \mathrm{cl}(C)$. But $x_3 \in \mathrm{cl}(C \cup \{x_2\})$ so $x_3 \in \mathrm{cl}(C)$ contradicting the fact that $x_3 \notin \mathrm{cl}(C)$, so we see that $C' \subsetneq X'$ and $Y' \subsetneq B'$. Let $\{e\} = B' - Y'$ (or equivalently $\{e\} = X' - C'$). Then as above, we have $e \in \mathrm{cl}^{(*)}(C')$ so $(X' \cup \{x_1, x_4, x_5\}, Y' \cup \{x_2\})$ is a 3–separation of $M \backslash x_3$ with $x_2 \in \mathrm{cl}_{M \backslash x_3}(X' \cup \{x_5\})$ and $x_2 \in \mathrm{cl}^*_{M \backslash x_3}(\{x_4, x_5\})$. Hence Y' is a 2–separator of $M \backslash x_3$ contradicting that $M \backslash x_3$ is 3–connected.

From the contradictions above, we see that either $M \backslash x_1$ or $M \backslash x_3$ is 4–connected up to 3–separators of size 5. □

5.2.7. *Theorem 1.2 holds if M has a type–H 3–separator.*

Proof. We label the elements $x_1, \ldots, x_5$ as shown below.



Suppose that $M \backslash x_1$ has a 3–separation (X, Y) where $|X|, |Y| \geq 6$. Then $x_1 \notin \mathrm{cl}(X)$ and $x_1 \notin \mathrm{cl}(Y)$ so without loss of generality $x_2, x_3 \in X$ and $x_4, x_5 \in Y$. Let $X' = X - \{x_2, x_3\}$ and let $Y' = Y - \{x_4, x_5\}$. Then as with 5.2.2, X' and Y' are 3–separators of M with $4 \leq |X'|, |Y'| \leq 5$. As with Lemma 5.2, $x_3 \in \mathrm{cl}(X' \cup \{x_2\})$, $x_2 \in \mathrm{cl}(X' \cup \{x_3\})$, $x_4 \in \mathrm{cl}(Y' \cup \{x_5\})$, and $x_5 \in \mathrm{cl}(Y' \cup \{x_4\})$.

Suppose that $M \backslash x_2$ has a 3–separation (B, C) where $|B|, |C| \geq 6$. Then $x_2 \notin \mathrm{cl}(B)$ and $x_2 \notin \mathrm{cl}(C)$ so $x_i, x_j \in C$ and $x_k, x_l \in B$ where $\{x_i, x_j, x_k, x_l\} = \{x_1, x_3, x_4, x_5\}$. Let $C' = C - \{x_i, x_j\}$ and let $B' = B - \{x_k, x_l\}$. Then as with 5.2.2, B' and C' are 3–separating in M with $4 \leq |B'|, |C'| \leq 5$. As with Lemma 5.2, $x_i \in \mathrm{cl}(C' \cup \{x_j\})$, $x_j \in \mathrm{cl}(C' \cup \{x_i\})$, $x_k \in \mathrm{cl}(B' \cup \{x_l\})$, and $x_l \in \mathrm{cl}(B' \cup \{x_k\})$. We now compare the 3–separators X', Y', B', and

C' to show that we may assume that $X' = B'$ and $Y' = C'$. There are three cases to check.

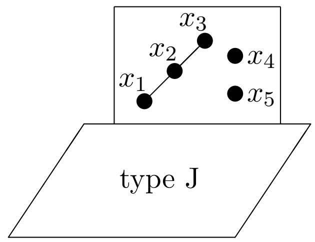
(1) If $|X'| = |Y'| = 5$, then we must have $B' = X'$ or $B' = Y'$ otherwise we obtain a similar contradiction to the one in 5.2.2.
(2) If $|X'| = 4$ and $|Y'| = 5$ then we can assume that $|C'| = 4$ and $|B'| = 5$. Either $C' \subseteq Y'$ or $B' = Y'$ otherwise we obtain a similar contradiction to the one in 5.2.2. And if $C' \subseteq Y'$ with $\{e\} = Y' - C'$, then as with 5.2.6, $e \in \mathrm{cl}^{(*)}(C')$ so $(Y' \cup \{x_i, x_j\}, X' \cup \{x_k, x_l\})$ is a 3–separation of $M\backslash x_2$ where $|Y' \cup \{x_i, x_j\}|, |X' \cup \{x_k, x_l\}| \geq 6$. A similar argument applies if $|X'| = 5$ and $|Y'| = 4$.
(3) If $|X'| = |Y'| = 4$, then $|B' \cap X'| \neq 2$ otherwise we obtain a similar contradiction to the one in 5.2.2. And if $|B' \cap X'| = 3$ then $B' \cup X'$ is 3–separating in M with $|B' \cup X'| = 5$. Let $\{e\} = B' - X'$ and $\{f\} = X' - B'$. Then since X' and $X' \cup \{e\}$ are 3–separators of M, we have $e \in \mathrm{cl}^{(*)}(X')$. And since $e \in \mathrm{cl}^{(*)}(X')$, we have $Y' - \{e\}$ is a 3–separator of M with $e \in \mathrm{cl}^{(*)}(Y' - \{e\})$. Similarly, $e \in \mathrm{cl}^{(*)}(B' - \{e\})$, $f \in \mathrm{cl}^{(*)}(X' - \{f\})$, and $f \in \mathrm{cl}^{(*)}(C' - \{f\})$. So we see that $(X' \cup \{x_k, x_l\}, Y' \cup \{x_i, x_j\})$ is a 3–separation of $M\backslash x_2$ where $|X' \cup \{x_k, x_l\}|, |Y' \cup \{x_i, x_j\}| \geq 6$. A similar argument applies if $|B' \cap Y'| = 3$.

The upshot of the three arguments above is that we can assume without loss of generality that $X' = B'$ and $Y' = C'$. Now, $M\backslash x_1$ has the 3–separation $(X' \cup \{x_2, x_3\}, Y' \cup \{x_4, x_5\})$ where $x_1 \notin \mathrm{cl}(X' \cup \{x_2, x_3\})$ and $x_1 \notin \mathrm{cl}(Y' \cup \{x_4, x_5\})$, and $M\backslash x_2$ has the 3–separation $(X' \cup \{x_k, x_l\}, Y' \cup \{x_i, x_j\})$ where $x_2 \notin \mathrm{cl}(X' \cup \{x_k, x_l\})$ and $x_2 \notin \mathrm{cl}(Y' \cup \{x_i, x_j\})$. We see from $M\backslash x_1$ that $x_2 \in \mathrm{cl}(X' \cup \{x_3\})$ so $x_k \neq x_3$ and $x_l \neq x_3$. Suppose that $Y' \cup \{x_i, x_j\} = Y' \cup \{x_3, x_4\}$, then $x_3 \in \mathrm{cl}(Y' \cup \{x_4\})$ and $x_1 \in \mathrm{cl}(Y' \cup \{x_3, x_4, x_5\})$, and it follows that $x_1 \in \mathrm{cl}(Y' \cup \{x_4, x_5\})$. This contradicts the fact that $x_1 \notin \mathrm{cl}(Y' \cup \{x_4, x_5\})$, so we see that $\{x_i, x_j\} \neq \{x_3, x_4\}$. Similarly $\{x_i, x_j\} \neq \{x_3, x_5\}$ so the 3–separation of $M\backslash x_2$ must be $(X' \cup \{x_4, x_5\}, Y' \cup \{x_1, x_3\})$.

Now consider $M\backslash x_5$, and suppose that it is not 4–connected up to 3–separators of size 5. Then a similar argument to the one above shows that $M\backslash x_5$ has a 3–separation of the form $(X' \cup \{x_m, x_n\}, Y' \cup \{x_p, x_q\})$ where $\{x_m, x_n, x_p, x_q\} = \{x_1, x_2, x_3, x_4\}$ and $x_5 \notin \mathrm{cl}(X' \cup \{x_m, x_n\})$ and $x_5 \notin \mathrm{cl}(Y' \cup \{x_p, x_q\})$. Then from $M\backslash x_1$ we know that $x_5 \in \mathrm{cl}(Y' \cup \{x_4\})$, so $x_4 \notin \{x_p, x_q\}$. But from $M\backslash x_2$ we know that $x_5 \in \mathrm{cl}(X' \cup \{x_4\})$ so $x_4 \notin \{x_m, x_n\}$. This is a contradiction since $x_4 \in \{x_m, x_n, x_p, x_q\}$, so we see that one of $M\backslash x_1$, $M\backslash x_2$ and $M\backslash x_5$ is 4–connected up to 3–separators of size 5. □

5.2.8. *Theorem 1.2 holds if M has a type–J 3–separator.*

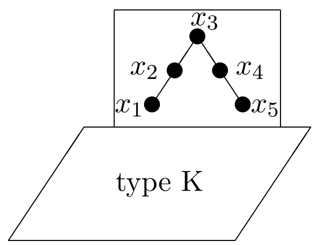
Proof. We label the elements $x_1, \dots, x_5$ as shown below. If we look at $M\backslash x_1$, $M\backslash x_2$ and $M\backslash x_3$ then we obtain a similar proof to 5.2.7.



□

5.2.9. *Theorem 1.2 holds if M has a type–K 3–separator.*

Proof. We label the elements $x_1, \dots, x_5$ as shown below.



Suppose that $M\backslash x_3$ has a 3–separation (X,Y) where $|X|,|Y| \geq 6$. Then $x_3 \notin \mathrm{cl}(X)$ and $x_3 \notin \mathrm{cl}(Y)$ so without loss of genarality $x_1, x_4 \in X$ and $x_2, x_5 \in Y$. Let $X' = X - \{x_1, x_4\}$ and let $Y' = Y - \{x_2, x_5\}$. Then as with 5.2.2, X' and Y' are 3–separators of M with $4 \leq |X'|,|Y'| \leq 5$. As with Lemma 5.2, we have $x_1 \in \mathrm{cl}(X' \cup \{x_4\})$, $x_4 \in \mathrm{cl}(X' \cup \{x_1\})$, $x_2 \in \mathrm{cl}(Y' \cup \{x_5\})$, and $x_5 \in \mathrm{cl}(Y' \cup \{x_2\})$.

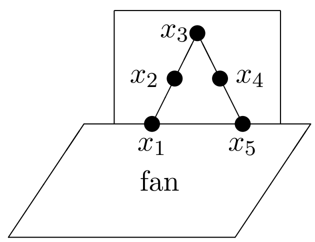
Suppose that $M\backslash x_1$ has a 3–separation (B,C) where $|B|,|C| \geq 6$. Then $x_1 \notin \mathrm{cl}(B)$ and $x_1 \notin \mathrm{cl}(C)$ so we have two possibilities. In the first case $x_2, x_i \in C$ and $x_3, x_j \in B$ where $\{x_i, x_j\} = \{x_4, x_5\}$, and in the second case $x_2 \in C$ and $\{x_3, x_4, x_5\} \subseteq B$. We consider the first case. Let $C' = C - \{x_2, x_i\}$ and let $B' = B - \{x_3, x_j\}$. As with 5.2.2, B' and C' are 3–separators of M with $4 \leq |B'|,|C'| \leq 5$. Then by a similar argument to the one in 5.2.7, we can assume that $C' = X'$ or $C' = Y'$. Suppose that $x_i = x_4$ and $x_j = x_5$ so that our 3–separation of $M\backslash x_1$ is $(C' \cup \{x_2, x_4\}, B' \cup \{x_3, x_5\})$. Now, $x_2 \notin \mathrm{cl}(B' \cup \{x_3, x_5\})$ otherwise x_1 would be in $\mathrm{cl}(B' \cup \{x_3, x_5\})$, and $x_5 \notin \mathrm{cl}(C' \cup \{x_2, x_4\})$ otherwise x_1 would be in $\mathrm{cl}(C' \cup \{x_2, x_4\})$. Then from $M\backslash x_3$, $x_2 \in \mathrm{cl}(Y' \cup \{x_5\})$ so $Y' \neq B'$. But $x_5 \in \mathrm{cl}(Y' \cup \{x_2\})$ so $Y' \neq C'$. This contradiction shows us that $x_i \neq x_4$ and $x_j \neq x_5$. So we see that $x_i = x_5$ and $x_j = x_4$. From $M\backslash x_3$, we see that $x_1 \in \mathrm{cl}(X' \cup \{x_4\})$ so $X' \neq B'$, and our 3–separation in $M\backslash x_1$ is $(X' \cup \{x_2, x_5\}, Y' \cup \{x_3, x_4\})$. Now, $x_5 \in \mathrm{cl}(\{x_3, x_4\})$ and $x_2 \in \mathrm{cl}(Y' \cup \{x_5\})$ so $x_2 \in \mathrm{cl}(Y' \cup \{x_3, x_4\})$. But $x_1 \in \mathrm{cl}(Y' \cup \{x_2, x_3, x_4\})$ so $x_1 \in \mathrm{cl}(Y' \cup \{x_3, x_4\})$ contradicting that $x_1 \notin \mathrm{cl}(B)$. Therefore, it is not the case that $x_2, x_i \in C$ and $x_3, x_j \in B$.

Now we consider the second case where $x_2 \in C$ and $\{x_3, x_4, x_5\} \subseteq B$. Let $C' = C - \{x_2\}$ and let $B' = B - \{x_3, x_4, x_5\}$, then by a similar argument to 5.2.3, C' is 3–separating in M and $|C'| = 5$. If $X' \nsubseteq C'$ and $Y' \nsubseteq C'$ then we obtain a similar contradiction to the one in 5.2.2, so we see that either $X' \subseteq C'$ or $Y' \subseteq C'$. If $Y' \subseteq C'$ then from $M\backslash x_3$ we know that $x_5 \in \mathrm{cl}(Y' \cup \{x_2\})$ so $x_5 \in \mathrm{cl}(C' \cup \{x_2\})$. And as with Lemma 5.2, we have $x_3 \in \mathrm{cl}(B' \cup \{x_4\})$. But $B' \subseteq X'$ so $x_3 \in \mathrm{cl}(X' \cup \{x_4\})$ contradicting that $x_3 \notin \mathrm{cl}(X)$. So we see that $X' \subseteq C'$. Now, either $C' - X' = \emptyset$ or $C' - X' = \{e\}$ for some $e \in E(M)$. If $C' - X' = \{e\}$ then by a similar argument to 5.2.6, $e \in \mathrm{cl}^{(*)}(B')$. In either case, we see that $(X' \cup \{x_2\}, Y' \cup \{x_3, x_4, x_5\})$ is a 3–separation of $M\backslash x_1$. But $x_2 \in \mathrm{cl}^*_{M\backslash x_1}(\{x_3, x_4, x_5\})$ and $x_2 \in \mathrm{cl}(Y' \cup \{x_5\})$ so X' is a 2–separator of $M\backslash x_1$. This contradicts the fact that $M\backslash x_1$ is 3–connected.

As a result of the contradictions above, we see that $M\backslash x_1$ or $M\backslash x_3$ is 4–connected up to 3–separators of size 5. □

5.2.10. *Theorem 1.2 holds if M has a fan.*

Proof. We label the elements of the fan $x_1, \dots, x_5$ as shown below.



Suppose that $M\backslash x_3$ has a 3–separation (X, Y) where $|X|, |Y| \geq 6$. Then $x_3 \notin \mathrm{cl}(X)$ and $x_3 \notin \mathrm{cl}(Y)$. If $x_1, x_4 \in X$ and $x_2, x_5 \in Y$ then it is easily checked that $(X \cup \{x_2\} - \{x_4\}, Y \cup \{x_4\} - \{x_2\})$ is also a 3–separation of $M\backslash x_3$ where $|X \cup \{x_2\} - \{x_4\}|, |Y \cup \{x_4\} - \{x_2\}| \geq 6$. But $x_3 \in \mathrm{cl}(X \cup \{x_2\} - \{x_4\})$ and $x_3 \in \mathrm{cl}(Y \cup \{x_4\} - \{x_2\})$ contradicting that M is 4–connected up to 3–separators of size 5, so we see that $x_1, x_5 \in X$ and $x_2, x_4 \in Y$. Let $X' = X - \{x_1, x_5\}$ and let $Y' = Y - \{x_2, x_4\}$. By the same argument as in Lemma 5.2, Y' is 3–separating in M, $|Y'| = 5$, $x_2 \in \mathrm{cl}(Y' \cup \{x_4\})$ and $x_4 \in \mathrm{cl}(Y' \cup \{x_2\})$. Since $|X| \geq 6$, $|X'| \geq 4$ so $14 \leq |E(M)| \leq 15$.

Suppose that $M\backslash x_1$ has a 3–separation (B, C) where $|B|, |C| \geq 6$, then $x_1 \notin \mathrm{cl}(B)$ and $x_1 \notin \mathrm{cl}(C)$. There are two possibilities. In the first case $x_2 \in B$ and $\{x_3, x_4, x_5\} \subseteq C$ and in the second case $|B \cap A| = |C \cap A| = 2$. We consider the first case. Let $B' = B - \{x_2\}$ and let $C' = C - \{x_3, x_4, x_5\}$. By a similar argument to the one in 5.2.3, B' is 3–separating in M and $|B'| = 5$. We now compare the 3–separators B' and Y'. There are four cases to consider.

(1) If $2 \leq |B' \cap Y'| \leq 4$ then we obtain a similar contradiction to the one in 5.2.2.

(2) If $|B' \cap Y'| = 0$ then $B' = X'$ and $C' = Y'$. Since $x_2 \in \mathrm{cl}(Y' \cup \{x_4\})$ we see that $x_2 \in \mathrm{cl}(C)$. But $x_1 \in \mathrm{cl}(C \cup \{x_2\})$ so $x_1 \in \mathrm{cl}(C)$ contradicting the fact that $x_1 \notin \mathrm{cl}(C)$.

(3) If $|B' \cap Y'| = 1$, let $B' \cap Y' = \{e\}$. Then by a similar argument to 5.2.6, $e \in \mathrm{cl}^{(*)}(Y' - \{e\})$ so $(B - \{e\}, C \cup \{e\})$ is a 3–separation of $M\backslash x_1$ where $Y' \cup \{x_4\} \subseteq C \cup \{e\}$. Now, $x_2 \in \mathrm{cl}(Y' \cup \{x_4\})$ so $x_2 \in \mathrm{cl}(C \cup \{e\})$. But $x_2 \in \mathrm{cl}*(C \cup \{e\})$ so $B' - \{e\}$ is a 2–separator of $M\backslash x_1$ contradicting that $M\backslash x_1$ is 3–connected.

(4) If $B' = Y'$ then $X' = C'$, and since $x_4 \in \mathrm{cl}(Y' \cup \{x_2\})$, we have $x_4 \in \mathrm{cl}(B)$. Also $x_3 \in \mathrm{cl}^*(B \cup \{x_4\})$ and $x_1 \in \mathrm{cl}(B \cup \{x_3, x_4\})$ so $C' \cup \{x_5\}$ is a 3–separator of M. By a similar argument to 5.2.2, $|C'| = 4$ and $x_5 \in \mathrm{cl}(C')$. At this stage we need to consider $M\backslash x_5$. Suppose that (D, F) is a 3–separation of $M\backslash x_5$ where $|D|, |F| \geq 6$. Let $D' = D - A$ and let $F' = F - A$. Then as with earlier cases, at least one of D' and F' is 3–separating in M, so we can assume that D' is 3–separating in M with $4 \leq |D'| \leq 5$. Now, since $x_5 \notin \mathrm{cl}(D)$ and $x_5 \notin \mathrm{cl}(F)$, we have $C' \nsubseteq D'$ and $C' \nsubseteq F'$. But now we obtain a similar contradiction to the one in 5.2.2 by looking at the sizes of $D' \cup B'$, $D' \cup C'$, and $D' \cup C' \cup \{x_5\}$.

These contradictions rule out the first case. Now we consider the second case where $|B \cap A| = |C \cap A| = 2$. Then $x_2, x_i \in B$ and $x_3, x_j \in C$ where $\{x_i, x_j\} = \{x_4, x_5\}$. Let $B' = B - \{x_2, x_i\}$ and let $C' = C - \{x_3, x_j\}$. Then as with 5.2.2, B' and C' are 3–separating in M with $4 \leq |B'|, |C'| \leq 5$. We now compare X', Y', B', and C'. There are three possible situations.

(1) If $B' \nsubseteq Y'$ and $C' \nsubseteq Y'$ then we obtain a similar contradiction to the one in 5.2.2.

(2) If $C' \subseteq Y'$ then $X' \subseteq B'$, and by a similar argument to 5.2.6, $(X' \cup \{x_2, x_i\}, Y' \cup \{x_3, x_j\})$ is a 3–separation of $M\backslash x_1$. And $x_i \in \mathrm{cl}(\{x_3, x_j\})$ so $(X' \cup \{x_2\}, Y' \cup \{x_3, x_4, x_5\})$ is a 3–separation of $M\backslash x_1$. Now $x_2 \in \mathrm{cl}(Y' \cup \{x_4\})$ and $x_2 \in \mathrm{cl}^*(\{x_3, x_4\})$, so X' is a 2–separator of $M\backslash x_1$ contradicting that $M\backslash x_1$ is 3–connected.

(3) If $B' \subseteq Y'$ then $X' \subseteq C'$, and by a similar argument to 5.2.6, $(Y' \cup \{x_2, x_i\}, X' \cup \{x_3, x_j\})$ is a 3–separation of $M\backslash x_1$, and $x_i \in \mathrm{cl}(\{x_3, x_j\})$ so $(Y' \cup \{x_2\}, X' \cup \{x_3, x_4, x_5\})$ is a 3–separation of $M\backslash x_1$ where $|Y' \cup \{x_2\}|, |X' \cup \{x_3, x_4, x_5\}| \geq 6$. But this is just an instance of the first case above, where we obtained a contradiction by looking at $M\backslash x_5$.

As a result of the contradictions above, we see that one of $M\backslash x_1$, $M\backslash x_5$, and $\mathrm{co}(M\backslash x_3)$ is 4–connected up to 3–separators of size 5. □

It follows from 5.2.2,5.2.3,…,5.2.10 that M contains an element x such that $\mathrm{co}(M\backslash x)$ or $\mathrm{si}(M/x)$ is 4–connected up to 3–separators of size 5, and has a cardinality of $|E(M)| - 1$ or $|E(M)| - 2$. □

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