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Traces, extensions, co-normal derivatives and solution regularity of elliptic systems with smooth and non-smooth coefficients

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Abstract

For functions from the Sobolev space $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, definitions of non-unique generalised and unique canonical co-normal derivative are considered, which are related to possible extensions of a partial differential operator and its right hand side from the domain Ω, where they are prescribed, to the domain boundary, where they are not. Revision of the boundary value problem settings, which makes them insensitive to the co-normal derivative inherent non-uniqueness are given. Some new facts about trace operator estimates, Sobolev spaces characterisations, and solution regularity of PDEs with non-smooth coefficients are also presented.

Keywords. Partial differential equation systems, Sobolev spaces, Classical, generalised and canonical co-normal derivatives, Weak BVP settings.

1 Introduction

While considering a second order partial differential equation for a function from the Sobolev space $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, with a right-hand side from $H^{s-2}(\Omega)$, the *strong* co-normal derivative of u defined on the boundary in the trace sense, does not generally exist. Instead, a *generalised* co-normal derivative operator can be defined by the first Green identity. However this definition is related to an extension of the PDE operator and its right hand side from the domain Ω, where they are prescribed, to the domain boundary, where they are not. Since the extensions are non-unique, the generalised co-normal derivative appears to be a non-unique and non-linear unless a linear relation between the PDE solution and the extension of its right hand side is enforced. This leads to a revision of the boundary value problem settings, which makes them insensitive to the co-normal derivative inherent non-uniqueness. For functions u from a subspace of $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, which can be mapped by the PDE operator into the space $\widetilde{H}^t(\Omega)$, $t \geq -\frac{1}{2}$, one can define a *canonical* co-normal derivative, which is unique, linear in u and coincides with the co-normal derivative in the trace sense if the latter does exist.

These notions were developed, to some extent, in [14, 15] for a PDE with an infinitely smooth coefficient on a domain with an infinitely smooth boundary, and a right hand side from $H^{s-2}(\Omega)$, $1 \leq s < \frac{3}{2}$, or extendable to $\widetilde{H}^t(\Omega)$, $t \geq -1/2$. In [16] the analysis was generalised to the co-normal derivative operators for some scalar PDE with a Hölder coefficient and right hand side from $H^{s-2}(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, on a bounded Lipschitz domain Ω.

In this paper, we extend the previous results on the co-normal derivatives to strongly elliptic second order PDE systems on bounded or unbounded Lipschitz domains with infinitely smooth or

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Hölder-Lipschitz coefficients, with complete proofs. To obtain these results, some new facts about trace operator estimates, Sobolev spaces characterisations, and solution regularity of PDEs with non-smooth coefficients are also proved in the paper.

The paper is arranged as follows. Section 2 provides a number of auxiliary facts on Sobolev spaces, traces and extensions, some of which might be new for Lipschitz domains. Particularly, we proved Lemma 2.1 on two-side estimates and unboundedness of the trace operator, Lemma 2.2 on boundedness of extension operators from boundary to the domain for a wider range of spaces, Theorem 2.4 on characterisation of the Sobolev space $H_0^s(\Omega) = \widetilde{H}^s(\Omega)$ on the (larger than usual) interval $\frac{1}{2} < s < \frac{3}{2}$, Theorem 2.5 on characterisation of the space $H^t_{\partial\Omega}$, $t > -\frac{3}{2}$, Theorem 2.6 on equivalence of $H_0^s(\Omega)$ and $H^s(\Omega)$ for $s \le \frac{1}{2}$, Lemma 2.7 and Theorem 2.8 on extension of $H^s(\Omega)$ to $\widetilde{H}^s(\Omega)$ for $s < \frac{1}{2}$, $s \neq \frac{1}{2} - k$.

The results of Section 2 are applied in Section 3 to introduce and analyse the generalised and canonical co-normal derivative operators on bounded and unbounded Lipschitz domains, associated with strongly elliptic systems of second order PDEs with infinitely smooth coefficients and right hand side from $H^{s-2}(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$. The weak settings of Dirichlet, Neumann and mixed problems (revised versions for the latter two) are considered and it is shown that they are well posed in spite of the inherent non-uniqueness of the generalised co-normal derivatives. It is proved that the canonical co-normal derivative coincides with the classical (strong) one for the cases when they both do exist.

In Section 4 we extend the well know result about the local regularity of elliptic PDE solution to the case of relaxed smoothness of the PDE coefficients. This is used then in Section 5, where all results of Section 3 are generalised to non-smooth coefficients.

2 Sobolev spaces, trace operators and extensions

Suppose $\Omega = \Omega^+$ is a bounded or unbounded open domain of $\mathbb{R}^n$, which boundary $\partial\Omega$ is a simply connected, closed, Lipschitz $(n-1)-$dimensional set. Let $\overline{\Omega}$ denote the closure of Ω and $\Omega^- = \mathbb{R}^n \backslash \overline{\Omega}$ its complement. In what follows $\mathcal{D}(\Omega) = C^\infty_{comp}(\Omega)$ denotes the space of Schwartz test functions, and $\mathcal{D}^*(\Omega)$ denotes the space of Schwartz distributions; $H^s(\mathbb{R}^n) = H^s_2(\mathbb{R}^n)$, $H^s(\partial\Omega) = H^s_2(\partial\Omega)$ are the Sobolev (Bessel potential) spaces, where $s \in \mathbb{R}$ is an arbitrary real number (see, e.g., [12]).

We denote by $\widetilde{H}^s(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^s(\mathbb{R}^n)$, which can be characterised as $\widetilde{H}^s(\Omega) = \{g : g \in H^s(\mathbb{R}^n),\ \mathrm{supp}\ g \subset \overline{\Omega}\}$, see e.g. [13, Theorem 3.29]. The space $H^s(\Omega)$ consists of restrictions on Ω of distributions from $H^s(\mathbb{R}^n)$, $H^s(\Omega) := \{g|_\Omega :\ g \in H^s(\mathbb{R}^n)\}$, and $H_0^s(\Omega)$ is closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$. We recall that $H^s(\Omega)$ coincide with the Sobolev–Slobodetski spaces $W_2^s(\Omega)$ for any non-negative s. We denote $H^s_{loc}(\Omega) := \{g : \phi g \in H^s(\Omega)\ \forall\phi \in \mathcal{D}(\Omega)\}$.

Note that distributions from $H^s(\Omega)$ and $H^s_0(\Omega)$ are defined only in Ω, while distributions from $\widetilde{H}^s(\Omega)$ are defined in $\mathbb{R}^n$ and particularly on the boundary $\partial\Omega$. For $s \ge 0$, we can identify $\widetilde{H}^s(\Omega)$ with the set of functions from $H^s(\Omega)$, whose extensions by zero outside Ω belong to $H^s(\mathbb{R}^n)$, cf. [13, Theorem 3.33], i.e., identify functions $u \in \widetilde{H}^s(\Omega)$ with their restrictions, $u|_\Omega$. However generally we will distinguish distributions $u \in \widetilde{H}^s(\Omega)$ and their restrictions $u|_\Omega$, especially for $s \le -\frac{1}{2}$.

We denote by $H^s_{\partial\Omega}$ the subspace of $H^s(\mathbb{R}^n)$ (and of $\widetilde{H}^s(\Omega)$), which elements are supported on $\partial\Omega$, i.e., $H^s_{\partial\Omega} := \{g :\ g \in H^s(\mathbb{R}^n),\ \mathrm{supp}\ g \subset \partial\Omega\}$. To simplify notations for vector-valued functions, $u : \Omega \to \mathbb{C}^m$, we will often write $u \in H^s(\Omega)$ instead of $u \in H^s(\Omega)^m = H^s(\Omega; \mathbb{C}^m)$, etc.

As usual (see e.g. [12, 13]), for two elements from dual complex Sobolev spaces the dual product $\langle\cdot,\cdot\rangle_\Omega$ associated with the inner product $(\cdot,\cdot)_{L_2(\Omega)}$ is defined as

$$\langle u, v\rangle_{\mathbb{R}^n} := \int_{\mathbb{R}^n} [\mathcal{F}^{-1}u](\xi)[\mathcal{F}v](\xi)d\xi =: (\mathcal{F}\bar{u}, \mathcal{F}v)_{L_2(\mathbb{R}^n)} =: (\overline{u}, v)_{L_2(\mathbb{R}^n)},$$
$$u \in H^s(\mathbb{R}^n),\ v \in H^{-s}(\mathbb{R}^n), \quad (2.1)$$

$$\langle u,v\rangle_\Omega := \langle u,V\rangle_{\mathbb{R}^n} =: (\overline{u},v)_{L_2(\Omega)} \text{ if } u\in \widetilde{H}^s(\mathbb{R}^n),\ v\in H^{-s}(\Omega),\ v=V|_\Omega \text{ with } V\in H^{-s}(\mathbb{R}^n), \quad (2.2)$$

$$\langle u,v\rangle_\Omega := \langle U,v\rangle_{\mathbb{R}^n} =: (\overline{u},v)_{L_2(\Omega)} \text{ if } u\in H^s(\mathbb{R}^n),\ v\in \widetilde{H}^{-s}(\Omega),\ u=U|_\Omega \text{ with } U\in H^s(\mathbb{R}^n) \quad (2.3)$$

for $s\in\mathbb{R}$, where $\bar{g}$ is the complex conjugate of g, while $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the distributional Fourier transform operator and its inverse, respectively, that for integrable functions take form

$$\hat{g}(\xi)=[\mathcal{F}g](\xi):=\int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}g(x)dx,\quad g(x)=[\mathcal{F}^{-1}\hat{g}](x):=\int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\hat{g}(\xi)d\xi.$$

For vector-valued elements $u\in H^s(\mathbb{R}^n)^m$, $v\in H^{-s}(\mathbb{R}^n)^m$, $s\in\mathbb{R}$, definition (2.1) should be understood as

$$\langle u,v\rangle_{\mathbb{R}^n} := \int_{\mathbb{R}^n}\hat{u}(\xi)\cdot\hat{v}(\xi)d\xi = \int_{\mathbb{R}^n}\hat{u}(\xi)^\top\hat{v}(\xi)d\xi =: (\overline{\hat{u}},\hat{v})_{L_2(\mathbb{R}^n)} =: (\overline{u},v)_{L_2(\mathbb{R}^n)}.$$

where $\hat{u}\cdot\hat{v}=\hat{u}^\top\hat{v}=\sum_{k=1}^m \hat{u}_k\hat{v}_k$ is the scalar product of two vectors.

Let $\mathcal{J}^s$ be the Bessel potential operator defined as

$$[\mathcal{J}^s g](x)=\mathcal{F}^{-1}_{\xi\to x}\{(1+|\xi|^2)^{s/2}\hat{g}(\xi)\}.$$

The inner product in $H^s(\Omega)$, $s\in\mathbb{R}$, is defined as follows,

$$(u,v)_{H^s(\mathbb{R}^n)} := (\mathcal{J}^s u,\mathcal{J}^s v)_{L_2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}(1+\xi^2)^s\overline{\hat{u}(\xi)}\hat{v}(\xi)d\xi = \left\langle\overline{u},\mathcal{J}^{2s}v\right\rangle_{\mathbb{R}^n},\quad u,v\in H^s(\mathbb{R}^n), \quad (2.4)$$

$$(u,v)_{H^s(\Omega)} := ((I-P)U,(I-P)V)_{H^s(\mathbb{R}^n)},\quad u=U|_\Omega,\ v=V|_\Omega,\quad U,V\in H^s(\mathbb{R}^n).$$

Here $P:H^s(\mathbb{R}^n)\to\widetilde{H}^s(\mathbb{R}^n\backslash\bar{\Omega})$ is the orthogonal projector, see e.g. [13, p. 77].

To introduce generalised co-normal derivatives in Section 3, we will need several facts about traces and extensions in Sobolev spaces on Lipschitz domain. First of all, it is well known [6, Lemma 3.7], that the trace operators $\gamma: H^s(\mathbb{R}^n)\to H^{s-\frac12}(\partial\Omega)$ and $\gamma^\pm: H^s(\Omega^\pm)\to H^{s-\frac12}(\partial\Omega)$ are continuous for $\frac12<s<\frac32$ on any Lipschitz domain Ω. Let $\gamma^*: H^{\frac12-s}(\partial\Omega)\to H^{-s}(\mathbb{R}^n)$ denote the operator adjoined to the trace operator,

$$\langle\gamma^* v,w\rangle=\langle v,\gamma w\rangle\quad\forall\ w\in H^s(\mathbb{R}^n).$$

Now we can prove a statement about the trace operator unboundedness (cf. [12, Chapter 1, Theorem 9.5] for domains with infinitely smooth boundary) that follows from two-side estimates for the trace operator and its adjoined.

LEMMA 2.1. *Let Ω be a Lipschitz domain and $\frac12<s\le 1$. Then*

$$C'\sqrt{C_s}\|v\|_{H^{\frac12-s}(\partial\Omega)}\le\|\gamma^* v\|_{H^{-s}(\mathbb{R}^n)}\le C''\sqrt{C_s}\|v\|_{H^{\frac12-s}(\partial\Omega)}\quad\forall v\in H^{\frac12-s}(\partial\Omega) \quad (2.5)$$

and thus

$$C'\sqrt{C_s}\le\|\gamma\|_{H^s(\mathbb{R}^n)\to H^{s-\frac12}(\partial\Omega)}=\|\gamma^*\|_{H^{\frac12-s}(\mathbb{R}^{n-1})\to H^{-s}(\mathbb{R}^n)}\le C''\sqrt{C_s}, \quad (2.6)$$

where $C_s:=\int_{-\infty}^\infty(1+\eta^2)^{-s}d\eta$, C' and C'' are positive constants independent of s and v. The norm of the trace operator $\gamma: H^s(\mathbb{R}^n)\to H^{s-\frac12}(\partial\Omega)$ tends to infinity as $s\searrow\frac12$ since $C_s\to\infty$, while the operator $\gamma: H^{\frac12}(\mathbb{R}^n)\to L_2(\partial\Omega)$ is unbounded.

Proof. Let first consider the lemma for the half-space, $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, where $x = \{x', x_n\}$, $x' \in \mathbb{R}^{n-1}$. For $v \in H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})$, the distributional Fourier transform gives

$$\mathcal{F}_{x\to\xi}\{\gamma^* v\} = \mathcal{F}_{x'\to\xi'}\{v(x')\} =: \hat{v}(\xi').$$

Then for $s > \frac{1}{2}$ we have,

$$\|\gamma^* v\|^2_{H^{-s}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} |\hat{v}(\xi')|^2 d\xi$$
$$= \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} (1+|\xi'|^2+|\xi_n|^2)^{-s} d\xi_n \right] |\hat{v}(\xi')|^2 d\xi' = C_s \|v\|^2_{H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})}, \quad (2.7)$$

where the substitution $\xi_n = (1+|\xi'|^2)^{\frac{1}{2}}\eta$ was used, cf. [5, Chap. 2, Proposition 4.6]. Thus

$$\|\gamma\|_{H^s(\mathbb{R}^n)\to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} = \|\gamma^*\|_{H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})\to H^{-s}(\mathbb{R}^n)} = \sqrt{C_s} \to \infty \quad \text{as } s \searrow \frac{1}{2}.$$

On the other hand, by (2.7) the norm $\|\gamma^* v\|_{H^{-\frac{1}{2}}(\mathbb{R}^n)}$ is not finite for any non-zero v. This means the operator $\gamma^* : H^0(\mathbb{R}^{n-1}) \to H^{-\frac{1}{2}}(\mathbb{R}^n)$ and thus the operator $\gamma : H^{\frac{1}{2}}(\mathbb{R}^n) \to H^0(\mathbb{R}^{n-1})$ is not bounded, which completes the lemma for $\Omega = \mathbb{R}^n_+$ with $C' = C'' = 1$.

For a general Lipschitz domain Ω, let $\{\omega_j\}_{j=1}^J \subset \mathbb{R}^n$ be a finite cover of $\partial\Omega$ and $\{\phi_j(x) \in \mathcal{D}(\omega_j)\}_{j=1}^J$ be a partition of unity subordinate to it, $\sum_{j=1}^J \phi_j(x) = 1$ for any $x \in \partial\Omega$. For any j there exists a half-space domain Ω_j such that $\omega_j \bigcap \Omega_j = \omega_j \bigcap \Omega$ and Ω_j can be transformed by a rigid translation to a Lipschitz hypograph $\tilde{\Omega}_j = \{x' \in \mathbb{R}^{n-1} : x_n > \zeta_j(x')\}$, where ζ_j are some uniformly Lipschitz functions, and $\kappa_j : \mathbb{R}^n \to \mathbb{R}^n$ is the Lipschitz-smooth invertible function such that $\mathbb{R}^n_+ \ni x \mapsto \kappa_j(x) \in \Omega_j$, $D_j(x')$ is the Jacobian of the corresponding boundary mapping $\mathbb{R}^{n-1} \ni x' \mapsto \kappa_j(x') \in \partial\Omega_j$, and $D_j \in L_\infty(\mathbb{R}^{n-1})$. For $v \in L_2(\partial\Omega)$, $w \in \mathcal{D}(\mathbb{R}^n)$, we have,

$$\langle \gamma^* v, w\rangle_{\mathbb{R}^n} = \langle v, \gamma w\rangle_{\partial\Omega} = \int_{\partial\Omega} v(x)w(x)d\sigma(x) = \sum_{j=1}^J \int_{\partial\Omega} \phi_j(x)v(x)w(x)d\sigma(x) =$$
$$\sum_{j=1}^J \int_{\mathbb{R}^{n-1}} [(\phi_j v)\circ\kappa_j](x')[w\circ\kappa_j](x')D_j(x')dx' =$$
$$\sum_{j=1}^J \langle D_j(\phi_j v)\circ\kappa_j, \gamma_0[w\circ\kappa_j]\rangle_{\mathbb{R}^{n-1}} = \sum_{j=1}^J \langle \gamma_0^*[D_j(\phi_j v)\circ\kappa_j], w\circ\kappa_j\rangle_{\mathbb{R}^n},$$

where γ_0, γ_0^* are the trace operator on $\mathbb{R}^n_+$ and its adjoined, respectively. Taking into account density of $\mathcal{D}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$ and $L_2(\partial\Omega)$ in $H^{\frac{1}{2}-s}(\partial\Omega)$, we have,

$$\|\gamma^* v\|_{H^{-s}(\mathbb{R}^n)} = \sup_{w\in H^s(\mathbb{R}^n)} \frac{|\langle\gamma^* v, w\rangle_{\mathbb{R}^n}|}{\|w\|_{H^s(\mathbb{R}^n)}} = \sup_{w\in H^s(\mathbb{R}^n)} \left| \sum_{j=1}^J \left\langle \gamma_0^*[D_j(\phi_j v)\circ\kappa_j], \frac{w\circ\kappa_j}{\|w\|_{H^s(\mathbb{R}^n)}} \right\rangle_{\mathbb{R}^n} \right| \quad (2.8)$$

for any $v \in H^{\frac{1}{2}-s}(\partial\Omega)$.

It is well known (see e.g. [13, Theorem 3.23 and p. 98]) that

$$\|v\|^2_{H^{\frac{1}{2}-s}(\partial\Omega)} = \sum_{j=1}^J \|D_j(\phi_j v)\circ\kappa_j\|^2_{H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})}, \quad \frac{1}{2} < s \le \frac{3}{2}, \quad (2.9)$$

$$\tilde{C}'\|w\|_{H^s(\mathbb{R}^n)} \le \|w\circ\kappa_j\|_{H^s(\mathbb{R}^n)} \le \tilde{C}''\|w\|_{H^s(\mathbb{R}^n)}, \quad j=1,...,J, \quad 0\le s\le 1, \tag{2.10}$$

where $\tilde{C}', \tilde{C}''$ are some positive constants independent of s. By (2.7) and (2.9),

$$\|\gamma_0^*[D_j(\phi_j v)\circ\kappa_j]\|_{H^{-s}(\mathbb{R}^n)} = \sqrt{C_s}\|D_j(\phi_j v)\circ\kappa_j\|_{H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})} \le \sqrt{C_s}\|v\|_{H^{\frac{1}{2}-s}(\partial\Omega)}.$$

Then (2.8) and (2.10) imply

$$\|\gamma^* v\|_{H^{-s}(\mathbb{R}^n)} \le \tilde{C}'' J\sqrt{C_s}\|v\|_{H^{\frac{1}{2}-s}(\partial\Omega)} \quad \forall v\in H^{\frac{1}{2}-s}(\partial\Omega),$$

which is the right inequality in (2.5).

On the other hand, we have for $v\in L_2(\partial\Omega)$, $w\in\mathcal{D}(\mathbb{R}^n)$,

$$\langle\phi_j\gamma^* v, w\rangle_{\mathbb{R}^n} = \langle v, \gamma(\phi_j w)\rangle_{\partial\Omega} = \int_{\partial\Omega} v(x)\phi_j(x)w(x)d\sigma(x) =$$
$$\int_{\partial\Omega\cap\omega_j} v(x)\phi_j(x)w(x)d\sigma(x) = \int_{\mathbb{R}^{n-1}} [(\phi_j v_j)\circ\kappa_j](x')[w\circ\kappa_j](x')D_j(x')dx' =$$
$$\langle D_j[(\phi_j v_j)\circ\kappa_j], \gamma_0[w\circ\kappa_j]\rangle_{\mathbb{R}^{n-1}} = \langle\gamma_0^*\{D_j[(\phi_j v_j)\circ\kappa_j]\}, w\circ\kappa_j\rangle_{\mathbb{R}^n}.$$

By (2.10) this implies,

$$\|\phi_j\gamma^* v\|_{H^{-s}(\mathbb{R}^n)} = \sup_{w\in H^s(\mathbb{R}^n)}\left|\left\langle\gamma_0^*\{D_j[(\phi_j v)\circ\kappa_j]\}, \frac{w\circ\kappa_j}{\|w\|_{H^s(\mathbb{R}^n)}}\right\rangle_{\mathbb{R}^n}\right| =$$
$$\sup_{w\in H^s(\mathbb{R}^n)}\left|\left\langle\gamma_0^*\{D_j[(\phi_j v)\circ\kappa_j]\}, \frac{w\circ\kappa_j}{\|w\circ\kappa_j\|_{H^s(\mathbb{R}^n)}}\right\rangle_{\mathbb{R}^n} \frac{\|w\circ\kappa_j\|_{H^s(\mathbb{R}^n)}}{\|w\|_{H^s(\mathbb{R}^n)}}\right| \ge$$
$$\tilde{C}'\sup_{w\in H^s(\mathbb{R}^n)}\left|\left\langle\gamma_0^*\{D_j[(\phi_j v)\circ\kappa_j]\}, \frac{w\circ\kappa_j}{\|w\circ\kappa_j\|_{H^s(\mathbb{R}^n)}}\right\rangle_{\mathbb{R}^n}\right| = \tilde{C}'\|\gamma_0^*\{D_j[(\phi_j v)\circ\kappa_j]\}\|_{H^{-s}(\mathbb{R}^n)}, \tag{2.11}$$

that is by (2.7) and (2.9),

$$\sum_{j=1}^J \|\phi_j\gamma^* v\|^2_{H^{-s}(\mathbb{R}^n)} \ge \tilde{C}'^2\sum_{j=1}^J \|\gamma_0^*\{D_j[(\phi_j v)\circ\kappa_j]\}\|^2_{H^{-s}(\mathbb{R}^n)} =$$
$$\tilde{C}'^2 C_s\sum_{j=1}^J \|D_j[(\phi_j v)\circ\kappa_j]\|^2_{H^{\frac{1}{2}-s}(\mathbb{R}^{n-1})} = \tilde{C}'^2 C_s\|v\|^2_{H^{\frac{1}{2}-s}(\partial\Omega}. \tag{2.12}$$

Since

$$\tilde{C}_j\|\gamma^* v\|_{H^{-s}(\mathbb{R}^n)} \ge \|\phi_j\gamma^* v\|_{H^{-s}(\mathbb{R}^n)} \tag{2.13}$$

for $\phi_j\in\mathcal{D}(\mathbb{R}^n)$, (2.12) gives the left inequality in (2.5).

Obviously, (2.5) implies (2.6) for γ^* and thus for γ.

As was shown in the first paragraph of the proof, the functional $\gamma_0^*\{D_j[(\phi_j v)\circ\kappa_j]\}$ is not bounded on $H^{\frac{1}{2}}(\mathbb{R}^n)$ for any non-zero v, then (2.11), (2.13) imply that the operator $\gamma^*: H^0(\partial\Omega)\to H^{-\frac{1}{2}}(\mathbb{R}^n)$ and thus the operator $\gamma: H^{\frac{1}{2}}(\mathbb{R}^n)\to H^0(\partial\Omega)$ is not bounded. □

LEMMA 2.2. *For a Lipschitz domain Ω there exists a linear bounded extension operator γ_{-1} : $H^{s-\frac{1}{2}}(\partial\Omega)\to H^s(\mathbb{R}^n)$, $\frac{1}{2}\le s\le\frac{3}{2}$, which is right inverse to the trace operator γ, i.e., $\gamma\gamma_{-1}g = g$ for any $g\in H^{s-\frac{1}{2}}(\partial\Omega)$. Moreover, $\|\gamma_{-1}\|_{H^{s-\frac{1}{2}}(\partial\Omega)\to H^s(\mathbb{R}^n)}\le C$, where C is independent of s.*

Proof. For Lipschitz domains and $\frac{1}{2} < s \le 1$, the boundedness of the extension operator is well known, see e.g. [13, Theorem 3.37].

To prove it for the whole range $\frac{1}{2} \le s \le \frac{3}{2}$, let us consider the classical single layer potential $V_\Delta\varphi$ with a density $\varphi = \mathcal{V}_\Delta^{-1} g \in H^{s-\frac{3}{2}}(\partial\Omega)$, solving the Laplace equation in Ω^+ with the Dirichlet boundary data g, where $\mathcal{V}_\Delta$ is the direct value of the operator V_Δ on the boundary. The operators $\mathcal{V}_\Delta^{-1} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega)$ and $V_\Delta : H^{s-\frac{3}{2}}(\partial\Omega) \to H^s_{loc}(\mathbb{R}^n)$ are continuous for $\frac{1}{2} \le s \le \frac{3}{2}$ as stated in [11, 10, 9, 19, 6]. Thus it suffice to take $\gamma_{-1} = \chi V_\Delta \mathcal{V}_\Delta^{-1}$, where $\chi \in \mathcal{D}(\mathbb{R}^n)$ is a cut-off function such that $\chi = 1$ in $\bar{\Omega}^+$. The estimate $\|\gamma_{-1}\|_{H^{s-\frac{1}{2}}(\partial\Omega)\to H^s(\mathbb{R}^n)} \le C$, where C is independent of s, then follows. □

Note that for $s = \frac{1}{2}$ the trace operator γ is understood in the non-tangential sense, and continuity of the operator γ was not needed in the proof.

To characterise the space $H_0^s(\Omega) = \widetilde{H}^s(\Omega)$ for $\frac{1}{2} < s < \frac{3}{2}$, we will need the following statement.

LEMMA 2.3. *If Ω is a Lipschitz domain and $u \in H^s(\Omega)$, $0 < s < \frac{1}{2}$, then*

$$\int_\Omega dist(x, \partial\Omega)^{-2s} |u(x)|^2 dx \le C\|u\|^2_{H^s(\Omega)}. \tag{2.14}$$

Proof. Note first that the lemma claim holds true for $u \in \mathcal{D}(\overline{\Omega})$, see [13, Lemma 3.32]. To prove it for $u \in H^s(\Omega)$, let first the domain Ω be such that

$$\mathrm{dist}(x, \partial\Omega) < C_0 < \infty \tag{2.15}$$

for all $x \in \Omega$, which holds true particularly for bounded domains. Let $\{\phi_k\} \in \mathcal{D}(\overline{\Omega})$ be a sequence converging to u in $H^s(\Omega)$. If we denote $w(x) = \mathrm{dist}(x, \partial\Omega)^{-2s}$, then $w(x) > C_0^{-2s} > 0$. Since (2.14) holds for functions from $\mathcal{D}(\overline{\Omega})$, the sequence $\{\phi_k\} \in \mathcal{D}(\overline{\Omega})$ is fundamental in the weighted space $L^2(\Omega, w)$, which is complete, implying that $\phi_k \in \mathcal{D}(\overline{\Omega})$ converges in this space to a function $u' \in L^2(\Omega, w)$. Since both $L^2(\Omega, w)$ and $H^s(\Omega)$ are continuously imbedded in the non-weighted space $L^2(\Omega)$, the sequence $\{\phi_k\}$ converges in $L^2(\Omega)$ implying the limiting functions u and u' belong to this space and thus coincide.

If the condition (2.15) is not satisfied, let $\chi(x) \in \mathcal{D}(\mathbb{R}^n)$ be a cut-off function such that $0 \le \chi(x) \le 1$ for all x, $\chi(x) = 1$ near $\partial\Omega$, while $w(x) < 1$ for $x \in$ supp $(1 - \chi)$. Then (2.15) is satisfied in $\Omega \bigcap \mathrm{supp}\, \chi(x)$ and

$$\int_\Omega w(x)|u(x)|^2 dx = \int_\Omega (1 - \chi(x))w(x)|u(x)|^2 dx + \int_\Omega \chi(x)w(x)|u(x)|^2 dx \le$$
$$\|u\|^2_{L_2(\Omega)} + \int_\Omega w(x)|\sqrt{\chi(x)}u(x)|^2 dx \le \|u\|^2_{H^s(\Omega)} + C\|\sqrt{\chi(x)}u\|^2_{H^s(\Omega)} \le C_1\|u\|^2_{H^s(\Omega)}.$$

due to the previous paragraph. □

Lemma 2.3 allows now extending the following statement known for $\frac{1}{2} < s \le 1$, see [13, Theorem 3.40(ii)], to a wider range of s.

THEOREM 2.4. *If Ω is a Lipschitz domain and $\frac{1}{2} < s < \frac{3}{2}$, then $H_0^s(\Omega) = \widetilde{H}^s(\Omega) = \{u \in H^s(\Omega) : \gamma^+ u = 0\}$.*

Proof. The first equality, $H_0^s(\Omega) = \widetilde{H}^s(\Omega)$, is well known for $\frac{1}{2} < s < \frac{3}{2}$, see e.g. the last part of Theorem 3.33 in [13]. The second equality for $\frac{1}{2} < s \le 1$ is stated in [13, Theorem 3.40(ii)].

Let $1 < s < \frac{3}{2}$. If $u \in \widetilde{H}^s(\Omega)$, then evidently $\gamma^+ u = 0$ since $\mathcal{D}$ is dense in $\widetilde{H}^s(\Omega)$ and the trace operator γ^+ is bounded in $H^s(\mathbb{R}^n)$. To prove the second equality of the theorem, it remains, due to the first part of Theorem 3.33 in [13], to prove that for any $u \in H^s(\Omega)$ such that $\gamma^+ u = 0$, its extension, $\tilde{u}$, by zero outside Ω, belongs to $H^s(\mathbb{R}^n)$. We remark first of all that $\tilde{u} \in H^1(\mathbb{R}^n)$ due to the previous paragraph and then make estimates similar to those in the proof of [13, Theorem 3.33],

$$\|\tilde{u}\|^2_{H^s(\mathbb{R}^n)} \sim \|\tilde{u}\|^2_{W^1_2(\mathbb{R}^n)} + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\nabla\tilde{u}(x) - \nabla\tilde{u}(y)|^2}{|x-y|^{2(s-1)+n}}\, dx\, dy$$
$$= \|u\|^2_{W^1_2(\Omega)} + \int_\Omega\int_\Omega \frac{|\nabla u(x) - \nabla u(y)|^2}{|x-y|^{2(s-1)+n}}\, dx\, dy$$
$$+ \int_{\mathbb{R}^n\setminus\Omega}\int_\Omega \frac{|\nabla u(x)|^2}{|x-y|^{2(s-1)+n}}\, dx\, dy + \int_\Omega\int_{\mathbb{R}^n\setminus\Omega} \frac{|\nabla u(y)|^2}{|x-y|^{2(s-1)+n}}\, dx\, dy$$
$$= \|u\|^2_{W^s_2(\Omega)} + 2\int_\Omega |w_{s-1}(x)\nabla u(x)|^2\, dx,$$

where

$$w_{s-1}(x) := \int_{\mathbb{R}^n\setminus\Omega} \frac{dy}{|x-y|^{2(s-1)+n}}, \quad x \in \Omega,$$

and $W^s_2(\Omega)$ is the Sobolev-Slobodetski space. Introducing spherical coordinates with x as an origin, we obtain, $w_{s-1}(x) \le C\,\mathrm{dist}(x,\partial\Omega)^{-2(s-1)}$ for $x \in \Omega$. Then, taking into account that $\nabla u \in H^{s-1}(\Omega)$ and $\|\nabla u\|_{H^{s-1}(\Omega)} \le \|u\|_{H^s(\Omega)}$, we have by Lemma 2.3,

$$\|\tilde{u}\|^2_{H^s(\mathbb{R}^n)} \le \|u\|^2_{W^s_2(\Omega)} + 2C\|u\|^2_{H^s(\Omega)} \le C_s\|u\|^2_{H^s(\Omega)}\ .$$

□

Let us now give a characterisation of the space $H^t_{\partial\Omega}$.

THEOREM 2.5. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$.*
(i) If $t \ge -\frac{1}{2}$, then $H^t_{\partial\Omega} = \{0\}$.
(ii) If $-\frac{3}{2} < t < -\frac{1}{2}$, then $g \in H^t_{\partial\Omega}$ if and only if $g = \gamma^* v$, i.e.,

$$\langle g, W\rangle_{\mathbb{R}^n} = \langle v, \gamma W\rangle_{\partial\Omega} \quad \forall\ W \in H^{-t}(\mathbb{R}^n), \tag{2.16}$$

with $v = \gamma^*_{-1} g \in H^{t+\frac{1}{2}}(\partial\Omega)$, i.e.,

$$\langle v, w\rangle_{\partial\Omega} \quad = \langle g, \gamma_{-1} w\rangle_{\mathbb{R}^n} \quad \forall\ w \in H^{-t-\frac{1}{2}}(\partial\Omega), \tag{2.17}$$

where v is independent of the choice of the non-unique operators γ_{-1}, γ^*_{-1}, and the estimate $\|v\|_{H^{t+\frac{1}{2}}(\partial\Omega)} \le C\|g\|_{H^t(\mathbb{R}^n)}$ holds with C independent of t.

Proof. We will follow an idea in the proof of Lemma 3.39 in [13] (see also [5, Proposition 4.8]), extending it from a half-space to a Lipschitz domain Ω.

Let $\Omega^+ = \Omega$ and $\Omega^- = \mathbb{R}^n\setminus\bar{\Omega}$. For any $\phi \in \mathcal{D}(\mathbb{R}^n)$, let us define

$$\phi^\pm(x) = \begin{cases} \phi(x) & \text{if } x \in \Omega^\pm, \\ 0 & \text{if } x \notin \Omega^\pm. \end{cases}$$

Let $t > -\frac{1}{2}$. Then $\phi^\pm \in \widetilde{H}^{-t}(\Omega^\pm)$ (see e.g. [13, Theorems 3.33, 3.40] for $-\frac{1}{2} < t \le 0$, for greater t it then follows by embedding), $\|\phi - \phi^+ - \phi^-\|_{H^{-t}(\mathbb{R}^n)} = 0$, and there exist sequences $\{\phi^\pm_k\} \in \mathcal{D}(\Omega^\pm)$ converging to $\phi^\pm$ in $\widetilde{H}^{-t}(\Omega^\pm)$ as $k \to \infty$. Hence $\langle g, \phi\rangle = \lim_{k\to\infty}\langle g, \phi^+_k + \phi^-_k\rangle = 0$ for any $g \in H^t_{\partial\Omega}$, $t > -\frac{1}{2}$, proving (i) for such t.

Let us prove (ii). For $g \in H^t_{\partial\Omega}$, $-\frac{3}{2} < t < -\frac{1}{2}$, let $v \in H^{t+\frac{1}{2}}(\partial\Omega)$ be defined by (2.17), where existence and continuity of $\gamma_{-1} : H^{-t-\frac{1}{2}}(\partial\Omega) \to H^{-t}(\Omega)$ is proved in Lemma 2.2. Observe that

$$|\langle v, w\rangle_{\partial\Omega}| \le \|g\|_{H^t(\mathbb{R}^n)}\|w\|_{H^{-t-\frac{1}{2}}(\partial\Omega)}\|\gamma_{-1}\|_{H^{-t-\frac{1}{2}}(\partial\Omega)\to H^{-t}(\mathbb{R}^n)},$$

so $\|v\|_{H^{t+\frac{1}{2}}(\partial\Omega)} \le \|\gamma_{-1}\|_{H^{-t-\frac{1}{2}}(\partial\Omega)\to H^{-t}(\mathbb{R}^n)} \|g\|_{H^t(\mathbb{R}^n)} \le C\|g\|_{H^t(\mathbb{R}^n)}$, where C is independent of t due to Lemma 2.2 if γ_{-1} is chosen as in that lemma. We also have that

$$\langle g, W\rangle_{\mathbb{R}^n} - \langle v, \gamma W\rangle_{\partial\Omega} = \langle g, \rho\rangle_{\mathbb{R}^n} \quad \forall\, W \in H^{-t}(\mathbb{R}^n),$$

where

$$\rho = W - \gamma_{-1}\gamma W \in H^{-t}(\mathbb{R}^n).$$

Then we have $\gamma\rho = 0$, which due to Theorem 2.4 implies $\tilde\rho^\pm \in \widetilde{H}^{-t}(\Omega^\pm)$, where $\tilde\rho^\pm$ are extensions of $\rho|_{\Omega^\pm}$ by zero outside $\Omega^\pm$ and $\rho = \tilde\rho^+ + \tilde\rho^-$. Thus there exist sequences $\{\rho^\pm_k\} \in \mathcal{D}(\Omega^\pm)$ converging to $\tilde\rho^\pm$ in $\widetilde{H}^{-t}(\Omega^\pm)$, implying $\langle g, \rho\rangle_{\mathbb{R}^n} = 0$ since $g \in H^t_{\partial\Omega}$, and thus ansatz (2.16). To prove that v is uniquely determined by g , i.e., independent of γ_{-1}, let us consider v' and v'' corresponding to different operators γ'_{-1} and γ''_{-1}. Then by (2.16),

$$\begin{aligned}\langle v' - v'', w\rangle_{\partial\Omega} &= \langle \gamma^{*\prime}_{-1} g - \gamma^{*\prime\prime}_{-1} g, w\rangle_{\partial\Omega} = \langle g, \gamma'_{-1}w - \gamma''_{-1}w\rangle_{\mathbb{R}^n}\\ &= \langle v', \gamma(\gamma'_{-1}w - \gamma''_{-1}w)\rangle_{\partial\Omega} = 0 \quad \forall\, w \in H^{-t-\frac{1}{2}}(\partial\Omega).\end{aligned}$$

It remains to deal with the case $t = -\frac{1}{2}$ in (i). Let $g \in H^{-\frac{1}{2}}_{\partial\Omega}$. Since $H^{-\frac{1}{2}}_{\partial\Omega} \subset H^t_{\partial\Omega}$ for $-\frac{3}{2} < t < -\frac{1}{2}$, then $g = \gamma^* v$ for some $v \in H^{t+\frac{1}{2}}(\partial\Omega)$ $\forall t \in (-\frac{3}{2}, -\frac{1}{2})$, and$\|g\|_{H^t_{\partial\Omega}} = \|\gamma^* v\|_{H^t_{\partial\Omega}} \ge C'\sqrt{C_{-t}}\,\|v\|_{H^{\frac{1}{2}+t}(\partial\Omega)}$ owing to Lemma 2.1. Since $C_{-t} \to \infty$ as $t \nearrow -\frac{1}{2}$, this means $\|v\|_{H^{\frac{1}{2}+t}(\partial\Omega)} \to 0$ as $t \nearrow -\frac{1}{2}$ implying $v = 0$. □

THEOREM 2.6. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and $s \le \frac{1}{2}$. Then $\mathcal{D}(\Omega)$ is dense in $H^s(\Omega)$, i.e., $H^s(\Omega) = H^s_0(\Omega)$.*

Proof. The proof for $0 \le s \le \frac{1}{2}$ is available in [13, Theorem 3.40(i)]. To prove the statement for any $s \le \frac{1}{2}$ we remark that if $w \in H^s(\Omega)^* = \widetilde{H}^{-s}(\Omega)$ satisfies $\langle w, \phi\rangle = 0$ for all $\phi \in \mathcal{D}(\Omega)$, then $w \in H^{-s}_{\partial\Omega}$ and Theorem 2.5 implies $w = 0$. Hence, $\mathcal{D}(\Omega)$ is dense in $H^s(\Omega)$, i.e., $H^s(\Omega) = H^s_0(\Omega)$. □

The following two statements give conditions when distributions from $H^s(\Omega)$ can be extended to distributions from $\widetilde{H}^s(\Omega)$ and when the extension can be written in terms of a linear bounded operator.

LEMMA 2.7. *Let Ω be a Lipschitz domain, $s < \frac{1}{2}$, $s \ne \frac{1}{2} - k$ for any integer $k > 0$. Then for any $g \in H^s(\Omega)$ there exists $\tilde g \in \widetilde{H}^s(\Omega)$ such that $g = \tilde g|_\Omega$ and $\|\tilde g\|_{\widetilde{H}^s(\Omega)} \le C\|g\|_{H^s(\Omega)}$, where $C > 0$ does not depend on g.*

Proof. Any distribution $g \in H^s(\Omega)$ is a bounded linear functional on $\widetilde{H}^{-s}(\Omega)$. On the other hand, $\widetilde{H}^{-s}(\Omega) = H^{-s}_0(\Omega) \subset H^{-s}(\Omega)$ for $s \le 0$ by [13, Theorem 3.33]. The latter holds true also for $0 < s < \frac{1}{2}$ since then $\widetilde{H}^{-s}(\Omega) = [H^s(\Omega)]^* = [H^s_0(\Omega)]^* = [\widetilde{H}^s(\Omega)]^* = H^{-s}(\Omega)$ by e.g. [13, Theorems 3.33 and 3.40]. Thus by the Hahn-Banach theorem g can be extended to a functional $\tilde g \in [H^{-s}(\Omega)]^* = \widetilde{H}^s(\Omega)$ such that $\|\tilde g\|_{\widetilde{H}^s(\Omega)} = \|\tilde g\|_{[H^{-s}(\Omega)]^*} = \|g\|_{[H^{-s}_0(\Omega)]^*} \le C\|g\|_{[\widetilde{H}^{-s}(\Omega)]^*} = C\|g\|_{H^s(\Omega)}$. □

THEOREM 2.8. *Let Ω be a Lipschitz domain and $-\frac{3}{2} < s < \frac{1}{2}$, $s \ne -\frac{1}{2}$. There exists a bounded linear extension operator $\widetilde{E}^s : H^s(\Omega) \to \widetilde{H}^s(\Omega)$, such that $\widetilde{E}^s g|_\Omega = g$, $\forall\, g \in H^s(\Omega)$.*

Proof. If $0 \le s < \frac{1}{2}$, then $\widetilde{H}^s(\Omega) = H^s(\Omega)$, see e.g. [13, Theorems 3.33 and 3.40], which implies $\widetilde{E}^s$ can be taken as the identity (imbedding) operator.

Let $-\frac{1}{2} < s < 0$. Since $\widetilde{H}^{-s}(\Omega) = H^{-s}(\Omega)$, we have $H^s(\Omega) = [\widetilde{H}^{-s}(\Omega)]^* = [H^{-s}(\Omega)]^* = \widetilde{H}^s(\Omega)$ due to the previous paragraph. The asterisk denotes the dual space. This implies $\widetilde{E}^s$ can be taken as the identity (imbedding) operator also in this case.

Let now $-\frac{3}{2} < s < -\frac{1}{2}$. For s in this range, the trace operator $\gamma^+ : H^{-s}(\Omega) \to H^{-s-\frac{1}{2}}(\partial\Omega)$ is bounded due to [6, Lemma 3.6] (see also [13, Theorem 3.38]), and there exists a bounded right inverse to the trace operator $\gamma_{-1} : H^{-s-\frac{1}{2}}(\partial\Omega) \to H^{-s}(\Omega)$, see Lemma 2.2. Then due to Theorem 2.4, $(I - \gamma_{-1}\gamma^+)$ is a bounded projector from $H^{-s}(\Omega)$ to $H_0^{-s}(\Omega) = \widetilde{H}^{-s}(\Omega)$. Thus any functional $v \in H^s(\Omega)$ can be continuously mapped into the functional $\tilde{v} \in \widetilde{H}^s(\Omega)$ such that $\tilde{v}u = v(I - \gamma_{-1}\gamma^+)u$ for any $u \in H^{-s}(\Omega)$. Since $\tilde{v}u = vu$ for any $u \in \widetilde{H}^{-s}(\Omega)$, we have, $\widetilde{E}^s = (I - \gamma_{-1}\gamma^+)^* : H^s(\Omega) \to \widetilde{H}^s(\Omega)$ is a bounded extension operator. □

Note that for $-\frac{1}{2} < s < \frac{1}{2}$ Theorem 2.5 implies that the extension operator $\widetilde{E}^s : H^s(\Omega) \to \widetilde{H}^s(\Omega)$ is unique, and we will call it *canonical extension operator*. For $-\frac{3}{2} < s < -\frac{1}{2}$, on the other hand, the operator $\gamma_{-1} : H^{-s-\frac{1}{2}}(\partial\Omega) \to H^{-s}(\Omega)$ in the proof of Theorem 2.8 is not unique, implying non-uniqueness of $\widetilde{E}^s : H^s(\Omega) \to \widetilde{H}^s(\Omega)$.

3 Partial differential operator extensions and co-normal derivatives for infinitely smooth coefficients

Let us consider in Ω a system of m complex linear differential equations of the second order with respect to m unknown functions $\{u_i\}_{i=1}^m = u : \Omega \to \mathbb{C}^m$, which for sufficiently smooth u has the following strong form,

$$Lu(x) := -\sum_{i,j=1}^{n} \partial_i[a_{ij}(x)\,\partial_j u(x)] + \sum_{j=1}^{n} b_j(x)\,\partial_j u(x) + c(x)u(x) = f(x), \quad x \in \Omega, \tag{3.1}$$

where $f : \Omega \to \mathbb{C}^m$, $\partial_j := \partial/\partial x_j$ $(j = 1, 2, ..., n)$, $a(x) = \{a_{ij}(x)\}_{i,j=1}^n = \{\{a_{ij}^{kl}(x)\}_{k,l=1}^m\}_{i,j=1}^n$, $b(x) = \{\{b_i^{kl}(x)\}_{k,l=1}^m\}_{i=1}^n$ and $c(x) = \{c^{kl}(x)\}_{k,l=1}^m$, i.e., $a_{ij}, b_i, c : \Omega \to \mathbb{C}^{m\times m}$ for fixed indices i, j. If $m = 1$, then (3.1) is a scalar equation. In this section we assume that $a, b, c \in C^\infty(\overline{\Omega})$; the case of non-smooth coefficients will be addressed in Sections 4, 5.

The operator L is (uniformly) strongly elliptic in an open domain Ω if there exists a bounded $m \times m$ matrix-valued function $\theta(x)$ such that

$$\mathrm{Re}\{\bar{\zeta}^\top \theta(x) \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j\zeta\} \ge C|\xi|^2|\zeta|^2 \tag{3.2}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$ and $\zeta \in \mathbb{C}^m$, where C is a positive constant, see e.g. [8, Definition 3.6.1] and references therein. We say that the operator L is uniformly strongly elliptic in a closed domain $\bar{\Omega}$ if its is uniformly strongly elliptic in an open domain $\Omega' \supset \bar{\Omega}$. We will need the strong ellipticity in relation with the solution regularity, starting from Theorem 3.10.

3.1 Partial differential operator extensions and generalised co-normal derivative

For $u \in H^s(\Omega)$, $f \in H^{s-2}(\Omega)$, $s \in \mathbb{R}$, equation (3.1) is understood in the distribution sense as

$$\langle Lu, v\rangle_\Omega = \langle f, v\rangle_\Omega \quad \forall v \in \mathcal{D}(\Omega), \tag{3.3}$$

where $v : \Omega \to \mathbb{C}^m$ and

$$\langle Lu, v\rangle_\Omega := \mathcal{E}(u, v) \quad \forall v \in \mathcal{D}(\Omega), \tag{3.4}$$

$$\mathcal{E}(u, v) = \mathcal{E}_\Omega(u, v) := \sum_{i,j=1}^n \langle a_{ij}\partial_j u, \partial_i v\rangle_\Omega + \sum_{j=1}^n \langle b_j\partial_j u, v\rangle_\Omega + \langle cu, v\rangle_\Omega\,. \tag{3.5}$$

Bilinear form (3.5) is well defined for any $v \in \mathcal{D}(\Omega)$ and moreover, the bilinear functional $\mathcal{E} : \{H^s(\Omega), \widetilde{H}^{2-s}(\Omega)\} \to \mathbb{C}$ is bounded for any $s \in \mathbb{R}$. Since the set $\mathcal{D}(\Omega)$ is dense in $\widetilde{H}^{2-s}(\Omega)$, expression (3.4) defines then a bounded linear operator $L : H^s(\Omega) \to H^{s-2}(\Omega) = [\widetilde{H}^{2-s}(\Omega)]^*$, $s \in \mathbb{R}$,

$$\langle Lu, v\rangle_\Omega := \mathcal{E}(u, v) \quad \forall v \in \widetilde{H}^{2-s}(\Omega). \tag{3.6}$$

Note that by (2.3) one can rewrite (3.6) also as

$$(Lu, v)_{L_2(\Omega)} := \Phi(u, v) \quad \forall v \in \widetilde{H}^{2-s}(\Omega),$$

where $\Phi(u, v) = \overline{\mathcal{E}(u, \bar{v})}$ is the sesquilinear form.

Let $\frac{1}{2} < s < \frac{3}{2}$. In addition to the operator L defined by (3.6), let us consider also the *aggregate* partial differential operator $\check{L} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega) = [H^{2-s}(\Omega)]^*$, defined as,

$$\langle \check{L}u, v\rangle_\Omega := \mathcal{E}(u, v) \quad \forall v \in H^{2-s}(\Omega). \tag{3.7}$$

The aggregate operator is evidently bounded since $\partial_i v \in H^{1-s}(\Omega) = \tilde{H}^{1-s}(\Omega)$, $v \in H^{2-s}(\Omega) \subset H^{1-s}(\Omega) = \tilde{H}^{1-s}(\Omega) \subset \tilde{H}^{-s}(\Omega)$, cf. the arguments in the proof of Theorem 2.8. For any $u \in H^s(\Omega)$, the functional $\check{L}u$ belongs to $\widetilde{H}^{s-2}(\Omega)$ and is an extension of the functional $Lu \in H^{s-2}(\Omega)$ from the domain of definition $\widetilde{H}^{2-s}(\Omega) \subset H^{2-s}(\Omega)$ to the domain of definition $H^{2-s}(\Omega)$.

The distribution $\check{L}u$ is not the only possible extension of the functional Lu, and any functional of the form

$$\check{L}u + g, \quad g \in H^{s-2}_{\partial\Omega} \tag{3.8}$$

gives another extension. On the other hand, any extension of the domain of definition of the functional Lu from $\widetilde{H}^{2-s}(\Omega)$ to $H^{2-s}(\Omega)$ has evidently form (3.8). The existence of such extensions is provided by Lemma 2.7.

For $u \in H^s(\Omega)$, $s > \frac{3}{2}$, the strong (classical) co–normal derivative operator

$$T_c^+ u(x) := \sum_{i,j=1}^n a_{ij}(x)\, \gamma^+[\partial_j u(x)]\nu_i(x), \tag{3.9}$$

is well defined on $\partial\Omega$ in the sense of traces. Here $\nu(x)$ is the outward (to Ω) unit normal vector at the point $x \in \partial\Omega$.

We can extend the definition of the generalised co–normal derivative, given in [13, Lemma 4.3] for $s = 1$, to a range of Sobolev spaces as follows.

DEFINITION 3.1. *Let Ω be a Lipschitz domain, $\frac{1}{2} < s < \frac{3}{2}$, $u \in H^s(\Omega)$, and $Lu = \tilde{f}|_\Omega$ in Ω for some $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$. Let us define* the generalised co–normal derivative $T^+(\tilde{f}, u) \in H^{s-\frac{3}{2}}(\partial\Omega)$ *as*

$$\left\langle T^+(\tilde{f}, u)\,,\, w\right\rangle_{\partial\Omega} := \mathcal{E}(u, \gamma_{-1}w) - \langle \tilde{f}, \gamma_{-1}w\rangle_\Omega = \langle \check{L}u - \tilde{f}, \gamma_{-1}w\rangle_\Omega \quad \forall\, w \in H^{\frac{3}{2}-s}(\partial\Omega), \tag{3.10}$$

where $\gamma_{-1} : H^{\frac{3}{2}-s}(\partial\Omega) \to H^{2-s}(\Omega)$ is a bounded right inverse to the trace operator.

The notation $T^+(\tilde{f}, u)$ corresponds to the notation $\widetilde{T}^+(\tilde{f}, u)$ in [16].

LEMMA 3.2. *Under hypotheses of Definition 3.1, the generalised co–normal derivative $T^+(\tilde f, u)$ is independent of the operator γ_{-1}, the estimate*

$$\|T^+(\tilde f, u)\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \le C_1\|u\|_{H^s(\Omega)} + C_2\|\tilde f\|_{\widetilde H^{s-2}(\Omega)} \tag{3.11}$$

takes place, and the first Green identity holds in the following form,

$$\Big\langle T^+(\tilde f, u)\,,\,\gamma^+ v\Big\rangle_{\partial\Omega} = \mathcal{E}(u,v) - \langle \tilde f, v\rangle_\Omega = \langle \check L u - \tilde f, v\rangle_\Omega \quad \forall\, v \in H^{2-s}(\Omega). \tag{3.12}$$

Proof. For $s = 1$ the lemma proof is available in [13, Lemma 4.3], which idea is extended here to the whole range $\frac{1}{2} < s < \frac{3}{2}$.

By Lemma 2.2, a bounded operator $\gamma_{-1} : H^{\frac{3}{2}-s}(\partial\Omega) \to H^{2-s}(\Omega)$ does exist. Then estimate (3.11) follows from (3.10).

To prove independence of the co-normal derivative $T^+(\tilde f, u)$ of γ_{-1}, let us consider two co-normal derivatives generated by two different operators γ'_{-1} and γ''_{-1}. Then their difference is

$$\langle T'^+(\tilde f, u) - T''^+(\tilde f, u), w\rangle_{\partial\Omega} = \langle \check L u - \tilde f, \gamma'_{-1}w - \gamma''_{-1}w\rangle_\Omega \quad \forall\, w \in H^{\frac{3}{2}-s}(\partial\Omega).$$

By definition, $\check L u - \tilde f \in H^{s-2}_{\partial\Omega}$, which by Theorem 2.5 means there exists $w_0 \in H^{s-\frac{3}{2}}(\partial\Omega)$ such that

$$\langle \check L u - \tilde f, \gamma'_{-1}w - \gamma''_{-1}w\rangle_\Omega = \langle w_0, \gamma^+\gamma'_{-1}w - \gamma^+\gamma''_{-1}w\rangle_{\partial\Omega} = \langle w_0, w - w\rangle_{\partial\Omega} = 0 \quad \forall\, w \in H^{\frac{3}{2}-s}(\partial\Omega).$$

To prove (3.12), consider the function $v_0 = v - \gamma_{-1}\gamma^+ v$. Since $\gamma^+ v_0 = 0$, we have $v_0 \in \widetilde H^{2-s}(\Omega)$ by Theorem 2.4, thus there exists a sequence $\{\phi_j\} \in \mathcal{D}(\Omega)$ converging to v_0 in $\widetilde H^{2-s}(\Omega)$. Hence, the equality $(Lu)|_\Omega = \tilde f|_\Omega \in H^{s-2}(\Omega)$ implies,

$$\mathcal{E}(u, v_0) = \lim_{j\to\infty} \mathcal{E}(u,\phi_j) = \lim_{j\to\infty} \langle \tilde f|_\Omega, \phi_j\rangle_\Omega = \lim_{j\to\infty} \langle \tilde f, \phi_j\rangle_\Omega = \langle \tilde f, v_0\rangle_\Omega\ .$$

Then taking into account definition (3.10), we have,

$$\begin{aligned}\mathcal{E}(u,v) = \mathcal{E}(u,v_0) + \mathcal{E}(u,\gamma_{-1}\gamma^+ v) = \langle \tilde f, v_0\rangle_\Omega + \Big\langle T^+(\tilde f, u)\,,\,\gamma^+ v\Big\rangle_{\partial\Omega} + \langle \tilde f, \gamma_{-1}\gamma^+ v\rangle_\Omega \\ = \langle \tilde f, v\rangle_\Omega + \Big\langle T^+(\tilde f, u)\,,\,\gamma^+ v\Big\rangle_{\partial\Omega}\end{aligned}$$

as required. □

Because of the involvement of $\tilde f$, the generalised co-normal derivative $T^+(\tilde f, u)$ is generally *non-linear* in u. It becomes linear if a linear relation is imposed between u and $\tilde f$ (including behaviour of the latter on the boundary $\partial\Omega$), thus fixing an extension of $\tilde f|_\Omega$ into $\widetilde H^{s-2}(\Omega)$. For example, $\tilde f|_\Omega$ can be extended as $\check f := \check L u$, which generally does not coincide with $\tilde f$. Then obviously, $T^+(\check f, u) = T^+(\check L u, u) = 0$, meaning that the co-normal derivatives associated with any other possible extension $\tilde f$ appears to be aggregated in $\check f$ as

$$\langle \check f, v\rangle_\Omega = \langle \tilde f, v\rangle_\Omega + \Big\langle T^+(\tilde f, u)\,,\,\gamma^+ v\Big\rangle_{\partial\Omega} \tag{3.13}$$

due to (3.12). This justifies the term *aggregate* for the extension $\check f$, and thus for the operator $\check L u$.

In fact, for a given function $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, any distribution $t \in H^{s-\frac{3}{2}}(\partial\Omega)$ may be nominated as a co-normal derivative of u, by an appropriate extension $\tilde f$ of the distribution $Lu \in H^{s-2}(\Omega)$ into $\widetilde H^{s-2}(\Omega)$. This extension is again given by the second Green formula (3.12) re-written as follows (cf. [2, Section 2.2, item 4] for $s = 1$),

$$\langle \tilde f, v\rangle_\Omega := \mathcal{E}(u,v) - \big\langle t, \gamma^+ v\big\rangle_{\partial\Omega} = \langle \check L u - \gamma^{+*} t, v\rangle_\Omega \quad \forall\, v \in H^{2-s}(\Omega). \tag{3.14}$$

Here the operator $\gamma^{+*} : H^{s-\frac{3}{2}}(\partial\Omega) \to \widetilde{H}^{s-2}(\Omega)$ is adjoined to the trace operator, $\langle\gamma^{+*}t, v\rangle_\Omega := \langle t, \gamma^+ v\rangle_{\partial\Omega}$ for all $t \in H^{s-\frac{3}{2}}(\partial\Omega)$ and $v \in H^{2-s}(\Omega)$. Evidently, the distribution $\tilde{f}$ defined by (3.14) belongs to $\widetilde{H}^{s-2}(\Omega)$ and is an extension of the distribution Lu into $\widetilde{H}^{s-2}(\Omega)$ since $\gamma^+ v = 0$ for $v \in \widetilde{H}^{2-s}(\Omega)$.

For $u \in C^1(\overline{\Omega}) \subset H^1(\Omega)$, one can take t equal to the strong co-normal derivative, $T_c^+ u \in L_\infty(\partial\Omega)$, and relation (3.14) can be considered as the *classical extension* of $f = Lu \in H^{-1}(\Omega)$ to $\tilde{f}_c \in \widetilde{H}^{-1}(\Omega)$, which is evidently linear.

3.2 Boundary value problems

Consider the BVP weak settings for PDE (3.1) on Lipschitz domain for $\frac{1}{2} < s < \frac{3}{2}$.

The Dirichlet problem: for $f \in H^{s-2}(\Omega)$, $\varphi_0 \in H^{s-\frac{1}{2}}(\partial\Omega)$, find $u \in H^s(\Omega)$ such that

$$\langle Lu, v\rangle_\Omega = \langle f, v\rangle_\Omega \quad \forall v \in \widetilde{H}^{2-s}(\Omega), \tag{3.15}$$

$$\gamma^+ u = \varphi_0 \quad \text{on } \partial\Omega. \tag{3.16}$$

The Neumann problem: for $\check{f} \in \widetilde{H}^{s-2}(\Omega)$, find $u \in H^s(\Omega)$ such that

$$\langle \check{L}u, v\rangle_\Omega = \langle \check{f}, v\rangle_\Omega \quad \forall v \in H^{2-s}(\Omega). \tag{3.17}$$

Here Lu and $\check{L}u$ are defined by (3.4) and (3.7), respectively.

To set the mixed problem, let $\partial_D\Omega$ and $\partial_N\Omega = \partial\Omega\backslash\overline{\partial_D\Omega}$ be nonempty, open sub–manifolds of $\partial\Omega$, and $H_0^s(\Omega, \partial_D\Omega) = \{w \in H^s(\Omega) : \gamma^+ w = 0 \text{ on } \partial_D\Omega\}$. We introduce the *mixed aggregate* operator $\check{L}_{\partial_D\Omega} : H^s(\Omega) \to [H_0^{2-s}(\Omega, \partial_D\Omega)]^*$, defined as

$$\langle \check{L}_{\partial_D\Omega} u, v\rangle_\Omega := \mathcal{E}(u, v) \quad \forall\ v \in H_0^{2-s}(\Omega, \partial_D\Omega).$$

The mixed operator $\check{L}_{\partial_D\Omega}$ is bounded by the same argument as the aggregate operator $\check{L}$. For any $u \in H^s(\Omega)$, the distribution $\check{L}_{\partial_D\Omega} u$ belongs to $[H_0^{2-s}(\Omega, \partial_D\Omega)]^*$ and is an extension of the functional $Lu \in H^{s-2}(\Omega)$ from the domain of definition $\widetilde{H}^{2-s}(\Omega) = H_0^{2-s}(\Omega) \subset H_0^{2-s}(\Omega, \partial_D\Omega)$ to the domain of definition $H_0^{2-s}(\Omega, \partial_D\Omega)$, and a restriction of the functional $\check{L}u \in \widetilde{H}^{s-2}(\Omega)$ from the domain of definition $H^{2-s}(\Omega) \supset H_0^{2-s}(\Omega, \partial_D\Omega)$ to the domain of definition $H_0^{2-s}(\Omega, \partial_D\Omega)$.

For $v \in H_0^{2-s}(\Omega, \partial_D\Omega)$, the trace $\gamma^+ v$ belongs to $\widetilde{H}^{s-\frac{1}{2}}(\partial_N\Omega)$. If $Lu = \tilde{f}|_\Omega$ in Ω for some $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$, then the first Green identity (3.12) gives,

$$\begin{aligned}&\langle \check{L}_{\partial_D\Omega} u, v\rangle_\Omega = \mathcal{E}(u, v) = \langle \check{f}_m, v\rangle_\Omega,\\ &\langle \check{f}_m, v\rangle_\Omega = \langle \tilde{f}, v\rangle_\Omega + \left\langle T^+(\tilde{f}, u)\,,\, \gamma^+ v\right\rangle_{\partial_N\Omega} \quad \forall\ v \in H_0^{2-s}(\Omega, \partial_D\Omega),\end{aligned} \tag{3.18}$$

where, evidently, $\check{f}_m \in [H_0^{2-s}(\Omega, \partial_D\Omega)]^*$. This leads to the following weak setting.

The mixed (Dirichlet-Neumann) problem: for $\check{f}_m \in [H_0^{2-s}(\Omega, \partial_D\Omega)]^*$, $\varphi_0 \in H^{s-\frac{1}{2}}(\partial_D\Omega)$, find $u \in H^s(\Omega)$ such that

$$\langle \check{L}_{\partial_D\Omega} u, v\rangle_\Omega = \langle \check{f}_m, v\rangle_\Omega \quad \forall v \in H_0^{2-s}(\Omega, \partial_D\Omega), \tag{3.19}$$

$$\gamma^+ u = \varphi_0 \quad \text{on } \partial_D\Omega. \tag{3.20}$$

The Neumann and the mixed problems are formulated in terms of the aggregate right hand sides $\check{f}$ and $\check{f}_m$, respectively, prescribed on their own, i.e., without necessary splitting them into the right hand side inside the domain Ω and the part related with the prescribed co-normal derivative. If a right hand side extension $\tilde{f}$ and an associated non-zero generalised co-normal derivative $T^+(\tilde{f}, u)$ are prescribed instead, then $\check{f}$ and $\check{f}_m$ can be expressed through them by relations (3.13), (3.18).

Thus the co-normal derivative does not enter, in fact, the weak settings of the Dirichlet, Neumann or mixed problem, and the non-uniqueness of $T^+(\tilde{f}, u)$ for a given function $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, does not influence the BVP weak settings, (cf. [2, Section 2.2, item 4] for $s = 1$). On the other hand, for a given $u \in H^s(\Omega)$ the aggregate right hand sides $\check{f}$ and $\check{f}_m$ are uniquely determined by (3.17), (3.19), as are, of course, f and φ_0 by (3.15), (3.16)/(3.20).

Note that one can take $v = \bar{w}$ to make the settings (3.15)-(3.16), (3.17) and (3.19)-(3.20) look closer to the usual variational formulations, cf. e.g. [12].

3.3 Canonical co-normal derivative

As we have seen above, for an arbitrary $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, the co-normal derivative $T^+(\tilde{f}, u)$ is not generally uniquely determined by u. An exception is $T^+(\check{L}u, u) \equiv 0$ but such co-normal derivative evidently differs from the strong co-normal derivative $T_c^+ u$, given by (3.9) for sufficiently smooth u. Another one way of making generalised co-normal derivative unique in $u \in H^1(\Omega)$ was presented in [8, Lemma 5.1.1] and is in fact associated with an extension of $Lu \in H^{-1}(\Omega)$ to $\tilde{f} \in \widetilde{H}^{-1}(\Omega)$, such that $\tilde{f}$ is orthogonal in $H^{-1}(\mathbb{R}^n)$ to $H^{-1}_{\partial\Omega} \subset H^{-1}(\mathbb{R}^n)$. However it appears (see Lemma A.1), that even for infinitely smooth functions f such extension $\tilde{f}$ does not generally belong to $L_2(\mathbb{R}^n)$, which implies that the so-defined co-normal derivative τu from [8, Lemma 5.1.1] does also not generally lead to the strong co-normal derivative.

Nevertheless, it is still possible to point out some subspaces of $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, where a unique definition of the co-normal derivative by u is possible and leads to the strong co-normal derivative for sufficiently smooth u. We define below one such sufficiently wide subspace.

DEFINITION 3.3. *Let $s \in \mathbb{R}$ and $L_* : H^s(\Omega) \to \mathcal{D}^*(\Omega)$ be a linear operator. For $t \geq -\frac{1}{2}$, we introduce a space $H^{s,t}(\Omega; L_*) := \{g : \ g \in H^s(\Omega),\ L_*g|_\Omega = \tilde{f}_g|_\Omega,\ \tilde{f}_g \in \widetilde{H}^t(\Omega)\}$ equipped with the graphic norm, $\|g\|^2_{H^{s,t}(\Omega;L_*)} := \|g\|^2_{H^s(\Omega)} + \|\tilde{f}_g\|^2_{\widetilde{H}^t(\Omega)}$.*

The distribution $\tilde{f}_g \in \widetilde{H}^t(\Omega)$, $t \geq -\frac{1}{2}$, in the above definition is an extension of the distribution $L_*g|_\Omega \in H^t(\Omega)$, and the extension is unique (if it does exist), since otherwise the difference between any two extensions belongs to $H^t_{\partial\Omega}$ but $H^t_{\partial\Omega} = \{0\}$ for $t \geq -\frac{1}{2}$ due to the Theorem 2.5. The uniqueness implies that the norm $\|g\|_{H^{s,t}(\Omega;L_*)}$ is well defined. Note that another subspace of such kind, where $L_*g|_\Omega$ belongs to $L_p(\Omega)$ instead of $H^t(\Omega)$, was presented in [7, p. 59].

If $s_1 \leq s_2$ and $t_1 \leq t_2$, then we have the embedding, $H^{s_2,t_2}(\Omega; L_*) \subset H^{s_1,t_1}(\Omega; L_*)$.

REMARK 3.4. *If $s \in \mathbb{R}$, $-\frac{1}{2} < t < \frac{1}{2}$, and $L_* : H^s(\Omega) \to H^t(\Omega)$ is a linear continuous operator, then $H^{s,t}(\Omega; L_*) = H^s(\Omega)$ by Theorem 2.8.*

LEMMA 3.5. *Let $s \in \mathbb{R}$. If a linear operator $L_* : H^s(\Omega) \to \mathcal{D}^*(\Omega)$ is continuous, then the space $H^{s,t}(\Omega; L_*)$ is complete for any $t \geq -\frac{1}{2}$.*

Proof. Let $\{g_k\}$ be a Cauchy sequence in $H^{s,t}(\Omega; L_*)$. Then there exists a Cauchy sequence $\{\tilde{f}_k\}$ in $\widetilde{H}^t(\Omega)$ such that $\tilde{f}_k|_\Omega = L_*g_k|_\Omega$. Since $H^s(\Omega)$ and $\widetilde{H}^t(\Omega)$ are complete, there exist elements $g_0 \in H^s(\Omega)$ and $\tilde{f}_0 \in \widetilde{H}^t(\Omega)$ such that $\|g_k - g_0\|_{H^s(\Omega)} \to 0$, $\|\tilde{f}_k - \tilde{f}_0\|_{\widetilde{H}^t(\Omega)} \to 0$ as $k \to \infty$. On the other hand, continuity of L_* implies that $|\langle L_*(g_k - g_0), \phi\rangle| \to 0$ for any $\phi \in \mathcal{D}(\Omega)$. Taking into account that $L_*g_k|_\Omega = \tilde{f}_k|_\Omega$, we obtain

$$\begin{aligned} |\langle \tilde{f}_0 - L_*g_0, \phi\rangle| &\leq |\langle \tilde{f}_0 - \tilde{f}_k, \phi\rangle| + |\langle \tilde{f}_k - L_*g_0, \phi\rangle| \\ &\leq \|\tilde{f}_0 - \tilde{f}_k\|_{\widetilde{H}^t(\Omega)} \|\phi\|_{H^{-t}(\Omega)} + |\langle L_*(g_k - g_0), \phi\rangle| \to 0, \quad k \to \infty \quad \forall \phi \in \mathcal{D}(\Omega), \end{aligned}$$

i.e., $L_*g_0|_\Omega = \tilde{f}_0|_\Omega \in H^t(\Omega)$, which implies $g_0 \in H^{s,t}(\Omega; L_*)$. □

We will further use the space $H^{s,t}(\Omega; L_*)$ for the case when the operator L_* is the operator L from (3.4) or the operator L^* formally adjoined to it (see Section 3.4).

DEFINITION 3.6. *Let $s \in \mathbb{R}$, $t \geq -\frac{1}{2}$. The operator L^0 mapping functions $u \in H^{s,t}(\Omega; L)$ to the extension of the distribution $Lu \in H^t(\Omega)$ to $\widetilde{H}^t(\Omega)$ will be called* the canonical extension *of the operator L.*

REMARK 3.7. *If $s \in \mathbb{R}$, $t \geq -\frac{1}{2}$, then $\|L^0 u\|_{\widetilde{H}^t(\Omega)} \leq \|u\|_{H^{s,t}(\Omega;L)}$ by definition of the space $H^{s,t}(\Omega; L)$, i.e., the linear operator $L^0 : H^{s,t}(\Omega; L) \to \widetilde{H}^t(\Omega)$ is continuous. Moreover, if $-\frac{1}{2} < t < \frac{1}{2}$, then by Theorem 2.8 and uniqueness of the extension of $H^t(\Omega)$ to $\widetilde{H}^t(\Omega)$, we have the representation $L^0 := \widetilde{E}^t L$.*

As in in [16, Definition 3] for scalar PDE, let us define the *canonical* co-normal derivative operator. This extends [7, Theorem 1.5.3.10] and [6, Lemma 3.2] where co-normal derivative operators acting on functions from $H^{1,0}_p(\Omega; \Delta)$ and $H^{1,0}(\Omega; L)$, respectively, were defined.

DEFINITION 3.8. *For $u \in H^{s,-\frac{1}{2}}(\Omega; L)$, $\frac{1}{2} < s < \frac{3}{2}$, we define the* canonical co-normal derivative *as $T^+u := T^+(L^0u, u) \in H^{s-\frac{3}{2}}(\partial\Omega)$, i.e.,*

$$\left\langle T^+u\,,\, w\right\rangle_{\partial\Omega} := \mathcal{E}(u, \gamma_{-1}w) - \langle L^0u, \gamma_{-1}w\rangle_\Omega = \langle \check{L}u - L^0u, \gamma_{-1}w\rangle_\Omega \quad \forall\, w \in H^{\frac{3}{2}-s}(\partial\Omega), \tag{3.21}$$

where $\gamma_{-1} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)$ is a bounded right inverse to the trace operator.

Lemma 3.2 for the generalised co-normal derivative and Definition 3.3 imply the following statement.

LEMMA 3.9. *Under hypotheses of Definition 3.8, the canonical co-normal derivative T^+u is independent of the operator γ_{-1}, the operator $T^+ : H^{s,-\frac{1}{2}}(\Omega; L) \to H^{s-\frac{3}{2}}(\partial\Omega)$ is continuous, and the first Green identity holds in the following form,*

$$\begin{aligned}\left\langle T^+u\,,\, \gamma^+v\right\rangle_{\partial\Omega} = \left\langle T^+(L^0u, u)\,,\, \gamma^+v\right\rangle_{\partial\Omega} &= \mathcal{E}(u, v) - \langle L^0u, v\rangle_\Omega \\ &= \langle \check{L}u - L^0u, v\rangle_\Omega \quad \forall\, v \in H^{2-s}(\Omega).\end{aligned}$$

Thus unlike the generalised co-normal derivative, the canonical co-normal derivative is uniquely defined by the function u and the operator L only, uniquely fixing an extension of the latter on the boundary.

Definitions 3.1 and 3.8 imply that the generalised co-normal derivative of $u \in H^{s,-\frac{1}{2}}(\Omega; L)$, $\frac{1}{2} < s < \frac{3}{2}$, for any other extension $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ of the distribution $Lu|_\Omega \in H^{-\frac{1}{2}}(\Omega)$ can be expressed as

$$\left\langle T^+(\tilde{f}, u)\,,\, w\right\rangle_{\partial\Omega} = \left\langle T^+u\,,\, w\right\rangle_{\partial\Omega} + \langle L^0u - \tilde{f}, \gamma_{-1}w\rangle_\Omega \quad \forall\, w \in H^{\frac{3}{2}-s}(\partial\Omega). \tag{3.22}$$

Note that the distributions $\check{L}u - \tilde{f}$, $\check{L}u - L^0u$ and $L^0 - \tilde{f}$ belong to $H^{2-s}_{\partial\Omega}$ since L^0u, $\check{L}u$, $\tilde{f}$ belong to $\widetilde{H}^{2-s}(\Omega)$, while $L^0u|_\Omega = \check{L}u|_\Omega = \tilde{f}|_\Omega = Lu|_\Omega \in H^{s-2}(\Omega)$.

To give conditions when the canonical co-normal derivative T^+u coincides with the strong co-normal derivative $T^+_c u$, if the latter does exist in the trace sense, we prove in Lemma 3.11 below that $\mathcal{D}(\overline{\Omega})$ is dense in $H^{s,t}(\Omega; L)$. The proof is based on the following local regularity theorem well known for the case of infinitely smooth coefficients, see e.g. [18, 1, 12]; its counterpart for the case of Hölder coefficients, Theorem 4.5, is proved in Section 4.

THEOREM 3.10. *Let Ω be an open set in $\mathbb{R}^n$, $s_1 \in \mathbb{R}$, function $u \in H^{s_1}_{loc}(\Omega)^m$, $m \geq 1$, satisfy strongly elliptic system* (3.1) *in Ω with $f \in H^{s_2}_{loc}(\Omega)^m$, $s_2 > s_1 - 2$, and infinitely smooth coefficients. Then $u \in H^{s_2+2}_{loc}(\Omega)^m$.*

LEMMA 3.11. *If $s \in \mathbb{R}$, $-\frac{1}{2} \leq t < \frac{1}{2}$ and the operator L is strongly elliptic on $\overline{\Omega}$, then $\mathcal{D}(\overline{\Omega})$ is dense in $H^{s,t}(\Omega; L)$.*

Proof. We modify appropriately the proof from [7, Lemma 1.5.3.9] given for another space of such kind.

For every continuous linear functional l on $H^{s,t}(\Omega; L)$ there exist distributions $\tilde{h} \in \widetilde{H}^{-s}(\Omega)$ and $g \in H^{-t}(\Omega)$ such that

$$l(u) = \langle \tilde{h}, u \rangle_\Omega + \langle g, L^0 u \rangle_\Omega.$$

To prove the lemma claim, it suffice to show that any l, which vanishes on $\mathcal{D}(\overline{\Omega})$, will vanish on any $u \in H^{s,t}(\Omega; L)$. Indeed, if $l(\phi) = 0$ for any $\phi \in \mathcal{D}(\overline{\Omega})$, then

$$\langle \tilde{h}, \phi \rangle_\Omega + \langle g, L^0 \phi \rangle_\Omega = 0. \tag{3.23}$$

Let us consider the case $-\frac{1}{2} < t < \frac{1}{2}$ first and extend g outside Ω to $\tilde{g} \in \widetilde{H}^{-t}(\Omega)$ (cf. the proof of Lemma 2.8). If $t \leq s - 2$, then evidently $\tilde{g} \in \widetilde{H}^{2-s}(\Omega)$. If $t > s - 2$, then equation (3.23) gives

$$\langle \tilde{h}, \phi \rangle_{\Omega'} + \langle \tilde{g}, L\phi \rangle_{\Omega'} = \langle \tilde{h}, \phi \rangle_\Omega + \langle g, L^0 \phi \rangle_\Omega = 0 \tag{3.24}$$

for any $\phi \in \mathcal{D}(\Omega')$ on some domain $\Omega' \supset \overline{\Omega}$, where the operator L is still strongly elliptic. This means $L^* \tilde{g} = -\tilde{h}$ in Ω' in the sense of distributions, where L^* is the operator formally adjoint to L, see (3.29). Then Theorem 3.10 implies $\tilde{g} \in H^{2-s}_{loc}(\Omega')$ and consequently $\tilde{g} \in \widetilde{H}^{2-s}(\Omega)$.

In the case $t = -\frac{1}{2}$, one can extend $g \in H^{\frac{1}{2}}(\Omega)$ outside $\overline{\Omega}$ by zero to $\tilde{g} \in \widetilde{H}^{\frac{1}{2}-\epsilon}(\Omega)$, $0 < \epsilon$, and prove as in the previous paragraph that $\tilde{g} \in \widetilde{H}^{2-s}(\Omega)$.

If $-\frac{1}{2} < t < \frac{1}{2}$ or $[t = -\frac{1}{2},\ s \leq \frac{3}{2}]$, let us denote $q := \max\{-t, 2 - s\}$ and $\{g_k\} \in \mathcal{D}(\Omega)$ be a sequence converging, as $k \to \infty$, to $\tilde{g}$ in $\widetilde{H}^q(\Omega)$ and thus in $\widetilde{H}^{-t}(\Omega)$ and in $\widetilde{H}^{2-s}(\Omega)$. Then for any $u \in H^{s,t}(\Omega; L)$, we have,

$$l(u) = \lim_{k\to\infty} \left\{ \langle -L^* g_k, u \rangle_\Omega + \langle g_k, L^0 u \rangle_\Omega \right\} = \lim_{k\to\infty} \left\{ \langle -L^* g_k, u \rangle_\Omega + \langle g_k, L u \rangle_\Omega \right\} = 0 \tag{3.25}$$

since $L^* g_k \in \widetilde{H}^{-s}(\Omega)$ and $L^* g_k \to L^* \tilde{g}$ in $\widetilde{H}^{-s}(\Omega)$. Thus l is identically zero.

On the other hand, if $t = -\frac{1}{2}$, $s > \frac{3}{2}$, let $\{g_k\} \in \mathcal{D}(\Omega)$ be a sequence converging, as $k \to \infty$, to g in $H_0^{\frac{1}{2}}(\Omega) = H^{\frac{1}{2}}(\Omega)$, cf. Theorem 2.6, and thus to $\tilde{g}$ in $\widetilde{H}^{2-s}(\Omega)$. Employing then the same reasoning about (3.25), as in the preceding paragraph, we complete the proof. □

LEMMA 3.12. *Let $u \in H^{s,-\frac{1}{2}}(\Omega; L)$, $\frac{1}{2} < s < \frac{3}{2}$, and $\{u_k\} \in \mathcal{D}(\overline{\Omega})$ be a sequence such that*

$$\|u_k - u\|_{H^{s,-\frac{1}{2}}(\Omega;L)} \to 0 \quad \text{as } k \to \infty. \tag{3.26}$$

Then $\|T_c^+ u_k - T^+ u\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \to 0$ as $k \to \infty$.

Proof. By Lemma 3.11 the sequence satisfying (3.26) does always exist. Using the definition of $T^+ u$ and the classical first Green identity for u_k, we have for any $w \in H^{\frac{3}{2}-s}(\partial\Omega)$,

$$\begin{aligned}
\left\langle T^+ u, w \right\rangle_{\partial\Omega} &= \mathcal{E}(u, \gamma_{-1} w) - \langle L^0 u, \gamma_{-1} w \rangle_\Omega = \mathcal{E}(u - u_k, \gamma_{-1} w) + \mathcal{E}(u_k, \gamma_{-1} w) - \langle L^0 u, \gamma_{-1} w \rangle_\Omega \\
&= \mathcal{E}(u - u_k, \gamma_{-1} w) + \int_{\partial\Omega} w \ T_c^+ u_k \ d\Gamma + \langle L^0 u_k, \gamma_{-1} w \rangle_\Omega - \langle L^0 u, \gamma_{-1} w \rangle_\Omega \\
&= \mathcal{E}(u - u_k, \gamma_{-1} w) + \left\langle T_c^+ u_k, w \right\rangle_{\partial\Omega} - \langle L^0 (u - u_k), \gamma_{-1} w \rangle_\Omega \to \left\langle T_c^+ u_k, w \right\rangle_{\partial\Omega},
\end{aligned}$$

as $k \to \infty$ due to (3.26). Since $T^+ u$ is uniquely determined by u, this implies existence of the limit of the right hand side and its independence of the sequence $\{u_k\}$. □

The following statement gives the equivalence of the classical co-normal derivative (in the trace sense) and the canonical co-normal derivative, for functions from $H^s(\Omega)$, $s > \frac{3}{2}$.

COROLLARY 3.13. *If $u \in H^s(\Omega)$, $s > \frac{3}{2}$, then $T^+u = T_c^+u$.*

Proof. If $u \in H^s(\Omega)$, $s > \frac{3}{2}$, then $u \in H^{s,-\frac{1}{2}}(\Omega; L) \subset H^{1,-\frac{1}{2}}(\Omega; L)$ by Remark 3.4. Let $\{u_k\} \in \mathcal{D}(\overline{\Omega})$ be a sequence such that $\|u_k - u\|_{H^s(\Omega)} \to 0$ and thus $\|u_k - u\|_{H^{1,-\frac{1}{2}}(\Omega;L)} \to 0$ as $k \to \infty$. Then

$$\|T^+u - T_c^+u\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq \|T^+u - T_c^+u_k\|_{H^{-\frac{1}{2}}(\partial\Omega)} + \|T_c^+(u_k - u)\|_{H^{-\frac{1}{2}}(\partial\Omega)}, \tag{3.27}$$

where the first norm in the right hand side vanishes as $k \to \infty$ by Lemma 3.12, while for the second norm we have,

$$\|T_c^+(u_k - u)\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq \| \sum_{i,j=1}^{n} a_{ij}\gamma^+[\partial_j(u_k - u)]n_j\|_{L_2}(\partial\Omega) \leq \max_{\partial\Omega} |a| \; \|u_k - u\|_{H^s(\Omega)} \to 0, \quad k \to \infty.$$

□

Let us prove now that the classical and canonical co-normal derivatives coincide also in another case, when the both do exist. First note, that $C^1(\overline{\Omega}) \subset H^1(\Omega)$ for bounded domain Ω and $C^1(\overline{\Omega'}) \subset H^1(\Omega')$ for any bounded subdomain Ω' of unbounded domain Ω .

COROLLARY 3.14. *Let $u \in C^1(\overline{\Omega}) \bigcap H^{1,t}_{loc}(\Omega; L)$ for some $t \in (-\frac{1}{2}, \frac{1}{2})$ and $\partial\Omega \in C^1$. Then $T^+u = T_c^+u$.*

Proof. Evidently $T_c^+u, T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$. Let the subdomain Ω'_ϵ be such that $\bar{\Omega}'_\epsilon \subset \Omega$ and its boundary $\partial\Omega'_\epsilon$ is equidistant from $\partial\Omega$, namely, $\partial\Omega'_\epsilon = \{\ y - \epsilon n(y)\ :\ y \in \partial\Omega\ \}$, where $\epsilon > 0$ is sufficiently small. Let also $\Omega_\epsilon := \Omega \setminus \overline{\Omega'}_\epsilon$ be the layer between $\partial\Omega$ and $\partial\Omega'_\epsilon$. For any $v \in \mathcal{D}(\overline{\Omega})$ we have,

$$\langle T_c^+u - T^+u, \gamma^+v\rangle_{\partial\Omega} = [\langle T_c^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T_c^+u, \gamma^+v\rangle_{\partial\Omega'_\epsilon}] - [\langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T_c^+u, \gamma^+v\rangle_{\partial\Omega'_\epsilon}]. \tag{3.28}$$

The first square bracket in the right hand side tends to zero as $\epsilon \to 0$ due to continuity of ∇u and v and to the chosen form of $\partial\Omega'_\epsilon \to \partial\Omega$. The membership $u \in H^{1,t}_{loc}(\Omega; L)$ implies $u \in H^{t+2}(\Omega'_\epsilon)$ by Theorem 3.10. Then $T_c^+u = T^+u$ on $\partial\Omega'_\epsilon$ by Corollary 3.13 and for the second bracket in (3.28) we have,

$$\begin{aligned}
[\langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T_c^+u, \gamma^+v\rangle_{\partial\Omega'_\epsilon}] &= \langle T^+u, \gamma^+v\rangle_{\partial\Omega_\epsilon} = \mathcal{E}_{\Omega_\epsilon}(u, v) - \langle L^0u, v\rangle_{\Omega_\epsilon} \\
&\leq C_1 \left[\|u\|_{H^1(\Omega_\epsilon)}\|v\|_{H^1(\Omega_\epsilon)} + \|Lu\|_{H^t(\Omega_\epsilon)}\|v\|_{H^{-t}(\Omega_\epsilon)}\right] \\
&\leq C_2 \left[\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega_\epsilon)} + \|Lu\|_{H^t(\Omega)}\|v\|_{H^1(\Omega_\epsilon)}\right] \to 0 \qquad \epsilon \to 0
\end{aligned}$$

since $L^0u = Lu \in H^t(\Omega) = \widetilde{H}^t(\Omega)$ for $-\frac{1}{2} < t < \frac{1}{2}$ and the Lebesgue measure of Ω_ϵ tends to zero.

□

REMARK 3.15. *Note that the operator $\check{L} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega)$ defined by (3.7), is not generally defined for $s = 1 \pm \frac{1}{2}$. Thus the generalised co-normal derivative $T^+(\tilde{f}, u) \in H^{s-\frac{3}{2}}(\partial\Omega)$ and the canonical co-normal derivative $T^+u \in H^{s-\frac{3}{2}}(\partial\Omega)$ expressed through $\check{L}$ by (3.10) and (3.21), respectively, are not well defined for $s = 1 \pm \frac{1}{2}$ either. On the other hand for $s \geq \frac{3}{2}$, evidently, the operator $\check{L} : H^s(\Omega) \to \widetilde{H}^{\sigma-2}(\Omega)$ is bounded for any $\sigma \in (\frac{1}{2}, \frac{3}{2})$, implying that $T^+(\tilde{f}, u) \in H^{\sigma-\frac{3}{2}}(\partial\Omega)$ is well defined if $u \in H^s(\Omega)$, $\tilde{f} \in H^{\sigma-2}(\Omega)$, while $T^+u \in H^{\sigma-\frac{3}{2}}(\partial\Omega)$ is well defined if $u \in H^{s,-\frac{1}{2}}(\Omega; L)$. Moreover $T^+u = T_c^+u \in H^{s-\frac{3}{2}}(\partial\Omega)$ for $\frac{3}{2} < s < \frac{5}{2}$ (on Lipschitz boundary $\partial\Omega$) if $u \in H^s(\Omega)$, as follows from Corollary 3.13.*

3.4 Formally adjoined PDE and the second Green identity

The PDE system formally adjoined to (3.1) is given in the strong form as

$$L^*v(x) := -\sum_{i,j=1}^{n} \partial_i[\bar{a}_{ji}^\top(x)\,\partial_j v(x)] - \sum_{j=1}^{n} \partial_j[\,\bar{b}_j^\top(x)v(x)] + \bar{c}^\top(x)v(x) = f(x), \quad x \in \Omega. \tag{3.29}$$

Similar to the operator L, for any $v \in H^{2-s}(\Omega)$, $s \in \mathbb{R}$, the weak form of the operator L^* is

$$\langle L^*v, u\rangle_\Omega := \mathcal{E}^*(v,u) \quad \forall u \in \widetilde{H}^s(\Omega), \tag{3.30}$$

where

$$\mathcal{E}^*(v,u) = \overline{\mathcal{E}(\bar{u},\bar{v})} = \Phi(\bar{u}, v) \tag{3.31}$$

is the bilinear form and so defined operator $L^* : H^{2-s}(\Omega) \to H^{-s}(\Omega) = [\widetilde{H}^s(\Omega)]^*$ is bounded for any $s \in \mathbb{R}$.

For $\frac{1}{2} < s < \frac{3}{2}$ let us consider also the *aggregate* operator $\check{L}^* : H^{2-s}(\Omega) \to \widetilde{H}^{-s}(\Omega) = [H^s(\Omega)]^*$, defined as,

$$\langle \check{L}^*v, u\rangle_\Omega := \mathcal{E}^*(v,u) \quad \forall u \in H^s(\Omega), \tag{3.32}$$

which is evidently bounded. For any $v \in H^{2-s}(\Omega)$, the distribution $\check{L}^*v$ belongs to $\widetilde{H}^{-s}(\Omega)$ and is an extension of the functional $L^*v \in H^{-s}(\Omega)$ from the domain of definition $\widetilde{H}^s(\Omega)$ to the domain of definition $H^s(\Omega)$.

Relations (3.32), (3.31) and (3.7) lead to the *aggregate second Green identity*,

$$\langle \check{L}u, \bar{v}\rangle_\Omega = \langle u, \overline{\check{L}^*v}\rangle_\Omega, \quad u \in H^s(\Omega), \quad v \in H^{2-s}(\Omega), \quad \frac{1}{2} < s < \frac{3}{2}. \tag{3.33}$$

For a sufficiently smooth function v, let

$$\widetilde{T}_c^+ v(x) := \sum_{i,j=1}^{n} \bar{a}_{ji}^\top(x)\,\gamma^+[\partial_j v(x)]\nu_i(x) + \sum_{i=1}^{n} \bar{b}_i^\top(x)\gamma^+ v(x)\nu_i \tag{3.34}$$

be the strong (classical) modified co-normal derivative (it corresponds to $\widetilde{\mathfrak{B}}_\nu v$ in [13]), associated with the operator L^*.

If $v \in H^{2-s}(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, and $L^*v = \tilde{f}_*|_\Omega$ in Ω for some $\tilde{f}_* \in \widetilde{H}^{-s}(\Omega)$, we define *the generalised modified co–normal derivative* $\widetilde{T}^+(\tilde{f}_*, v) \in H^{\frac{1}{2}-s}(\partial\Omega)$, associated with the operator L^*, similar to Definition 3.1, as

$$\left\langle \widetilde{T}^+(\tilde{f}_*, v)\,,\, w\right\rangle_{\partial\Omega} := \mathcal{E}^*(v, \gamma_{-1}w) - \langle \tilde{f}_*, \gamma_{-1}w\rangle_\Omega \quad \forall\, w \in H^{s-\frac{1}{2}}(\partial\Omega).$$

As in Lemma 3.2, this leads to the following first Green identity for the function v,

$$\left\langle \widetilde{T}^+(\tilde{f}_*, v)\,,\, u^+\right\rangle_{\partial\Omega} = \mathcal{E}^*(v,u) - \langle \tilde{f}_*, u\rangle_\Omega \quad \forall\, u \in H^s(\Omega), \tag{3.35}$$

which by (3.31) implies

$$\left\langle u^+, \overline{\widetilde{T}^+(\tilde{f}_*, v)}\right\rangle_{\partial\Omega} = \mathcal{E}(u, \bar{v}) - \langle u, \bar{\tilde{f}}_*\rangle_\Omega \quad \forall\, u \in H^s(\Omega). \tag{3.36}$$

If, in addition, $Lu = \tilde{f}|_\Omega$ in Ω, where $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$, then combining (3.36) and the first Green identity (3.12) for u, we arrive at the following generalised second Green identity,

$$\langle \tilde{f}, \bar{v}\rangle_\Omega \; - \langle u, \bar{\tilde{f}}_*\rangle_\Omega = \left\langle u^+, \overline{\widetilde{T}^+(\tilde{f}_*, v)}\right\rangle_{\partial\Omega} - \left\langle T^+(\tilde{f}, u)\,,\, \overline{v^+}\right\rangle_{\partial\Omega}. \tag{3.37}$$

Taking in mind (3.35), (3.32) and (3.12), (3.7), this, of course, leads to the aggregate second Green identity (3.33).

If $\frac{1}{2} < s < \frac{3}{2}$ and $v \in H^{2-s,-\frac{1}{2}}(\Omega; L^*)$, then similar to Definitions 3.6 and 3.8 we can introduce the *canonical extension* L^{*0} of the operator L^*, and the *canonical modified co-normal derivative* $\widetilde{T}^+v := \widetilde{T}^+(L^{*0}v, v) \in H^{\frac{1}{2}-s}(\partial\Omega)$. In this case the second Green identity (3.37) takes form

$$\langle \tilde{f}, \bar{v}\rangle_\Omega \; - \left\langle u, \overline{L^{*0}v} \right\rangle_\Omega = \left\langle u^+, \overline{\widetilde{T}^+v} \right\rangle_{\partial\Omega} - \left\langle T^+(\tilde{f}, u), \overline{v^+} \right\rangle_{\partial\Omega} \tag{3.38}$$

for $u \in H^s(\Omega)$, $Lu = \tilde{f}|_\Omega$ in Ω, where $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$. This form was a starting point in formulation and analysis of the extended boundary-domain integral equations in [14].

If, moreover, $u \in H^{s,-\frac{1}{2}}(\Omega; L)$, we obtain from (3.38) the second Green identity for the canonical extensions and canonical co-normal derivatives,

$$\left\langle L^0 u, \bar{v} \right\rangle_\Omega - \left\langle u, \overline{L^{*0}v} \right\rangle_\Omega = \left\langle u^+, \overline{\widetilde{T}^+v} \right\rangle_{\partial\Omega} - \left\langle T^+u \,,\, \overline{v^+} \right\rangle_{\partial\Omega}. \tag{3.39}$$

Particularly, if $u, v \in H^{1,0}(\Omega; L)$, then (3.39) takes the familiar form, cf. [6, Lemma 3.4],

$$\int_\Omega [\, \overline{v(x)} Lu(x) - u(x) \overline{L^*v(x)} \,] dx = \left\langle u^+, \overline{\widetilde{T}^+v} \right\rangle_{\partial\Omega} - \left\langle T^+u \,,\, \overline{v^+} \right\rangle_{\partial\Omega}.$$

4 Local solution regularity for strongly elliptic system with Hölder-Lipschitz coefficients

In this section, after introducing some Hölder-Lipschitz type spaces for coefficients and giving statements on boundedness of the considered PDEs with such coefficients, we extend the well know result about the local regularity of elliptic PDE solution, Theorem 3.10, to the case of relaxed smoothness of the PDE coefficients. This will be used then to prove a counterparts of Lemma 3.11 and Corollary 3.14 in Section 5, where all results of Section 3 are generalised to non-smooth coefficients.

For an open set Ω let $W^\mu_\infty(\Omega)$, $\mu \geq 0$, be the Sobolev-Slobodetskii space with the norm

$$\|g\|_{W^\mu_\infty(\Omega)} := \sum_{0 \leq |\alpha| \leq \mu} \|\partial^\alpha g\|_{L_\infty(\Omega)} < \infty$$

for integer μ, and

$$\|g\|_{W^\mu_\infty(\Omega)} := \|g\|_{W^{[\mu]}_\infty(\Omega)} + |g|_{W^\mu_\infty(\Omega)} < \infty, \quad |g|_{W^\mu_\infty(\Omega)} := \sum_{|\alpha|=\mu} \left\| \frac{\partial^\alpha g(x) - \partial^\alpha g(y)}{|x-y|^{\mu-[\mu]}} \right\|_{L_\infty(\Omega \times \Omega)}$$

for non-integer μ. Evidently $W^0_\infty(\Omega) = L_\infty(\Omega)$, while (possibly after adjusting functions on zero measure sets) $W^\mu_\infty(\Omega)$ is the usual Hölder space $C^{0,\mu}(\Omega)$ for $0 < \mu < 1$, $W^\mu_\infty(\Omega) = C^{[\mu],\mu-[\mu]}(\Omega)$ for non-integer $\mu > 1$, and $W^\mu_\infty(\Omega)$ is the Lipschitz space $C^{\mu-1,1}(\Omega)$ for integer $\mu \geq 1$, where $[\mu]$ is the integer part of μ.

Let us denote by $\mathbb{R}_+(s)$ the set of all non-negative numbers if s is integer and of all positive numbers otherwise.

DEFINITION 4.1. *For an open set Ω and $\mu \geq 0$ let $\bar{C}^\mu(\overline{\Omega})$ be the set of restrictions on Ω of all functions from $W^\mu_\infty(\mathbb{R}^n)$, equipped with the norm $\|v\|_{\bar{C}^\mu(\overline{\Omega})} = \inf_{V|_\Omega = v} \|V\|_{W^\mu_\infty(\mathbb{R}^n)}$. Evidently $\|v\|_{W^\mu_\infty(\Omega)} \leq \|v\|_{\bar{C}^\mu(\overline{\Omega})}$, $\bar{C}^0(\overline{\Omega}) = L_\infty(\Omega)$, $\bar{C}^\mu(\overline{\Omega}) \subset W^\mu_\infty(\Omega)$ for $\mu > 0$.*

The set $\bar{C}^\mu_+(\overline{\Omega})$ is defined as $\bar{C}^\mu(\overline{\Omega})$ for integer non-negative μ and as $\bigcup_{\nu>\mu} \bar{C}^\nu(\overline{\Omega})$ for non-integer nonnegative μ. Evidently $g \in \bar{C}^\mu_+(\overline{\Omega})$ if and only if $g \in \bar{C}^{\mu+\epsilon}(\overline{\Omega})$ for some $\epsilon \in \mathbb{R}_+(\mu)$.

THEOREM 4.2. *Let Ω be an open set, $s \in \mathbb{R}$, $g_1 \in \bar{C}^\mu(\overline{\Omega})$, $\mu - |s| \in \mathbb{R}_+(s)$. Then $g_1 g_2 \in H^s(\Omega)$ for every $g_2 \in H^s(\Omega)$, and*

$$\|g_1 g_2\|_{H^s(\Omega)} \leq C\|g_1\|_{\bar{C}^\mu(\overline{\Omega})}\|g_2\|_{H^s(\Omega)}.$$

Proof. Note that the theorem is close to the statement given in [7, Theorem 1.4.1.1] without proof.

Let first $\Omega = \mathbb{R}^n$. The case $s = 0$ is evident. For $s > 0$ the estimate is obtained from [20, Theorem 2(b)] with parameters $s_1 = \mu$, $s_2 = s$, $p_1 = \infty$, $q_1 = p_2 = q_2 = p = q = 2$ there (see also [7, Theorem 1.4.4.2]). A simpler proof for all $s \in \mathbb{R}$ is available in [3, Theorems 11-13].

When $\Omega \neq \mathbb{R}^n$, let $G_2 \in H^s(\mathbb{R})$ and $G_1 \in W^s_\infty(\mathbb{R})$ are such that $g_2 = G_2|_\Omega$, $\|G_2\|_{H^s(\mathbb{R}^n)} = \|g_2\|_{H^s(\Omega)}$, $g_1 = G_1|_\Omega$, $\|G_1\|_{W^\mu_\infty(\mathbb{R}^n)} < 2\|g_1\|_{\bar{C}^\mu(\overline{\Omega})}$. Then

$$\|g_1 g_2\|_{H^s(\Omega)} \leq \|G_1 G_2\|_{H^s(\mathbb{R}^n)} \leq C\|G_1\|_{W^\mu_\infty(\mathbb{R}^n)}\|G_2\|_{H^s(\mathbb{R}^n)} < 2C\|g_1\|_{\bar{C}^\mu(\overline{\Omega})}\|g_2\|_{H^s(\Omega)}.$$

□

Note that the condition on g_1 in Theorem 4.2 is equivalent to the membership $g_1 \in \bar{C}^{|s|}_+(\overline{\Omega})$.

Let $u \in H^s(\Omega)$, $s \in \mathbb{R}$, $a, b \in \bar{C}^{\mu_a}_+(\overline{\Omega})$, $c \in \bar{C}^{\mu_c}_+(\overline{\Omega})$, where $\mu_a = 0$ if $s \geq 1$ and $\mu_a = |s-1|$ otherwise, $\mu_c = 0$ if $s \geq 0$ and $\mu_c = |s|$ otherwise. Then equation (3.1) in the distributional sense has the same form (3.3)-(3.5), where the bilinear form $\mathcal{E}(u, v)$, given by (3.5), is well defined for $v \in \mathcal{D}(\Omega)$.

DEFINITION 4.3. *We will say that the coefficients of equation* (3.1) *belong to the class $\mathcal{C}^\sigma_+(\overline{\Omega})$, i.e. $\{a, b, c\} \in \mathcal{C}^\sigma_+(\overline{\Omega})$, if $a \in \bar{C}^{|\sigma|}_+(\overline{\Omega})$, $b \in \bar{C}^{\mu_b(\sigma)}_+(\overline{\Omega})$, $\mu_b(\sigma) = \max(0, |\sigma - \frac{1}{2}| - \frac{1}{2})$, $c \in \bar{C}^{\mu_c(\sigma)}_+(\overline{\Omega})$, $\mu_c(\sigma) = \max(0, |\sigma| - 1)$.*

For an open set Ω, as usual, $\{a, b, c\} \in \mathcal{C}^\sigma_{+loc}(\Omega)$ means that $\{a, b, c\} \in \mathcal{C}^\sigma_+(\overline{\Omega'})$ for any $\overline{\Omega'} \subset \Omega$.

By Theorem 4.2 we immediately have the following statement.

THEOREM 4.4. *If $s \in \mathbb{R}$ and $\{a, b, c\} \in \mathcal{C}^{s-1}_+(\overline{\Omega})$, then bilinear form* (3.5), *$\mathcal{E} : \{H^s(\Omega), \widetilde{H}^{2-s}(\Omega)\} \to \mathbb{C}$ is bounded and expressions* (3.6), (3.30) *define bounded linear operators $L : H^s(\Omega) \to H^{s-2}(\Omega)$, $L^* : H^{2-s}(\Omega) \to H^{-s}(\Omega)$.*

The local regularity of solution to PDEs (3.1) and (3.29) for the case of infinitely smooth coefficients, Theorem 3.10, is well known (see e.g. [18, 1, 12]). The case $a, b, c \in C^{k,1}(\bar{\Omega})$, $s_1 = 1$, $s_2 = k$ with integer $k \geq 0$ can be found in [13, Theorem 4.16], and the case $a \in C^{0,1}(\bar{\Omega})$, $b = 0$, $c = const$, $s_2 \in (-3/2, -1/2)$ in [17, Theorem 4], extended in [4] to general elliptic systems with all coefficients from $C^{0,1}(\bar{\Omega})$. For arbitrary Hölder coefficients the corresponding result formulated below seems to be new.

THEOREM 4.5. *Let Ω be an open set in $\mathbb{R}^n$, $s_1 \in \mathbb{R}$, $m \geq 1$, $u \in H^{s_1}_{loc}(\Omega)^m$, $f \in H^{s_2}_{loc}(\Omega)^m$, $s_2 > s_1 - 2$. If u satisfies*

(a) strongly elliptic system (3.1), *$Lu = f$, in Ω with $\{a, b, c\} \in \mathcal{C}^{s_1-1}_{+loc}(\Omega) \bigcap \mathcal{C}^{s_2+1}_{+loc}(\Omega)$ or*

(b) strongly elliptic system (3.29), *$L^*u = f$, in Ω with $\{a, b, c\} \in \mathcal{C}^{1-s_1}_{+loc}(\Omega) \bigcap \mathcal{C}^{-s_2-1}_{+loc}(\Omega)$,*

then $u \in H^{s_2+2}_{loc}(\Omega)^m$.

Note that the theorem hypothesis $s_2 > s_1 - 2$ implies that either $s_1 \neq 1$ or $s_2 \neq -1$ and thus $a \in \bar{C}^\mu_{loc}(\Omega)$ for some $\mu > 0$ and particularly, $a \in C(\Omega)$ (maybe after adjusting a on a zero measure set, that we will assume to be done). To prove the theorem, we need first to prove Lemma 4.6 and Corollary 4.7 below.

LEMMA 4.6. *Let $s, t \in \mathbb{R}$, $w \in H^s(\mathbb{R}^n)^m$, $g \in W^{\sigma+\varepsilon}_\infty(\mathbb{R}^n)^m$, $\sigma = |s - \frac{t-1}{2}| + |\frac{|t|-1}{2}| + 1$ and $\varepsilon \in \mathbb{R}_+(\sigma)$. Then $\mathcal{J}^t(gw) - g\mathcal{J}^t w \in H^{s-t+1}(\mathbb{R}^n)^m$ and*

$$\|\mathcal{J}^t(gw) - g\mathcal{J}^t w\|_{H^{s-t+1}(\mathbb{R}^n)^m} \leq C|t|\|g\|_{W^{\sigma+\varepsilon}_\infty(\mathbb{R}^n)^m}\|w\|_{H^s(\mathbb{R}^n)^m}. \tag{4.1}$$

Proof. The proof below is given for $m = 1$, generalisation to the vector case, $m > 1$, is evident.

$$K(\xi) := \mathcal{F}[\mathcal{J}^t(gw) - g\mathcal{J}^t w](\xi) = (1+|\xi|^2)^{t/2}(\hat{g} * \widehat{w})(\xi) - (\hat{g} * \mathcal{F}[\mathcal{J}^t w])(\xi) =$$
$$\int_{\mathbb{R}^n} [(1+|\xi|^2)^{t/2} - (1+|\xi-\eta|^2)^{t/2}]\hat{g}(\eta)\widehat{w}(\xi-\eta)d\eta = \int_{\mathbb{R}^n} [(\eta\cdot\xi + \eta\cdot(\xi-\eta)]f(\xi,\xi-\eta)\hat{g}(\eta)\widehat{w}(\xi-\eta)d\eta$$
$$= \int_{\mathbb{R}^n} \widehat{\nabla g}(\eta)\cdot\left(\frac{\xi\widehat{w}(\xi-\eta)}{2\pi i} - \frac{\widehat{\nabla w}(\xi-\eta)}{4\pi^2}\right) f(\xi,\xi-\eta)d\eta.$$

Here

$$f(\xi,\xi-\eta) := \frac{(1+|\xi|^2)^{t/2} - (1+|\xi-\eta|^2)^{t/2}}{|\xi|^2 - |\xi-\eta|^2}$$

and we took into account that $|\xi|^2 - |\xi-\eta|^2 = \eta\cdot\xi + \eta\cdot(\xi-\eta)$.

Using the inequality $|c_1^\beta - c_2^\beta| \le |\beta||c_1 - c_2|(c_1^{\beta-1} + c_2^{\beta-1})$, for any $c_1, c_2 > 0$, $\beta \in \mathbb{R}$, and denoting $p(\xi) = (1+|\xi|^2)^{1/2}$, we have,

$$|f(\xi,\xi-\eta)| \le |t|\frac{p^{t-1}(\xi) + p^{t-1}(\xi-\eta)}{p(\xi) + p(\xi-\eta)} \le |t|\frac{p^{t-1}(\xi) + p^{t-1}(\xi-\eta)}{p(\xi-\eta)},$$

for any $t \in \mathbb{R}$, and the left inequality implies also

$$|\xi f(\xi,\xi-\eta)| \le |t|[p^{t-1}(\xi) + p^{t-1}(\xi-\eta)].$$

Then

$$\begin{aligned}
|K(\xi)| \;\le\; & \frac{|t|}{2\pi}\int_{\mathbb{R}^n}\left[p^{t-1}(\xi) + p^{t-1}(\xi-\eta)\right]|\widehat{\nabla g}(\eta)\widehat{w}(\xi-\eta)|d\eta \\
& + \frac{|t|}{4\pi^2}\int_{\mathbb{R}^n}\left[p^{t-1}(\xi) + p^{t-1}(\xi-\eta)\right]p^{-1}(\xi-\eta)|\widehat{\nabla g}(\eta)\cdot\widehat{\nabla w}(\xi-\eta)|d\eta \\
= \; & \frac{|t|}{2\pi}\int_{\mathbb{R}^n}\left[p^{t-1}(\xi)|\widehat{\nabla g}(\eta)\widehat{w}(\xi-\eta)| + |\widehat{\nabla g}(\eta)p^{t-1}(\xi-\eta)\hat{w}(\xi-\eta)|\right]d\eta \\
& + \frac{|t|}{4\pi^2}\int_{\mathbb{R}^n}\left[p^{t-1}(\xi)|\widehat{\nabla g}(\eta)\cdot[p^{-1}(\xi-\eta)\widehat{\nabla w}(\xi-\eta)]| + |\widehat{\nabla g}(\eta)\cdot[p^{t-2}(\xi-\eta)\widehat{\nabla w}(\xi-\eta)]|\right]d\eta \\
= \; & \frac{|t|}{2\pi}\left[p^{t-1}(\xi)\{|\widehat{\nabla g} * \widehat{w}|\}(\xi) + \{|\widehat{\nabla g} * (p^{t-1}\hat{w})|\}(\xi)\right] \\
& + \frac{|t|}{4\pi^2}\left[p^{t-1}(\xi)\{|\widehat{\nabla g} * (p^{-1}\widehat{\nabla w})|\}(\xi) + \{|\widehat{\nabla g} * (p^{t-2}\widehat{\nabla w})|\}(\xi)\right].
\end{aligned}$$

Taking into account Theorem 4.2, we obtain,

$$\begin{aligned}
& \|\mathcal{J}^t(gw) - g\mathcal{J}^t w\|_{H^{s-t+1}(\mathbb{R}^n)} = \|p^{s-t+1}K\|_{L_2(\mathbb{R}^n)} \\
& \le \frac{|t|}{2\pi}\left\|p^s|\widehat{\nabla g} * \widehat{w}| + p^{s-t+1}|\widehat{\nabla g} * (p^{t-1}\hat{w})|\right\|_{L_2(\mathbb{R}^n)} \\
& \qquad + \frac{|t|}{4\pi^2}\left\|p^s|\widehat{\nabla g} * (p^{-1}\widehat{\nabla w})| + p^{s-t+1}|\widehat{\nabla g} * (p^{t-2}\widehat{\nabla w})|\right\|_{L_2(\mathbb{R}^n)} \\
& = \frac{|t|}{2\pi}\left[\|(\nabla g)w\|_{H^s(\mathbb{R}^n)} + \|(\nabla g)(\mathcal{J}^{t-1}w)\|_{H^{s-t+1}(\mathbb{R}^n)}\right. \\
& \qquad \left. + \|(\nabla g)\cdot(\mathcal{J}^{-1}\nabla w)\|_{H^s(\mathbb{R}^n)} + \|(\nabla g)\cdot(\mathcal{J}^{t-2}\nabla w)\|_{H^{s-t+1}(\mathbb{R}^n)}\right] \\
& \le C_1|t|\left[\|g\|_{W_\infty^{|s|+1+\varepsilon_1}(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)} + \|g\|_{W_\infty^{|s-t+1|+1+\varepsilon_2}(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)}\right. \\
& \qquad \left. + \|g\|_{W_\infty^{|s|+1+\varepsilon_1}(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)} + \|g\|_{W_\infty^{|s-t+1|+1+\varepsilon_2}(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)}\right].
\end{aligned}$$

for any $\varepsilon_1 \in \mathbb{R}_+(s)$, $\varepsilon_2 \in \mathbb{R}_+(s-t)$. That is,

$$\|\mathcal{J}^t(gw) - g\mathcal{J}^t w\|_{H^{s-t+1}(\mathbb{R}^n)} \le 2C_1|t|(\|g\|_{W_\infty^{|s|+1+\varepsilon_1}(\mathbb{R}^n)} + \|g\|_{W_\infty^{|s-t+1|+1+\varepsilon_2}(\mathbb{R}^n)})\|w\|_{H^s(\mathbb{R}^n)}. \tag{4.2}$$

On the other hand, let us denote $v = \mathcal{J}^t w \in H^{s-t}(\mathbb{R}^n)$ and remark that by inequality (4.2), where t is replaced with $-t$ and s with $s-t$, we have,

$$\|\mathcal{J}^t(gw) - g\mathcal{J}^t w\|_{H^{s-t+1}(\mathbb{R}^n)} = \|\mathcal{J}^t[gJ^{-t}v - J^{-t}(gv)]\|_{H^{s-t+1}(\mathbb{R}^n)} = \|gJ^{-t}v - J^{-t}(gv)\|_{H^{s+1}(\mathbb{R}^n)}$$
$$\le 2C_1|t|(\|g\|_{W_\infty^{|s-t|+1+\varepsilon_2}(\mathbb{R}^n)} + \|g\|_{W_\infty^{|s+1|+1+\varepsilon_1}(\mathbb{R}^n)})\|v\|_{H^{s-t}(\mathbb{R}^n)}$$
$$\le 2C_1|t|(\|g\|_{W_\infty^{|s-t|+1+\varepsilon_2}(\mathbb{R}^n)} + \|g\|_{W_\infty^{|s+1|+1+\varepsilon_1}(\mathbb{R}^n)})\|w\|_{H^s(\mathbb{R}^n)}.$$

Inequality (4.1) follows if we remark that

$$\sigma = |s - \frac{t-1}{2}| + |\frac{|t|-1}{2}| + 1 = \min\{\max(|s|+1, |s-t+1|+1), \max(|s-t|+1, |s+1|+1)\}.$$

□

Let us denote by L_0 the principal part of the operator L from (3.1), i.e.,

$$L_0 u(x) := -\sum_{i,j=1}^n \partial_i[a_{ij}(x)\,\partial_j u(x)].$$

Taking in mind that the Bessel potential operator $\mathcal{J}$ commutate with differentiation, Lemma 4.6 implies the following statement.

COROLLARY 4.7. *Let $s, t \in \mathbb{R}$, $u \in H^s(\mathbb{R}^n)^m$, $a \in W_\infty^{\sigma+\varepsilon}(\mathbb{R}^n)^m$, $\sigma = |s - \frac{t+1}{2}| + |\frac{|t|-1}{2}| + 1$, $\varepsilon \in \mathbb{R}_+(\sigma)$. Then $\mathcal{J}^t(L_0 u) - L_0\mathcal{J}^t u \in H^{s-t-1}(\mathbb{R}^n)^m$ and*

$$\|\mathcal{J}^t(L_0 u) - L_0\mathcal{J}^t u\|_{H^{s-t-1}(\mathbb{R}^n)^m} \le 2C|t|\|a\|_{W_\infty^{\sigma+\varepsilon}(\mathbb{R}^n)^m}\|u\|_{H^s(\mathbb{R}^n)^m}.$$

Now we are in a position to return to Theorem 4.5.

PROOF OF THEOREM 4.5. We give only a proof for part (a) of the theorem, organised in several steps. The proof for part (b) is similar.

Step (i) As usual, c.f. [12, Chapter 2, Theorem 3.1], let us first consider the case $a = const$, $b = 0$, $c = 0$ and $\Omega = \mathbb{R}^n$. Suppose a function U satisfies the distributional form of equation (3.1), i.e., (3.3)-(3.4). Then the strong ellipticity condition (3.2) implies,

$$C_0|\xi|^2|\hat{U}(\xi)|^2 \le \mathrm{Re}\{\overline{\hat{U}^\top(\xi)}\,\theta\sum_{i,j=1}^n a_{ij}\xi_i\xi_j\hat{U}(\xi)\} = (2\pi)^{-2}\mathrm{Re}\{\overline{\hat{U}^\top(\xi)}\,\theta\hat{f}(\xi)\} \le (2\pi)^{-2}|\hat{U}(\xi)|\,|\theta|\,|\hat{f}(\xi)|,$$

where $C_0 > 0$. Therefore also

$$C_1^2(1+|\xi|^2)^2|\hat{U}(\xi)|^2 \le |\hat{f}(\xi)|^2 + 2C_1^2|\hat{U}(\xi)|^2,$$

implying

$$C_1^2\|U\|^2_{H^{s+2}(\mathbb{R}^n)} \le 2\|f\|^2_{H^s(\mathbb{R}^n)} + 2C_1^2\|U\|^2_{H^s(\mathbb{R}^n)} \tag{4.3}$$

for any $s \in \mathbb{R}$.

Step (ii) Let now the coefficients $\{a,b,c\} \in \mathcal{C}^{s_1-1}_{+loc}(\Omega)\bigcap\mathcal{C}^{s_2+1}_{+loc}(\Omega)$ are not generally constant, Ω is not generally $\mathbb{R}^n$, and $u \in H^{s_1}_{loc}(\Omega)$. Let $B_\rho = B_{y,\rho} \subset \Omega' \Subset \Omega$ denote an open ball of radius ρ centred at a point $y \in \Omega$. Let a, b, c and u be extended outside Ω' to $\{a',b',c'\} \in \mathcal{C}^{s_1-1}_{+}(\mathbb{R}^n)\bigcap\mathcal{C}^{s_2+1}_{+}(\mathbb{R}^n)$ and $u' \in H^{s_1}(\mathbb{R}^n)$, and we will further drop primes for brevity.

Let $\eta \in \mathcal{D}(B_\rho)$ be a cut-off function such that $\eta(x) = 1$ in $B_{\rho/2}$. Then $U(x) := \eta(x)u(x)$ is compactly supported in B_ρ and satisfies equation

$$L_{0y}U = \eta f + L_\eta u - L_0^- U \quad \text{in } \mathbb{R}^n. \tag{4.4}$$

Here L_{0y} is the principal part of the operator with the coefficient matrix $a(y)$, i.e. constant in x,

$$\begin{aligned} L_\eta u &= -\sum_{i,j=1}^n (\partial_i\eta)a_{ij}\partial_j u - \sum_{i,j=1}^n \partial_i[(\partial_j\eta)a_{ij}u] - \sum_{j=1}^n \eta b_j\partial_j u - \eta c u, \\ L_0^- U &= -\sum_{i,j=1}^n \partial_i(a_{ij}^-\partial_j U), \end{aligned} \tag{4.5}$$

where $a^-(x) = a(x) - a(y)$.

If $s_2+1 \le s_1 < s_2+2$, then by Theorem 4.2,

$$\begin{aligned} \|L_\eta u\|_{H^{s_2}(\mathbb{R}^n)} &\le C_2\left[\|(\nabla\eta)a\nabla u\|_{H^{s_2}(\mathbb{R}^n)} + \|(\nabla\eta)au\|_{H^{s_2+1}(\mathbb{R}^n)} + \|\eta b\nabla u\|_{H^{s_2}(\mathbb{R}^n)} + \|\eta c u\|_{H^{s_2}(\mathbb{R}^n)}\right] \\ &\le CC_2\left[\|\nabla\eta\|_{W^{|s_1-1|+\varepsilon_1}_\infty(\mathbb{R}^n)}\|a\nabla u\|_{H^{s_1-1}(\mathbb{R}^n)} + \|\nabla\eta\|_{W^{|s_2+1|+\varepsilon_2}_\infty(\mathbb{R}^n)}\|au\|_{H^{s_2+1}(\mathbb{R}^n)} + \right. \\ &\left. \|\eta\|_{W^{|s_2|+\varepsilon_2}_\infty(\mathbb{R}^n)}\|b\nabla u\|_{H^{s_2}(\mathbb{R}^n)} + \|\eta\|_{W^{|s_2|+\varepsilon_2}_\infty(\mathbb{R}^n)}\|cu\|_{H^{s_2}(\mathbb{R}^n)}\right] \le C_3(\rho)\|u\|_{H^{s_1}(\mathbb{R}^n)}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} C_3(\rho) := CC_2 &\left[\|\nabla\eta\|_{W^{|s_1-1|+\varepsilon_1}_\infty(\mathbb{R}^n)}\|a\|_{W^{|s_1-1|+\varepsilon_1}_\infty(\mathbb{R}^n)} + \|\nabla\eta\|_{W^{|s_2+1|+\varepsilon_2}_\infty(\mathbb{R}^n)}\|a\|_{W^{|s_2+1|+\varepsilon_2}_\infty(\mathbb{R}^n)}\right. \\ &\left. + \|\eta\|_{W^{|s_2|+\varepsilon_2}_\infty(\mathbb{R}^n)}\|b\|_{W^{\mu_b^0+\varepsilon_b^0}_\infty(\mathbb{R}^n)} + \|\eta\|_{W^{|s_2|+\varepsilon_2}_\infty(\mathbb{R}^n)}\|c\|_{W^{\mu_c^0+\varepsilon_c^0}_\infty(\mathbb{R}^n)}\right], \end{aligned} \tag{4.7}$$

where by Definition 4.3,

$$\begin{aligned} \mu_b^0 &= \min\{|s| : s_2 \le s \le s_1 - 1\} = \max\{\mu_b(s_1-1),\, \mu_b(s_2+1)\}, \\ \mu_c^0 &= \min\{|s| : s_2 \le s \le s_1\} = \max\{\mu_c(s_1-1),\, \mu_c(s_2+1)\}, \end{aligned}$$

and by the theorem hypothesis there exist $\varepsilon_1 \in \mathbb{R}_+(s_1)$, $\varepsilon_2 \in \mathbb{R}_+(s_2)$, $\varepsilon_b^0 \in \mathbb{R}_+(\mu_b^0)$, $\varepsilon_c^0 \in \mathbb{R}_+(\mu_c^0)$ such that the norms of the coefficients a,b,c are bounded in (4.7).

Since $a^- \in C(\bar{B}_\rho)$ let us define $a_0^-(x) = \begin{cases} a^-(x), & x \in \bar{B}_\rho \\ a^-(x\rho/|x|), & x \notin \bar{B}_\rho \end{cases}$. Then it is easy to see that $\|a_0^-\|_{W^\mu_\infty(\mathbb{R}^n)} = \|a_0^-\|_{C^\mu(\mathbb{R}^n)} = \|a^-\|_{C^\mu(\bar{B}_\rho)}$, $0 < \mu < 1$. Thus, since $\operatorname{supp} U \subset B$, we have by Theorem 4.2,

$$\begin{aligned} \|L_0^- U\|_{H_2^s(\mathbb{R}^n)} &\le C_4\|a^-\nabla U\|_{H^{s_2+1}(\mathbb{R}^n)} = C_4\|a_0^-\nabla U\|_{H^{s_2+1}(\mathbb{R}^n)} \le \\ &CC_4\|a_0^-\|_{W^{|s_2+1|+\varepsilon_2/2}_\infty(\mathbb{R}^n)}\|\nabla U\|_{H^{s_2+1}(\mathbb{R}^n)} \le 2CC_4\|a^-\|_{C^{|s_2+1|+\varepsilon_2/2}(\bar{B}_\rho)}C_5\|U\|_{H^{s_2+2}(\mathbb{R}^n)} \end{aligned} \tag{4.8}$$

for any $\varepsilon_2 \in \mathbb{R}_+(s_2)$ such that $|s_2+1| + \varepsilon_2/2 < 1$.

Applying estimate (4.3) to equation (4.4) and taking into account estimates (4.6) and (4.8), we then have for $s_2+1 \le s_1 < s_2+2$, $\quad 0 \le |s_2+1| + \varepsilon_2/2 < 1$,

$$C_6(\rho)\|U\|^2_{H^{s_2+2}(\mathbb{R}^n)} \le 4C_7^2(\rho)\|f\|^2_{H^{s_2}(B_\rho)} + 2C_8^2(\rho)\|u\|^2_{H^{s_1}(B_\rho)}, \tag{4.9}$$

$C_6(\rho) := C_1^2 - 8C^2C_4^2C_5^2\|a^-\|^2_{C^{|s_2+1|+\varepsilon_2/2}(\bar{B}_\rho)},\ C_7(\rho) := C\|\eta\|_{\bar{C}^{|s_2|+\varepsilon_2}(\bar{B}_\rho)},\ C_8^2(\rho) := C_3^2(\rho) + C_1^2C_7^2(\rho).$

Further in this step we prove the theorem under the following conditions (see also Fig. 1),

$$|s_2+1| < 1, \qquad s_2+1 \le s_1 < s_2+2. \tag{4.10}$$

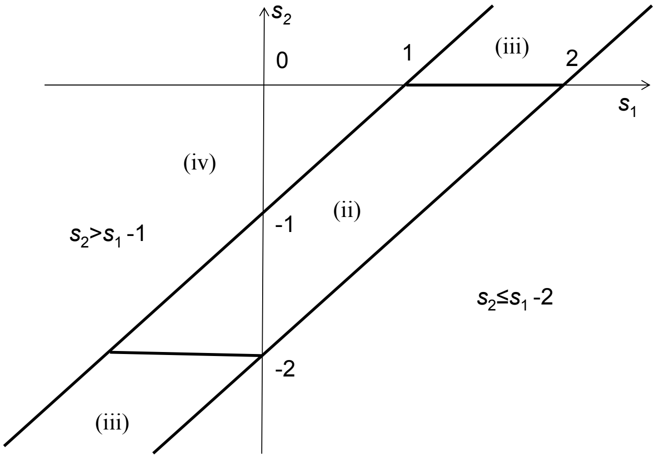

Figure 1: Zones of parameters s_1, s_2 with corresponding proof step numbers.

Let first $s_2 = -1$, and consider estimate (4.9) with $s_2+1 = \varepsilon_2 = 0$. Then for any sufficiently small $\rho > 0$, the norm $\|a^-\|_{C^{|s_2+1|+\varepsilon_2/2}(\bar{B}_\rho)} = \|a^-\|_{C(\bar{B}_\rho)}$ becomes small enough for $C_6(\rho)$ in (4.9) to be positive since $a^-(y) = 0$.

Let now $0 < |s_2+1| < 1$. Due to the theorem hypothesis there exists $\varepsilon_2 \in (0, 1-|s_2+1|)$ such that $a^- \in C^{|s_2+1|+\varepsilon_2}(\bar{B}_\rho)$. This implies that C_3 and thus C_8 are bounded and we have estimate

$$\begin{aligned} \|a^-\|_{C^{|s_2+1|+\varepsilon_2/2}(\bar{B}_\rho)} &\le \|a^-\|_{C(\bar{B}_\rho)} + (2\rho)^{\varepsilon_2/2}|a|_{C^{|s_2+1|+\varepsilon_2}(\bar{B}_\rho)}, \\ |a|_{C^{|s_2+1|+\varepsilon_2}(\bar{B}_\rho)} &:= \sup_{x,x'\in\bar{B}_\rho} \frac{|a(x)-a(x')|}{|x-x'|^{|s_2+1|+\varepsilon_2}} < \infty. \end{aligned}$$

Thus again for any sufficiently small $\rho > 0$, the norm $\|a^-\|_{C^{|s_2+1|+\varepsilon_2/2}(\bar{B}_\rho)}$ becomes small enough for $C_6(\rho)$ in (4.9) to be positive.

This means $U \in H^{s_2+2}(\mathbb{R}^n)$ implying $u \in H^{s_2+2}(B_{y,\rho/2})$. Since the point y is arbitrary, we thus proved the theorem under conditions (4.10).

Step (iii) Let us prove the theorem under conditions

$$|s_2+1| \ge 1, \qquad s_2+1 \le s_1 < s_2+2. \tag{4.11}$$

First of all, for arbitrary $\eta \in \mathcal{D}(\Omega)$ the function $u_\eta = \eta u \in H^{s_1}(\mathbb{R}^n)$ satisfies equation

$$L_0 u_\eta = f_\eta, \quad f_\eta = \eta f + L_\eta u,$$

where L_η is given by (4.5) and $L_\eta u \in H^{s_2}(\mathbb{R}^n)$ by estimate (4.6). This implies $f_\eta \in H^{s_2}(\mathbb{R}^n)$

Let $t = -1 - s_2$ and $v := \mathcal{J}^{-t}u_\eta$. Then $v \in H^{s_1+t}(\mathbb{R}^n)$ and satisfies equation

$$\mathcal{J}^t L_0(v) = f_v, \tag{4.12}$$

where $f_v = f_\eta - [L_0\mathcal{J}^t v - \mathcal{J}^t L_0 v]$.

If $s_2 \le -2$, we can employ Corollary 4.7 for v with $s = s_1 + t = s_1 - 1 - s_2$ and thus $\sigma = 1 - s_1$ due to conditions (4.11), and the theorem hypothesis imply $[L_0\mathcal{J}^t v - \mathcal{J}^t L_0 v] \in H^{s_1-1}(\mathbb{R}^n)$. Then taking in mind the second condition (4.11) again, we obtain $f_v \in H^{s_2}(\mathbb{R}^n)$.

If $s_2 \ge 0$, then similarly the hypothesis of Corollary 4.7 are satisfied for v with $s = 0$ and thus $\sigma = 1 + s_2$, which implies $[L_0\mathcal{J}^t v - \mathcal{J}^t L_0 v] \in H^{s_2}(\mathbb{R}^n)$ and $f_v \in H^{s_2}(\mathbb{R}^n)$.

Thus in both these cases (4.12) gives $L_0(v) = \mathcal{J}^{-t} f_v \in H^{-1}(\mathbb{R}^n)$ implying $v \in H^1_{loc}(\mathbb{R}^n)$ by Step (ii). On the other hand we have,

$$u_\eta = \eta u_\eta = \mathcal{J}^t\mathcal{J}^{-t}\eta u_\eta = \mathcal{J}^t\{\eta\mathcal{J}^{-t}u_\eta + [\mathcal{J}^{-t}(\eta u_\eta) - \eta\mathcal{J}^{-t}u_\eta]\} = \mathcal{J}^t(\eta v) + \mathcal{J}^t[\mathcal{J}^{-t}(\eta u_\eta) - \eta\mathcal{J}^{-t}u_\eta],$$

which by Lemma 4.6 and the second condition (4.11) means $u_\eta \in H^{s_2+2}(\mathbb{R}^n)\bigcup H^{s_1+1}(\mathbb{R}^n) = H^{s_2+2}(\mathbb{R}^n)$. This gives $u \in H^{s_2+2}_{loc}(\Omega)$, which implies the theorem claim under conditions (4.11).

Step (iv) Now we prove the theorem for

$$s_1 - 1 < s_2. \tag{4.13}$$

Since $f \in H^{s_2}_{loc}(\Omega)$, we have by (4.13) also $f \in H^{s_1-1}_{loc}(\Omega)$, i.e., we arrive at the situation covered by Steps (ii) and (iii) with $s_2 = s_1 - 1$, which implies $u \in H^{s_1+1}_{loc}(\Omega)$. If $s_1 < s_2$, we iterate this procedure, obtaining at the end $u \in H^{s_1+k}_{loc}(\Omega)$, where $k = s_2 - s_1 + 1$ if $s_2 - s_1$ is integer and k is the integer part of $s_2 - s_1 + 2$ otherwise. Recalling that $f \in H^{s_2}_{loc}(\Omega)$, we can apply Steps (ii) and (iii) again, which proves the theorem under condition (4.13). □

5 PDE extensions and co-normal derivatives for Hölder-Lipschitz coefficients

In this section we give further comments on validity of the statements of Section 3 when the PDE coefficients are not infinitely smooth.

Due to Theorem 4.2 and the same argument as for the infinitely smooth coefficients, we have the following statement.

THEOREM 5.1. *If $\frac{1}{2} < s < \frac{3}{2}$ and $\{a, b, c\} \in \mathcal{C}^{s-1}_+(\overline{\Omega})$, then expressions* (3.7), (3.32) *define bounded linear operators $\check{L} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega)$, $\check{L}^* : H^{2-s}(\Omega) \to \widetilde{H}^{-s}(\Omega)$, and the aggregate second Green identity* (3.33) *holds true.*

For $u \in H^s(\Omega)$, $s > \frac{3}{2}$, and $a \in C(\bar{\Omega})$, the strong co–normal derivative $T^+_c u$ given by (3.9) is well defined on $\partial\Omega$ in the sense of traces.

Let $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$ and $\{a, b, c\} \in \mathcal{C}^{s-1}_+(\overline{\Omega})$. Then we can still use Definition 3.1 of the generalised co–normal derivative $T^+(\tilde{f}, u)$, Lemma 3.2 holds true and the weak settings of the BVPs and conclusions about them made in Subsection 3.2 remain valid.

One can observe that the space $H^{s,-\frac{1}{2}}(\Omega; L)$ and thus the canonical extension L^0 are well defined by Definitions 3.3 and 3.6, respectively, when the operator L is well defined, which is particularly the case when $\frac{1}{2} < s < \frac{3}{2}$, $\{a, b, c\} \in \mathcal{C}^{s-1}_+(\overline{\Omega})$. Under these conditions the *canonical* co-normal derivative operator is also well defined by Definition 3.8 and Lemma 3.9 along with relation (3.22) hold true.

To consider the cases when the canonical co-normal derivative T^+u coincides with the strong co-normal derivative $T^+_c u$, we will need higher smoothness of the coefficients than needed for continuity of the PDEs in Theorem 5.1. First of all, we remark by Theorem 4.2 and Definition 4.3 that if $\{a, b, c\} \in \mathcal{C}^{t+1}_+(\overline{\Omega})$, $t \ge -\frac{1}{2}$, then $\mathcal{D}(\overline{\Omega}) \subset H^{s,t}(\Omega; L)$ (and moreover, $\mathcal{D}(\overline{\Omega}) \subset H^{s,t+\epsilon}(\Omega; L)$ for some $\epsilon \in \mathbb{R}_+(t)$) for any $s \in \mathbb{R}$. The following counterpart of Lemma 3.11 holds.

LEMMA 5.2. *If $s \in \mathbb{R}$, $-\frac{1}{2} \le t < \frac{1}{2}$, $\{a,b,c\} \in \mathcal{C}_+^{s-1}(\overline{\Omega}) \bigcap \mathcal{C}_+^{t+1}(\overline{\Omega})$ and the operator L is strongly elliptic on $\overline{\Omega}$, then $\mathcal{D}(\overline{\Omega})$ is dense in $H^{s,t}(\Omega;L)$.*

Proof. The proof coincides with the proof of Lemma 3.11. One should only remark that the hypothesis on the coefficients in Lemma 5.2 imply that since $\phi \in \mathcal{D}(\overline{\Omega})$, then $\phi \in H^{s,t}(\Omega;L)$ in (3.23) and Theorem 4.5(b) is applicable to conclude that (3.24) implies $\tilde{g} \in H^{2-s}_{loc}(\Omega')$. □

Let us prove an analog of Lemma 3.12 for non-smooth coefficients.

LEMMA 5.3. *Let $\frac{1}{2} < s < \frac{3}{2}$, $\{a,b,c\} \in \mathcal{C}_+^{s-1}(\overline{\Omega}) \bigcap \mathcal{C}_+^{\frac{1}{2}}(\overline{\Omega})$, $u \in H^{s,-\frac{1}{2}}(\Omega;L)$, and $\{u_k\} \in \mathcal{D}(\overline{\Omega})$ be a sequence such that $\|u_k - u\|_{H^{s,-\frac{1}{2}}(\Omega;L)} \to 0$ as $k \to \infty$. Then $\|T_c^+ u_k - T^+ u\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \to 0$ as $k \to \infty$.*

Proof. Using the definition of the canonical co-normal derivative by (3.21), we have for any $w \in H^{\frac{3}{2}-s}(\partial\Omega)$,

$$\left\langle T^+ u, w \right\rangle_{\partial\Omega} = \mathcal{E}(u, \gamma_{-1}w) - \langle L^0 u, \gamma_{-1}w\rangle_\Omega = \mathcal{E}(u - u_k, \gamma_{-1}w) + \mathcal{E}(u_k, \gamma_{-1}w) - \langle L^0 u, \gamma_{-1}w\rangle_\Omega.$$

By the lemma hypothesis on the coefficients, there exists $\epsilon \in (0, s - \frac{1}{2})$ such that $\sum_{j=1}^n a_{ij}\partial_j u_k \in H^{\frac{1}{2}+\epsilon}(\Omega)$. Then there exist sequences $\{W_p\}_{p=1}^\infty, \{U_{qi}\}_{q=1}^\infty \in \mathcal{D}(\overline{\Omega})$ such that

$$\lim_{p\to\infty} \|\gamma_{-1}w - W_p\|_{H^{2-s}(\Omega)} = 0, \quad \lim_{q\to\infty} \|\sum_{j=1}^n a_{ij}\partial_j u_k - U_{qi}\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} = 0$$

and we have

$$\begin{aligned}
\mathcal{E}(u_k, \gamma_{-1}w) - \sum_{j=1}^n \langle b_j \partial_j u_k, \gamma_{-1}w\rangle_\Omega - \langle c u_k, \gamma_{-1}w\rangle &= \sum_{i,j=1}^n \langle a_{ij}\partial_j u_k, \partial_i \gamma_{-1}w\rangle_\Omega \\
= \lim_{p,q\to\infty} \sum_{i=1}^n \langle U_{qi}, \partial_i W_p\rangle_\Omega &= \lim_{p,q\to\infty} \sum_{i=1}^n \left\{ \int_{\partial\Omega} U_{qi}\nu_i W_p \, d\Gamma - \int_\Omega (\partial_i U_{qi}) W_p \, d\Omega \right\} \\
&= \sum_{i,j=1}^n \left\{ \int_{\partial\Omega} a_{ij}\partial_j u_k \nu_i w \, d\Gamma - \langle \partial_i(a_{ij}\partial_j u_k), \gamma_{-1}w\rangle_\Omega \right\} \\
&= \left\langle T_c^+ u_k, w \right\rangle_{\partial\Omega} + \langle L^0 u_k, \gamma_{-1}w\rangle_\Omega - \sum_{j=1}^n \langle b_j\partial_j u_k, \gamma_{-1}w\rangle_\Omega - \langle c u_k, \gamma_{-1}w\rangle .
\end{aligned}$$

Thus we obtain,

$$\begin{aligned}
\left\langle T^+ u, w \right\rangle_{\partial\Omega} &= \mathcal{E}(u - u_k, \gamma_{-1}w) + \left\langle T_c^+ u_k, w \right\rangle_{\partial\Omega} + \langle L^0 u_k, \gamma_{-1}w\rangle_\Omega - \langle L^0 u, \gamma_{-1}w\rangle_\Omega \\
&= \mathcal{E}(u - u_k, \gamma_{-1}w) + \left\langle T_c^+ u_k, w \right\rangle_{\partial\Omega} - \langle L^0(u - u_k), \gamma_{-1}w\rangle_\Omega \to \left\langle T_c^+ u_k, w \right\rangle_{\partial\Omega}
\end{aligned}$$

by the convergence of u_k to u as $k \to \infty$. Since T^+u is uniquely determined by u, this implies existence of the limit of the right hand side and its independence of the sequence $\{u_k\}$. □

Note that the class $\mathcal{C}_+^{s-1}(\overline{\Omega}) \bigcap \mathcal{C}_+^{\frac{1}{2}}(\overline{\Omega})$ in Lemma 5.3 coincides with $\mathcal{C}_+^{\frac{1}{2}}(\overline{\Omega})$ if $1 \le s < \frac{3}{2}$. Now we can prove the counterpart of Corollary 3.13.

COROLLARY 5.4. *If $\{a,b,c\} \in \mathcal{C}_+^{\frac{1}{2}}(\overline{\Omega})$ and $u \in H^s(\Omega)$, $s > \frac{3}{2}$, then $T^+u = T_c^+ u$.*

Proof. The proof coincides with the proof of Corollary 3.13 if we remark that $\mathcal{C}_+^0(\overline{\Omega}) \bigcap \mathcal{C}_+^{\frac{1}{2}}(\overline{\Omega}) = \mathcal{C}_+^{\frac{1}{2}}(\overline{\Omega})$ and the first norm in the right hand side of (3.27) vanishes as $k \to \infty$ by Lemma 5.3. □

Let us prove now the counterpart of Corollary 3.14 for non-smooth coefficients.

COROLLARY 5.5. *Let* $\{a,b,c\} \in \mathcal{C}_+^{\frac{1}{2}}(\overline{\Omega})$ *and* $u \in C^1(\overline{\Omega}) \bigcap H^{1,t}_{loc}(\Omega; L)$ *for some* $t \in (-\frac{1}{2}, \frac{1}{2})$ *and* $\partial\Omega \in C^1$. *Then* $T^+u = T^+_c u$.

Proof. The corollary hypothesis and Definition 4.3 imply that there exists $t' \in (-\frac{1}{2}, t)$ such that $\{a,b,c\} \in \mathcal{C}_+^{t'+1}(\overline{\Omega})$ and $u \in C^1(\overline{\Omega}) \bigcap H^{1,t'}_{loc}(\Omega; L)$. The proof then coincides with the proof of Corollary 3.14 if one replaces there t with t' and references to Theorem 3.10 and Corollary 3.13 by references to their counterparts, Theorem 4.5 and Corollary 5.4, respectively. □

For a sufficiently smooth function v, the strong (classical) modified co-normal derivative $\widetilde{T}^+_c v$ given by (3.34) is well defined if $a, b \in C(\overline{\Omega})$. One can readily check that the results of Section 3.4 on the modified co-normal derivatives and different forms of the second Green identity hold true under condition $\{a,b,c\} \in \mathcal{C}_+^{s-1}(\overline{\Omega})$.

A APPENDIX

LEMMA A.1. *There exist a distribution* $w \in H^{-1}_{\partial\Omega}$ *and a function* $f \in L_2(\mathbb{R}^n)$, $f = 0$ *on* Ω^-, *such that* $(w, f)_{H^{-1}(\mathbb{R}^n)} \neq 0$.

Proof. Under the definition (2.4) of the inner product in $H^s(\mathbb{R}^n)$,

$$(w, f)_{H^{-1}(\mathbb{R}^n)} = \langle \overline{w}, \mathcal{J}^{-2} f \rangle_{\mathbb{R}^n}. \tag{A.1}$$

By Theorem 2.5, for any distribution $\overline{w} \in H^{-1}_{\partial\Omega}$ there exists a distribution $v \in H^{-1/2}(\partial\Omega)$ such that

$$\langle \overline{w}, \mathcal{J}^{-2} f \rangle_{\mathbb{R}^n} = \langle v, \gamma \mathcal{J}^{-2} f \rangle_{\partial\Omega}, \tag{A.2}$$

where γ is the trace operator.

Denoting $\Phi = \mathcal{J}^{-2} f \in H^2(\mathbb{R}^n)$, we have, $\mathcal{J}^2 \Phi = f$ in $\mathbb{R}^n$, and taking in mind the explicit representation for the operator $\mathcal{J}^2$, the latter equation can be rewritten as

$$\mathcal{J}^2 \Phi \equiv -\frac{1}{4\pi^2} \Delta \Phi + \Phi = f \quad \text{in } \mathbb{R}^n \tag{A.3}$$

and its solution as

$$\mathcal{J}^{-2} f(y) = \Phi(y) = \mathcal{P} f := \int_\Omega F(x,y) f(x) dx, \quad y \in \mathbb{R}^n.$$

Here $\mathcal{P}$ is the Newton volume potential and $F(x,y)$ is the well known fundamental solution of equation (A.3). For example, for $n = 3$,

$$F(x,y) = C \frac{e^{-2\pi|x-y|}}{|x-y|} \,. \tag{A.4}$$

Then (A.1), (A.2) give,

$$(w, f)_{H^{-1}(\mathbb{R}^n)} = \langle v, \gamma \mathcal{J}^{-2} f \rangle_{\partial\Omega} = \langle v, \gamma \mathcal{P} f \rangle_{\partial\Omega}. \tag{A.5}$$

If we assume $(w, f)_{H^{-1}(\mathbb{R}^n)} = 0$ for any $w \in H^{-1}_{\partial\Omega}$, then (A.5) implies $\gamma \mathcal{P} f = 0$, which is not the case for arbitrary $f \in L_2(\Omega)$ and particularly for $f = 1$ in Ω due to (A.4). □

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