Gap Probabilities in Non-Hermitian Random Matrix Theory

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Abstract

We compute the gap probability that a circle of radius r around the origin contains exactly k complex eigenvalues. Four different ensembles of random matrices are considered: the Ginibre ensembles and their chiral complex counterparts, with both complex ($\beta = 2$) or quaternion real ($\beta = 4$) matrix elements. For general non-Gaussian weights we give a Fredholm determinant or Pfaffian representation respectively, depending on the non-Hermiticity parameter. At maximal non-Hermiticity, that is for rotationally invariant weights, the product of Fredholm eigenvalues for $\beta = 4$ follows from $\beta = 2$ by skipping every second factor, in contrast to the known relation for Hermitian ensembles. On additionally choosing Gaussian weights we give new explicit expressions for the Fredholm eigenvalues in the chiral case, in terms of Bessel-K and incomplete Bessel-I functions. This compares to known results for the Ginibre ensembles in terms of incomplete exponentials. Furthermore we present an asymptotic expansion of the logarithm of the gap probability for large argument r at large N in all four ensembles, up to including the third order linear term. We can provide strict upper and lower bounds and present numerical evidence for its conjectured values, depending on the number of exact zero eigenvalues in the chiral ensembles. For the Ginibre ensemble at $\beta = 2$ exact results were previously derived by Forrester.

1 Introduction

Non-Hermitian Random Matrix Theory (RMT) as introduced by Ginibre [1] is almost as old as the classical Wigner-Dyson ensembles. In the past decade it has enjoyed a revival of interest and many different applications of it have been made, where we refer to [2] for a recent review. However, most works have concentrated on computing the spectral correlation functions, culminating recently in the solution of the Ginibre ensembles of real asymmetric matrices [3].

In this article we will focus on the computation of gap probabilities from which the distribution of individual eigenvalues or spacings can be derived. Once the spectral correlations of an ensemble are known in terms of a kernel of (skew) orthogonal polynomials in the complex plane, one can in principle express the gap probabilities in terms of a Fredholm determinant involving this kernel, as we will show. However, such expressions are in general not very explicit, unless the Fredholm eigenvalues are known. On the other hand an expansion of the Fredholm determinant in terms of integrals over spectral correlation functions converges very rapidly (see e.g. [4]), but nevertheless becomes cumbersome because of having 2D integrals in the complex plane.

Explicit results for the gap probability were first derived for the $\beta = 2$ Ginibre ensemble with unitary invariance at maximal non-Hermiticity [5]. Here the probability $E_0^{(2)}(r)$ that a circle of radius r around the origin is empty is given in terms of a product over incomplete exponentials. The same quantity follows for the $\beta = 4$ Ginibre class [6], where every second term in the product is skipped. This result was then used in [5] for $\beta = 2$ to compute the so-called level spacing distribution in the complex plane, by placing one eigenvalue at the origin and computing the probability to find a second eigenvalue at radius r. Due to the translational invariance with respect to the large-N macroscopic density being constant on a disc the corresponding repulsion of complex eigenvalues is supposed to hold everywhere in the bulk [5].

The first aim of the present work is to extend the above results to the chiral complex ensembles at $\beta = 2$ and 4, computing explicitly the product of Fredholm eigenvalues at finite-N. These ensembles were introduced in [7] ($\beta = 2$) and [8] ($\beta = 4$). One motivation for this is the application of non-Hermitian RMT to Quantum Chromodynamics (QCD) with non-zero chemical potential, and we refer to [9] for references and a review. The RMT predictions in the complex plane can be compared to numerical solutions of QCD. The first comparison on $\mathbb{C}$ was done using the complex level spacing distribution of [5] of the $\beta = 2$ Ginibre ensemble in the bulk of the spectrum [10]. However, in QCD chiral symmetry is very important at the origin, and a successful comparison to individual complex eigenvalues there was made very recently [4]. Our results for the gap in the chiral complex $\beta = 2$ class valid at maximal non-Hermiticity were already announced there. We will give a derivation of this result and extend it to $\beta = 4$.

Apart from these explicit results we give a Fredholm determinant (Pfaffian) form valid for a general weight function and non-Hermiticity parameter at $\beta = 2$ (4). This is shown to be equivalent to a (matrix) eigenvalue equation involving the kernel of (skew) orthogonal polynomials. For rotationally invariant weights it implies that the relation between the gap at $\beta = 2$ and 4 skipping every second Fredholm eigenvalue holds in general, extending this relation known for the Gaussian Ginibre case [6].

The second part is then devoted to the asymptotic expansion of the logarithm of $E_0^{(\beta)}(r)$ in all four Gaussian ensembles at maximal non-Hermiticity. Our motivation here is to extend similar considerations done for Hermitian RMT. These include the so-called Widom-Dyson constant to the third order, and the first computation in [11] was made rigorous only very recently [12].

Our approach will be more heuristic in the sense that we give an exact derivation of the first two terms using standard asymptotic techniques as the Stirling formula. For our corresponding third order linear terms we can provide strict upper and lower bounds for their coefficients, as well as numerical evidence for their conjectured values. Our results are consistent with the exact results for the Ginibre ensemble at $\beta = 2$ given in [13] to the fourth order. The chiral ensembles which are our main focus depend explicitly on the number $\nu\frac{\beta}{2}$ of exact zero-eigenvalues in the third order linear term.

Our paper is organised as follows. In Section 2 we define our ensembles and correlation functions. Section 3 first introduces Fredholm theory in the generic non-rotationally invariant case, followed by a calculation of the new product formulae for $E_0^{(\beta)}(r)$ for both Gaussian chiral complex ensembles at maximal non-Hermiticity. Section 4 is devoted to a detailed analysis of our asymptotic expansion of $\log[E_0^{(\beta)}(r)]$ for Ginibre $\beta = 2$ at large N, and Section 5 summarises and compares the corresponding results for the other three ensembles, including a small radius expansion. Our concluding remarks in Section 6 are followed by several appendices where some technical results are collected.

2 Definitions

In this section we collect together the definitions of the random matrix ensembles to be considered, starting with the Gaussian Ginibre ensembles and then moving to their chiral complex counterparts. We then define all density correlation functions, gap probabilities and individual eigenvalue distributions and point out their mutual relationships.

2.1 Ginibre ensembles

The Ginibre ensembles depending on a non-Hermiticity parameter τ, also known as Ginibre-Girko or elliptic ensembles, are defined as (see e.g. [14, 15])

$$\mathcal{Z}_{Gin}^{(\beta)} \sim \int d\Phi \exp\left[-\frac{N}{1-\tau^2}\mathrm{Tr}\Big(\Phi^\dagger\Phi - \frac{\tau}{2}(\Phi^2 + \Phi^{\dagger\, 2})\Big)\right] \ , \quad \tau \in [0, 1) \ . \tag{2.1}$$

Here Φ is a complex non-Hermitian $N \times N$ matrix for $\beta = 2$ or quaternion real matrix for $\beta = 4$. This ensemble can also be thought of as a two-matrix model $\Phi = H_1 + ivH_2$, with $v = [(1-\tau)/(1+\tau)]^{\frac{1}{2}}$, being composed of two Hermitian (self dual) matrices $H_{1,2}$ with distribution $\exp[-N\mathrm{Tr}H_j^2/(1+\tau)]$. In the limit $\tau \to 1$ the Gaussian Unitary or Symplectic Ensemble is recovered.

In both ensembles one can go to a complex eigenvalues basis $z_{j=1,\ldots,N}$ of the matrix Φ

$$\mathcal{Z}_{Gin}^{(\beta)} \equiv \prod_{j=1}^{N}\int_{\mathbb{C}} d^2z_j \ w_{Gin}(z_j) \ \mathcal{J}_N^{(\beta)}(\{z\}) \tag{2.2}$$

with Jacobian

$$\mathcal{J}_N^{(\beta=2)}(\{z\}) \equiv \prod_{k>l}^{N}|z_k - z_l|^2 = |\Delta_N(\{z\})|^2 \ , \tag{2.3}$$

$$\mathcal{J}_N^{(\beta=4)}(\{z\}) \equiv \prod_{k>l}^{N}|z_k - z_l|^2 \ |z_k - z_l^*|^2 \prod_{h=1}^{N}|z_h - z_h^*|^2 = \prod_{h=1}^{N}(z_h - z_h^*)\Delta_{2N}(\{z, z^*\}) \ . \tag{2.4}$$

Both can be expressed through a Vandermonde determinant $\Delta_N(z)$, but in a way different from the Hermitian ensembles with real eigenvalues. In $\Delta_{2N}(\{z, z^*\})$ the variables are ordered as $z_1, z_1^*, z_2, \ldots$. The exponential weight function is given simply by

$$w_{Gin}(z) \equiv \exp\left[-\frac{N}{1-\tau^2}\Big(|z|^2 - \frac{\tau}{2}(z^2 + z^{*\,2})\Big)\right] \ , \tag{2.5}$$

for both $\beta = 2, 4$. In the case $\tau = 0$ called maximal non-Hermiticity it becomes rotationally invariant, a case we will study in great detail later.

At $\beta = 2$ the ensemble can be solved using Hermite polynomials as orthogonal polynomials in the complex plane, and we refer to [16] for an exhaustive discussion. At $\beta = 4$ the solution is given in terms of skew orthogonal polynomials in the complex plane, again constructed from Hermite polynomials, and a complete discussion is given in [17]. At $\tau = 0$ a basis of (skew) orthogonal polynomials is always constructed from monic powers instead of Hermite polynomials.

2.2 Chiral ensembles

Next we turn to the chiral complex counterparts of these Ginibre ensembles. They are defined as Gaussian two-matrix models [7, 8]

$$\mathcal{Z}_{ch}^{(\beta)} \sim \int d\Phi d\Psi \; \exp\left[-\frac{N(1+\mu^2)}{4\mu^2}\mathrm{Tr}(\Phi^\dagger\Phi \;+\; \Psi\Psi^\dagger) - \frac{N(1-\mu^2)}{4\mu^2}\mathrm{Tr}(\Psi\Phi \;+\; \Phi^\dagger\Psi^\dagger)\right] \;,\quad \mu \in (0,1] \;. \tag{2.6}$$

Here Φ and $\Psi^\dagger$ are two matrices of rectangular size $(N+\nu) \times N$ with either complex ($\beta = 2$) or quaternion real ($\beta = 4$) elements, without further symmetry properties. The two matrices can be composed from two matrices $\Phi = iH_1 + \mu H_2$, and $\Psi = iH_1^\dagger + \mu H_2^\dagger$ with $H_{1,2}$ being non-Hermitian (quaternion real) with distribution $\exp[-N\mathrm{Tr}H_j^\dagger H_j]$ each. In the limit $\mu \to 0$ leading to $\Phi = -\Psi^\dagger$ we recover the chiral Gaussian Unitary or Symplectic Ensemble [1].

We are interested in the complex eigenvalues of the matrix $\begin{pmatrix} 0 & \Phi \\ \Psi & 0 \end{pmatrix}$, or equivalently of $\Phi\Psi$. This change of variables is tedious, by first going to complex eigenvalues of Φ and Ψ each and then integrating out one set. With details given in [7] ($\beta = 2$) and [8] ($\beta = 4$) we only give the answer

$$\mathcal{Z}_{ch}^{(\beta)} \;\equiv\; \prod_{j=1}^{N} \int_{\mathbb{C}} d^2 z_j \; w_\nu^{(\beta)}(z_j)\; \mathcal{J}_N^{(\beta)}(\{z^2\}) \tag{2.7}$$

with the following non-Gaussian weight function

$$w_\nu^{(\beta)}(z) \;\equiv\; |z|^{\beta\nu+2} K_{\frac{1}{2}\beta\nu}\left(N|z|^2\frac{1+\mu^2}{2\mu^2}\right) \exp\left[N(z^2+z^{*\,2})\frac{1-\mu^2}{4\mu^2}\right] \;, \tag{2.8}$$

which now depends on β and the number of exact zero-eigenvalues $\nu \geq 0$. We note that the Jacobian $\mathcal{J}$ only differs from the Ginibre ensembles by inserting *squared* variables.

The chiral ensembles eq. (2.7) can be solved in terms of Laguerre polynomials as orthogonal [7, 18, 8] and skew orthogonal [8] polynomials in the complex plane for $\beta = 2$ and 4 respectively.

Regarding the weight we note that the Bessel-K function again becomes an exponential at half integer values of the index, e.g. $K_{\pm\frac{1}{2}}(x) = \sqrt{\frac{2\pi}{x}}\, e^{-x} \sim \lim_{x\to\infty} K_\nu(x)$, or in the asymptotic limit of large arguments which is also reached when taking $\mu \to 0$. In the opposite limit $\mu = 1$ at maximal non-Hermiticity we obtain again a rotationally invariant weight.

2.3 Correlation functions

Let us now turn to the correlation functions to be calculated, which we define for arbitrary weight functions. We define the k-th gap probability with respect to radial ordering to give the probability

[1] The non-Hermiticity parameters of the two sets of ensembles can be brought onto an equal footing by mapping $\mu^2 = (1-\tau)/(1+\tau)$.

that k (independent) complex eigenvalues lie inside the circle of radius r around the origin, and $N-k$ lie outside. Here we do not count those ν eigenvalues which are always zero by construction in the chiral ensembles. For the chiral ensembles the definition reads [2],

$$E_k^{(\beta)}(r) \equiv \frac{1}{\mathcal{Z}_{ch}^{(\beta)}} \frac{N!}{(N-k)!k!} \prod_{j=1}^{k} \int_0^r dr_j r_j \prod_{l=k+1}^{N} \int_r^\infty dr_l r_l \prod_{n=1}^{N} \int_0^{2\pi} d\theta_n w_\nu^{(\beta)}(z_n)\ \mathcal{J}_N^{(\beta)}(\{z^2\}) \ , \tag{2.9}$$

where we have switched to polar coordinates $z_n = r_n e^{i\theta_n}$. The quantities for the Ginibre ensembles are the same and simply obtained by replacing the last two factors by $w_{Gin}(z_n)\mathcal{J}_N^{(\beta)}(\{z\})$, the respective joint probability distribution function (jpdf).

In eq. (2.9) we could have given a more general definition, by choosing other sets (i.e. not necessarily concentric circles around the origin) or by including an angular dependence. While this was briefly sketched in [4] we focus here on quantities that we can compute most explicitly. We will come back to this point in the next section.

From the k-th gap probability defined above, the radial distribution of the k-th individual eigenvalue ordered with respect to radius follows recursively by a simple differentiation

$$\frac{\partial}{\partial r} E_k^{(\beta)}(r) \equiv r\left(p_k^{(\beta)}(r) - p_{k+1}^{(\beta)}(r)\right) \ , \tag{2.10}$$

setting $p_0^{(\beta)}(r) \equiv 0$. In particular this implies for the first eigenvalue that $\frac{\partial}{\partial r}E_0^{(\beta)}(r) = -rp_1^{(\beta)}(r)$. The relation eq. (2.10) is easily inverted to

$$p_k^{(\beta)}(r) = -\frac{1}{r}\frac{\partial}{\partial r}\sum_{l=0}^{k-1} E_l^{(\beta)}(r) \ . \tag{2.11}$$

Obviously it should hold that the sum over all individual eigenvalues gives the spectral density $R_1^{(\beta)}(z)$ to be defined below,

$$\int_0^{2\pi} d\theta R_1^{(\beta)}\left(z = re^{i\theta}\right) = \sum_{k=1}^{N} p_k^{(\beta)}(r) \ , \ \text{with} \ \int_0^\infty dr\, r p_k^{(\beta)}(r) = 1 \ \ \forall k \ . \tag{2.12}$$

Here we have integrated over the angle as the $p_k^{(\beta)}(r)$ only depend on the radius. They are normalised with the two-dimensional radial measure. This relation will be used later to illustrate and check our results for individual eigenvalues. We now come to all the k-point density correlation functions defined as

$$R_k^{(\beta)}(z_1, \ldots, z_k) \equiv \frac{1}{\mathcal{Z}_{ch}^{(\beta)}} \frac{N!}{(N-k)!} \prod_{j=k+1}^{N} \int_{\mathbb{C}} d^2 z_j \prod_{l=1}^{N} w_\nu^{(\beta)}(z_l)\ \mathcal{J}_N^{(\beta)}(\{z^2\}) \ , \tag{2.13}$$

with the same quantities for Ginibre defined using their respective jpdf. In particular for $k=1$ the spectral density is normalised to the number of eigenvalues: $\int_{\mathbb{C}} d^2 z R_1^{(\beta)}(z) = N$.

All k-point density functions are known explicitly for finite N for all four ensembles defined above:

$$R_k^{(\beta)}(z_1, \ldots, z_k) = (\mathrm{Q})\det_{1\le i,j\le k}[K_N^{(\beta)}(z_i, z_j^*)] \ , \tag{2.14}$$

where $K_N^{(2)}(z_i, z_j^*)$ is the kernel of the corresponding orthogonal polynomials in the complex plane for $\beta = 2$. In our Gaussian ensembles these are of Hermite [15] type for Ginibre, or Laguerre [7, 18]

[2] This definition of $E_k^{(\beta)}(r)$ differs from [19] by a factor $1/k!$.

for the chiral case. For $\beta = 4$ the quaternion determinant Qdet (or Pfaffian) is taken over the 2×2 matrix kernel $K_N^{(4)}$ expressed in terms of the pre-kernel $\kappa_N(z_i, z_j)$ of the corresponding skew orthogonal polynomials in the complex plane. For our examples these are given again by Hermite [17] (Ginibre) or Laguerre [8] (chiral) polynomials.

The k-th gap probability can be expressed in terms of n-point density correlation functions as follows

$$E_k^{(\beta)}(r) = \frac{1}{k!} \sum_{\ell=0}^{N-k} \frac{(-1)^\ell}{\ell!} \prod_{j=1}^{k+\ell} \int_0^r dr_j r_j \int_0^{2\pi} d\theta_j \; R_{k+\ell}(z_1, \ldots, z_{k+\ell}) \;, \tag{2.15}$$

as was pointed out in [4] for the chiral complex ensembles at $\beta = 2$. The term in the sum with $k+\ell = 0$ is set to unity. In fact this expansion holds for all 4 ensembles both at $\beta = 2$ and 4, independent of the structure of the jpdf (or even for real β in case of Hermitian RMT). This can easily be obtained using the same expansion as for real eigenvalues in [19], inserting merely the definition (2.9), and without using the solution eq. (2.14). Moreover one can define a generating function by generalising the zero-th gap probability to

$$\begin{aligned} E^{(\beta)}(r;\xi) \;\; &\equiv \;\; \frac{1}{\mathcal{Z}_{ch}^{(\beta)}} \prod_{j=1}^{N} \left(\int_0^\infty dr_j r_j - \xi \int_0^r dr_j r_j \right) \int_0^{2\pi} d\theta_j w_\nu^{(\beta)}(r_j e^{i\theta_j}) \; \mathcal{J}_N^{(\beta)}(\{z^2\}) \\ &= \;\; \sum_{\ell=0}^{N} \frac{(-\xi)^\ell}{\ell!} \prod_{j=1}^{\ell} \int_0^r dr_j r_j \int_0^{2\pi} d\theta_j \; R_\ell(z_1, \ldots, z_\ell) \;, \end{aligned} \tag{2.16}$$

leading to

$$E_k^{(\beta)}(r) = \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \xi^k} E^{(\beta)}(r;\xi)\Big|_{\xi=1}, \;\; \text{for} \;\; k = 0, 1, \ldots, N \;. \tag{2.17}$$

The importance of this relation will become evident when we give a product representation in terms of the eigenvalues of a certain determinant.

Note that although eq. (2.15) is explicit with all terms on the right hand side known, it contains multiple integrals of determinants of an increasing size. As an approximation the sum can be truncated after the first few terms, and it was found in [4] to converge rapidly for $k = 0$. However, even within this approximation the few integrals to be done numerically become rapidly cumbersome.

3 Results for the gap probabilities

In Subsections 3.1 and 3.2 we give some general results for $E_0^{(\beta)}(r)$ valid for generic non-Hermiticity. In the chiral case this includes a determinant or Pfaffian formula for recursively known one-dimensional integrals. Then in Subsections 3.3 and 3.4 we turn to maximal non-Hermiticity where the determinants or Pfaffians can be diagonalised and an explicit product representation is given for any matrix size.

3.1 General case $\beta = 2$

We will start the discussion with $\beta = 2$ for a general weight function in the eigenvalues representation, including the chiral and Ginibre case. First the gap probability $E_0^{(2)}(r)$ is treated, and then through the generating functional $E^{(2)}(r;\xi)$ all subsequent gaps. Our derivation is in complete analogy to the real case, see Section 6.3 [6], but we will repeat it here in a slightly modified version to prepare for $\beta = 4$ which is new. We have that

$$E_0^{(2)}(r) \equiv \frac{1}{\mathcal{Z}^{(2)ch}} \prod_{l=1}^{N} \int_{\mathbb{C}\backslash\mathcal{C}_r} d^2 z_l \; w_\nu^{(2)}(z_l) \; \mathcal{J}_N^{(2)}(\{z^2\}) \;\; = \;\; \prod_{l=1}^{N} \int_{\mathbb{C}\backslash\mathcal{C}_r} d^2 z_l \left| \det_{1\leq j,k,\leq N}[\varphi_{j-1}(z_k)] \right|^2 \;, \tag{3.1}$$

where $\mathcal{C}_r$ denotes the circle of radius r around the origin. While this will become important in Subsections 3.3 and 3.4, throughout this subsection we could choose the set to be general. The Ginibre case is trivially obtained by choosing the other weight and non-squared arguments inside the Jacobian.

Here we have also used that the normalising partition function is given as follows in terms of the squared norms h_j of the corresponding orthogonal polynomials $p_j(z)$, $\int d^2z\, w_\nu^{(2)}(z)p_j(z)p_k(z^*) = h_j\delta_{jk}$,

$$\mathcal{Z}_{ch}^{(2)} \;=\; N!\prod_{j=0}^{N-1} h_j \ . \tag{3.2}$$

Examples for the orthonormalised wave functions $\varphi_j(z) = w_\nu^{(2)}(z)^{\frac{1}{2}}h_j^{-\frac{1}{2}}p_j(z)$ are

$$\varphi_{j\,ch}(z) \;=\; w_\nu^{(2)}(z)^{\frac{1}{2}}h_j^{-\frac{1}{2}}L_j^\nu\left(\frac{Nz^2}{1-\mu^2}\right) \ , \tag{3.3}$$

$$\varphi_{j\,Gin}(z) \;=\; w_{Gin}^{(2)}(z)^{\frac{1}{2}}h_j^{-\frac{1}{2}}H_j\left(z\sqrt{\frac{N}{\tau}}\right) \ , \tag{3.4}$$

for the chiral ensemble and Ginibre ensemble, respectively. Applying Gram's result, e.g. see Appendix A.12. of [6], we obtain

$$E_0^{(2)}(r) = \det_{1\leq j,k,\leq N}\left[\delta_{jk} - \int_{\mathcal{C}_r} d^2z\ \varphi_{j-1}(z)\varphi_{k-1}(z^*)\right] = \prod_{i=0}^{N-1}\left(1-\lambda_i^{(2)}\right) \ . \tag{3.5}$$

This determinant has to be diagonalised. Although all integrals inside the matrix elements can be determined recursively for the chiral Gaussian case, see Appendix A for a slightly different framework, this does not allow us to compute the eigenvalues $\lambda_j^{(2)}$ explicitly. Alternatively these eigenvalues can be obtained from the following integral equation

$$\lambda\psi(u) \;=\; \int_{\mathcal{C}_r} d^2z\, K_N^{(2)}(u,z^*)\psi(z) \ , \tag{3.6}$$

$$K_N^{(2)}(u,z^*) \;\equiv\; \sum_{j=0}^{N-1}\varphi_j(u)\varphi_j(z^*) \ . \tag{3.7}$$

This is seen as follows. Because the $\varphi_j(u)$ form a basis the eigenfunction $\psi(u)$ can be expanded in terms of them. Here, we will use a slightly more formal argument than in [6], by using the projection property of the kernel,

$$\int_{\mathbb{C}} d^2zK_N^{(\beta)}(u,z^*)K_N^{(\beta)}(z,v^*) = K_N^{(\beta)}(u,v^*) \ \ , \ \beta = 2,4 \ \ . \tag{3.8}$$

As indicated this property will generalise to $\beta = 4$ (for the definition of the $\beta = 4$ kernel see eq. (3.20)). Integrating $K_N^{(2)}(v,u^*)$ times eq. (3.6) with respect to u and applying eq. (3.8) we have

$$\lambda\sum_{i=0}^{N-1}\varphi_i(v)\ c_i \;=\; \int_{\mathcal{C}_r} d^2z\, K_N^{(2)}(v,z^*)\psi(z) = \lambda\psi(v) \tag{3.9}$$

$$\text{with}\ \ c_i \;=\; \int_{\mathbb{C}} d^2z\,\varphi_i(u^*)\psi(u) \ , \tag{3.10}$$

and thus the desired expansion of the eigenfunction (for $\lambda \neq 0$). Plugging this expansion back into eq. (3.6) we have

$$\lambda \sum_{i=0}^{N-1} c_i \varphi_i(u) = \sum_{j=0}^{N-1} \varphi_j(u) \int_{\mathcal{C}_r} d^2z\ \varphi_j(z^*) \sum_{i=0}^{N-1} c_i \varphi_i(z)\ . \tag{3.11}$$

The projection onto the coefficients c_l can be achieved using the orthonormality of the $\varphi_i(u)$, integrating both sides with $\int_{\mathbb{C}} d^2u\, \varphi_l(u^*)$:

$$\lambda c_l = \sum_{i=0}^{N-1} c_i \int_{\mathcal{C}_r} d^2z\ \varphi_l(z^*) \varphi_i(z)\ . \tag{3.12}$$

This equation has a solution if

$$0 = \det_{1 \leq i,j \leq N} \left[\lambda \delta_{ij} - \int_{\mathcal{C}_r} d^2z\ \varphi_{i-1}(z^*) \varphi_{j-1}(z) \right]\ . \tag{3.13}$$

The determinant of this Hermitian matrix is given by a polynomial of degree N in term of its real eigenvalues $\lambda_i^{(2)}$

$$\det_{1 \leq i,j \leq N} \left[\lambda \delta_{ij} - \int_{\mathcal{C}_r} d^2z\ \varphi_{i-1}(z^*) \varphi_{j-1}(z) \right] \equiv \prod_{i=0}^{N-1} \left(\lambda - \lambda_i^{(2)} \right)\ . \tag{3.14}$$

This determinant is called the Fredholm determinant, and its eigenvalues are the Fredholm eigenvalues which obviously depend on r here. Setting $\lambda = 1$ we are back to eq. (3.5) as we wanted to prove.

Moving to the generating functional we can repeat all the steps. Multiplying all equations by ξ and redefining $\lambda \to \lambda \xi$ it trivially follows for the generating functional

$$E^{(2)}(r;\xi) = \prod_{i=0}^{N-1} \left(1 - \xi \lambda_i^{(2)} \right)\ . \tag{3.15}$$

Applying definition eq. (2.16) the following representation of the k-th gap probability holds:

$$E_k^{(\beta)}(r) \;\; = \;\; \prod_{j=0}^{N-1} \left(1 - \lambda_j^{(\beta)} \right) \sum_{\{j_i\}} \frac{\lambda_{j_1}^{(\beta)}}{1 - \lambda_{j_1}^{(\beta)}} \cdots \frac{\lambda_{j_k}^{(\beta)}}{1 - \lambda_{j_k}^{(\beta)}}\ , \quad \beta = 2, 4\ , \tag{3.16}$$

where we have anticipated that the same result holds for $\beta = 4$ (as will be shown next). The same result is equally true for Hermitian RMT, see [6] where it was derived in a different way. In eq. (3.16) the sum is over all possible permutations of subsets of k out of N indices, e.g. for $k = 1$ it reads:

$$E_1^{(\beta)}(r) \;\; = \;\; \prod_{j=0}^{N-1} \left(1 - \lambda_j^{(\beta)} \right) \sum_{\ell=0}^{N-1} \frac{\lambda_\ell^{(\beta)}}{1 - \lambda_\ell^{(\beta)}}\ . \tag{3.17}$$

3.2 General case $\beta = 4$

Let us turn to $\beta = 4$. To the best of our knowledge the relation between the Fredholm eigenvalues and an eigenvalue equation involving the kernel is new. Looking at the Jacobian eq. (2.4) and replacing the single Vandermonde by a determinant of monic, skew orthogonal polynomials and their complex conjugates (see eq. (3.22)) we can write for the gap probability

$$
\begin{aligned}
E_0^{(4)}(r) &\equiv \frac{1}{\mathcal{Z}_{ch}^{(4)}} \prod_{l=1}^{N} \int_{\mathbb{C}\setminus\mathcal{C}_r} d^2 z_l \; w_\nu^{(4)}(z_l) \; \mathcal{J}_N^{(4)}(\{z^2\}) \;, \\
&= \frac{1}{\mathcal{Z}_{ch}^{(4)}} \prod_{l=1}^{N} \int_{\mathbb{C}\setminus\mathcal{C}_r} d^2 z_l (z_l^2 - z_l^{*\,2}) w_\nu^{(4)}(z_l) \det_{1\le k\le N,\, 1\le j\le 2N,} \left[\begin{array}{c} q_{j-1}(z_k^2) \\ q_{j-1}(z_k^{2\,*}) \end{array} \right] .
\end{aligned}
\tag{3.18}
$$

The $2N$ rows alternate in the variables z_k and z_k^*. The N-fold integral can be reduced to a Pfaffian over single integrals using the de Bruijn integration formula, see e.g in [17], and we obtain

$$
E_0^{(4)}(r) = \frac{(2N)!}{\mathcal{Z}_{ch}^{(4)}} \mathrm{Pf}_{1\le k,l\le 2N} \left[\int_{\mathbb{C}\setminus\mathcal{C}_r} d^2 z (z^2 - z^{*\,2}) w_\nu^{(4)}(z) \Big(q_{k-1}(z) q_{l-1}(z^*) - q_{k-1}(z^*) q_{l-1}(z) \Big) \right] . \tag{3.19}
$$

This is the $\beta = 4$ result corresponding to eq. (3.5), valid for any weight function which we display explicitly here. The normalising partition function given by the squared norms, $\mathcal{Z}_{ch}^{(4)} = (2N)! \prod_{j=0}^{N-1} h_j$, as can be seen from the same calculation but over the full complex plane, using the definition (3.22). The Ginibre ensembles are trivially obtained by choosing the corresponding weights and non-squared arguments inside the Jacobian and in front of the weight.

Our next task is to relate the eigenvalues of the matrix inside the Pfaffian to eigenvalues involving the $\beta = 4$ matrix kernel given by [17]

$$
K_N^{(4)}(z, u^*) \equiv (z^{*\,2} - z^2)^{\frac{1}{2}} (u^{*\,2} - u^2)^{\frac{1}{2}} w_\nu^{(4)}(z)^{\frac{1}{2}} w_\nu^{(4)}(u)^{\frac{1}{2}} \left(\begin{array}{cc} \kappa_N(z^*, u) & -\kappa_N(z^*, u^*) \\ \kappa_N(z, u) & -\kappa_N(z, u^*) \end{array} \right) , \tag{3.20}
$$

where the pre-kernel is defined as

$$
\kappa_N(z, u) \equiv \sum_{k=0}^{N-1} h_k^{-1} [q_{2k+1}(z) q_{2k}(u) - q_{2k+1}(u) q_{2k}(z)] \;. \tag{3.21}
$$

The polynomials we choose to be skew orthogonal with respect to the following antisymmetric product:

$$
\int_{\mathbb{C}} d^2 z (z^{*\,2} - z^2) w_\nu^{(4)}(z) \Big[q_{2k+1}(z) q_{2l}(z^*) - q_{2k+1}(z^*) q_{2l}(z) \Big] = h_k \delta_{kl} \;, \tag{3.22}
$$

and zero whenever two even or two odd polynomials are contracted. The kernel eq. (3.20) defined in this way then satisfies the contraction property eq. (3.8) [17].

We can now state the matrix eigenvalue equation that is needed to solve eq. (3.19),

$$
\lambda \left(\begin{array}{c} \psi(u^*) \\ \psi(u) \end{array} \right) = \int_{\mathcal{C}_r} d^2 z (z^{*\,2} - z^2) w_\nu^{(4)}(z) \left(\begin{array}{cc} \kappa_N(u^*, z) & -\kappa_N(u^*, z^*) \\ \kappa_N(u, z) & -\kappa_N(u, z^*) \end{array} \right) \left(\begin{array}{c} \psi(z^*) \\ \psi(z) \end{array} \right) . \tag{3.23}
$$

The two components are trivially related by complex conjugation and thus give rise to the same λ. However, for the sequel it is useful to use this matrix form. As for $\beta = 2$ the generalisation to gaps with respect to other sets is trivial.

To proceed the expansion of the eigenfunctions in analogy to eq. (3.9) can be achieved by multiplying eq. (3.23) by the matrix in eq. (3.20) from the left, integrating over $(u^{*\,2} - u^2) w_\nu^{(4)}(u)$ and using the contraction eq. (3.8) to arrive at

$$
\lambda \int_{\mathbb{C}} d^2 u (u^{*\,2} - u^2) w_\nu^{(4)}(u) \left(\begin{array}{c} \kappa_N(v^*, u) \psi(u^*) - \kappa_N(v^*, u^*) \psi(u) \\ \kappa_N(v, u) \psi(u^*) - \kappa_N(v, u^*) \psi(u) \end{array} \right) = \lambda \left(\begin{array}{c} \psi(v^*) \\ \psi(v) \end{array} \right) . \tag{3.24}
$$

The desired expansion of the eigenfunction $\psi(v)$ in terms of the skew orthogonal polynomials thus reads

$$\psi(v) = \sum_{k=0}^{N-1} \frac{1}{h_k}[q_{2k+1}(v)c_{2k} - q_{2k}(v)c_{2k+1}] \tag{3.25}$$

with

$$c_k \;=\; \int_{\mathbb{C}} d^2u(u^{*\,2}-u^2)w_\nu^{(4)}(u)[q_k(u)\psi(u^*) - q_k(u^*)\psi(u)]\ . \tag{3.26}$$

Since the even skew orthogonal polynomials $q_{2k}(v)$ are not uniquely determined (see [17]), then neither is this expansion. However, after fixing the former there is no ambiguity in the definition of the c_k's.

Next we can project onto the coefficients c_l. Multiplying eq. (3.23) with $(-q_l(u), q_l(u^*))$ and integrating over u with the corresponding measure the skew orthogonality eq. (3.22) leads to

$$\begin{aligned}\lambda c_l \;=\; & \int_{\mathcal{C}_r} d^2z(z^{*\,2}-z^2)w_\nu^{(4)}(z)\sum_{k=0}^{N-1} h_k^{-1}\Big[[q_{2k+1}(z^*)c_{2k} - q_{2k}(z^*)c_{2k+1}]q_l(z) \\ & -[q_{2k+1}(z)c_{2k} - q_{2k}(z)c_{2k+1}]q_l(z^*)\Big]\ . \end{aligned}\tag{3.27}$$

Splitting into odd and even this equation can be written in matrix form as

$$\begin{aligned}\lambda\begin{pmatrix} c_{2l} \\ -c_{2l+1}\end{pmatrix} \;=\; & \int_{\mathcal{C}_r} d^2z(z^{*\,2}-z^2)w_\nu^{(4)}(z)\sum_{k=0}^{N-1} h_k^{-1} \qquad (3.28)\\ \times & \begin{pmatrix} q_{2k+1}(z^*)q_{2l}(z) - q_{2k+1}(z)q_{2l}(z^*) & q_{2k}(z^*)q_{2l}(z) - q_{2k}(z)q_{2l}(z^*) \\ q_{2k+1}(z^*)q_{2l+1}(z) - q_{2k+1}(z)q_{2l+1}(z^*) & q_{2k}(z^*)q_{2l+1}(z) - q_{2k}(z)q_{2l+1}(z^*)\end{pmatrix}\begin{pmatrix} c_{2k} \\ -c_{2k+1}\end{pmatrix}.\end{aligned}$$

It has a solution if the following determinant of an antisymmetric $2N\times 2N$ matrix vanishes (after normalising the $q_k(z)$ with $h_k^{-\frac12}$):

$$\begin{aligned} 0 \;=\; & \det_{l,k=0,N-1}\Bigg[\lambda\begin{pmatrix} 0 & -\delta_{kl} \\ \delta_{kl} & 0\end{pmatrix} - \int_{\mathcal{C}_r} d^2z(z^{*\,2}-z^2)w_\nu^{(4)}(z)\ h_k^{-\frac12}h_l^{-\frac12} \qquad (3.29)\\ & \times\begin{pmatrix} q_{2k}(z^*)q_{2l}(z) - q_{2k}(z)q_{2l}(z^*) & q_{2k+1}(z^*)q_{2l}(z) - q_{2k+1}(z)q_{2l}(z^*) \\ q_{2k}(z^*)q_{2l+1}(z) - q_{2k}(z)q_{2l+1}(z^*) & q_{2k+1}(z^*)q_{2l+1}(z) - q_{2k+1}(z)q_{2l+1}(z^*)\end{pmatrix}\Bigg] \\ =\; & \prod_{j=0}^{N-1}\left(\lambda-\lambda_j^{(4)}\right)^2\ . \qquad (3.30)\end{aligned}$$

Here we have swapped the columns of the 2×2 matrices which is allowed under the determinant. Let us compare to eq. (3.19): using the skew orthogonality and taking out the norms which cancels the normalising prefactor, the matrix inside the Pfaffian there agrees with the one in eq. (3.29), up to an overall irrelevant sign as $\mathrm{Pf}(A) = \mathrm{Pf}(A^T = -A)$. Because of the double degeneracy of the eigenvalues from eq. (3.30) we obtain as a final result for the gap probability and the generating functional

$$E_0^{(4)}(r) = \prod_{j=0}^{N-1}\left(1-\lambda_j^{(4)}\right)\quad,\quad E^{(4)}(r;\xi) = \prod_{j=0}^{N-1}\left(1-\xi\lambda_j^{(4)}\right)\ . \tag{3.31}$$

We have checked that the same result can be derived independently as a quaternion determinant, using orthogonal quaternions instead. It would be very interesting to relate the two gap probabilities eq. (3.15) for $\beta=2$ and eq. (3.31) for $\beta=4$ for a general weight function and at general non-Hermiticity. In the following subsection we will be able to find such a relation for maximal non-Hermiticity by diagonalising the determinant or Pfaffian. For the Gaussian weight we can then explicitly compute the $\lambda_j^{(\beta)}$ as a function of r.

3.3 Maximal non-Hermiticity $\beta = 2$

In this section we derive an exact product representation for the gap probabilities for an arbitrary rotationally invariant weight function. The corresponding one-dimensional integrals can be solved explicity for our Bessel-K weight function eq. (2.8) at $\mu = 1$, and have been computed previously for the Ginibre ensemble with exponential weight eq. (2.5) in [5, 6] at $\tau = 0$.

We again start with $\beta = 2$ and a general weight function, and consider the Gaussian chiral and Ginibre weights as examples at the end. For any rotationally invariant weight $w_\nu^{(2)}(|z|)$ (e.g. eq. (2.8) at $\mu = 1$) the orthogonal polynomials are given by monomial powers (we will use the chiral case with squared variables here, the Ginibre class trivially follows).

This can be seen from the orthogonality relation

$$\int_0^{2\pi} d\theta\; z^{2k} z^{*\,2l} \;=\; 2\pi\, t^{4k}\delta_{kl} \;\;,\;\; k,l = 0,1,\ldots \;\;, \tag{3.32}$$

where $z = te^{i\theta}$. Thus the wave functions in eq. (3.1) only differ by their norm, defined as

$$\int_{\mathbb{C}} d^2z w_\nu^{(2)}(|z|) z^{2k} z^{*\,2l} = 2\pi\,\delta_{kl} \int_0^\infty dt\, t\, t^{4k} w_\nu^{(2)}(t) \equiv \delta_{kl} h_k \tag{3.33}$$

with $\varphi_k(z) = h_k^{-\frac{1}{2}} w_\nu^{(2)}(|z|)^{\frac{1}{2}} z^{2k}$.

Consequently the determinant in eq. (3.5) becomes diagonal due to eq. (3.32) even though we only integrate over a circle of radius r. Putting the two results together we can read off the Fredholm eigenvalues for the gap probability

$$E_0^{(2)}(r) \;=\; \prod_{j=0}^{N-1}\left(1 - \lambda_j^{(2)}(r)\right) \;\;,\;\; \text{with}\;\; \lambda_j^{(2)}(r) \;\equiv\; \frac{\int_0^r dt\, t^{4j+1} w_\nu^{(2)}(t)}{\int_0^\infty dt\, t^{4j+1} w_\nu^{(2)}(t)} \;. \tag{3.34}$$

After this general result let us return to our two Gaussian matrix models as examples. The weight function of the chiral model eq. (2.8) at $\mu = 1$ contains a Bessel-K function and all integrals can be done explicitly. First we compute the norms using a standard integral, see e.g. eq. (6.561.16) [20],

$$\int_0^\infty ds\, s^{2k+\nu+1} K_\nu(s) \;=\; 2^{2k+\nu}(k+\nu)!k! \;, \tag{3.35}$$

after choosing the scaling variable $s = N|z|^2$.

The second integral needed is derived in Appendix A

$$\begin{aligned} F_\nu(k,x) \;\equiv\; \int_0^x ds s^{2k+\nu+1} K_\nu(s) \;&=\; 2^{2k+\nu}(k+\nu)!k!\left(1 - \frac{x^{2k+\nu+1}}{2^{2k+\nu}(k+\nu)!k!} K_{\nu+1}(x)\right. \\ &\left. - x\Big(I_{\nu+2}^{[k-2]}(x) K_{\nu+1}(x) + I_{\nu+1}^{[k-1]}(x) K_{\nu+2}(x)\Big)\right) . \end{aligned} \tag{3.36}$$

Here we have introduced the incomplete Bessel-I functions as

$$I_\nu^{[k]}(x) \;\equiv\; \sum_{l=0}^{k} \frac{1}{(l+\nu)!l!}\left(\frac{x}{2}\right)^{2l+\nu} \;\;,\;\; \text{for } k \geq 0 \;, \tag{3.37}$$

and zero for a negative upper summation index. The normalisation eq. (3.35) can be written as $F_\nu(k, x = \infty)$, and for the gap probability we thus have

$$E_0^{(2)}(r) \;=\; \prod_{k=0}^{N-1}\left(1 - \frac{F_\nu(k, Nr^2)}{F_\nu(k,\infty)}\right) . \tag{3.38}$$

Most explicitly we can thus write out one minus the Fredholm eigenvalues as

$$1-\lambda_k^{(2)}(r)=\frac{x^{2k+\nu+1}}{2^{2k+\nu}(k+\nu)!k!}K_{\nu+1}(x)+x\Big(I_{\nu+2}^{[k-2]}(x)K_{\nu+1}(x)+I_{\nu+1}^{[k-1]}(x)K_{\nu+2}(x)\Big)\ , \tag{3.39}$$

where $x = Nr^2$. Inserted into the generating functional eq. (3.15) this gives all gap probabilities for our chiral ensemble with $\beta = 2$. This holds both at finite N and infinite N, where the product extends to infinity and the combination $x = Nr^2$ is kept fixed. We also note that in the large-k limit the second term in eq. (3.39) relates to the Wronsky identity for (complete) Bessel-I functions,

$$0\ =\ 1-x\Big(I_{\nu+1}(x)K_\nu(x)+I_\nu(x)K_{\nu+1}(x)\Big)\ . \tag{3.40}$$

For completeness we give the kernel at $\mu = 1$ [18] to be inserted in eq. (3.6) the eigenvalues of which we have computed:

$$K_N(z,v^*)=|z|^{\nu+1}|v|^{\nu+1}K_\nu\Big(N|z|^2\Big)^{\frac12}K_\nu\Big(N|v|^2\Big)^{\frac12}\sum_{k=0}^{N-1}\frac{2N^{2k+2+\nu}}{\pi 2^{2k+\nu}k!\,(k+\nu)!}z^{2k}v^{*\,2k}\ . \tag{3.41}$$

Our second example is the known Ginibre ensemble at $\beta = 2$ [5]. Instead of eq. (3.34) we have

$$\lambda_{j\ Gin}^{(2)}(r)\ =\ \frac{\int_0^r dt\, t^{2j+1}w_{Gin}(t)}{\int_0^\infty dt\, t^{2j+1}w_{Gin}(t)} \tag{3.42}$$

from the weight eq. (2.5) at $\tau = 0$. Consequently each factor in the product for the gap probability can be written in terms of (upper) incomplete Gamma functions:

$$1-\lambda_{j\,Gin}^{(2)}(r)\ =\ \frac{\Gamma(j+1,x)}{\Gamma(j+1)}\ =\ e^{-x}\sum_{k=0}^{j}\frac{x^k}{k!}\ , \tag{3.43}$$

with $x = Nr^2$. This result was first derived in [5], and the corresponding kernel in eq. (3.6) reads [16]

$$K_N(z,v^*)=\frac{N}{\pi}e^{-\frac{N}{2}(|z|^2+|v|^2)}\sum_{k=0}^{N-1}\frac{N^k}{k!}z^k v^{*\,k}\ . \tag{3.44}$$

Before turning to $\beta = 4$ we would like to illustrate our new result eq. (3.38) by plotting the individual eigenvalues in the chiral case, using eq. (2.12). For that purpose we compare the first few eigenvalues and their sum with the large-N microscopic spectral density at strong non-Hermiticity. It is obtained from eq. (3.41) [18, 21]

$$\rho_\nu^{(2)}(s)\ \equiv\ \lim_{N\to\infty}\frac{1}{N}R_1^{(2)}\left(|z|=\sqrt{\frac{s}{N}}\right)\ =\ \frac{2}{\pi}sK_\nu(s)I_\nu(s) \tag{3.45}$$

and has a very simple form depending only on the rescaled modulus s. For large arguments it approaches the corresponding constant density of the Ginibre ensemble, $\rho_{Gin}^{(2)}(s)=\frac{1}{\pi}$. The corresponding gap probabilities are now given by an infinite product. Because of its fast convergence the individual eigenvalues can be computed by truncating the products at n, where we have used $n = 8$ in Figure 1 to compute the first 5 eigenvalues from eqs. (3.16) and (2.11). Note the normalisation eq. (2.12) of the $p_k^{(2)}(r)$. In Figure 1 we show π times the density versus $\frac12 p_k^{(2)}(r)$.

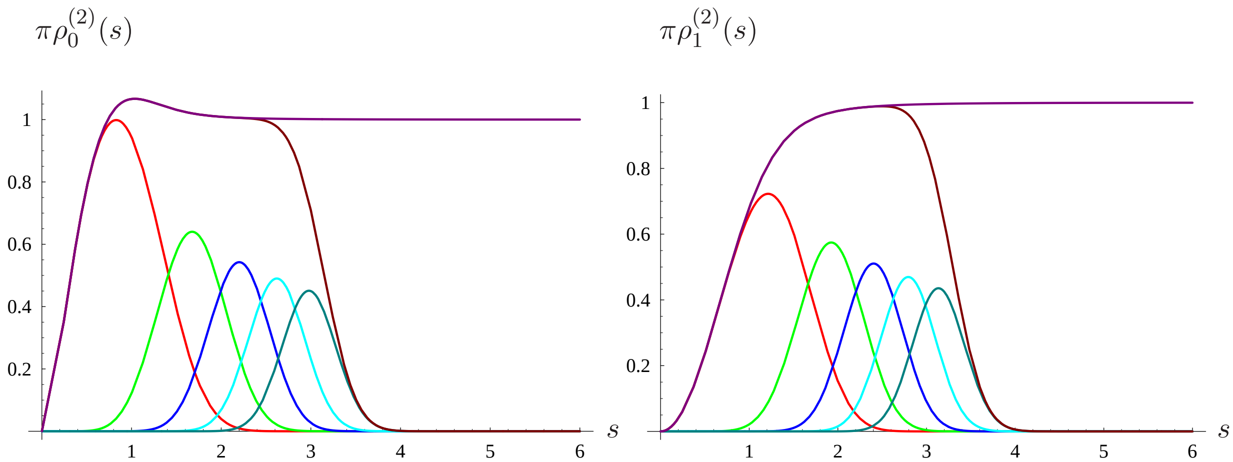

Figure 1: The spectral density eq. (3.45) of the chiral ensemble at $\beta = 2$ times π, and the corresponding distributions of the first five eigenvalues eq. (2.11), as well as their sum, for $\nu = 0$ (left) and $\nu = 1$ (right).

3.4 Maximal non-Hermiticity $\beta = 4$

Next we treat $\beta = 4$ at maximal non-Hermiticity. We choose the polynomials in eq. (3.18) as monomial powers here (these are not the skew orthogonal polynomials) as in the derivation [6]

$$\int_0^{2\pi} d\theta (z^{*\,2} - z^2)\Big(z^{2(k-1)} z^{*\,2(l-1)} - z^{2(l-1)} z^{*\,2(k-1)}\Big) = 4\pi\Big(t^{4k-4}\delta_{k-1,l} - t^{4k}\delta_{k,l-1}\Big), \tag{3.46}$$

where $z = te^{i\theta}$. Therefore the Pfaffian eq. (3.19) can be computed as follows (see also [6] chapter 15), where every other term contributes,

$$E_0^{(4)}(r) \;=\; \prod_{j=0}^{N-1}\Big(1 - \lambda_j^{(4)}(r)\Big) \;\;,\;\; \text{with} \;\; \lambda_j^{(4)}(r) \;\equiv\; \frac{\int_0^r dt\; t^{8j+5} w_\nu^{(4)}(t)}{\int_0^\infty dt\; t^{8j+5} w_\nu^{(4)}(t)} \;. \tag{3.47}$$

Here we have used the fact that the partition function $\mathcal{Z}_{ch}^{(4)}$ can be computed from eq. (3.19) at $r = 0$. This leads to the integrals in the denominator. Eq. (3.47) is of the same form as eq. (3.34), with the difference being that alternate powers are skipped. This leads to the relation between the gap probabilities at $\beta = 2$ and 4 true for general rotationally invariant weight functions[3]

$$E_0^{(4)}(r) \;=\; \prod_{j=0}^{N-1}\Big(1 - \lambda_j^{(4)}\Big) \;=\; \prod_{j=0}^{N-1}\Big(1 - \lambda_{2j+1}^{(2)}\Big) \tag{3.48}$$

where only the *odd* Fredholm eigenvalues from $\beta = 2$ contribute to $\beta = 4$. The same statement can be made relating the generating functionals, as well as for the Ginibre ensembles. This new relation can be compared to the relation known for Hermitian Gaussian RMT ($\tau = 1$) in the large N limit where one has $E_0^{(4)}(r) = \frac{1}{2}\Big(\prod_{i=0}^\infty(1 - \lambda_{2i}^{(2)}) + \prod_{i=0}^\infty(1 - \lambda_{2i+1}^{(2)})\Big)$, given as the *sum* of the product of even and odd eigenvalues (see e.g. [6] eq. (11.7.5)). It would be very interesting to generalise our relation eq. (3.48) to the general case where one could take limits $\mu \to 0$ or $\mu \to 1$. We can only conjecture that $\mu(\tau)$-dependent prefactors of a general linear combination of the product of even and odd terms

[3] We have to use the *same* weight for both β here, that is keeping $\beta\nu$ fixed in eq. (2.8).

could provide a form valid in the limiting cases. For completeness we also give the expression for the Fredholm eigenvalues in the Ginibre ensembles,

$$\lambda_{j\ Gin}^{(4)}(r) \ = \ \frac{\int_0^r dt\, t^{4j+3} w_{Gin}(t)}{\int_0^\infty dt\, t^{4j+3} w_{Gin}(t)} \ . \tag{3.49}$$

Switching back to our explicit example of the chiral Gaussian model with a Bessel-K weight we do not need to do a new computation and can simply use the integrals provided by eq. (3.39)

$$E_0^{(4)}(r) \ = \ \prod_{k=0}^{N-1}\left(1 - \frac{F_{2\nu}(2k+1, Nr^2)}{F_{2\nu}(2k+1,\infty)}\right). \tag{3.50}$$

In comparison to $\beta = 2$ we have to shift the index $\nu \to 2\nu$ here, due to the explicit β-dependence of the weight eq. (2.8). The corresponding pre-kernel to be inserted into eq. (3.23) is given by [8]

$$\kappa_N(z, v^*) = \frac{N^{2\nu+2}}{\pi 2^{2\nu+3}} \sum_{k=0}^{N-1} \sum_{j=0}^{k} \frac{k!\Gamma(k+\nu+1) N^{2k+2j+1}}{\Gamma(2k+2\nu+2)(2k+1)!} \frac{(z^{4k+2} v^{*\,4j} - z^{4j} v^{*\,4k+2})}{2^{4j} j! \Gamma(j+\nu+1)} \ . \tag{3.51}$$

We also give the corresponding expression for the Ginibre ensemble at $\beta = 4$ which was already derived in [6],

$$1 - \lambda_{j\,Gin}^{(4)}(r) \ = \ 1 - \lambda_{2j+1\,Gin}^{(2)}(r) \ = \ \frac{\Gamma(2j+2, x)}{\Gamma(2j+2)} \ = \ e^{-x} \sum_{k=0}^{2j+1} \frac{x^k}{k!} \ , \tag{3.52}$$

with $x = Nr^2$. The pre-kernel is here given by [17]

$$\kappa_N(z, v^*) = \frac{N^{\frac{3}{2}}}{2\pi} \sum_{k=0}^{N-1} \sum_{l=0}^{k} \frac{N^{k+l+\frac{1}{2}}}{(2k+1)!!(2l)!!} (z^{2k+1} v^{*\,2l} - z^{2l} v^{*\,2k+1}) \ . \tag{3.53}$$

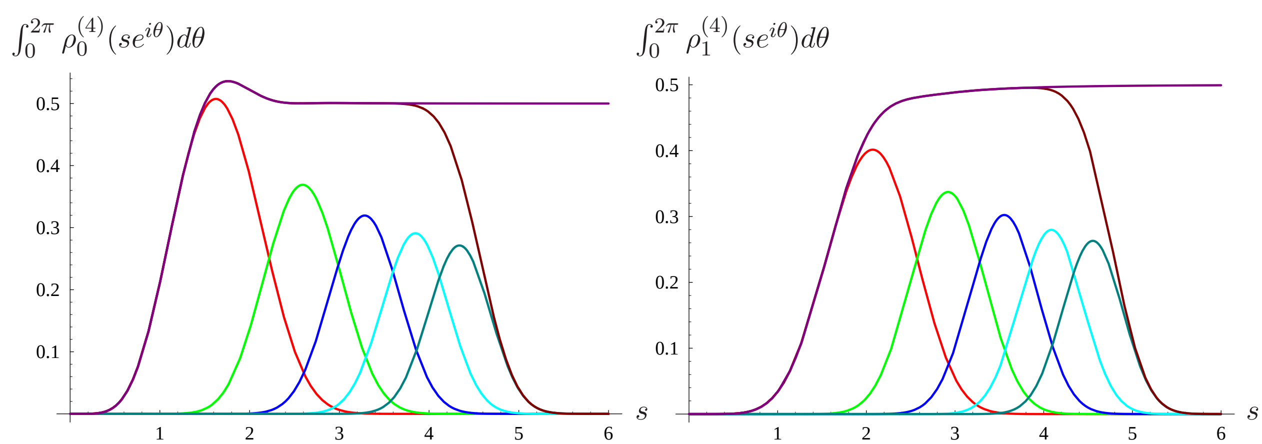

Figure 2: The integrated spectral density eq. (3.54) of the chiral ensemble at $\beta = 4$, and distributions of the first five eigenvalues eq. (2.11), as well as their sum, for $\nu = 0$ (left) and $\nu = 1$ (right).

As before we illustrate the new individual eigenvalue distributions we found for $\beta = 4$ in the chiral case resulting from eqs. (3.16) and (2.11). The microscopic spectral density in the large-N limit at maximal non-Hermiticity is given by [8] (note the radius of support $\beta/2$ and height depend on β)

$$\rho_\nu^{(4)}(\eta) \ = \ \frac{1}{4\pi}(\eta^{*\,2} - \eta^2)|\eta|^2 K_{2\nu}\left(|\eta|^2\right) \int_0^1 \frac{dt}{\sqrt{1-t^2}}\ I_{2\nu}(t|\eta|^2) \sinh\left(\frac{1}{2}\sqrt{1-t^2}\,(\eta^2 - \eta^{*\,2})\right), \tag{3.54}$$

where $\eta = \sqrt{N} r e^{i\theta}$. In contrast to $\beta = 2$ the density is no longer rotationally invariant and explicitly depends on the angle θ. In particular it vanishes along the real and imaginary axis, and for a more detailed discussion of its symmetries we refer to [8]. In order to be able to compare with the individual complex eigenvalues we have to integrate over the angle θ as indicated in eq. (2.12). The result for the first few eigenvalues is shown in Figure 2. Here we have again truncated the infinite product at $n = 8$ to display the first 5 eigenvalues and their sum.

4 Asymptotic expansion for the gap probability for Ginibre $\beta = 2$

We now turn to consider the large-r asymptotics for the gap probability $E_0^{(2)}(r)$ in the large N limit keeping $x = Nr^2$ fixed. Our method has the advantage that it can be easily extended to the chiral ensembles in the next section, although for this part it provides a weaker result than given in [13]. Here we use eq. (3.34), but with the Ginibre Fredholm eigenvalues from eq. (3.43). Starting at unity, as r increases $E_0^{(2)}(r)$ tends towards zero. Therefore, we define $P(x) > 0$ by

$$E_0^{(2)}(r) \equiv \exp\left[-P(x = Nr^2)\right], \tag{4.1}$$

and will analyse the large-x behaviour of $P(x)$, which is given by

$$P(x) \;=\; -\log \prod_{n=0}^{\infty} g_n(x) \;=\; -\sum_{n=0}^{\infty} \log g_n(x) \tag{4.2}$$

where

$$g_n(x) \equiv \sum_{k=0}^{n} t_k(x) \quad \text{and} \quad t_k(x) = e^{-x}\frac{x^k}{k!}. \tag{4.3}$$

The large N limit of the corresponding Fredholm equation (3.6) reads

$$\lambda\psi(\eta) = \int_0^x dt\, t \int_0^{2\pi} d\theta\; e^{-\frac{1}{2}(t^2+|\eta|^2)+\eta t e^{-i\theta}}\, \psi(te^{i\theta}) \,. \tag{4.4}$$

4.1 Statement of large x asymptotic results

We show that using our method

$$\boxed{P(x) \;=\; \frac{x^2}{4} + \frac{x\log x}{2} + \left(\frac{\log 2\pi}{2} - 1\right) x + O\left(\sqrt{x}\right)} \tag{4.5}$$

although the coefficient of x is partly conjectured. It was derived rigorously in [13] including the coefficient of the next order term. In more detail, we proceed by splitting the sum over n in eq. (4.2) into two parts, namely $S_1(\mathcal{N}, x)$ (in which n runs from 1 to $\mathcal{N}$) and $S_2(\mathcal{N}, x)$ (with n running from $\mathcal{N}+1$ to ∞), where $\mathcal{N}$ is some arbitrary (at this stage) parameter (which in general can depend on x) [4]. We then further split $S_1(\mathcal{N}, x)$ into two parts, denoted $S_{11}(\mathcal{N}, x)$ and $S_{12}(\mathcal{N}, x)$, so that we have

$$\begin{aligned} P(x) \;&=\; S_{11}(\mathcal{N}, x) + S_{12}(\mathcal{N}, x) + S_2(\mathcal{N}, x) \\ &\equiv\; -\sum_{n=0}^{\mathcal{N}} \log t_n(x) - \sum_{n=0}^{\mathcal{N}} \log r_n(x) - \sum_{n=\mathcal{N}+1}^{\infty} \log g_n(x) \end{aligned} \tag{4.6}$$

[4] $\mathcal{N}$ here is unrelated to the matrix size N which has been taken to infinity.

with $r_n(x)$ defined as

$$r_n(x) \equiv \frac{g_n(x)}{t_n(x)} = 1 + \frac{n}{x} + \frac{n(n-1)}{x^2} + ... + \frac{n!}{x^n} \tag{4.7}$$

and $t_n(x)$ and $g_n(x)$ defined above in eq. (4.3). Since $S_{12}(\mathcal{N}, x) < 0$, we will always work with $|S_{12}(\mathcal{N}, x)|$ in the following.

With the choice that $\mathcal{N} = x$ (which we justify in Subsection 4.5), we show that the following three properties *i) - iii)* hold:

$$i) \qquad S_{11}(x, x) \;=\; \frac{x^2}{4} + \frac{x\log x}{2} + \frac{\log 2\pi}{2}x + O(\log x) \tag{4.8}$$

$$ii) \qquad S_{12}^{LB}(x, x) < |S_{12}(x, x)| < S_{12}^{UB}(x, x) \tag{4.9}$$

where

$$S_{12}^{UB}(x, x) \;=\; x + O(\log x) \tag{4.10}$$

and

$$S_{12}^{LB}(x, x) \;=\; \left[1 - \log 2 + \frac{\pi^2}{24} + \frac{1}{2}\mathrm{Li}_2(-e^{-2})\right] x + O(1) \;\approx\; 0.653x + O(1) \tag{4.11}$$

with $\mathrm{Li}_2(x)$ being the dilogarithm

$$\mathrm{Li}_2(x) \equiv \sum_{k=1}^{\infty} \frac{x^k}{k^2}\ . \tag{4.12}$$

$$iii) \qquad S_2(x, x) \;=\; O(\sqrt{x})\ . \tag{4.13}$$

More precisely,

$$0 < S_2(x, x) < S_2^{UB}(x) \tag{4.14}$$

where

$$S_2^{UB}(x) \;=\; M\sqrt{x} + O(1) \tag{4.15}$$

and M is a constant given by

$$M \;\equiv\; -\int_0^{\infty} \log(\Phi(m))dm \;\approx\; 0.478 \tag{4.16}$$

and $\Phi(m)$ is the cumulative normal function

$$\Phi(m) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m} e^{-x^2/2}dx\ . \tag{4.17}$$

Furthermore, based on some preliminary numerical analysis, we *conjecture* the following somewhat stronger result for S_{12} (we comment on the bound for the sub-leading term later)

$$\boxed{ii)' \qquad -S_{12}(x,x) \;=\; x + O(\sqrt{x}) \qquad (4.18)}$$

and for S_2 we conjecture

$$\boxed{iii)' \qquad S_2(x,x) \;=\; M\sqrt{x} + C + o(1) \qquad (4.19)}$$

where M is defined as before (eq. (4.16)), and C is another constant, given by

$$C \;\equiv\; \frac{1}{6\sqrt{2\pi}} \int_0^\infty \frac{(m^2-1)e^{-m^2/2}}{\Phi(m)} dm \;-\; \log 2 \;\approx\; -0.716 \; . \tag{4.20}$$

4.2 Calculation of S_{11}

We begin by computing S_{11} containing the first two leading orders,

$$\begin{aligned} S_{11}(\mathcal{N},x) \;&=\; -\sum_{n=0}^{\mathcal{N}} \log\left(e^{-x}\frac{x^n}{n!}\right) \;=\; \sum_{n=0}^{\mathcal{N}} (x - n\log x + \log n!) \\ &=\; (\mathcal{N}+1)x - \frac{\mathcal{N}(\mathcal{N}+1)}{2}\log x + \sum_{n=1}^{\mathcal{N}} \log n! \end{aligned} \tag{4.21}$$

(since $\log 0! = \log 1 = 0$). We evaluate the sum of factorials as follows:

$$\begin{aligned} \sum_{n=1}^{\mathcal{N}} \log n! \;&=\; \sum_{n=1}^{\mathcal{N}}\sum_{k=1}^{n} \log k \;=\; \sum_{k=1}^{\mathcal{N}} (\mathcal{N}-k+1)\log k \;=\; (\mathcal{N}+1)\sum_{n=1}^{\mathcal{N}} \log n - \sum_{n=1}^{\mathcal{N}} n\log n \\ &=\; (\mathcal{N}+1)\log\mathcal{N}! - \sum_{n=1}^{\mathcal{N}} n\log n. \end{aligned} \tag{4.22}$$

For the first term, we use Stirling's formula

$$\log\mathcal{N}! = \left(\mathcal{N}+\frac{1}{2}\right)\log\mathcal{N} - \mathcal{N} + \frac{\log 2\pi}{2} + O\left(\frac{1}{\mathcal{N}}\right), \tag{4.23}$$

and for the second term, we apply the Euler-MacLaurin summation formula (as hinted in [5])

$$\sum_{n=1}^{\mathcal{N}} f(n) \;=\; \int_1^{\mathcal{N}} f(t)dt + \frac{f(1)+f(\mathcal{N})}{2} + O(f'(\mathcal{N})) \tag{4.24}$$

which gives

$$\begin{aligned} \sum_{n=1}^{\mathcal{N}} n\log n \;&=\; \int_1^{\mathcal{N}} t\log t dt + \frac{\mathcal{N}\log\mathcal{N}}{2} + O(\log\mathcal{N}) \;=\; \left[\frac{t^2\log t}{2} - \frac{t^2}{4}\right]_1^{\mathcal{N}} + \frac{\mathcal{N}\log\mathcal{N}}{2} + O(\log\mathcal{N}) \\ &=\; \frac{\mathcal{N}^2\log\mathcal{N}}{2} - \frac{\mathcal{N}^2}{4} + \frac{\mathcal{N}\log\mathcal{N}}{2} + O(\log\mathcal{N}). \end{aligned} \tag{4.25}$$

Inserting eqs. (4.23) and (4.25) into eq. (4.22) gives us

$$\begin{aligned} S_{11}(\mathcal{N},x) &= (\mathcal{N}+1)x - \frac{\mathcal{N}(\mathcal{N}+1)}{2}\log x + \frac{\mathcal{N}^2\log\mathcal{N}}{2} \\ &\quad - \frac{3\mathcal{N}^2}{4} + \mathcal{N}\log\mathcal{N} + \left(\frac{\log 2\pi}{2} - 1\right)\mathcal{N} + O(\log\mathcal{N}) \end{aligned} \tag{4.26}$$

and on setting $\mathcal{N} = x$, we arrive at

$$S_{11}(x,x) = \frac{x^2}{4} + \frac{x\log x}{2} + \frac{\log 2\pi}{2}x + O(\log x). \tag{4.27}$$

4.3 Calculation of S_{12}

We turn now to the large x behaviour of

$$|S_{12}(\mathcal{N},x)| \equiv \sum_{n=1}^{\mathcal{N}} \log r_n(x), \tag{4.28}$$

with $r_n(x)$ defined in eq. (4.7). We can write the sum starting at $n=1$, because $\log r_0(x) = 0$.

Unfortunately, we have found no simple method for calculating S_{12}, and so we determine some upper and lower bounds, and then conjecture the behaviour of S_{12} based on some numerical investigations. Note that, in our notation, S_{12}^{UB} (S_{12}^{LB}) is an upper (lower) bound for $|S_{12}|$ (rather than for S_{12} itself), since $S_{12} < 0$.

4.3.1 Upper bound for $|S_{12}|$

Let us first obtain an upper bound for $|S_{12}(\mathcal{N},x)|$, assuming $\mathcal{N} < x$. We have

$$\begin{aligned} r_n(x) &\equiv 1 + \frac{n}{x} + \frac{n(n-1)}{x^2} + \dots + \frac{n!}{x^n} \\ &< 1 + \frac{n}{x} + \left(\frac{n}{x}\right)^2 + \dots + \left(\frac{n}{x}\right)^n + \dots = \frac{1}{1-\frac{n}{x}} \end{aligned} \tag{4.29}$$

for $n < x$. Therefore

$$\log r_n(x) < -\log\left(1 - \frac{n}{x}\right) \tag{4.30}$$

from which it follows that (for $\mathcal{N} < x$)

$$|S_{12}(\mathcal{N},x)| < S_{12}^{UB}(\mathcal{N},x) \equiv -\sum_{n=1}^{\mathcal{N}} \log\left(1 - \frac{n}{x}\right). \tag{4.31}$$

We approximate the sum using the Euler-MacLaurin summation formula. Since our analysis is valid only for $\mathcal{N} < x$, we choose $\mathcal{N} = x-1$, and this gives

$$S_{12}^{UB}(x-1,x) = x - \frac{1}{2}\log x + O(1). \tag{4.32}$$

We consider the term $n = \mathcal{N}(= x)$ separately; each term in the definition of $r_n(x)$ is individually ≤ 1, and there are $n+1$ such terms, so

$$0 < \log r_n(x) < \log(n+1) = O(\log x). \tag{4.33}$$

Hence we have

$$S_{12}^{UB}(x,x) = x + O(\log x). \tag{4.34}$$

4.3.2 Lower bound for $|S_{12}|$

We have not been able to find a particularly tight lower bound for $|S_{12}|$ using elementary methods. Our best attempt involves the use of the following lower bound for $r_n(x)$ (denoted $r_n^{LB}(x)$) which is strong for low values of n (indeed, we have equality for $n = 1$ and 2), but rather weaker when n is closer to (but still less than) x:

$$
\begin{aligned}
r_n(x) &= 1 + \frac{n}{x} + \frac{n(n-1)}{x^2} + \ldots + \frac{n!}{x^n} \\
&\geq \frac{1}{2}\left(1 + \left(1 + \frac{2}{x}\right)^n\right) \equiv r_n^{LB}(x).
\end{aligned} \tag{4.35}
$$

Therefore

$$
\begin{aligned}
\log r_n^{LB}(x) &= \log\left(1 + \left(1 + \frac{2}{x}\right)^n\right) - \log 2 = \log\left(1 + \frac{2}{x}\right)^n + \log\left(1 + \left(1 + \frac{2}{x}\right)^{-n}\right) - \log 2 \\
&= n \log\left(1 + \frac{2}{x}\right) + \sum_{k=1}^{\infty} \frac{(-)^{k+1}}{k}\left(1 + \frac{2}{x}\right)^{-kn} - \log 2.
\end{aligned} \tag{4.36}
$$

Hence

$$
\begin{aligned}
-S_{12}(\mathcal{N}, x) &> S_{12}^{LB}(\mathcal{N}, x) \equiv \sum_{n=1}^{\mathcal{N}} \log r_n^{LB}(x) \\
&= \frac{\mathcal{N}(\mathcal{N}+1)}{2} \log\left(1 + \frac{2}{x}\right) + \sum_{n=1}^{\mathcal{N}} \sum_{k=1}^{\infty} \frac{(-)^{k+1}}{k}\left(1 + \frac{2}{x}\right)^{-kn} - \mathcal{N} \log 2.
\end{aligned} \tag{4.37}
$$

It is convenient at this stage to set $\mathcal{N}$ equal to x. Using the result that

$$
\left(1 + \frac{1}{a}\right)^a = e + O\left(\frac{1}{a}\right) \tag{4.38}
$$

we can easily evaluate the first term of eq. (4.37)

$$
\begin{aligned}
\frac{x(x+1)}{2} \log\left(1 + \frac{2}{x}\right) &= (x+1) \log\left(1 + \frac{2}{x}\right)^{x/2} = (x+1)(1 + O(x^{-1})) \\
&= x + O(1).
\end{aligned} \tag{4.39}
$$

The second term, which can be evaluated using the result (which we do not prove here)

$$
\sum_{n=1}^{\mathcal{N}} \sum_{k=1}^{\infty} \frac{(-)^{k+1}}{k}\left(1 + \frac{2}{\mathcal{N}}\right)^{-kn} = \left[\frac{\pi^2}{24} + \frac{1}{2}\mathrm{Li}_2(-e^{-2})\right] \mathcal{N} + O(1) \approx 0.346\mathcal{N} + O(1), \tag{4.40}
$$

is also linear, as indeed is the third term (which is simply $-x \log 2$).

Combining these results then gives

$$
S_{12}^{LB}(x, x) = \left[1 - \log 2 + \frac{\pi^2}{24} + \frac{1}{2}\mathrm{Li}_2(-e^{-2})\right] x + O(1) \tag{4.41}
$$

where Li is the dilogarithm defined in eq. (4.12). The term in square brackets is approximately equal to 0.653.

4.3.3 Conjecture for $|S_{12}|$ based on numerical analysis

In the absence of a concrete analytical proof for the asymptotic limit of S_{12} itself, we undertook some elementary numerical analysis. In particular, we considered

$$A_{12}(x) \equiv \frac{|S_{12}(x,x)|}{x} \tag{4.42}$$

for various (necessarily finite) values of x. The function $A_{12}(x)$ begins at ≈ 0.69, and is increasing. It flattens very quickly, and appears to be converging monotonically to 1. For example, we find that $A_{12}(200\,000) \approx 0.996$.

Assuming that the coefficient of x is indeed unity, we then numerically investigated

$$A^*_{12}(x) \equiv \frac{|S_{12}(x,x)| - x}{\sqrt{x}}. \tag{4.43}$$

for increasing x. It was not completely apparent that this converges (we again went as far as $x = 200\,000$), but it seems possible that it does, implying that the sub-leading term is $O(\sqrt{x})$.

Putting these together, we anticipate that

$$|S_{12}(x,x)| \quad = \quad x + O(\sqrt{x}), \tag{4.44}$$

(i.e. that the true asymptote actually equals our earlier *upper* bound, at least to leading order). Considering the sub-leading term, this might (at first sight) appear inconsistent with eq. (4.34), since we have replaced $O(\log x)$ with $O(\sqrt{x})$, but there is no contradiction. The more complete statement of our conjecture is, in fact, that

$$|S_{12}(x,x)| \quad = \quad x - c\sqrt{x} + o(\sqrt{x}) \tag{4.45}$$

where c is some *positive* constant.

4.4 Calculation of S_2 using an upper bound

We need only to provide an upper bound for $S_2(x)$, since our aim is merely to show that $S_2(x)$ is smaller than linear in x, and we know that $S_2(x) > 0$.

From eq. (4.6) we have

$$S_2(\mathcal{N}, x) \equiv -\sum_{n=\mathcal{N}+1}^{\infty} \log g_n(x). \tag{4.46}$$

For $n > x$, we have (see Appendix B for details) a lower bound for $g_n(x)$:

$$g_n(x) \quad > \quad g^{LB}(m,x) \ \equiv \ \Phi(m) - \sum_{i=1}^{\infty} \frac{c_i(m)}{x^{i/2}} - \alpha\sqrt{x}e^{-\beta x} - \frac{\gamma e^{-\delta x}}{\sqrt{x}} \tag{4.47}$$

where we have introduced a scaled variable

$$m \equiv \frac{n-x}{\sqrt{x}} \tag{4.48}$$

and $\Phi(m)$ is the cumulative normal function. The $c_i(m)$ are m-dependent numbers, and α, β, γ and δ are constants. Therefore we have (on setting $\mathcal{N} = x$)

$$\begin{aligned} S_2(x,x) \quad &< \quad S_2^{UB}(x) \\ &\equiv \ -\int_x^{\infty} \log(g^{LB}(m,x))\ dn \ = \ -\sqrt{x}\int_0^{\infty} \log(g^{LB}(m,x))\ dm \end{aligned} \tag{4.49}$$

and on factorising out the $\Phi(m)$ and then expanding the logarithm, we arrive at

$$S_2^{UB}(x) \;=\; -\sqrt{x}\int_0^\infty \log(\Phi(m))\;dm + O(1) \;\equiv\; M\sqrt{x} + O(1) \tag{4.50}$$

where

$$M \;\equiv\; -\int_0^\infty \log(\Phi(m))\;dm \;\approx\; 0.478\;, \tag{4.51}$$

see also eq. (27) in [13]. For our purposes, it is sufficient that we have shown $S_2(x,x) = O(\sqrt{x})$. However, some more detailed analysis (supported by numerical analysis) leads us to conjecture the following, much stronger, result:

$$S_2(x,x) \;=\; M\sqrt{x} + C + o(1) \tag{4.52}$$

where

$$C \;\equiv\; \frac{1}{6\sqrt{2\pi}}\int_0^\infty \frac{(m^2-1)e^{-m^2/2}}{\Phi(m)}dm - \log 2 \;\approx\; -0.716\;. \tag{4.53}$$

4.5 Choice of the 'split point' $\mathcal{N}$

Our final result must, of course, be independent of the choice of $\mathcal{N}$. However, if we had chosen $\mathcal{N} \ll x$, then we would not have been able to bound S_2 in the way that we did, and it would not have been of smaller order than S_1. Conversely, if we had chosen $\mathcal{N} \gg x$, then our argument for S_{12} would not have been applicable, and this term would then have been of higher order. So, setting $\mathcal{N} = x$ is a pragmatic choice which simplifies our analysis, giving $S_{11} \gg |S_{12}| \gg S_2$.

We can understand this in a little more detail if we analyse what happens to each of S_{11}, S_{12} and S_2 as we increase the value of $\mathcal{N}$ by 1. For S_{11}, we can estimate the effect of this by partially differentiating eq. (4.26) with respect to $\mathcal{N}$:

$$\frac{\partial S_{11}}{\partial \mathcal{N}} = \mathcal{N}\left(\log\mathcal{N} - \log x\right) - (\mathcal{N} - x) + \log\mathcal{N} - \frac{1}{2}\log x + O(1). \tag{4.54}$$

We can effectively 'minimise' this by setting $\mathcal{N} = x$, which has the effect of killing everything down to $O(\log x)$. So, by setting $\mathcal{N} = x+1$ instead of $\mathcal{N} = x$, we will only see an increase in S_{11} of $O(\log x)$, which we have already decided to call 'small'.

Of course, S_{12} must decrease by $O(\log x)$ to compensate, and the argument preceeding eq. (4.33) shows that this is precisely the case. (S_2 will change only by $O(1)$, and so can be ignored.)

So, with this choice of $\mathcal{N}$, we can be sure that all of the 'large' contributions are already included in S_{11}; consequently, it is the optimal choice.

5 Asymptotic expansion of $E_0^{(\beta)}(r)$ for the other ensembles

We now extend the problem to the $\beta = 4$ case, and to the corresponding chiral ensembles. For completeness, we will also include here the $\beta = 2$ Ginibre case which was discussed in fuller detail in the previous section.

The aim here, as earlier, is to determine $P(x)$ where $E_0(r) = \exp(-P(Nr^2))$. We established before that

$$g_n^{Gin}(x) \;=\; \sum_{k=0}^{n} e^{-x}\frac{x^k}{k!}, \tag{5.1}$$

$$g_{\nu,n}^{ch}(x) \;=\; \frac{x^{2n+\nu+1}K_{\nu+1}(x)}{2^{2n+\nu}n!(n+\nu)!} + x\left[K_{\nu+1}(x)I_{\nu+2}^{[n-2]}(x) + K_{\nu+2}(x)I_{\nu+1}^{[n-1]}(x)\right] \tag{5.2}$$

Ensemble	$\beta = 2$	$\beta = 4$
Ginibre	$P(x) = -\sum_{n=0}^{\infty} \log g_n^{Gin}(x)$	$P(x) = -\sum_{n=1,\ n \text{ odd}}^{\infty} \log g_n^{Gin}(x)$
Chiral	$P(x) = -\sum_{n=0}^{\infty} \log g_{\nu,n}^{ch}(x)$	$P(x) = -\sum_{n=1,\ n \text{ odd}}^{\infty} \log g_{2\nu,n}^{ch}(x)$

Table 1: Definition of the exponent $P(x)$ for all four ensembles.

where $I_\nu^{[n]}(x)$ is the incomplete I-Bessel function defined in eq. (3.37).

Our task is to find the asymptotic behaviour of each of the $P(x)$ which have the form given in Table 1. For $\beta = 2$ chiral the large N limit of the Fredholm equation (3.6) takes the compact form

$$\lambda\psi(\eta) = \int_0^x dt\, t \int_0^{2\pi} d\theta\ t|\eta| K_\nu(t^2)^{\frac{1}{2}} K_\nu(|\eta|^2)^{\frac{1}{2}} I_\nu(\eta t e^{-i\theta})\psi(te^{i\theta}) \ , \tag{5.3}$$

after inserting the radial Bessel kernel. For $\beta = 4$ a similar equation can be written, but the corresponding kernels from eqs. (3.51) and (3.53) do not simplify.

5.1 Statement of results

To proceed, we first identify the leading term (highest power of x) of each $g_n(x)$

$$t_n^{Gin}(x) \ \equiv\ e^{-x}\frac{x^n}{n!}, \tag{5.4}$$

$$t_{\nu,n}^{ch}(x) \ \equiv\ \sqrt{\pi} e^{-x} \frac{(\frac{x}{2})^{2n+\nu+\frac{1}{2}}}{n!(n+\nu)!} \tag{5.5}$$

and denote the ‘remainder’ as

$$r_n^{Gin}(x) \ \equiv\ \frac{g_n^{Gin}(x)}{t_n^{Gin}(x)} \left(= 1 + \frac{n}{x} + \frac{n(n-1)}{x^2} + \dots + \frac{n!}{x^n} \right), \tag{5.6}$$

$$r_{\nu,n}^{ch}(x) \ \equiv\ \frac{g_{\nu,n}^{ch}(x)}{t_{\nu,n}^{ch}(x)}. \tag{5.7}$$

We have used the asymptotic form for the K-Bessel function

$$K_\nu(x) \to \sqrt{\frac{\pi}{2x}}\, e^{-x} \tag{5.8}$$

as $x \to \infty$ when determining $t_{\nu,n}^{ch}(x)$ in eq. (5.5); however, the terms that we dropped in doing this are incorporated into $r_{\nu,n}^{ch}(x)$, so no approximation has been made. For notational convenience, we drop the ‘Gin/ch’ labels in what follows, and write everything generically.

Ensemble	Choice of $\mathcal{N}$	$S_{11}(\mathcal{N},x)$	$S_{12}(\mathcal{N},x)$	Coeff. of x in $P(x)$
Ginibre ($\beta=2$)	x	$A(x)$	$-\,x$	$\dfrac{\log 2\pi}{2}-1$
Ginibre ($\beta=4$)	x	$\dfrac{A(x)}{2}-\dfrac{x}{4}$	$-\,\dfrac{x}{2}$	$\dfrac{\log 2\pi}{4}-\dfrac{3}{4}$
Chiral ($\beta=2$)	$\dfrac{x}{2}$	$\dfrac{A(x)}{2}-\dfrac{\nu}{2}x$	$-\,\dfrac{x}{2}$	$\dfrac{\log 2\pi}{4}-\dfrac{\nu+1}{2}$
Chiral ($\beta=4$)	$\dfrac{x}{2}$	$\dfrac{A(x)}{4}-\left(\dfrac{2\nu}{4}+\dfrac{1}{4}\right)x$	$-\,\dfrac{x}{4}$	$\dfrac{\log 2\pi}{8}-\dfrac{2\nu+2}{4}$

Table 2: Explicit results for the exponent $P(x)$ for all four ensembles.

We can write $P(x)$ as the sum of three sums:

$$S_{11}(\mathcal{N},x) \equiv -\sum_{0\le n\le\mathcal{N}} \log t_n(x), \tag{5.9}$$

$$S_{12}(\mathcal{N},x) \equiv -\sum_{0\le n\le\mathcal{N}} \log r_n(x), \tag{5.10}$$

$$S_2(\mathcal{N},x) \equiv -\sum_{n>\mathcal{N}} \log g_n(x) \tag{5.11}$$

where $\mathcal{N}$ is a suitably chosen 'split point', and n is restricted to the odd integers for the $\beta=4$ ensembles. To simplify the appearance of our results, we introduce $A(x)$, defined as

$$A(x) \equiv \frac{x^2}{4}+\frac{x\log x}{2}+\frac{\log 2\pi}{2}x\ . \tag{5.12}$$

Our results are then summarised in Table 2 where we omit terms of smaller order than linear in x. For S_{11} these terms are $O(\log x)$, and for S_{12} they are $O(\sqrt{x})$. In all cases, $S_2(x)=O(\sqrt{x})$.

5.2 Derivation by relating the ensembles

The method for each of the 3 other ensembles is broadly similar to that for the original (Ginibre $\beta=2$) case, and so we do not give details here. Essentially, the same techniques can be used to determine the relevant bounds, and we have undertaken numerical analysis to support the results that we claim above.

However, it is quite instructive to understand the relationships *between* the results for the different symmetry classes, so we will provide here some alternative (and quite concise) proofs of the S_{11}'s which highlight these relationships.

5.2.1 From Ginibre to chiral

Assuming the Ginibre results, we will show how to derive the corresponding chiral results (for $\nu = 0$ only), starting with the $\beta = 2$ case. We begin by stating the following exact relationship which follows immediately from the definitions:

$$\log t_{0,n}^{ch}(x) = 2 \log t_n^{Gin}\left(\frac{x}{2}\right) + \frac{1}{2}\log\frac{x}{2} + \frac{1}{2}\log \pi. \tag{5.13}$$

Therefore

$$\begin{aligned} S_{11}^{ch,\beta=2}(x) &\equiv -\sum_{0\le n\le x/2} \log t_{0,n}^{ch}(x) \\ &= -2\sum_{0\le n\le x/2} \log t_n^{Gin}\left(\frac{x}{2}\right) - \frac{1}{2}\sum_{0\le n\le x/2}\left(\log\frac{x}{2} + \log\pi\right) \\ &\equiv 2S_{11}^{Gin,\beta=2}\left(\frac{x}{2}\right) - \frac{1}{2}\sum_{0\le n\le x/2}\left(\log\frac{x}{2} + \log\pi\right) \\ &= 2A\left(\frac{x}{2}\right) - \frac{x}{4}\left(\log\frac{x}{2} + \log\pi\right) + O(\log x) \end{aligned} \tag{5.14}$$

where $A(x)$ was defined in eq. (5.12). But it easily follows from the definition of $A(x)$ that

$$2A\left(\frac{x}{2}\right) = \frac{A(x)}{2} + \frac{x}{4}\left(\log\frac{x}{2} + \log\pi\right) + O(\log x) \tag{5.15}$$

where the scaling of the leading term is explained by the fact that $A(x)$ is essentially quadratic in x. Hence we have

$$\begin{aligned} S_{11}^{ch,\beta=2}(x) &= \frac{A(x)}{2} + O(\log x) \\ &= \frac{S_{11}^{Gin,\beta=2}(x)}{2} + O(\log x), \end{aligned} \tag{5.16}$$

i.e. the chiral sum is half the Ginibre sum. This is the result given in Table 2 (when $\nu = 0$).

A similar argument gives the corresponding result for $\beta = 4$ (at $\nu = 0$):

$$S_{11}^{ch,\beta=4}(x) = \frac{S_{11}^{Gin,\beta=4}(x)}{2} - \frac{x}{8} + O(\log x). \tag{5.17}$$

Note that here we pick up an extra term which is linear in x, so it is not true in general that the chiral case is always half the Ginibre case (at least, not when we consider the terms linear in x).

5.2.2 From $\beta = 2$ to $\beta = 4$

The $\beta = 4$ case involves summing alternate (odd n) terms from the same sequence used for $\beta = 2$. We would therefore expect the total for $\beta = 4$ to be approximately half that for $\beta = 2$, since $-\log g_n(x)$ is a 'smooth' function of n (for fixed x). However, since $-\log g_n(x)$ is monotonic decreasing as a function of n, we would expect the sum of the odd terms (starting at 1) to be slightly less than the sum of the even terms (which start at zero), and indeed we do see this bias.

Let us now quantify this; we will do this first (and in detail) for the Ginibre ensembles. We have

$$
\begin{aligned}
S_{11}^{Gin,\beta=2}(x) &\equiv -\sum_{0\le n\le x} \log\left(e^{-x}\frac{x^n}{n!}\right) \\
&= -\sum_{\substack{1\le n\le x \\ n \text{ odd}}} \left\{\left(\log e^{-x}\frac{x^{n-1}}{(n-1)!}\right) + \log\left(e^{-x}\frac{x^n}{n!}\right)\right\} (\,+\,O(\log x) \text{ if } x \text{ even}) \\
&= -\sum_{\substack{1\le n\le x \\ n \text{ odd}}} \left\{2\left(\log e^{-x}\frac{x^n}{n!}\right) - \log\left(\frac{x}{n}\right)\right\} \quad (\,+\text{ ditto})
\end{aligned}
\tag{5.18}
$$

and thus

$$
\begin{aligned}
S_{11}^{Gin,\beta=2}(x) &\equiv 2S_{11}^{Gin,\beta=4}(x) + \sum_{\substack{1\le n\le x \\ n \text{ odd}}} \log\frac{x}{n} \quad (\,+\text{ ditto}) \\
&= 2S_{11}^{Gin,\beta=4}(x) + \sum_{0\le n\le x/2} \log x - \sum_{0\le n\le x/2} \log(2n) + O(\log x) \\
&= 2S_{11}^{Gin,\beta=4}(x) + \frac{x}{2} + O(\log x)
\end{aligned}
\tag{5.19}
$$

where the last summation has been approximated using the Euler-MacLaurin summation formula. A trivial rearrangement then gives[5]

$$
S_{11}^{Gin,\beta=4}(x) = \frac{S_{11}^{Gin,\beta=2}(x)}{2} - \frac{x}{4} + O(\log x) \tag{5.20}
$$

which is the result given in Table 2.

An almost identical argument gives a similar result for the chiral ensembles (for $\nu = 0$)[5]:

$$
S_{11}^{ch,\beta=4}(x) = \frac{S_{11}^{ch,\beta=2}(x)}{2} - \frac{x}{4} + O(\log x)\ . \tag{5.21}
$$

5.3 Illustration of our results

Writing eq. (5.1) in terms of gamma functions (see eq. (3.43)) allows us to generalise $g_n^{Gin}(x)$ to non-integer n. For the case when x is large, we can then easily show that

$$
g_{\nu,n}^{ch}(x) \approx g_{2n+\nu+\frac{1}{2}}^{Gin}(x)\ . \tag{5.22}
$$

The $P(x)$ for all of the four ensembles can therefore be written as products of the logarithms of certain $g_n^{Gin}(x)$ as given in Table 3.

We can easily see from this why the Ginibre $\beta = 4$ case is slightly less than half of the Ginibre $\beta = 2$ case, for example. There are half as many dots in the second row as in the first, but they are systematically shifted to the right (this is the same argument we presented above in Section 5.2.2).

More interesting is the comparison of the chiral and Ginibre cases for fixed β. For the chiral $\beta = 2$ case (for $\nu = 0$, i.e. no exact zero eigenvalues) there are also only half as many dots, but there is no systematic shift compared with the Ginibre $\beta = 2$ case as the chiral dots take the average value, e.g. $\frac{1}{2} = (0 + 1/2)$ etc, and so the chiral $S_1(x)$ is exactly half that of the Ginibre case.

[5]We could have changed the proportionality factor from $\frac{1}{2} \to 1$ by modifying the weight to $\exp[-\frac{\beta}{2}N\mathrm{Tr}\ldots]$.

Ensemble	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$\frac{11}{2}$	6	$\frac{13}{2}$	7	*etc.*
Ginibre $\beta = 2$	•		•		•		•		•		•		•		•	
Ginibre $\beta = 4$			•				•				•				•	
Chiral $\beta = 2$ ($\nu = 0$)		•				•				•				•		
($\nu = 1$)				•				•				•				
($\nu = 2$)						•				•				•		
Chiral $\beta = 4$ ($\nu = 0$)						•								•		

Table 3: Pictorial illustration of the Fredholm eigenvalues taken by the four ensembles.

What about the case where we know there is exactly *one* eigenvalue at the origin? For the Ginibre $\beta = 2$ case, we must then consider the probability that, *given there is one eigenvalue at the origin*, there are no other eigenvalues within a distance r of the origin. It turns out that this conditional probability is calculated simply by removing the first ($n = 0$) term from the sum that defines S_1 (see also [5]). Compare this modification of the top row of Table 3 with the chiral $\beta = 2$ case for $\nu = 1$ (i.e. when there is again precisely one exactly-zero eigenvalue). It will be seen that there are half as many dots in the latter case as in the former, and there is again no systematic shift. So, for precisely one exact zero eigenvalue, it is also the case that the chiral case is exactly half the Ginibre case.

It is not possible to do a similar comparison for $\beta = 4$ directly, since the Ginibre ensemble has zero probability to find an eigenvalue at the origin.

5.4 Low x asymptotics for the four ensembles

The probability that one eigenvalue is at the origin and a second one at radial distance r can be used to compute the spacing distribution $p(s)$ in the complex plane [5]. Following their argument the mean of the macroscopic large-N density is constant on a disc for the Ginibre ensemble $\beta = 2$ and one can assume that the probability calculated in this way is translation invariant, giving the spacing everywhere in the bulk.

However, for all the other ensembles this is not true and the origin is special. What can be compared is the strength of repulsion between two complex eigenvalues to leading order: for the Ginibre ensemble $\beta = 2$ everywhere in the bulk following [5], and for the chiral ensembles at the origin by placing an exact zero eigenvalue at zero.

Therefore we also give the expansion of the gap probabilities close to the origin. Note that the

Ensemble	$\beta = 2$	$\beta = 4$
Ginibre	$1 - x + \frac{x^3}{2} - \frac{5x^4}{12} + \frac{7x^5}{24} + \ldots$	$1 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{6} + \frac{x^5}{15} + \ldots$
Chiral ($\nu = 0$)	$1 + \frac{x^2}{2} \log x + \ldots$	$1 + \frac{x^4}{16} \log x + \ldots$
($\nu = 2/\beta$)	$1 - \frac{x^2}{4} + \ldots$	$1 - \frac{x^4}{64} + \ldots$

Table 4: Expansion of the gap probabilities for small radii $x = Nr^2 \ll 1$.

first product begins at zero, unlike in [5], as we give the gap probability here in Table 4, compared to the conditional probability to have one eigenvalue at zero and one at radius r there. The first two equations agree with [6] eqs. (15.1.19) and (15.2.18).

The leading power of the spacing distribution $p(s)$ can be reproduced for Ginibre $\beta = 2$ as follows. Multiplying the first line in Table 4 with e^x to remove the $n = 0$ contribution also removes the linear term $-x$, and we obtain $1 - \frac{1}{2}x^2 + \ldots$. The spacing is then obtained by first reinserting $x = Nr^2$ and then differentiating with respect to r: $p(s) \sim s^3$. The same cubic repulsion was found in [22] for other examples of rotationally invariant weights, as well as in [16] for strong non-Hermiticity by expanding the gap to second order as in eq. (2.15), and it is thus considered to be universal. Comparing this to the chiral ensemble with one exact zero eigenvalue $\nu = 1$, we find once more a cubic repulsion, adding a further ensemble to this universality class.

While for $\beta = 4$ a comparison to Ginibre is not possible we can at least compare the two chiral ensembles, finding that the repulsion is much stronger for $\beta = 4$ than for $\beta = 2$: here placing *one* eigenvalue at the origin corresponds to $\nu = \frac{1}{2}$. This indicates that there exists a different universality class $p(s) \sim s^7$ of spacing distributions for $\beta = 4$ at the origin.

As a final observation we note that when comparing ensembles in Table 4 the first power in x doubles when going from Ginibre to chiral, or from $\beta = 2$ to $\beta = 4$. This is in contrast to the coefficient in the exponent going down by $\frac{1}{2}$ for these comparisons of ensembles in the large-x asymptotics in Table 2.

6 Conclusions

We have investigated the gap probabilities and distributions of individual eigenvalues with respect to radial ordering in the complex plane in non-Hermitian Random Matrix Theory (RMT).

After setting up a general framework in terms of Fredholm determinants and Pfaffians for general non-Gaussian RMT with unitary ($\beta = 2$) and symplectic ($\beta = 4$) invariance we turned to maximal non-Hermiticity. For general weights with rotational invariance we found that the product representation of the gap probabilities in terms of Fredholm eigenvalues are related for $\beta = 2$ and 4, and we gave explicit new expressions for two ensembles of Gaussian chiral RMT. This relation between the gap probabilities is different from the one for Hermitian RMT. It would be very interesting to extend our relation to intermediate Hermiticity, interpolating between the two limiting cases. This may be compared to the Hermitian limit of the spectral correlators. For $\beta = 4$ these are given in terms of a 2×2 matrix kernel containing a single pre-kernel of skew orthogonal polynomials in the complex plane which is much simpler than the three pre-kernels in the Hermitian limit following from a Taylor expansion of the complex one.

We then derived an asymptotic expansion for the gap probability at large radii for the Gaussian Ginibre and chiral complex ensembles, both at $\beta = 2$ and 4. In particular this included a detailed discussion how to get from Ginibre to chiral and from $\beta = 2$ to 4 in these ensembles. Our results are consistent with the known results for Ginibre at $\beta = 2$ of Forrester, for the other ensembles they were new. It would be very interesting to sharpen our strict upper and lower bounds for the linear coefficients, and we have given numerical evidence for their conjectured values. Expanding for small radii we found that the chiral complex $\beta = 2$ ensemble also displays a cubic level repulsion at the origin, in contrast to its $\beta = 4$ counterpart.

While in this paper we have focused on the $\beta = 2$ and 4 ensembles one could try to generalise our results to the recently solved $\beta = 1$ Ginibre and chiral ensemble with real asymmetric matrix entries. Because these two ensembles have both real and complex eigenvalues, several different gap probabilities can be defined and computed, and we leave these open questions for future work.

Acknowledgements: Financial support by an EPSRC doctoral training grant (M.J.P.) and EPSRC first grant EP/D031613/1 (G.A. and L.S.), as well as the European Community Network grant ENRAGE MRTN-CT-2004-005616 (G.A.) is gratefully acknowledged. We thank Peter Forrester for kindly providing a reference to a little known paper of his after we had written up this work.

A Integrals over Bessel functions

In this appendix we derive several integrals over Bessel functions, including the one given in eq. (3.36).

A.1 General case

First we compute the matrix elements of a determinant yielding the gap probability $E_0^{(2)}(r)$ in the general elliptic case of the chiral ensembles. For that purpose a slightly different representation than eq. (3.1) is more convenient. Instead of choosing orthonormal wave functions in eq. (3.1) one can simply keep the original monomials in the Vandermonde determinant, leading to

$$E_0^{(2)}(r) = \frac{N!}{\mathcal{Z}_{ch}^{(2)}} \det_{1\leq k,j\leq N} \left[\int_{\mathbb{C}\setminus\mathcal{C}_r} d^2z\, w_\nu^{(2)}(z) z^{2(k-1)} z^{*\,2(j-1)} \right] \tag{A.1}$$

instead of eq. (3.5) (the same identity was shown in a different way in [6] eq. (15.1.13), without squared arguments). While this seems tailored to the rotationally invariant case at maximal non-Hermiticity [6] we can also use it here in the general case. We have to compute the following integrals:

$$\begin{aligned} &\int_r^\infty ds\, s \int_0^{2\pi} d\theta s^{2\nu+2} K_\nu(as^2) \exp\left[bs^2\cos(2\theta)\right] s^{2(k+j-2)} \exp[2i\theta(k+j-2)] \;= \\ =\;& \pi \int_{r^2}^\infty dt\, t^{k+j+\nu-1} K_\nu(at) I_{k+j-2}(bt) \qquad\qquad (\text{A.2})\\ &\text{with} \quad a = \frac{N(1+\mu^2)}{2\mu^2} \quad , \; b = \frac{N(1-\mu^2)}{2\mu^2} \;, \qquad\qquad (\text{A.3}) \end{aligned}$$

and we have $a > b \geq 0$ ensuring convergence. While we were unable to give a general result we can provide a recursive prescription, starting with the known integral

$$\int_{r^2}^\infty dt\, tK_0(at)I_0(bt) = \frac{1}{a^2-b^2}\left[br^2 I_1(br^2)K_0(ar^2) + ar^2 I_0(br^2)K_1(ar^2)\right] \equiv f(a,b;r^2) \;, \tag{A.4}$$

for $\nu = 0$ and $m \equiv k+j-2 = 0$. Any value of $m > 0$ and $\nu > 0$ can be obtained from this by applying the following Bessel identities for $a, b \neq 0$

$$\begin{aligned} \partial_b I_m(bt) - \frac{m}{b} I_m(bt) &= tI_{m+1}(bt) \\ -\partial_a K_\nu(bt) + \frac{\nu}{a} K_\nu(at) &= tK_{\nu+1}(at) \;. \end{aligned} \tag{A.5}$$

We thus obtain as a final result for $m = k+j-2$

$$\begin{aligned} &\int_{r^2}^\infty dt\, t^{m+\nu+1} K_\nu(at) I_m(bt) \;= \\ =\;& \left(\partial_b - \frac{m}{b}\right)\left(\partial_b - \frac{m-1}{b}\right)\cdots\partial_b\left(-\partial_a + \frac{\nu}{a}\right)\left(-\partial_a + \frac{\nu-1}{a}\right)\cdots(-\partial_a) f(a,b;r^2) \;. \end{aligned} \tag{A.6}$$

All differentiations can be carried out recursively and are algebraic, acting on the expression in the middle of eq. (A.4). For $b = 0$ or $\mu = 1$ we have rotational invariance and all integrals can be computed in a closed form, see Subsection 3.3 and this appendix below.

We end this part on the elliptic case by sketching how to proceed for $\beta = 4$. Keeping monic powers instead of skew orthogonal polynomials in eq. (3.19) we arrive at

$$E_0^{(4)}(r) = \frac{(2N)!}{\mathcal{Z}^{(4)}}\mathrm{Pf}_{1\le k,l\le 2N}\left[\int_{\mathbb{C}\setminus\mathcal{C}_r} d^2z(z^2-z^{*\,2})w_\nu^{(4)}(z)\Big(z^{2k-2}z^{*\,2l-2}-z^{*\,2k-2}z^{2l-2}\Big)\right] \tag{A.7}$$

After multiplying out we have to compute the same types of integrals as already done for $\beta = 2$, with the index 2ν of the Bessel-K function and weight now increasing in steps of two instead.

A.2 Maximal non-Hermiticity

In the second part of this appendix we compute the integral needed in the rotationally invariant case $\mu = 1$ ($b = 0$) given in eq. (3.36). In this case the angular integration diagonalises the matrix, see (A.1), and the Bessel-I functions from above become simple powers. We have to show that:

$$\begin{aligned}
F_\nu(k,x) &\equiv \int_0^x ds\, s^{2k+\nu+1}K_\nu(s) &\text{(A.8)}\\
&= 2^{2k+\nu}(k+\nu)!k!\Bigg(1-\frac{x^{2k+\nu+1}}{2^{2k+\nu}(k+\nu)!k!}K_{\nu+1}(x) &\text{(A.9)}\\
&\quad -x\sum_{l=0}^{k-2}\frac{1}{(l+\nu+2)!l!}\left(\frac{x}{2}\right)^{2l+\nu+2}K_{\nu+1}(x)-x\sum_{l=0}^{k-1}\frac{1}{(l+\nu+1)!l!}\left(\frac{x}{2}\right)^{2l+\nu+1}K_{\nu+2}(x)\Bigg),
\end{aligned}$$

where the sums $\sum_{l=0}^{-2}$, $\sum_{l=0}^{-1}$ are set to zero. For $k = 0$ this integral is standard, see e.g. eq. (6.561.8) in [20]

$$\int_0^x ds\, s^{\nu+1}K_\nu(s) \;=\; 2^\nu\nu! - x^{\nu+1}K_{\nu+1}(x)\ , \tag{A.10}$$

and in this case the sums giving the incomplete Bessel-I functions in the second line are absent. Using the identities

$$\Big(s^{\nu+1}K_{\nu+1}(s)\Big)' \;=\; -s^{\nu+1}K_\nu(s) \;\;\text{and}\;\; \Big(s^{-\nu}K_\nu(s)\Big)' \;=\; -s^{-\nu}K_{\nu+1}(s)\ , \tag{A.11}$$

we can show that the following recursion holds:

$$\begin{aligned}
F_\nu(k+1,x) &= -\int_0^x ds\, s^{2k+2}\Big(s^{\nu+1}K_{\nu+1}(s)\Big)' &\text{(A.12)}\\
&= -x^{2k+2+\nu+1}K_{\nu+1}(x)-2(k+1)\int_0^x ds\, s^{2k+2+2\nu}\Big(s^{-\nu}K_\nu(s)\Big)'\\
&= -x^{2k+3+\nu}K_{\nu+1}(x)-2(k+1)x^{2k+2+\nu}K_\nu(x)+4(k+1)(k+1+\nu)F_\nu(k,x)\ .
\end{aligned}$$

It is easy to verify that the explicit expression eq. (A.9) satisfies this relation

$$\begin{aligned}
F_\nu(k+1,x) &= 2^{2k+2+\nu}(k+1+\nu)!(k+1)!\Bigg(1-\frac{x^{2k+2+\nu+1}}{2^{2k+2+\nu}(k+1+\nu)!(k+1)!}K_{\nu+1}(x)\\
&\quad -x\sum_{l=0}^{k-2}\frac{1}{(l+\nu+2)!l!}\left(\frac{x}{2}\right)^{2l+\nu+2}K_{\nu+1}(x)-x\frac{\left(\frac{x}{2}\right)^{2k-2+\nu+2}}{(k-1+\nu+2)!(k-1)!}K_{\nu+1}(x)\\
&\quad -x\sum_{l=0}^{k-1}\frac{1}{(l+\nu+1)!l!}\left(\frac{x}{2}\right)^{2l+\nu+1}K_{\nu+2}(x)-x\frac{1}{(k+\nu+1)!k!}\left(\frac{x}{2}\right)^{2k+\nu+1}K_{\nu+2}(x)\Bigg).
\end{aligned}$$

In a last step we have to apply the Bessel-K identity

$$2(\nu+1)K_{\nu+1}(x) - xK_{\nu+2}(x) \;=\; -xK_\nu(x)\ , \tag{A.13}$$

to arrive at

$$\begin{aligned} F_\nu(k+1,x) \;&=\; 4(k+1+\nu)(k+1)\Big(F_\nu(k,x) + x^{2k+\nu+1}K_{\nu+1}(x)\Big) - x^{2k+3+\nu}K_{\nu+1}(x) \\ &\quad -4(k+1)kx^{2k+\nu+1}K_{\nu+1}(x) - 2(k+1)x^{2k+\nu+1}K_{\nu+2}(x) \end{aligned} \tag{A.14}$$

which finishes our proof by induction. As before the same integrals computed here apply to $\beta = 4$.

B A lower bound for $g_n^{Gin}(x)$ when $n > x$

We derive a lower bound for $g_n^{Gin}(x)$, valid when $n > x$. In order to do this, we first need to determine an *upper* bound for the individual terms in $g_n^{Gin}(x)$, denoted $t_k(x)$, for $k > n > x$. We will actually do this for $k \geq x$.

B.1 An upper bound for the reciprocal of a factorial

We have that [23]

$$n! > \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{B.1}$$

and thus

$$\frac{1}{n!} < \frac{1}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)^n . \tag{B.2}$$

B.2 An upper bound for $t_k(x)$

We define

$$t_k(x) \equiv \frac{e^{-x}x^k}{k!} \tag{B.3}$$

and wish to determine an upper bound for this when $k \geq x$. It proves useful in what follows to introduce a scaled variable

$$p \equiv \frac{k-x}{\sqrt{x}} \geq 0. \tag{B.4}$$

We use eq. (B.2), and write k in terms of p as follows:

$$\begin{aligned} t_k(x) \;&<\; \frac{e^{-x}x^k}{\sqrt{2\pi k}}\left(\frac{e}{k}\right)^k \;=\; \frac{1}{\sqrt{2\pi x}}\exp\left[p\sqrt{x} - \left(x + p\sqrt{x} + \frac{1}{2}\right)\log\left(1 + \frac{p}{\sqrt{x}}\right)\right] \\ &\equiv\; \frac{1}{\sqrt{2\pi x}}\exp[E(p,x)] \end{aligned} \tag{B.5}$$

Now, keeping x fixed, we consider three different regimes for k (or equivalently for p).

First, for $x \leq k < \frac{3x}{2}$ (equivalently $0 \leq p < \frac{\sqrt{x}}{2}$), we can expand the logarithm as an absolutely convergent series, and collect powers of $x^{-1/2}$:

$$\begin{aligned} E(p,x) \;&=\; p\sqrt{x} - \left(x + p\sqrt{x} + \frac{1}{2}\right)\left\{\frac{p}{\sqrt{x}} - \frac{p^2}{2x} + \frac{p^3}{3x^{3/2}} - \frac{p^4}{4x^2} + \dots\right\} \\ &=\; -\frac{p^2}{2} + \sum_{i=1}^{\infty}\frac{a_i(p)}{x^{i/2}} \end{aligned} \tag{B.6}$$

where the coefficients $a_i(p)$ are polynomials in p:

$$a_i(p) \equiv (-)^{i+1} p^i \left[\frac{p^2}{(i+1)(i+2)} - \frac{1}{2i} \right]. \tag{B.7}$$

Hence

$$t_k(x) \quad < \quad \frac{1}{\sqrt{2\pi x}} \exp\left\{ -\frac{p^2}{2} + \sum_{i=1}^{\infty} \frac{a_i(p)}{x^{i/2}} \right\} \quad = \quad \frac{1}{\sqrt{2\pi x}} e^{-p^2/2} \left\{ 1 + \sum_{i=1}^{\infty} \frac{b_i(p)}{x^{i/2}} \right\} \quad \equiv \quad \bar{t}^{(1)}(k,x) \tag{B.8}$$

where the $b_i(p)$ are also polynomials in p which we do not give explicitly.

Second, for $\frac{3x}{2} \leq k < e^2 x$ (equivalently $\frac{\sqrt{x}}{2} \leq p < (e^2-1)\sqrt{x}$), we use the fact that $t_k(x)$ is monotonic decreasing in k (for $k \geq x$ and x fixed), to give

$$t_k(x) \quad \leq \quad t_{\frac{3x}{2}}(x) \quad = \quad \frac{1}{\sqrt{2\pi x}} \sqrt{\frac{2}{3}} \exp\left\{ \left(\frac{1}{2} - \frac{3}{2} \log \frac{3}{2} \right) x \right\} \quad \equiv \quad \bar{t}^{(2)}(x) \tag{B.9}$$

which is (clearly) p-independent. The coefficient of x in the exponent is negative.

Finally, for $k \geq e^2 x$ (equivalently $p \geq (e^2-1)\sqrt{x}$), we have

$$\log\left(1 + \frac{p}{\sqrt{x}}\right) \geq 2 \tag{B.10}$$

and hence

$$t_k(x) \quad < \quad \frac{e^{-2x-1}}{\sqrt{2\pi x}} e^{-\sqrt{x}\,p} \quad \equiv \quad \bar{t}^{(3)}(k,x) \tag{B.11}$$

in which we emphasise the p-dependence, since we will be integrating over p.

B.3 A lower bound for $g_n^{Gin}(x)$

Using the previous definitions of $g_n(x)$ and $t_k(x)$ (eqs. (4.3) and (B.3) respectively), we have (for $n > x$)

$$\begin{aligned} g_n^{Gin}(x) \quad &\equiv \quad \sum_{k=0}^{n} t_k(x) \quad = \quad 1 - \sum_{k=n+1}^{\infty} t_k(x) \\ &> \quad \begin{cases} 1 - \int_n^{3x/2} \bar{t}^{(1)}(k,x)dk - \int_{3x/2}^{e^2 x} \bar{t}^{(2)}(x)dk - \int_{e^2 x}^{\infty} \bar{t}^{(3)}(k,x)dk & (x < n < 3x/2) \\ 1 - \int_n^{e^2 x} \bar{t}^{(2)}(x)dk - \int_{e^2 x}^{\infty} \bar{t}^{(3)}(k,x)dk & (3x/2 \leq n < e^2 x) \\ 1 - \int_n^{\infty} \bar{t}^{(3)}(k,x)dk & (e^2 x \leq n) \end{cases} \qquad \text{(B.12)} \\ &> \quad 1 - \int_n^{\infty} \bar{t}^{(1)}(k,x)dk - \int_{3x/2}^{e^2 x} \bar{t}^{(2)}(x)dk - \int_{e^2 x}^{\infty} \bar{t}^{(3)}(k,x)dk \end{aligned}$$

$$\tag{B.13}$$

where the $\bar{t}^{(i)}(k,x)$ are the upper bounds for $t_k(x)$ derived in Appendix B.2. We have also used the fact $t_k(x)$ is monotonic decreasing as a function of k.

In the same way that we introduced p as a scaled proxy for k (eq. (B.4)), we similarly introduce m as a scaled proxy for n:

$$m \equiv \frac{n-x}{\sqrt{x}} \tag{B.14}$$

The first integral can then be written

$$\begin{aligned}\int_n^\infty \bar{t}_1(k,x)dk &= \frac{1}{\sqrt{2\pi}}\int_m^\infty e^{-p^2/2}\left\{1+\sum_{i=1}^\infty \frac{b_i(p)}{x^{i/2}}\right\}dp \\ &= 1-\Phi(m)+\sum_{i=1}^\infty \frac{c_i(m)}{x^{i/2}} \end{aligned} \tag{B.15}$$

where $\Phi(m)$ is the cumulative normal function, and the $c_i(m)$ have no x-dependence.

The second and third integrals have the forms $\alpha x^{1/2}e^{-\beta x}$ and $\gamma x^{-1/2}e^{-\delta x}$ respectively, where α, β, γ and δ are constants:

$$\begin{aligned}\alpha &= \frac{e^2-\frac{3}{2}}{\sqrt{3\pi}}, \\ \beta &= \frac{3}{2}\log\left(\frac{3}{2}\right)-\frac{1}{2}>0, \\ \gamma &= \frac{1}{e\sqrt{2\pi}}, \\ \delta &= 1+e^2>0. \end{aligned} \tag{B.16}$$

The important thing to note is that these two integrals are exponentially small when compared with the first integral.

Hence, we can write (as an exact result)

$$g_n^{Gin}(x)\equiv g_{x+m\sqrt{x}}^{Gin}(x) \;>\; \Phi(m)-\sum_{i=1}^\infty \frac{c_i(m)}{x^{i/2}}-\alpha\sqrt{x}e^{-\beta x}-\frac{\gamma e^{-\delta x}}{\sqrt{x}}. \tag{B.17}$$

This is true for any $m>0$ (equivalently, for any $n>x$). However, we see that the right-hand side has a well-defined asymptotic limit as $x\to\infty$ *only for fixed* m (i.e. and not for fixed n).

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