On the factorization of a class of Wiener-Hopf kernels

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February 1995

Abstract

The Wiener-Hopf technique is a powerful aid for solving a wide range of problems in mathematical physics. The key step in its application is the factorization of the Wiener-Hopf kernel into the product of two functions which have different regions of analyticity. The traditional approach to obtaining these factors gives formulae which are not particularly easy to compute. In this article a novel approach is used to derive an elegant form for the product factors of a specific class of Wiener-Hopf kernels. The method utilizes the known solution to a difference equation and the main advantage of this approach is that, without recourse to the Cauchy integral, the product factors are expressed in terms of simple, finite range integrals which are easy to compute.

1 Introduction

The Wiener-Hopf technique is a powerful analytical tool for solving a wide range of problems in physics and engineering. Applications include the analysis of crystal growth (Kuiken, 1985), scattering of acoustic or water waves (Leppington, 1968), viscous and inviscid fluid flows (Lewis and Carrier, 1949; Davis, 1990), crack propagation in elastic media (Abrahams and Wickham, 1992), and radiation and neutron transport problems. The latter topic was the original application for which the Wiener-Hopf technique was devised (Wiener & Hopf, 1931). This method can be applied to all linear boundary value problems defined in an infinite strip and which have two-part boundary conditions on one or both of the infinite faces. It has successfully been employed on hyperbolic, elliptic and parabolic governing equations.

The reader is referred to the book by Noble (1958) for full details of the solution method and several examples. This paper will concentrate on an examination of the key-step of the Wiener-Hopf procedure, that is, the factorization of the Wiener-Hopf kernel. The kernel, $Q(\alpha)$ say, is a function of a complex variable, α, and in general has singularities (both poles and branch cuts) and zeros in the whole of the complex α-plane except for a finite width strip containing the line $\Im(\alpha) = 0$. The strip of regularity is called $\mathcal{D}$. The precise form of $Q(\alpha)$ is determined by the governing equation and boundary conditions for a given problem. In order to solve any Wiener-Hopf problem it is necessary to decompose the kernel into a product of two functions, one regular and zero free in the upper half of the complex α-plane including $\mathcal{D}$, labelled $\mathcal{D}_+$, and the other regular and zero free in the lower half plane including $\mathcal{D}$, namely $\mathcal{D}_-$. Thus,

$$Q(\alpha) = Q_+(\alpha)Q_-(\alpha), \qquad \alpha \in \mathcal{D}, \tag{1.1}$$

where $Q_\pm(\alpha)$ is regular in $\mathcal{D}_\pm$. It is a simple matter to obtain a sum factorization of any function by application of Cauchy's theorem (Noble, 1958 pp 13). Hence, taking the sum factorization of the logarithm of $Q(\alpha)$, and then obtaining the exponential of this term gives

$$Q_\pm(\alpha) = \exp\left\{\frac{\pm 1}{2\pi i}\int_{-\infty}^{\infty} \log[Q(\zeta)]\frac{d\zeta}{\zeta - \alpha}\right\}, \quad \Im(\alpha) \gtrless \Im(\zeta), \tag{1.2}$$

where the symmetry of these integral definitions implies that

$$Q_+(\alpha) = Q_-(-\alpha) \tag{1.3}$$

if $Q(\alpha)$ is an even function. The form of this expression has several difficulties, highlighted in section 3, which makes its computation both slow and cumbersome. A slow numerical algorithm is highly undesirable because the final solution to any physical problem will inevitably be expressed as an infinite integral of Fourier type, the integrand of which contains one of the product factors. Examples of such integrals are shown later in this article. The primary aim of this paper (section 4) is to offer a new direct approach to factorizing Wiener-Hopf kernels of a specified class. This not only avoids utilizing the Cauchy integral, (1.2), but results in a computationally simple, and highly efficient form for the product factors. In section 5 numerical timings for both the Cauchy integral and the alternative representations are presented. The Appendix verifies the results of section 4, employing a derivation based on standard but little used integral techniques.

2 The specific class of Wiener-Hopf kernels

Perhaps the major areas of mathematical physics that have exploited the Wiener-Hopf technique are those involved with the propagation or diffraction of time-harmonic waves. These include the subjects of electromagnetism, acoustics and elastodynamics, which, for steady-state oscillations yield the reduced wave equation (often referred to as Helmholtz' equation). The class of Wiener-Hopf kernels, $Q(\alpha)$, to be examined herein results from, amongst other problems, the following boundary value system:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+k^2\phi = 0, \quad -\infty<x<\infty, y>0, \tag{2.1}$$

$$\mathbf{A}\frac{\partial\phi}{\partial y}+\mathbf{B}\phi = 0, \quad y=0, x<0, \tag{2.2}$$

$$\mathbf{C}\frac{\partial\phi}{\partial y}+\mathbf{D}\phi = 0, \quad y=0, x>0, \tag{2.3}$$

where $\phi(x,y)$ is a physical dependent variable such as a pressure or stress function etc., k is a constant wavenumber and $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ are differential operators of the form

$$\mathbf{A}=\sum_{n=0}^{N_a} a_n\frac{\partial^{2n}}{\partial x^{2n}},\ldots,\mathbf{D}=\sum_{n=0}^{N_b} d_n\frac{\partial^{2n}}{\partial x^{2n}}. \tag{2.4}$$

Note that $N_a,\ldots,N_d$ are any non-negative integers, $a_n,\ldots,d_n$ are arbitrary coefficients dependent on the physics, and the presence of only even derivatives in x is a requirement on most boundary conditions in wave theory.

Omitting all details, it can easily be shown that $Q(\alpha)$ for the boundary value problem (2.1)–(2.3) may be cast into the form

$$Q(\alpha)=C\frac{\prod_{j=1}^{N}\gamma(\alpha)-ik\sin(X_j)}{\prod_{j=1}^{M}\gamma(\alpha)-ik\sin(Y_j)} \tag{2.5}$$

where

$$\gamma(\alpha)=(\alpha^2-k^2)^{1/2}, \tag{2.6}$$

C is a constant which will henceforth be omitted for brevity, N and M are odd integers related to the derivative orders $N_a,\ldots,N_d$, and X_j and Y_j are complex constants related to the coefficients $a_n,\ldots,d_n$. The double valued complex function $\gamma(\alpha)$ has branch points at $\pm k$ and its Riemann surface is chosen such that

$$\gamma(0)=-ik. \tag{2.7}$$

In scattering problems it is usual, initially, to allow k to have a small positive imaginary part in order to give a strip $\mathcal{D}$ of width $2\Im(k)$. Thus, the branch cuts emanating from $+k$, $-k$ go off infinity within $\mathcal{D}_+$, $\mathcal{D}_-$ respectively, and without loss of generality are here taken along $+k\to+\infty+\Im(k)$ and $-k\to-\infty-\Im(k)$. Finally, for reasons to be made clear later, and without loss of generality, X_j and Y_j are chosen to lie within

$$-\pi<\Re(X_j),\Re(Y_j)\leq\pi/2, \quad \Im(X_j),\Im(Y_j)\geq 0, \tag{2.8}$$

or

$$0\leq\Re(X_j),\Re(Y_j)<\pi/2, \quad \Im(X_j),\Im(Y_j)<0. \tag{2.9}$$

Note that $\gamma(\alpha)-ik\sin(X_j)$ has a zero in the Riemann surface defined by (2.7) only when (for k real) $-\pi<\Re(X_j)\leq 0,\Im(X_j)>0$, or $\Re(X_j)=0,\Im(X_j)<0$.

A specific example from the class of boundary value problems, defined by (2.1)–(2.4), is an acoustic fluid lying above a semi-infinite elastic plate set in a semi-infinite rigid baffle. Then, from Cannell (1975) the kernel is given by

$$Q(\alpha)=\frac{\prod_{j=1}^{5}\gamma(\alpha)-ik\sin(X_j)}{\gamma(\alpha)} \tag{2.10}$$

where the $\sin(X_j)$ satisfy

$$\sin^5(X_j) - 2\sin^3(X_j) - (1 - \mu^4/k^4)\sin(X_j) + i\tau/k^5 = 0 \tag{2.11}$$

in which μ and τ are parameters dependent on the plate and fluid variables.

It can be noted that there is one root, X_1 say, which is purely negative imaginary, one root lies in region (2.9), and the other three roots lie respectively in the strips $-\pi < \Re(X) \leq -\pi/2, -\pi/2 < \Re(X) \leq 0, 0 < \Re(X) \leq \pi/2$, $\Im(X) > 0$. Also, $Y_1 = 0$ so that $\gamma(\alpha)$ appears in the denominator on its own. This term is trivial to factorize and so in what follows only the case $|X_j| \neq 0$ will be examined.

The product factor which is regular in $\mathcal{D}_+$ for the general kernel written in (2.5) is

$$Q_+(\alpha) = \frac{\prod_{j=1}^{N} K_+(\alpha, X_j)}{\prod_{j=1}^{M} K_+(\alpha, Y_j)} \tag{2.12}$$

where the Cauchy's integral formula gives

$$K_+(\alpha, X) = \exp\left\{\frac{1}{2\pi i}\lim_{R\to\infty}\int_{-R}^{R}\log[\gamma(\zeta) - ik\sin(X)]\frac{d\zeta}{\zeta - \alpha}\right\}, \quad \Im(\alpha) > \Im(\zeta). \tag{2.13}$$

The main thrust of this article is the novel derivation of an expression for $K_+(\alpha, X)$ in terms of simple, finite range integrals. However, to emphasize the need for such representation, the the next section is concerned with describing the difficulties commonly encountered with numerical evaluation of (2.13). Note that, as $\Im(k) \to 0$, the integration path passes above the branch cut lying along $-k$ to $-\infty$ and below the cut along k to ∞. The path runs below the pole at α but logarithmic branch points can lie in any part of the complex α-plane. Note, finally, that the definition of the product factorization (2.13) implies

$$K_+(\alpha, X) = K_-(-\alpha, X). \tag{2.14}$$

3 Numerical evaluation of the Cauchy integral representation

This section will concentrate on the numerical evaluation of the integral representation of $K_+(\alpha, X)$ written in (2.13). There are several difficulties associated with this expression; these obstacles are discussed and practical procedures for overcoming them are outlined. First, by inspection of (2.13), it is clear that the integral converges by virtue of cancellation in the integrand as $\zeta \to \pm\infty$. This situation can be improved upon by changing ζ to $-\zeta$ on the negative part of the integration range and combining both half-range integrals. Thus, after allowing $R \to \infty$

$$K_+(\alpha, X) = \exp\{J(\alpha, X)\}, \quad \Im(\alpha) > \Im(\zeta), \tag{3.1}$$

where

$$J(\alpha, X) = \frac{\alpha}{\pi i}\int_0^\infty \log[\gamma(\zeta) - ik\sin(X)]\frac{d\zeta}{\zeta^2 - \alpha^2}. \tag{3.2}$$

This is now a convergent expression but it can be improved slightly by removing the logarithmic behaviour for large ζ:

$$\begin{aligned} J(\alpha, X) &= \frac{\alpha}{\pi i}\int_0^1 \log[\gamma(\zeta) - ik\sin(X)]\frac{d\zeta}{\zeta^2 - \alpha^2} + \frac{\alpha}{\pi i}\int_1^\infty \log[(\gamma(\zeta) - ik\sin(X))/\zeta]\frac{d\zeta}{\zeta^2 - \alpha^2} \\ &+ \frac{\alpha}{4\pi i}\Phi(\alpha^2, 2, .5), \quad \Im(\alpha) > \Im(\zeta), \end{aligned} \tag{3.3}$$

where $\Phi(z,s,a)$ is the Lerch transcendent, which is related to the Riemann Zeta function, and is defined as

$$\Phi(z,s,a)=\sum_{i=0}^{\infty}\frac{z^i}{(a+i)^s}, \tag{3.4}$$

see Gradshteyn & Ryzhik (1965), section (9.55). The second integral in (3.3) can now be turned into a finite range expression by recourse to the substitution $\zeta \to 1/\zeta$ and so $J(\alpha,X)$ can now be computed provided α or the logarithmic branch point is not near the integration path.

This leads to the second difficulty, which is to compute $J(\alpha,X)$ when α lies on or near the real line. In practice, $Q_+(\alpha)$ (and hence $K_+(\alpha,X)$) is almost always required along the whole of the real line, and so this case **must** be tackled. As α lies in the upper half plane it is clear that the integration path in ζ is indented below the point $\zeta=\alpha$. Without detail, it can be shown that the singularities can be removed by rewriting $J(\alpha,X)$ as

$$\begin{aligned} J(\alpha,X) &= \frac{\alpha}{\pi i}\int_0^1 \log\left[\frac{\gamma(\zeta)-ik\sin(X)}{\beta(\alpha)-ik\sin(X)}\right]\frac{d\zeta}{\zeta^2-\alpha^2} \\ &+ \frac{\alpha}{\pi i}\int_1^{\infty}\log\left[\frac{(\gamma(\zeta)-ik\sin(X))\epsilon\alpha}{(\beta(\alpha)-ik\sin(X))\zeta}\right]\frac{d\zeta}{\zeta^2-\alpha^2}+\frac{\alpha}{4\pi i}\Phi(\alpha^2,2,.5) \\ &- \frac{1}{2\pi i}\log(\epsilon\alpha)\log\left(\frac{1+\alpha}{1-\alpha}\right)+\frac{1}{2}\log(\beta(\alpha)-ik\sin(X)),\quad \Im(\alpha)>\Im(\zeta), \end{aligned} \tag{3.5}$$

where $\epsilon=\mathrm{sgn}\{\Re(\alpha)\}$. Also

$$\beta(\alpha) = \gamma(\alpha), \qquad |\Re(\alpha)|\le k, \tag{3.6}$$

$$\beta(\alpha) = \bar{\gamma}(\epsilon\alpha), \qquad |\Re(\alpha)|> k, \tag{3.7}$$

in which $\bar{\gamma}(\alpha)$ is a complementary function to $\gamma(\alpha)$ with a branch-cut lying between $\pm k$, and $\bar{\gamma}(\alpha)\to\alpha$ as $\alpha\to\infty$ in the whole of the complex plane.

Expressions (3.3) and (3.5) can be combined numerically to allow $J(\alpha,X)$ to be computed for all α in the upper half plane, as long as $\gamma(\alpha)-ik\sin(X)$ is not zero near the contour of integration. When X takes a complex value (see the constraints (2.8), (2.9)) near to, or on, any of the lines

$$\Re(X)=-\pi,\Im(X)>0;\quad \Re(X)=0,\Im(X)<0;\quad \Re(X)=0,-\pi<\Im(X)<0 \tag{3.8}$$

then this zero **will** lie close to the path of integration and so must be dealt with. It is straightforward to show that the logarithmic singularity thus appearing in (3.3) can be removed by writing $K_+(\alpha,X)$ in the form

$$K_+(\alpha,X)=i(\alpha+\cos X)\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\log\left[\frac{\gamma(\zeta)-ik\sin(X)}{\zeta^2-\cos^2(X)}\right]\frac{d\zeta}{\zeta-\alpha}\right\},\quad \Im(\alpha)>\Im(\zeta). \tag{3.9}$$

The explicit zero at $\alpha=-\cos(X)$ lies in the lower half plane, as required, as long as X does not take a value within the quadrant $-\pi/2<\Re(X)<0,\Im(X)>0$. This is consistent with only choosing this factorization representation when X is close to one of the lines defined in (3.8). Finally, in a similar fashion to that demonstrated above, a better form for $K_+(\alpha,X)$ in this case is

$$\begin{aligned} K_+(\alpha,X) &= i(\alpha+\cos X)\exp\left\{\frac{\alpha}{\pi i}\int_0^1\log\left[\frac{\gamma(\zeta)-ik\sin(X)}{\zeta^2-\cos^2(X)}\right]\frac{d\zeta}{\zeta^2-\alpha^2}\right. \\ &+ \left.\frac{\alpha}{\pi i}\int_1^{\infty}\log\left[\frac{(\gamma(\zeta)-ik\sin(X))\zeta}{\zeta^2-\cos^2(X)}\right]\frac{d\zeta}{\zeta^2-\alpha^2}-\frac{\alpha}{4\pi i}\Phi(\alpha^2,2,.5)\right\},\Im(\alpha)>\Im(\zeta). \end{aligned} \tag{3.10}$$

It has now been shown that the Cauchy integral representation for the product factorization (2.13) must be modified to deal with the various computational difficulties, namely bad convergence and singularities near to the integration path. A case not explicitly dealt with here, but one which needs to be addressed in many problems, is when both α and $\cos(X)$ lie close to or on the real line. This requires a combination of the techniques used in the representations (3.5) and (3.10). In the following section an elegant form for $K_+(\alpha, X)$ will be derived which will be shown to hold uniformly for α everywhere in the upper half plane and for all X in the region (2.8) and (2.9). It will also be shown that the new representation is a great deal faster to compute.

4 New representation of the kernel factorization

In this section a novel procedure is examined by which the product factor $K_+(\alpha, X)$ can be obtained in terms of simple finite range integrals. This new procedure hinges on the use of functional difference equations and the factorization is achieved without recourse to the Cauchy integral. All Wiener-Hopf kernels of the type defined by (2.5) may be factorized in this way and the method may give insight into factorizing other kernels.

It is convenient to introduce the transformation

$$\alpha = -k \sin s \tag{4.1}$$

where the upper (lower) half of the α-plane is chosen to map to the strip $-3\pi/2 \leq \Re(s) \leq -\pi/2$, $\Im(s) > 0$ ($\Im(s) < 0$), and

$$\gamma(-k \sin s) = ik \cos s. \tag{4.2}$$

Using these substitutions the kernel may be written as

$$K(-k \sin s, X) = K_+(-k \sin s, X) K_-(-k \sin s, X) = ik\{\cos(s) - \sin(X)\}. \tag{4.3}$$

This may be formulated as a difference equation by writing

$$\frac{K_+(-k \sin(s), X) K_-(-k \sin(s), X)}{K_+(-k \sin(s+\pi), X) K_-(-k \sin(s+\pi), X)} = \frac{\cos(s) - \sin(X)}{\cos(s) + \sin(X)}. \tag{4.4}$$

Note that, in the complex α-plane the function $K(\alpha, X)$ is an even function of α and thus (2.14) applies. However, in the complex s-plane it is found that $K_+(-k \sin(s+\pi), X) \neq K_-(-k \sin(s), X)$. The reason for this becomes apparent when one considers exactly how the two branches of the complex function $\gamma(\alpha)$ map into the complex s-plane (see section 2). That defined by $\gamma(0) = -ik$ maps into the strip $-3\pi/2 < \Re(s) < -\pi/2$, $-\infty < \Im(s) < \infty$ whilst the alternative surface, defined by $\gamma(0) = ik$, maps into a strip of width π adjacent to this. The full complex function $\gamma(\alpha)$ (that is both Riemann surfaces) maps into a strip of the complex s-plane of width 2π. Thus, it is reasonable to seek an expression for $K_+(-k \sin(s), X)$ in terms of the function $f(s)$ which satisfies

$$\frac{f(s+2\pi)}{f(s)} = \frac{\cos(s) - \sin(X)}{\cos(s) + \sin(X)}. \tag{4.5}$$

The solution to this difference equation is well documented (see Maliuzhinets, 1958) and can be written as

$$f(s) = \frac{1}{M(s - X + \pi) M(s + X)} \tag{4.6}$$

where $M(s)$ has the integral representation

$$M(s) = \exp\left\{ \frac{1}{4\pi} \int_0^s \frac{2u - \pi \sin u}{\cos u} \, du \right\}. \tag{4.7}$$

Note that the function $M(s)$ is a special case of the Maliuzhinets function, $M_\beta(s)$, as defined by Abrahams & Lawrie (1994). This function arises in the solution of the wave equation in a wedge of angle 2β with Robins or higher order boundary conditions. Hence, for Wiener-Hopf problems, which have planar boundaries, $\beta = \pi/2$ and so $M(s) = M_{\pi/2}(s)$. The Maliuzhinets' function satisfies the symmetry property

$$M_\beta(-s) = M_\beta(s) \tag{4.8}$$

and the following difference equation:

$$M_\beta(s+\pi/2)M_\beta(s-\pi/2) = M_\beta^2(\pi/2)\cos(\frac{\pi s}{4\beta}). \tag{4.9}$$

Expression (4.6) is easily justified by direct substitution of $f(s)$ into (4.5) making use of (4.9).

It is now possible to write (4.4) as

$$\frac{K_+(-k\sin(s),X)K_-(-k\sin(s),X)}{K_+(-k\sin(s+\pi),X)K_-(-k\sin(s+\pi),X)} = \frac{f(s+2\pi)}{f(s+\pi)}\frac{f(s+\pi)}{f(s)}. \tag{4.10}$$

Note the the quantity $f(s+\pi)$ is introduced both in the numerator and denominator of the right hand side in order to imitate the difference in argument of both the functions of the left hand side. The equation can now be 'separated' into two independent difference equations. Without loss of generality choose

$$\frac{K_+(-k\sin(s),X)}{K_+(-k\sin(s+\pi),X)} = \frac{f(s+\pi)}{f(s)}, \tag{4.11}$$

$$\frac{K_-(-k\sin(s),X)}{K_-(-k\sin(s+\pi),X)} = \frac{f(s+2\pi)}{f(s+\pi)}. \tag{4.12}$$

This choice of separation is entirely arbitrary; should the right hand sides of (4.11) and (4.12) be interchanged then the following steps would produce the factorization of $[K(-k\sin(s),X)]^{-1}$ from which, of course, the desired factorization is easily deduced. Note that it is not necessary, indeed undesirable, to introduce any eigensolutions when separating the difference equations. Such eigensolutions must satisfy $e(s+\pi) = e(s)$, which would introduce spurious zeros and/or poles into the product factors $K_\pm(-k\sin(s),X)$. Equations (4.11) and (4.12) are trivial to solve in terms of $f(s)$:

$$K_+(-k\sin(s),X) = \frac{E_1(s)}{f(s)} \tag{4.13}$$

$$K_-(-k\sin(s),X) = \frac{E_2(s)}{f(s+\pi)} \tag{4.14}$$

where $E_1(s)$, $E_2(s)$ are 2π-periodic eigensolutions. The functions $E_1(s)$ and $E_2(s)$ are chosen such that $K_\pm(-k\sin(s),X)$ have no zeros or poles in $-3\pi/2 < \Re(s) < -\pi/2$, $\Im(s) \gtrless 0$. Referring to Abrahams & Lawrie (1994), the function $f(s)$ has poles at $s = X+\pi/2+2m\pi$, $X-5\pi/2-2m\pi$, $-X+3\pi/2+2m\pi$, $-X-3\pi/2-2m\pi$, $m = 0,1,2,\ldots$, and zeros given by $s = X+3\pi/2+2m\pi$, $X-7\pi/2-2m\pi$, $-X+5\pi/2+2m\pi$, $-X-5\pi/2-2m\pi$, $m = 0,1,2,\ldots$. For X in the regions described by (2.8) and (2.9), it is found, on considering each family of zeros and poles, that $f(s)$ is in fact analytic and non-zero in the region $-3\pi/2 < \Re(s) \leq -\pi/2$, $\Im(s) > 0$. Thus, $E_1(s)$ must be a constant, say c_1. Similarly, it can be shown that

$$E_2(s) = c_2\frac{\cos(s)-\sin(X)}{\cos(s)+\sin(X)}, \tag{4.15}$$

where c_2 is constant. It remains only to select c_1 and c_2, which is conveniently achieved by enforcing conditions (2.14) at $\alpha = 0$, that is, at $s = -\pi$, and (4.3) at $s = -\pi/2 - X$. It is found that $c_1 = c_2 = \sqrt{2k}e^{-i\pi/4}M^{-2}(\pi/2)$. Hence, the factorization is complete:

$$K_+(-k\sin s, X) = \sqrt{2k}e^{-i\pi/4}M^{-2}(\pi/2)M(s - X + \pi)M(s + X) \tag{4.16}$$

this is valid for all s in the region $-3\pi/2 \leq \Re(s) \leq -\pi/2$, $\Im(s) > 0$ as long as X takes a value in the regions defined in (2.8) or $0 < \Re(X) < \pi/2$, $\Im(X) < 0$. For $\Im(X) < 0$ and $\Re(X) = 0$ there is a zero in $K_+(-k\sin s, X)$ situated on the line $\Re(s) = -3\pi/2$, $\Im(s) > 0$ and so the representation (4.16) will not be valid at this point due to the divergence of the function $M(s + X)$. This can be overcome by analytically continuing (4.16) into another region of the s-plane by application of (4.9). Thus, $K_+(-k\sin s, X)$ may be rewritten as

$$K_+(-k\sin s, X) = \sqrt{2k}e^{-i\pi/4}\cos\frac{1}{2}(s + X + \pi/2)\frac{M(s - X + \pi)}{M(s + X + \pi)} \tag{4.17}$$

which can easily be shown to be well defined for all s lying in $-2\pi \leq \Re(s) \leq -\pi/2$, $\Im(s) > 0$ if X satisfies (2.8) or (2.9). Note that successive applications of (4.9) will give a continuation of the product factorization into any region of the complex s-plane desired. For example,

$$K_+(-k\sin s, X) = \sqrt{2k}e^{-i\pi/4}M^2(\pi/2)\frac{\cos\frac{1}{2}(s - X + \pi/2)\cos\frac{1}{2}(s + X - \pi/2)}{M(s - X)M(s + X - \pi)} \tag{4.18}$$

can be employed in the region $-\pi/2 \leq \Re(s) < \pi/2$, $\Im(s) > 0$ or $\pi/2 \leq \Re(s) < 3\pi/2$, $\Im(s) < 0$. This result has application beyond the scope of this article (see Abrahams & Lawrie, 1994).

Finally, it is illustrative to consider the elastic plate kernel defined by (2.10). Comparing this with (2.10) it is clear that $M = 1$ and $Y_1 = 0$ so that the denominator is simply $\gamma(\alpha)$ and this must be factorized to satisfy (1.3). Then using (2.12) and (4.16) it can be shown that

$$Q_+(-k\sin s) = \frac{4k^2}{M^{10}(\pi/2)\sin\frac{1}{2}(s - \pi/2)}\prod_{j=1}^{5}M(s - X_j + \pi)M(s + X_j) \tag{4.19}$$

where $M(s)$ is given by (4.7). This was, in fact, the form of the factorization which produced the fastest result when computing the integral (5.1) in section 5. It should be noted that, whilst the above approach has not previously been documented, similar finite range integral representations for kernels of this class have been derived by Weinstein (1969) using a method similar to that outlined in the appendix.

5 Numerical timings and discussion

It is clear that the representation (4.16) (or 4.17) for $K_+(-k\sin s, X)$ has a significantly simpler form than that given in (3.1) using (3.3), (3.5) or (3.10). To demonstrate that it is also very much more efficient for computational purposes, a few timings are given for various calculations with different values of X. As mentioned previously, $Q_+(\alpha)$, and hence every $K_+(\alpha, X)$, is usually required along the real line (passing above the left hand branch cut and below the right hand cut) as well as at a number of discrete points in the upper half plane. For example, figure 2 gives a plot of $|K_+(\alpha, -\pi/4 + i)|$ along the real line between $-10 < \alpha < 10$ for $k = 1$. To compute this curve on a 90MHz Pentium PC using the Mathematica graphing call 'ParametricPlot', expression (3.5) took 441.6 secs whilst (4.16) took 41.1 secs. A similar plot for $X = -i$ was completed in 389.1 secs using the Cauchy representation suitably modified, and in 18.1 secs using (4.17).

A physically relevant example to indicate the advantage of the new representation is to compute the integral

$$\int_{-\infty}^{\infty} \frac{\exp\{i\zeta x_0 - \gamma(\zeta)y_0\}}{Q_+(\zeta)(\zeta-\alpha)}\, d\zeta \tag{5.1}$$

where $Q_+(\alpha)$ is a product factorization of the elastic plate kernel (2.10) and α takes any value in the upper half plane. Note that the contour is indented above a simple pole arising from a zero of $Q_+(\alpha)$ on the negative real line. This integral is typical of the type arising either as the final solution for scattering problems forced by a plane wave (see, for example Noble 1958, equation (2.84a)), or as the inner integral when a delta function forcing (at position (x_0, y_0)) is applied. Choosing, arbitrarily, $x_0 = y_0 = 1$, $\alpha = 1+i$ and the plate constants $k=1$, $\tau = 2$, $\mu = .5$ it is found that the Mathematica routine 'NIntegrate' computes (5.1) to be $-0.13985 - 0.472107i$ in just 3 minutes using (4.16) (in the combined form (4.19)), and in excess of 50 minutes employing the method described in section 3. As a matter of interest, the calculation of (5.1) using the Cauchy integral representation on a 386SX personal computer takes approximately 13 hours!

Insert figure 2 about here

To conclude, this paper has offered a simple, convenient and efficient form, (4.16), for the product factors of the general Wiener-Hopf kernel class (2.12). Great saving in computation time has been achieved, and this will be further compounded in problems where there are double integrals (both of infinite range) containing $Q_\pm(\alpha)$. The new representation of the kernel factorization allows a numerical experiment, over several parameter ranges, to be completed quickly and easily. In the case of the thin elastic plate, for example, the outgoing unattenuated plate wave coefficient could conveniently be plotted as a function of wavenumber, plate density, fluid loading etc. To obtain the same set of results using the Cauchy integral approach would require computation times of the order of weeks.

The difference equation approach to factorizing Wiener-Hopf kernels can be extended to other classes of kernel provided that they contain a branch point. However, it should be recognized that such an approach will require the derivation of the solution to a difference equation; in this case that was known *apriori*.

A Appendix

This appendix outlines an alternative, and more standard, approach by which expression (4.19) can be obtained. As before, it is convenient to introduce the transformation (4.1) and (4.2), thus (2.13) can be expressed as

$$K_+(-k\sin s, X) = \exp\left\{\frac{1}{2\pi i} I(s)\right\}, \quad \Im(\alpha) > \Im(\zeta), \tag{A.1}$$

where

$$I(s) = \int_{-\infty}^{\infty} \log\left(\gamma(\zeta) - ik\sin X\right) \frac{1}{\zeta + k\sin s}\, d\zeta, \tag{A.2}$$

and the functional dependence on X is implied. Note that, as described in section 3, this integral exists in a limiting sense.

Insert figure 1 about here

Attention is now given to writing $I(s)$ in terms of simple finite range integrals. Firstly, it is expedient to differentiate $I(s)$ with respect to s and to integrate once by parts. Rearranging this expression and transforming the integration variables to $\zeta = -k \sin t$, $\gamma(\zeta) = ik \cos t$ gives

$$\frac{dI}{ds}(s) = \frac{\cos s}{\sin^2 s - \cos^2 X} \int_{\mathcal{C}_1} \left[\frac{\sin^2 s}{\sin^2 t - \sin^2 s} - \frac{\cos^2 X}{\sin^2 t - \cos^2 X} \right] (\cos t + \sin X)\ dt. \tag{A.3}$$

where the contour $\mathcal{C}_1$ is illustrated in figure 1. For simplicity, only the location of the poles of the integrand for complex values of X satisfying (2.9) are shown in figure 1. By a simple substitution (A.3) can be rewritten as the loop integral

$$\frac{dI}{ds}(s) = \frac{\cos s}{\sin^2 s - \cos^2 X} \int_{\mathcal{C}_1+\mathcal{C}_2} \left[\frac{\sin^2 s}{\sin^2 t - \sin^2 s} - \frac{\cos^2 X}{\sin^2 t - \cos^2 X} \right] \left(\frac{1}{2}\cos t - \frac{t}{\pi}\sin X \right)\ dt. \tag{A.4}$$

The integral can now be evaluated by picking up the residue contributions from inside the loop $\mathcal{C}_1 + \mathcal{C}_2$, see figure 1. After a good deal of algebra it is found that

$$\frac{dI}{ds}(s) = \frac{i}{\sin^2 s - \cos^2 X} \left[\frac{\pi}{2}\sin(2s) - (2s+\pi)\sin s \sin X + (\pi - 2X)\cos s \cos X \right], \tag{A.5}$$

for **any** location of X in the range (2.8), (2.9), and this can further be expressed as

$$\frac{dI}{ds}(s) = \frac{i}{2} \left[\frac{2u - \pi \sin u}{\cos u} + \frac{2v - \pi \sin v}{\cos v} \right] \tag{A.6}$$

in which $u = s - X + \pi$ and $v = s + X$.

To obtain $I(s)$, (A.6) is in a suitable form to integrate, with the limits chosen as s and $-\pi$. Using u and v as suitable substitution variables, it is found that

$$I(s) = I(-\pi) + \frac{i}{2} \left\{ \int_0^{s-X+\pi} + \int_0^{s+X} - \int_0^{-X} - \int_0^{X-\pi} \right\} \frac{2u - \pi \sin u}{\cos u}\ du, \tag{A.7}$$

and so $K_+(-k\sin s, X)$ can be written, from (A.1), as

$$K_+(-k\sin s, X) = \frac{M(s - X + \pi)M(s+X)}{M(-X)M(X-\pi)} \exp\left\{ \frac{1}{2\pi i} I(-\pi) \right\}, \tag{A.8}$$

where the function $M(s)$ is defined by (4.7). Simplification of expression (A.8) may be achieved by using (2.14) at $s = -\pi$, that is $\alpha = 0$, to give

$$K_+^2(0, X) = \gamma(0) - ik \sin X. \tag{A.9}$$

Hence

$$\exp\left\{ \frac{1}{2\pi i} I(-\pi) \right\} = \sqrt{2k} e^{-i\pi/4} \cos \frac{1}{2}(X - \pi/2). \tag{A.10}$$

Properties (4.8) and (4.9) with $\beta = \pi/2$ can now be used to simplify (A.8) thus:

$$K_+(-k\sin s, X) = \sqrt{2k} e^{-i\pi/4} M^{-2}(\pi/2) M(s - X + \pi) M(s + X) \tag{A.11}$$

which agrees with (4.16).

The above technique may be adapted to other kernels of physical interest, for example, those of the form $\gamma(\alpha)\sinh(\gamma(\alpha)) - ik\sin X$.

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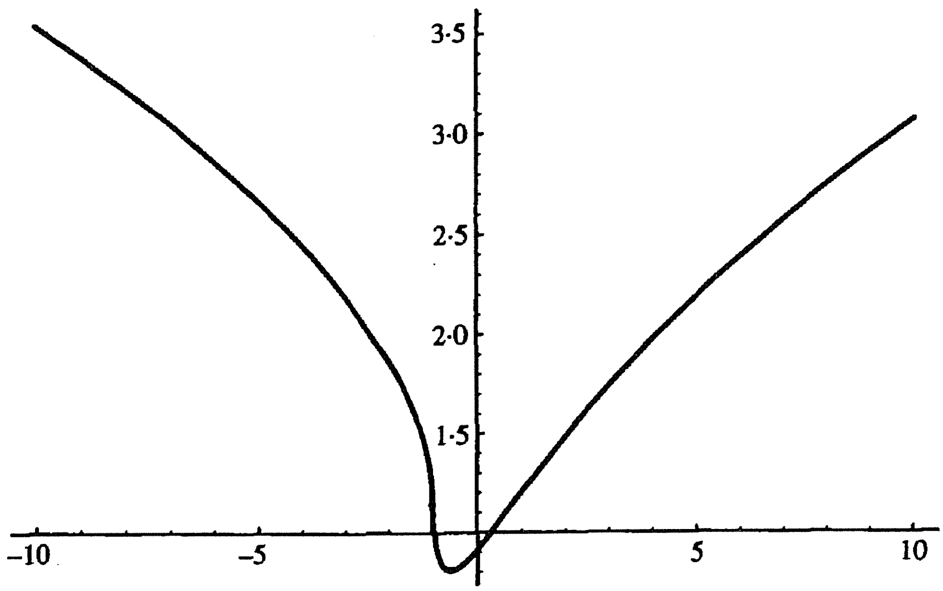

FIG. 1. Plot of $|K_+(\alpha, -\pi/4 + \mathrm{i})|$ along the real line for $k = 1$.

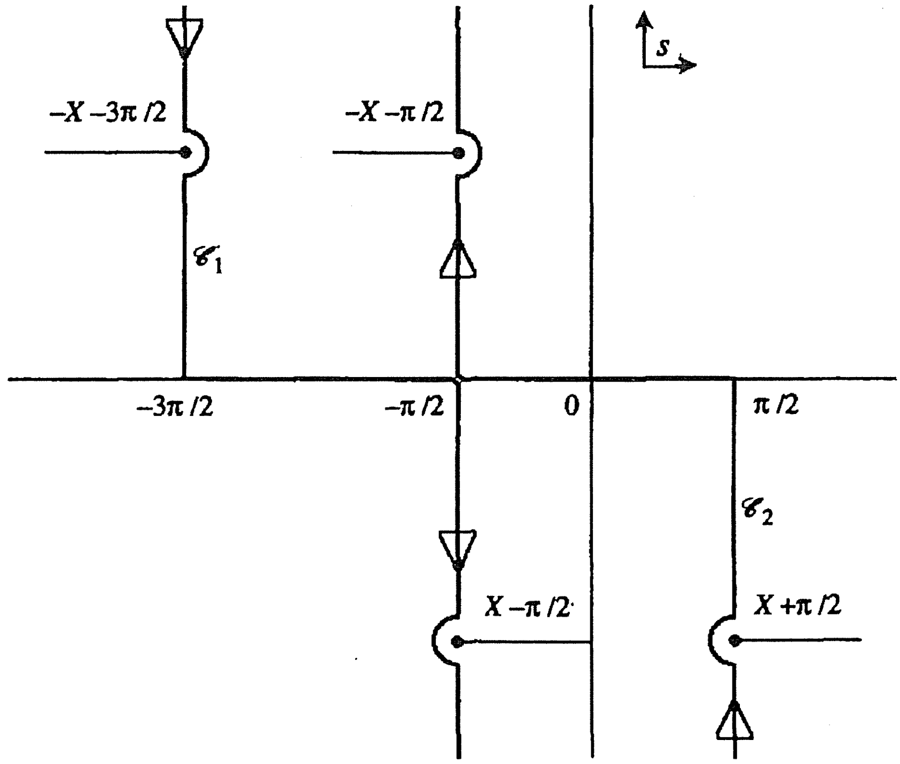

FIG. 2. The complex s-plane, showing the integration contours $\mathscr{C}_1$ and $\mathscr{C}_2$. The pole locations of the integrand of (A.4), in the vicinity of the integral paths, are shown for X lying in $0 \leqslant \mathrm{Re}(X) < \pi/2$, $\mathrm{Im}(X) < 0$.