End For j_2

End For j_1

— Make the set of ± 1 column vectors of length a_c such as $F = \{f \in \mathbf{R}^{a_c} : |f_j| = 1,\ \text{for } j = 1, \ldots, a_c\}$, which has cardinality of 2^{a_c}.

— Make the set of ± 1 column vectors of length b_c such as $G = \{g \in \mathbf{R}^{b_c} : |g_j| = 1, \text{for } j = 1, \ldots, b_c\}$, which has cardinality of 2^{b_c}.

— For $k_1 = 1, \ldots, 2^{a_c}$

— Select ± 1 column vector from the set F, f^{k_1}.

• For $k_2 = 1, \ldots, a_c$

if $f_{k_2}^{k_1} = 1$, then $l = a_{k_2}^{row}$, $m = a_{k_2}^{column}$; $a_{l,m} = \overline{a_{l,m}}$

else if $f_{k_2}^{k_1} = -1$, then $l = a_{k_2}^{row}$, $m = a_{k_2}^{column}$; $a_{l,m} = \underline{a_{l,m}}$

end if

• End For k_2

• For $k_4 = 1, \ldots, 2^{b_c}$

— Select ± 1 column vector from the set G, g^{k_4}.

◇ For $k_5 = 1, \ldots, b_c$

if $g_{k_5}^{k_4} = 1$, then $l = b_{k_5}^{row}$, $m = b_{k_5}^{column}$; $b_{l,m} = \overline{b_{l,m}}$

else if $g_{k_5}^{k_4} = -1$, then $l = b_{k_5}^{row}$, $m = b_{k_5}^{column}$; $b_{l,m} = \underline{b_{l,m}}$

end if

◇ End For k_5

— Calculate the maximum singular value, σ, using A and B matrices generated so far.

— Choose $\sigma^{\dagger} = \max\{\sigma, \sigma^{\dagger}\}$

• End For k_4

— End For k_1

• **Step 4:** Repeat **Step 1**, **Step 2** and **Step 3** for all i_1, i_2, i_3, and i_4.

• **Step 5:** Select $\sigma^{\dagger}$ as the maximum singular value of the complex interval matrix.

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On Nonlinear H_∞ Filtering for Discrete-Time Stochastic Systems With Missing Measurements

Bo Shen, Zidong Wang, Huisheng Shu, and Guoliang Wei

***Abstract*—In this paper, the H_∞ filtering problem is investigated for a general class of nonlinear discrete-time stochastic systems with missing measurements. The system under study is not only corrupted by state-dependent white noises but also disturbed by exogenous inputs. The measurement output contains randomly missing data that is modeled by a Bernoulli distributed white sequence with a known conditional probability. A filter of very general form is first designed such that the filtering process is stochastically stable and the filtering error satisfies H_∞ performance constraint for all admissible missing observations and nonzero exogenous disturbances under the zero-initial condition. The existence conditions of the desired filter are described in terms of a second-order nonlinear inequality. Such an inequality can be decoupled into some auxiliary ones that can be solved independently by taking special form of the Lyapunov functionals. As a consequence, a linear time-invariant filter design problem is discussed for the benefit of practical applications, and some simplified conditions are obtained. Finally, two numerical simulation examples are given to illustrate the main results of this paper.**

***Index Terms*—Discrete-time systems, H_∞ filtering, missing measurements, nonlinear systems, stochastic stability, stochastic systems.**

I. INTRODUCTION

H_∞ filtering or state estimation has long been one of the foundational problems in signal processing and control systems. The so-called

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H_∞ filtering problem can be briefly described as the design of an estimator for a given system in order to estimate an unknown state combination such that the L_2 gain from the exogenous disturbance to the estimation error is less than a prescribed level $\gamma > 0$. In the past decades, significant advances have been made in the research area of nonlinear H_∞ filtering/control since Zames' original work [28], see e.g., [3], [4], [9], [10], [22], [26], [27], [30].

Nonlinearity and stochasticity are arguably two of the main resources in reality that have resulted in considerable system complexity [8]. In the past few years, the H_∞ filtering problems for nonlinear and/or stochastic systems have received increasing research attention, and a great deal of results have been available in the literature. For some recent representative work on this general topic in the *deterministic* case, we refer the reader to [16], [17], [19] and the references therein. With respect to the *stochastic* case, the nonlinear filtering problem has been studied by many researchers. For example, the H_∞ filtering problem has been dealt with in [24], [26], [27] for nonlinear stochastic time-delay systems. In [5], an H_∞ filtering theory has been developed, from the dissipation point of view, for a large class of continuous-time stochastic nonlinear systems. In [18], the application of the unscented Kalman filter (UKF) has been considered for continuous-time filtering problems, where both the state and measurement processes are modeled as stochastic differential equations. The filtering problem has been dealt with in [11] for stochastic nonlinear differential systems driven by standard Wiener processes, where a filter has been presented that is a generalization of the classical Extended Kalman-Bucy filter. In [29], the stochastic H_∞ filtering problem has been studied for system modeled by Itô-type stochastic differential equation where the addressed filter is of a very general nonlinear form, and the H_∞ filter proposed in [29] can be obtained by solving second-order nonlinear Hamilton-Jacobi inequalities. It should be noted that all the literature mentioned above has been concerned with continuous-time systems, and the corresponding results for discrete-time case are relatively few.

It is quite common in practice that the measurement output of a *discrete-time* stochastic system is not consecutive but contains missing observations due to a variety of causes such as sensor temporal failure and network-induced packet loss, see, e.g., [1], [2], and [15]. Therefore, it is not surprising that the filtering problem for system with missing measurements has recently attracted much attention. For example, a binary switching sequence has been used in [21], [23], and [25], which can be viewed as a Bernoulli distributed white sequence taking values of 0 and 1, to model the missing measurements phenomena. A Markovian jumping process has been employed in [20] to reflect the measurement missing problem. In [6] and [7], the date missing (dropout) rate has been converted into the signal transmission delay that has both the upper and lower bounds. Unfortunately, to the best of our knowledge, the H_∞ filtering problem for *general nonlinear discrete-time stochastic systems* with or without missing measurements has not been fully investigated despite its potential in practical applications, and the purpose of this paper is therefore to shorten such a gap by providing a rather general framework.

In this paper, we aim to investigate the H_∞ filtering problem for a class of general nonlinear discrete-time stochastic systems with missing measurements. A Bernoulli distributed white sequence with a known conditional probability is used for modeling the missing measurements. It is worth mentioning that we first consider a very general form of the filter and then discuss the linear filter as a special case. Specifically, a sufficient condition is derived in the form of a second-order nonlinear inequality, which guarantees that the filtering process is stochastically stable and the filtering error satisfies H_∞ performance constraint for all possible missing observations and all nonzero exogenous disturbances under the zero-initial condition. Based on this condition, the second-order nonlinear inequality is then decoupled into two inequalities that can be solved independently by selecting special Lyapunov functionals. Some corollaries with much simplified conditions are given to facilitate the filter design. Moreover, the linear filter design problem is discussed for the addressed nonlinear stochastic systems and our main results are specialized to this case readily. Finally, we demonstrate the usefulness and applicability of the developed theory by means of two numerical simulation examples.

Notation: The notation used here is fairly standard except where otherwise stated. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\|A\|$ refers to the norm of a matrix A defined by $\|A\| = \sqrt{\mathrm{trace}(A^TA)}$. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are real symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). M^T represents the transpose of the matrix M. I denotes the identity matrix of compatible dimension. If A is a matrix, $\lambda_{\min}(A)$ (respectively, $\lambda_{\max}(A)$) stands for the smallest (respectively, largest) eigenvalue of A. Moreover, we may fix a probability space $(\Omega, \mathcal{F}, \mathrm{Prob})$, where Prob, the probability measure, has total mass 1. $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x with respect to the given probability measure Prob. The set of all nonnegative integers is denoted by $\mathbb{I}^+$ and the set of all nonnegative real numbers is represented by $\mathbb{R}^+$. CK denotes the class of all continuous nondecreasing convex functions $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(0) = 0$ and $\mu(r) > 0$ for $r > 0$. $C^2(\mathbb{R}^n)$ denotes the class of functions $V(x)$ twice continuously differential with respect to $x \in \mathbb{R}^n$ except possibly at the origin. Finally, we let $V_x(x) = ((\partial V(x)/\partial x_1), (\partial V(x)/\partial x_2), \ldots, (\partial V(x)/\partial x_n))^T$ and $V_{xx}(x) = \left(\partial^2 V(x)/\partial x_i \partial x_j\right)_{n\times n}$.

II. Problem Formulation and Preliminaries

Consider the following nonlinear discrete-time stochastic system with missing measurements:

$$\begin{cases} x_{k+1} = f(x_k) + g(x_k)v_k + (h(x_k) + s(x_k)v_k)w_k \\ y_k = r_k(l(x_k) + k(x_k)v_k) \\ z_k = m(x_k) \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $z_k \in \mathbb{R}^m$ is the state combination to be estimated, and w_k is a one-dimensional, zero-mean Gaussian white noise sequence on a probability space $(\Omega, \mathcal{F}, \mathrm{Prob})$ with $\mathbb{E}w_k^2 = \theta^2$. Let $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_k\}_{k\in\mathbb{I}^+},\ \mathrm{Prob})$ be a filtered probability space where $\{\mathcal{F}_k\}_{k\in\mathbb{I}^+}$ is the family of sub σ-algebras of $\mathcal{F}$ generated by $\{w_k\}_{k\in\mathbb{I}^+}$. In fact, each $\mathcal{F}_k$ is assumed to be the minimal σ-algebras generated by $\{w_i\}_{0\leq i\leq k-1}$ while $\mathcal{F}_0$ is assumed to be some given sub σ-algebras of $\mathcal{F}$, independent of $\mathcal{F}_k$ for all $k > 0$. The exogenous disturbance input $v_k \in \mathbb{R}^q$, it is assumed that $\{v_k\}_{k\in\mathbb{I}^+} \in l_2([0,\infty), \mathbb{R}^q)$, where $l_2([0,\infty), \mathbb{R}^q)$ is the space of nonanticipatory square-summable stochastic process $\{v_k\}_{k\in\mathbb{I}^+}$ with respect to $(\mathcal{F}_k)_{k\in\mathbb{I}^+}$.

Remark 1: In the model (1), v_k is an exogenous input that describes the external disturbance, and w_k means the inner disturbance of system.

The nonlinear functions $f_{n\times 1}$, $g_{n\times q}$, $h_{n\times 1}$, $s_{n\times q}$, $l_{r\times 1}$, $k_{r\times q}$ and $m_{m\times 1}$ are smooth matrix-valued functions with $f_{n\times 1}(0) = 0$, $h_{n\times 1}(0) = 0$, $l_{r\times 1}(0) = 0$ and $m_{m\times 1}(0) = 0$. $y_k \in \mathbb{R}^r$ is the measured output vector with probabilistic missing data. The stochastic variable $r_k \in \mathbb{R}$, which describes the measurement missing phenomena, takes values of 1 and 0 with

$$\begin{aligned} \mathrm{Prob}\{r_k = 1\} &= \beta \\ \mathrm{Prob}\{r_k = 0\} &= 1 - \beta \end{aligned} \tag{2}$$

where $\beta \in [0, 1]$ is a known constant. r_k is assumed to be independent of the Gaussian white noise sequence w_k, and the initial value x_0 is a known vector.

We start with designing the following *general* filter for system (1):

$$\begin{cases} \hat{x}_{k+1} = \hat{f}(\hat{x}_k) + \hat{g}(\hat{x}_k)y_k \\ \hat{z}_k = \hat{m}(\hat{x}_k),\ \hat{f}(0) = 0, \hat{m}(0) = 0, \hat{x}_0 = 0 \end{cases} \tag{3}$$

where $\hat{x}_k$ is the state estimate; $\hat{z}_k$ is an estimate for z_k; $\hat{f}$, $\hat{g}$ and $\hat{m}$, which are matrices of appropriate dimensions with sufficient smoothness, are filter parameters to be determined.

Setting $\eta_k = [x_k^T \ \ \hat{x}_k^T]^T$, we obtain an augmented system as follows:

$$\begin{cases} \eta_{k+1} = f_e(\eta_k) + \eta_k + g_e(\eta_k)v_k + (h_e(\eta_k) + s_e(\eta_k)v_k)w_k \\ \tilde{z}_k := z_k - \hat{z}_k = m(x_k) - \hat{m}(\hat{x}_k) \end{cases} \tag{4}$$

where

$$f_e(\eta_k) = \begin{bmatrix} f(x_k) - x_k \\ \hat{f}(\hat{x}_k) + r_k\hat{g}(\hat{x}_k)l(x_k) - \hat{x}_k \end{bmatrix},$$
$$g_e(\eta_k) = \begin{bmatrix} g(x_k) \\ r_k\hat{g}(\hat{x}_k)k(x_k) \end{bmatrix}, \quad h_e(\eta_k) = \begin{bmatrix} h(x_k) \\ 0 \end{bmatrix},$$
$$s_e(\eta_k) = \begin{bmatrix} s(x_k) \\ 0 \end{bmatrix}. \tag{5}$$

Remark 2: The form of the augmented system is appreciably different from those in the literature such as [29]. With the augmented system (4) and (5), it would be more convenient to state the problem to be investigated and derive our main results. Moreover, since (4) is inherently a stochastic system because of both $w(k)$ and $r(k)$, we need to introduce the notion of stochastic stability.

The following definition is considered as a discrete-time version of that in [12].

Definition 1: The solution $\eta_k = 0$ of the augmented system (4) with $v_k = 0$ is said to be *stochastically stable* if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\mathbb{E}\{\|\eta_k\|\} < \varepsilon \tag{6}$$

whenever $k \in \mathbb{I}^+$ and $\|\eta_0\| < \delta$.

We are now in a position to state the nonlinear stochastic H_∞ filtering problem as follows. We are interested in finding filter gain matrices $\hat{f}(\hat{x}_k)$, $\hat{g}(\hat{x}_k)$ and $\hat{m}(\hat{x}_k)$ in (3) such that the following requirements are met simultaneously:

a) The zero-solution of the augmented system (4) with $v_k = 0$ is stochastically stable.
b) Under the zero-initial condition, the filtering error $\tilde{z}_k$ satisfies
$$\sum_{k=0}^{\infty}\mathbb{E}\left\{\|\tilde{z}_k\|^2\right\} < \gamma^2\sum_{k=0}^{\infty}\mathbb{E}\left\{\|v_k\|^2\right\} \tag{7}$$
for all nonzero v_k where $\gamma > 0$ is a given disturbance attenuation level.

The nonlinear stochastic H_∞ filtering problem addressed will be solved in next section, and the results will be specialized to several special cases for practical convenience.

III. Main Results

Let us start with introducing a lemma that will be used in the proof of our main results.

Lemma 1: If there exist a Lyapunov functional $V(\eta) \in C^2(\mathbb{R}^{2n})$ and a function $a(r) \in CK$ satisfying the following conditions:

$$V(0) = 0, \tag{8a}$$
$$a(\|\eta\|) \le V(\eta), \tag{8b}$$
$$\mathbb{E}\{V(\eta_{k+1})\} - \mathbb{E}\{V(\eta_k)\} < 0, \quad k \in \mathbb{I}^+ \tag{8c}$$

then the solution $\eta_k = 0$ of the system (4) with $v_k = 0$ is stochastically stable.

Proof: First of all, we note that $V(0) = 0$ and $V(\eta)$ is continuous. Therefore, for any $\varepsilon > 0$, there exists a scalar $\delta > 0$ such that $\|\eta_0\| < \delta \to V(\eta_0) < a(\varepsilon)$.

We claim that every solution η_k with $\|\eta_0\| < \delta$ implies $\mathbb{E}\{\|\eta_k\|\} < \varepsilon$ for all $k > 0$. Let us now prove our claim by contradiction. Suppose that, for a solution η_k satisfying $\|\eta_0\| < \delta$, there exists a $k_1 \in \mathbb{I}^+$ such that $\mathbb{E}\{\|\eta_{k_1}\|\} \ge \varepsilon$. By Jensen Inequality, one can get the fact of $a(\mathbb{E}\{\|\eta_{k_1}\|\}) \le \mathbb{E}\{a(\|\eta_{k_1}\|)\}$. And then from (8c), it follows readily that $\mathbb{E}\{V(\eta_{k_1})\} < \mathbb{E}\{V(\eta_0)\}$ and

$$\begin{aligned} a(\varepsilon) &\le a(\mathbb{E}\{\|\eta_{k_1}\|\}) \le \mathbb{E}\{a(\|\eta_{k_1}\|)\} \\ &\le \mathbb{E}\{V(\eta_{k_1})\} < \mathbb{E}\{V(\eta_0)\} < a(\varepsilon) \end{aligned}$$

which is a contradiction. Therefore, it follows easily from Definition 1 that the solution $\eta_k = 0$ of the augmented system (4) with $v_k = 0$ is stochastically stable. The proof is complete. ■

The following theorem provides sufficient conditions under which the augmented system (4) with $v_k = 0$ is stochastically stable and the filtering error $\tilde{z}_k$ satisfies (7) for all nonzero v_k under the zero-initial condition.

Theorem 1: Given a disturbance attenuation level $\gamma > 0$. If there exists a Lyapunov functional $V(\eta) \in C^2(\mathbb{R}^{2n})$ satisfying the inequalities in (9), shown at the bottom of the next page, where $A(\eta,\eta_\alpha)$, $B(\eta,\eta_\alpha)$, and $D(\eta,\eta_\alpha)$ are defined in (10)–(12), also shown at the bottom of the next page, for some matrices $\hat{f}$, $\hat{g}$ and $\hat{m}$ of suitable dimensions, then the stochastic H_∞ filtering problem for system (1) is solved by (3).

Proof: Let $V(\eta) \in C^2(\mathbb{R}^{2n})$ be a Lyapunov functional satisfying (9) and the difference of the Lyapunov functional be defined by

$$\Delta V(\eta_k) = \mathbb{E}\{V(\eta_{k+1})|\eta_k\} - V(\eta_k). \tag{13}$$

First, let us now show that augmented system (4) satisfies H_∞ robustness performance constraint for all nonzero exogenous disturbances under the zero-initial condition. Using Taylor's formula, there exists a $\alpha_k \in [0,1]$ such that

$$\begin{aligned} &\mathbb{E}\{\Delta V(\eta_k)\} + \mathbb{E}\{\|\tilde{z}_k\|^2\} - \gamma^2\mathbb{E}\{\|v_k\|^2\} \\ &\quad = \mathbb{E}\{V(\eta_{k+1})\} - \mathbb{E}\{V(\eta_k)\} + \mathbb{E}\{\|\tilde{z}_k\|^2\} - \gamma^2\mathbb{E}\{\|v_k\|^2\} \\ &\quad = \mathbb{E}\Bigg\{V_\eta^T(\eta_k)(\eta_{k+1} - \eta_k) + \frac{1}{2}(\eta_{k+1} - \eta_k)^T \\ &\qquad\qquad \times V_{\eta\eta}(\eta_k + \alpha_k(\eta_{k+1} - \eta_k))(\eta_{k+1} - \eta_k)\Bigg\} \\ &\qquad + \mathbb{E}\{\|\tilde{z}_k\|^2\} - \gamma^2\mathbb{E}\{\|v_k\|^2\}. \end{aligned} \tag{14}$$

For simplicity, we denote $\eta_{\alpha_k} := \eta_k + \alpha_k(\eta_{k+1} - \eta_k)$ and then it follows from $\mathbb{E}w_k = 0$, $\mathbb{E}w_k^2 = \theta^2$ and (4) that

$$\begin{aligned} &\mathbb{E}\{\Delta V(\eta_k)\} + \mathbb{E}\{\|\tilde{z}_k\|^2\} - \gamma^2\mathbb{E}\{\|v_k\|^2\} \\ &\quad = \mathbb{E}\Bigg\{ - v_k^T\Bigg(\gamma^2 I - \frac{1}{2}g_e^T(\eta_k)V_{\eta\eta}(\eta_{\alpha_k})g_e(\eta_k) \\ &\qquad - \frac{1}{2}\theta^2 s_e^T(\eta_k)V_{\eta\eta}(\eta_{\alpha_k})s_e(\eta_k)\Bigg)v_k \\ &\qquad + \Bigg(V_\eta^T(\eta_k)g_e(\eta_k) + \theta^2 h_e^T(\eta_k)V_{\eta\eta}(\eta_{\alpha_k})s_e(\eta_k) \\ &\qquad + f_e^T(\eta_k)V_{\eta\eta}(\eta_{\alpha_k})g_e(\eta_k)\Bigg)v_k \\ &\qquad + V_\eta^T(\eta_k)f_e(\eta_k) + \frac{1}{2}f_e^T(\eta_k)V_{\eta\eta}(\eta_{\alpha_k})f_e(\eta_k) \\ &\qquad + \frac{1}{2}\theta^2 h_e^T(\eta_k)V_{\eta\eta}(\eta_{\alpha_k})h_e(\eta_k) + \|\tilde{z}_k\|^2\Bigg\}. \end{aligned} \tag{15}$$

By a series of computations and noting that $\mathbb{E}r_k = \mathbb{E}r_k^2 = \beta$, we can conclude that (15) is equal to (16), as shown at the bottom of the page, where $v_k^* = (1/2)A^{-1}(\eta_k, \eta_{\alpha_k})B^T(\eta_k, \eta_{\alpha_k})$. Therefore, it can be seen that we have (17), also shown at the bottom of the page, and then it follows from (9) that

$$\mathbb{E}\{\Delta V(\eta_k)\} + \mathbb{E}\{\|\tilde{z}_k\|^2\} - \gamma^2\mathbb{E}\{\|v_k\|^2\} < 0. \quad (18)$$

Summing up (18) from 0 to positive integer N with respect to k yields

$$\sum_{k=0}^{N}\{\mathbb{E}\{\Delta V(\eta_k)\} + \mathbb{E}\{\|\tilde{z}_k\|^2\} - \gamma^2\mathbb{E}\{\|v_k\|^2\}\} < 0 \quad (19)$$

i.e.,

$$\sum_{k=0}^{N}\mathbb{E}\{\|\tilde{z}_k\|^2\} < \gamma^2\sum_{k=0}^{N}\mathbb{E}\{\|v_k\|^2\} + \mathbb{E}\{V(0)\} - \mathbb{E}\{V(\eta_{N+1})\}. \quad (20)$$

Considering $\mathbb{E}\{V(\eta_{N+1})\} \geq 0$, $V(0) = 0$ and letting $N \to +\infty$, we obtain

$$\sum_{k=0}^{\infty}\mathbb{E}\{\|\tilde{z}_k\|^2\} < \gamma^2\sum_{k=0}^{\infty}\mathbb{E}\{\|v_k\|^2\} \quad (21)$$

$$\begin{cases} \mathbb{H}(\eta, \eta_\alpha) = \frac{1}{4}B(\eta, \eta_\alpha)A^{-1}(\eta, \eta_\alpha)B^T(\eta, \eta_\alpha) + \frac{\partial V^T}{\partial x}(\eta)(f(x) - x) + \frac{\partial V^T}{\partial \hat{x}}(\eta)(\hat{f}(\hat{x}) - \hat{x}) \\ \qquad +\beta\frac{\partial V^T}{\partial \hat{x}}(\eta)\hat{g}(\hat{x})l(x) + D(\eta, \eta_\alpha) + \frac{1}{2}\theta^2 h^T(x)\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)h(x) + \|\tilde{z}\|^2 < 0, \\ \qquad \text{for any } \eta \neq 0, \eta_\alpha \in \mathbb{R}^{2n}, \\ a(\|\eta\|) \leq V(\eta) \text{ with } a(r) \in CK, \\ A(\eta, \eta_\alpha) > 0, \text{ for any } \eta, \eta_\alpha \in \mathbb{R}^{2n}, \ V(0) = 0 \end{cases} \quad (9)$$

$$\begin{aligned} A(\eta, \eta_\alpha) &= \gamma^2 I - \frac{1}{2}g^T(x)\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)g(x) - \frac{1}{2}\beta k^T(x)\hat{g}^T(\hat{x})\frac{\partial^2 V}{\partial \hat{x}^2}(\eta_\alpha)\hat{g}(\hat{x})k(x) - \beta k^T(x)\hat{g}^T(\hat{x})\frac{\partial^2 V}{\partial x^T\partial \hat{x}}(\eta_\alpha) \\ &\quad \times g(x) - \frac{1}{2}\theta^2 s^T(x)\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)s(x) \end{aligned} \quad (10)$$

$$\begin{aligned} B(\eta, \eta_\alpha) &= \frac{\partial V^T}{\partial x}(\eta)g(x) + \beta\frac{\partial V^T}{\partial \hat{x}}(\eta)\hat{g}(\hat{x})k(x) + \theta^2 h^T(x)\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)s(x) + (f(x) - x)^T\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)g(x) \\ &\quad + (\hat{f}(\hat{x}) + \beta\hat{g}(\hat{x})l(x) - \hat{x})^T\frac{\partial^2 V}{\partial x^T\partial \hat{x}}(\eta_\alpha)g(x) + \beta(f(x) - x)^T\frac{\partial^2 V}{\partial \hat{x}^T\partial x}(\eta_\alpha)\hat{g}(\hat{x})k(x) \\ &\quad + \beta(\hat{f}(\hat{x}) - \hat{x})^T\frac{\partial^2 V}{\partial \hat{x}^2}(\eta_\alpha)\hat{g}(\hat{x})k(x) + \beta l^T(x)\hat{g}^T(\hat{x})\frac{\partial^2 V}{\partial \hat{x}^2}(\eta_\alpha)\hat{g}(\hat{x})k(x) \end{aligned} \quad (11)$$

$$\begin{aligned} D(\eta, \eta_\alpha) &= \frac{1}{2}(f(x_k) - x_k)^T\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)(f(x) - x) + (f(x) - x)^T\frac{\partial^2 V}{\partial \hat{x}^T\partial x}(\eta_\alpha)(\hat{f}(\hat{x}) + \beta\hat{g}(\hat{x})l(x_k) - \hat{x}) \\ &\quad + \frac{1}{2}(\hat{f}(\hat{x}) - \hat{x})^T\frac{\partial^2 V}{\partial \hat{x}^2}(\eta_\alpha)(\hat{f}(\hat{x}) - \hat{x}) + \beta(\hat{f}(\hat{x}) - \hat{x})^T\frac{\partial^2 V}{\partial \hat{x}^2}(\eta_\alpha)\hat{g}(\hat{x})l(x) \\ &\quad + \frac{1}{2}\beta l^T(x)\hat{g}^T(\hat{x})\frac{\partial^2 V}{\partial \hat{x}^2}(\eta_\alpha)\hat{g}(\hat{x})l(x) \end{aligned} \quad (12)$$

$$\begin{aligned} \mathbb{E}\Bigg\{ &-(v_k - v_k^*)^T A(\eta_k, \eta_{\alpha_k})(v_k - v_k^*) + \frac{1}{4}B(\eta_k, \eta_{\alpha_k})A^{-1}(\eta_k, \eta_{\alpha_k})B^T(\eta_k, \eta_{\alpha_k}) + \frac{\partial V^T}{\partial x}(\eta_k)(f(x_k) - x_k) \\ &+ D(\eta_k, \eta_{\alpha_k}) + \frac{\partial V^T}{\partial \hat{x}}(\eta_k)(\hat{f}(\hat{x}_k) + \beta\hat{g}(\hat{x}_k)l(x_k) - \hat{x}_k) + \frac{1}{2}\theta^2 h^T(x_k)\frac{\partial^2 V}{\partial x^2}(\eta_{\alpha_k})h(x_k) + \|\tilde{z}_k\|^2 \Bigg\} \end{aligned} \quad (16)$$

$$\begin{aligned} &\mathbb{E}\{\Delta V(\eta_k)\} + \mathbb{E}\{\|\tilde{z}_k\|^2\} - \gamma^2\mathbb{E}\{\|v_k\|^2\} \\ &\quad \leq \mathbb{E}\Bigg\{\frac{1}{4}B(\eta_k, \eta_{\alpha_k})A^{-1}(\eta_k, \eta_{\alpha_k})B^T(\eta_k, \eta_{\alpha_k}) + \frac{\partial V^T}{\partial x}(\eta_k)(f(x_k) - x_k) + \frac{\partial V^T}{\partial \hat{x}}(\eta_k)(\hat{f}(\hat{x}_k) - \hat{x}_k) \\ &\qquad + \beta\frac{\partial V^T}{\partial \hat{x}}(\eta_k)\hat{g}(\hat{x}_k)l(x_k) + D(\eta_k, \eta_{\alpha_k}) + \frac{1}{2}\theta^2 h^T(x_k)\frac{\partial^2 V}{\partial x^2}(\eta_{\alpha_k})h(x_k) + \|\tilde{z}_k\|^2\Bigg\} \\ &\quad := \mathbb{E}\{\mathbb{H}(\eta_k, \eta_{\alpha_k})\} \end{aligned} \quad (17)$$

which means the desired H_∞ performance requirement is met. Next, we show that the augmented system (4) with $v_k = 0$ is stochastically stable. It is not difficult to see that (9) implies

$$\frac{\partial V^T}{\partial x}(\eta)(f(x)-x)+\frac{\partial V^T}{\partial \hat{x}}(\eta)(\hat{f}(\hat{x})+\beta\hat{g}(\hat{x})l(x)-\hat{x}) +D(\eta,\eta_\alpha)+\frac{1}{2}\theta^2 h^T(x)\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)h(x) < \mathbb{H}(\eta,\eta_\alpha). \quad (22)$$

Using Taylor's formula again, we obtain (23), as shown at the bottom of the page. Then, it follows readily from Lemma 1 that the augmented system (4) with $v_k = 0$ is stochastically stable and the proof of Theorem 1 is complete. ■

Remark 3: From the proof of Theorem 1, it can be seen that we have only used the Taylor expansion approach and the "completing the square" technique which would not lead to much conservatism. Note that the condition of Theorem 1 is dependent on the probability β. Therefore, the possible conservatism of identifying the probability β has an important impact on the overall results. In view of this, the identified probability β should be obtained as accurately as possible.

Remark 4: In this paper, the expectation of the stochastic variable r_k is used to scale the missing degree of the measurement data. Such an approach is efficient for online application at the cost of reducing the preciseness of the filtering performance, especially when the probability β is low. An alternative approach to improve the preciseness, which is suitable for off-line implementations, is to develop some algorithms to estimate the missing data and then use the estimated signal in the filter design. This will be one of our future research topics.

In Theorem 1, a *very general* condition is given that can guarantee the H_∞ performance as well as the stochastic stability of the filtering process. To gradually reduce the difficulty of verifying such a condition, we are going to introduce a number of corollaries which provide *simplified* by choosing different forms of the Lyapunov functionals. For this purpose, we need the following assumption which is often used in the literature concerning stochastic stability [14].

Assumption 1: $V^{(1)}(x) \in C^2(\mathbb{R}^n)$ and $V^{(2)}(\hat{x}) \in C^2(\mathbb{R}^n)$ are two Lyapunov functionals satisfying

$$V^{(1)}(x) \geq c_1\|x\|^2, \quad V^{(2)}(\hat{x}) \geq c_2\|\hat{x}\|^2 \quad (24)$$

for some positive scalars c_1 and c_2.

Note that the existence conditions of the desired filter given in Theorem 1 are described in terms of a second-order nonlinear inequality. We first show that such a seemingly complicated inequality can be decoupled into two auxiliary ones that can be solved independently by taking special form of the Lyapunov functionals. For this purpose, we take the Lyapunov functional $V(\eta)$ as $V(\eta) = V^{(1)}(x) + V^{(2)}(\hat{x})$ where $V^{(1)}(x) \in C^2(\mathbb{R}^n)$ and $V^{(2)}(\hat{x}) \in C^2(\mathbb{R}^n)$ satisfy Assumption 1, and the following corollary can be obtained from Theorem 1.

Corollary 1: Given the disturbance attenuation level $\gamma > 0$ and the filter parameters $\hat{f}$, $\hat{g}$ and $\hat{m}$. If there exist two Lyapunov functionals $V^{(1)}(x) \in C^2(\mathbb{R}^n)$ $(V^{(1)}(0) = 0)$ and $V^{(2)}(\hat{x}) \in C^2(\mathbb{R}^n)$ $(V^{(2)}(0) = 0)$ satisfying Assumption 1, the inequality in (25), shown at the bottom of the page, for any x, $\hat{x}$, x_α, $\hat{x}_\alpha \in \mathbb{R}^n$, and the inequality in (26), also shown at the bottom of the page, for any $x \neq 0$, $\hat{x} \neq 0$, x_α, $\hat{x}_\alpha \in \mathbb{R}^n$, where

$$\begin{aligned} B(\eta,\eta_\alpha) = {} & V_x^{(1)^T}(x)g(x) + \beta V_{\hat{x}}^{(2)^T}(\hat{x})\hat{g}(\hat{x})k(x) \\ & + \theta^2 h^T(x)V_{xx}^{(1)}(x_\alpha)s(x) \\ & + (f(x)-x)^T V_{xx}^{(1)}(x_\alpha)g(x) \\ & + \beta(\hat{f}(\hat{x})-\hat{x})^T V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)\hat{g}(\hat{x})k(x) \\ & + \beta l^T(x)\hat{g}^T(\hat{x})V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)\hat{g}(\hat{x})k(x) \end{aligned} \quad (27)$$

then the stochastic H_∞ filtering problem for system (1) can be solved by (3).

$$\begin{aligned} &\mathbb{E}\{V(\eta_{k+1})\} - \mathbb{E}\{V(\eta_k)\} \\ &= \mathbb{E}\left\{V_\eta^T(\eta_k)(\eta_{k+1}-\eta_k) + \frac{1}{2}(\eta_{k+1}-\eta_k)^T V_{\eta\eta}(\eta_{\alpha_k})(\eta_{k+1}-\eta_k)\right\} \\ &= \mathbb{E}\left\{V_\eta^T(\eta_k)f_e(\eta_k) + \frac{1}{2}f_e^T(\eta_k)V_{\eta\eta}(\eta_{\alpha_k})f_e(\eta_k) + \frac{1}{2}\theta^2 h_e^T(\eta_k)V_{\eta\eta}(\eta_{\alpha_k})h_e(\eta_k)\right\} \\ &= \mathbb{E}\left\{\frac{\partial V^T}{\partial x}(\eta_k)(f(x_k)-x_k) + \frac{\partial V^T}{\partial \hat{x}}(\eta_k)(\hat{f}(\hat{x}_k)+\beta\hat{g}(\hat{x}_k)l(x_k)-\hat{x}_k) \right. \\ &\qquad \left. + D(\eta_k,\eta_{\alpha_k}) + \frac{1}{2}\theta^2 h^T(x_k)\frac{\partial^2 V}{\partial x^2}(\eta_{\alpha_k})h(x_k)\right\} \\ &< \mathbb{E}\{\mathbb{H}(\eta_k,\eta_{\alpha_k})\} < 0 \end{aligned} \quad (23)$$

$$A(\eta,\eta_\alpha) = \gamma^2 I - \frac{1}{2}g^T(x)V_{xx}^{(1)}(x_\alpha)g(x) - \frac{1}{2}\beta k^T(x)\hat{g}^T(\hat{x})V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)\hat{g}(\hat{x})k(x) - \frac{1}{2}\theta^2 s^T(x)V_{xx}^{(1)}(x_\alpha)s(x) > 0 \quad (25)$$

$$\begin{aligned} \mathbb{H}(\eta,\eta_\alpha) = {} & \frac{1}{4}B(\eta,\eta_\alpha)A^{-1}(\eta,\eta_\alpha)B^T(\eta,\eta_\alpha) + V_x^{(1)^T}(x)(f(x)-x) + V_{\hat{x}}^{(2)^T}(\hat{x})(\hat{f}(\hat{x})+\beta\hat{g}(\hat{x})l(x)-\hat{x}) \\ & + \frac{1}{2}\theta^2 h^T(x)V_{xx}^{(1)}(x_\alpha)h(x) + \frac{1}{2}(f(x_k)-x_k)^T V_{xx}^{(1)}(x_\alpha)(f(x)-x) \\ & + \frac{1}{2}(\hat{f}(\hat{x})-\hat{x})^T V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)(\hat{f}(\hat{x})-\hat{x}) + \beta(\hat{f}(\hat{x})-\hat{x})^T V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)\hat{g}(\hat{x})l(x) \\ & + \frac{1}{2}\beta l^T(x)\hat{g}^T(\hat{x})V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)\hat{g}(\hat{x})l(x) + \|\tilde{z}\|^2 < 0 \end{aligned} \quad (26)$$

Proof: By Theorem 1, we only need to set $V(\eta) = V^{(1)}(x) + V^{(2)}(\hat{x})$, where $\eta = [\, x^T \quad \hat{x}^T]^T$. It can be easily seen from (24) that $V(\eta) \geq \min(c_1, c_2)\|\eta\|^2 \in CK$. Furthermore, since

$$\frac{\partial V^T}{\partial x}(\eta) = V_x^{(1)^T}(x), \quad \frac{\partial V^T}{\partial \hat{x}}(\eta) = V_{\hat{x}}^{(2)^T}(\hat{x}),$$
$$\frac{\partial^2 V}{\partial x^2}(\eta) = V_{xx}^{(1)}(x), \quad \frac{\partial^2 V}{\partial \hat{x}^2}(\eta) = V_{\hat{x}\hat{x}}^{(2)}(\hat{x}),$$
$$\frac{\partial^2 V}{\partial x^T \partial \hat{x}}(\eta) = \frac{\partial^2 V}{\partial \hat{x}^T \partial x}(\eta) = 0$$

then (9)–(12) of Theorem 1 reduce to (25)–(27) immediately. Therefore, the proof of Corollary 1 follows directly from Theorem 1 and is therefore omitted.

Before giving the next corollary, we introduce a lemma which will be frequently used hereafter. ■

Lemma 2: Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $\varepsilon > 0$. Then we have

$$2x^T y \leq \varepsilon x^T x + \varepsilon^{-1} y^T y.$$

Under the standard assumption of $k^T(x)k(x) \equiv I$ (see, e.g., [13]), the conditions of Corollary 1 can be further decoupled into four inequalities that can be solved independently.

Corollary 2: Given the disturbance attenuation level $\gamma > 0$ and the filter parameters $\hat{f}$, $\hat{g}$ and $\hat{m}$. The stochastic H_∞ filtering problem for system (1) is solved by (3) if there exist two positive constants μ_1, μ_2 and two Lyapunov functionals $V^{(1)}(x) \in C^2(\mathbb{R}^n)$ $(V^{(1)}(0) = 0)$ and $V^{(2)}(\hat{x}) \in C^2(\mathbb{R}^n)$ $(V^{(2)}(0) = 0)$ satisfying (24) and the conditions in (28) and (29), as shown at the bottom of the page, for any $x, \hat{x}, x_\alpha, \hat{x}_\alpha \in \mathbb{R}^n$, and the conditions in (30) and (31), also shown at the bottom of the page, for any $x \neq 0,\ \hat{x} \neq 0,\ x_\alpha,\ \hat{x}_\alpha \in \mathbb{R}^n$.

Proof: It is easily seen from (24) that $V(\eta) \geq \min(c_1, c_2)\|\eta\|^2 \in CK$. Now, using the elementary inequality $\|a + b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$, we can obtain

$$\|\tilde{z}\|^2 = \|m(x) - \hat{m}(\hat{x})\|^2 \leq 2\|m(x)\|^2 + 2\|\hat{m}(\hat{x})\|^2. \tag{32}$$

Considering (28) and (29), it follows from (27) that we have (33), as shown at the bottom of the page. By means of Lemma 2, we have

$$V_{\hat{x}}^{(2)^T}(\hat{x})\hat{g}(\hat{x})l(x) \leq \frac{1}{2}\left\|V_{\hat{x}}^{(2)^T}(\hat{x})\hat{g}(\hat{x})\right\|^2 + \frac{1}{2}\|l(x)\|^2 \tag{34}$$

and

$$(\hat{f}(\hat{x}) - \hat{x})^T V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)\hat{g}(\hat{x})l(x) \leq \frac{1}{2}\left\|(\hat{f}(\hat{x}) - \hat{x})^T V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)\hat{g}(\hat{x})\right\|^2 + \frac{1}{2}\|l(x)\|^2. \tag{35}$$

Obviously, it follows from (28) and (32)–(35) that we get (36), as shown at the bottom of the next page, and the rest of the proof follows directly from Corollary 1. ■

In what follows, we take more special form of the Lyapunov functionals in order to deduce more simplified conditions under which the stochastic H_∞ filtering problem is solvable. Let us now consider the case where $V(\eta)$ is set as $V(\eta) = x^T P x + \hat{x}^T Q \hat{x}$ and we have the following corollary.

Corollary 3: Given the disturbance attenuation level $\gamma > 0$ and the filter parameters $\hat{f}$, $\hat{g}$ and $\hat{m}$. The stochastic H_∞ filtering problem for system (1) is solved by (3) if there exist two positive definite matrices $P = P^T > 0$ and $Q = Q^T > 0$ satisfying the following conditions:

$$g^T(x)Pg(x) + \beta k^T(x)\hat{g}^T(\hat{x})Q\hat{g}(\hat{x})k(x) + \theta^2 s^T(x)Ps(x) < \gamma^2 I \tag{37}$$

$$\hat{g}^T(\hat{x})V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)\hat{g}(\hat{x}) \leq \mu_1 I \tag{28}$$

$$\gamma^2 I - \frac{1}{2}g^T(x)V_{xx}^{(1)}(x_\alpha)g(x) - \frac{1}{2}\theta^2 s^T(x)V_{xx}^{(1)}(x_\alpha)s(x) > \left(\frac{1}{2}\beta\mu_1 + \mu_2\right)I \tag{29}$$

$$\begin{aligned}\mathbb{H}_1(\eta, \eta_\alpha) = {} & \frac{3}{2\mu_2}\left(\left\|V_x^{(1)^T}(x)g(x)\right\|^2 + \theta^4\left\|h^T(x)V_{xx}^{(1)}(x_\alpha)s(x)\right\|^2 + \left\|(f(x) - x)^T V_{xx}^{(1)}(x_\alpha)g(x)\right\|^2\right) \\ & + V_x^{(1)^T}(x)(f(x) - x) + \frac{1}{2}\theta^2 h^T(x)V_{xx}^{(1)}(x_\alpha)h(x) + \frac{1}{2}(f(x) - x)^T V_{xx}^{(1)}(x_\alpha)(f(x) - x) \\ & + \left(\frac{3\beta^2\mu_1^2}{2\mu_2} + \frac{\beta\mu_1}{2} + \beta\right)\|l(x)\|^2 + 2\|m(x)\|^2 < 0\end{aligned} \tag{30}$$

$$\begin{aligned}\mathbb{H}_2(\eta, \eta_\alpha) = {} & \left(\frac{3\beta^2}{2\mu_2} + \frac{\beta}{2}\right)\left\|V_{\hat{x}}^{(2)^T}(\hat{x})\hat{g}(\hat{x})\right\|^2 + V_{\hat{x}}^{(2)^T}(\hat{x})(\hat{f}(\hat{x}) - \hat{x}) + \frac{1}{2}(\hat{f}(\hat{x}) - \hat{x})^T V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)(\hat{f}(\hat{x}) - \hat{x}) \\ & + \left(\frac{3\beta^2}{2\mu_2} + \frac{\beta}{2}\right)\left\|(\hat{f}(\hat{x}) - \hat{x})^T V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)\hat{g}(\hat{x})\right\|^2 + 2\|\hat{m}(\hat{x})\|^2 < 0.\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{1}{4}B(\eta, \eta_\alpha)A^{-1}(\eta, \eta_\alpha)B^T(\eta, \eta_\alpha) < \frac{3}{2\mu_2}\Bigg(& \left\|V_x^{(1)^T}(x)g(x)\right\|^2 + \beta^2\left\|V_{\hat{x}}^{(2)^T}(\hat{x})\hat{g}(\hat{x})\right\|^2 + \theta^4\left\|h^T(x)V_{xx}^{(1)}(x_\alpha)s(x)\right\|^2 \\ & + \left\|(f(x) - x)^T V_{xx}^{(1)}(x_\alpha)g(x)\right\|^2 + \beta^2\left\|(\hat{f}(\hat{x}) - \hat{x})^T V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)\hat{g}(\hat{x})\right\|^2 \\ & + \beta^2\mu_1^2\|l(x)\|^2\Bigg)\end{aligned} \tag{33}$$

for any $x, \hat{x} \in \mathbb{R}^n$ and the condition (38), as shown at the bottom of the page, for any nonzero $x,\ \hat{x} \in \mathbb{R}^n$, where

$$B(x,\hat{x}) = 2f^T(x)Pg(x) + 2\beta \hat{f}^T(\hat{x})Q\hat{g}(\hat{x})k(x) + 2\beta l^T(x)\hat{g}^T(\hat{x})Q\hat{g}(\hat{x})k(x) + 2\theta^2 h^T(x)Ps(x). \quad (39)$$

Proof: Set $V^{(1)}(x) = x^T Px$ and $V^{(2)}(\hat{x}) = \hat{x}^T Q\hat{x}$. Obviously, $V(\eta) \geq \min\{\lambda_{\min}(P), \lambda_{\min}(Q)\}\|\eta\|^2 \in CK$ where $\eta = [x^T \quad \hat{x}^T]^T$. On the other hand, in view of $V_x^{(1)^T}(x) = 2x^T P$, $V_{\hat{x}}^{(2)^T}(\hat{x}) = 2\hat{x}^T Q$, $V_{xx}^{(1)}(x) = 2P$ and $V_{\hat{x}\hat{x}}^{(2)}(\hat{x}) = 2Q$, it is easy to verify that (37)–(39) can be obtained from (25)–(27), respectively. Therefore, the proof of Corollary 3 can been easily accomplished from Corollary 1. ■

Similarly, when $V(\eta) = x^T Px + \hat{x}^T Q\hat{x}$, we have the following corollary from Corollary 2.

Corollary 4: Given the disturbance attenuation level $\gamma > 0$ and the filter parameters $\hat{f}$, $\hat{g}$ and $\hat{m}$. If there exist two positive constants μ_1, μ_2 and two positive definite matrices $P = P^T > 0$ and $Q = Q^T > 0$ satisfying the inequalities in (40) and (41), as shown at the bottom of the page, for any $x, \hat{x} \in \mathbb{R}^n$ and the inequalities in (42) and (43), also shown on the bottom of the page, for any nonzero $x,\ \hat{x} \in \mathbb{R}^n$, then the stochastic H_∞ filtering problem for system (1) can be solved by (3).

Proof: After tedious calculation, one can obtain from the proof of Corollary 2 that we get (44), shown at the bottom of the page. Therefore, the proof of this corollary follows immediately from that of Corollary 3. ■

$$\begin{aligned}
\mathbb{H}(\eta,\eta_\alpha) < & \frac{3}{2\mu_2}\left(\left\|V_x^{(1)^T}(x)g(x)\right\|^2 + \theta^4\left\|h^T(x)V_{xx}^{(1)}(x_\alpha)s(x)\right\|^2 + \left\|(f(x)-x)^T V_{xx}^{(1)}(x_\alpha)g(x)\right\|^2\right) \\
& + V_x^{(1)^T}(x)(f(x)-x) + V_{\hat{x}}^{(2)^T}(\hat{x})(\hat{f}(\hat{x})-\hat{x}) + \left(\frac{3\beta^2}{2\mu_2}+\frac{\beta}{2}\right)\left\|V_{\hat{x}}^{(2)^T}(\hat{x})\hat{g}(\hat{x})\right\|^2 \\
& + \frac{1}{2}\theta^2 h^T(x)V_{xx}^{(1)}(x_\alpha)h(x) + \frac{1}{2}(f(x)-x)^T V_{xx}^{(1)}(x_\alpha)(f(x)-x) \\
& + \frac{1}{2}(\hat{f}(\hat{x})-\hat{x})^T V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)(\hat{f}(\hat{x})-\hat{x}) + \left(\frac{3\beta^2}{2\mu_2}+\frac{\beta}{2}\right)\left\|(\hat{f}(\hat{x})-\hat{x})^T V_{\hat{x}\hat{x}}^{(2)}(\hat{x}_\alpha)\hat{g}(\hat{x})\right\|^2 \\
& + \left(\frac{3\beta^2\mu_1^2}{2\mu_2}+\frac{\beta\mu_1}{2}+\beta\right)\|l(x)\|^2 + 2\|m(x)\|^2 + 2\|\hat{m}(\hat{x})\|^2 \\
= & \, \mathbb{H}_1(\eta,\eta_\alpha) + \mathbb{H}_2(\eta,\eta_\alpha) < 0
\end{aligned} \quad (36)$$

$$\begin{aligned}
\overline{\mathbb{H}}(x,\hat{x}) = & \frac{1}{4}B(x,\hat{x})(\gamma^2 I - g^T(x)Pg(x) - \beta k^T(x)\hat{g}^T(\hat{x})Q\hat{g}(\hat{x})k(x) - \theta^2 s^T(x)Ps(x))^{-1}B^T(x,\hat{x}) \\
& + (f(x)+x)^T P(f(x)-x) + (\hat{f}(\hat{x})+\hat{x})^T Q(\hat{f}(\hat{x})-\hat{x}) + \beta l^T(x)\hat{g}^T(\hat{x})Q\hat{g}(\hat{x})l(x) \\
& + 2\beta\hat{f}^T(\hat{x})Q\hat{g}(\hat{x})l(x) + \theta^2 h^T(x)Ph(x) + \|\tilde{z}\|^2 < 0
\end{aligned} \quad (38)$$

$$\hat{g}^T(\hat{x})Q\hat{g}(\hat{x}) \leq \frac{\mu_1}{2}I \quad (40)$$

$$\gamma^2 I - g^T(x)Pg(x) - \theta^2 s^T(x)Qs(x) > \left(\frac{1}{2}\beta\mu_1 + \mu_2\right)I \quad (41)$$

$$\begin{aligned}
\overline{\mathbb{H}}_1(x) = & \frac{4}{\mu_2}(\|f^T(x)Pg(x)\|^2 + \theta^4\|h^T(x)Ps(x)\|^2) + \theta^2 h^T(x)Ph(x) + (f(x)+x)^T P(f(x)-x) \\
& + \left(\frac{\mu_1^2\beta^2}{\mu_2}+\frac{\mu_1\beta}{2}+\beta\right)\|l(x)\|^2 + 2\|m(x)\|^2 < 0
\end{aligned} \quad (42)$$

$$\overline{\mathbb{H}}_2(\hat{x}) = \left(\frac{4\beta^2}{\mu_2}+\beta\right)\|\hat{f}^T(\hat{x})Q\hat{g}(\hat{x})\|^2 + (\hat{f}(\hat{x})+\hat{x})^T Q(\hat{f}(\hat{x})-\hat{x}) + 2\|\hat{m}(\hat{x})\|^2 < 0 \quad (43)$$

$$\begin{aligned}
\overline{\mathbb{H}}(x,\hat{x}) < & \frac{4}{\mu_2}(\|f^T(x)Pg(x)\|^2 + \theta^4\|h^T(x)Ps(x)\|^2) \\
& + \left(\frac{4\beta^2}{\mu_2}+\beta\right)\|\hat{f}^T(\hat{x})Q\hat{g}(\hat{x})\|^2 + \theta^2 h^T(x)Ph(x) \\
& + (f(x)+x)^T P(f(x)-x) + (\hat{f}(\hat{x})+\hat{x})^T Q(\hat{f}(\hat{x})-\hat{x}) + \left(\frac{\mu_1^2\beta^2}{\mu_2}+\frac{\mu_1\beta}{2}+\beta\right)\|l(x)\|^2 \\
& + 2\|m(x)\|^2 + 2\|\hat{m}(\hat{x})\|^2 \\
= & \, \overline{\mathbb{H}}_1(x) + \overline{\mathbb{H}}_2(\hat{x}) < 0.
\end{aligned} \quad (44)$$

Remark 5: Note that we have obtained a series of analysis results in Theorem 1, Corollary 1, Corollary 2, Corollary 3, and Corollary 4. Based on the assumption that the filter structure is nonlinear, these analysis results offer sufficient conditions under which the filtering process is stochastically stable and the filtering error satisfies H_∞ performance constraint for all admissible missing observations and nonzero exogenous disturbances under the zero-initial condition. However, in practice, one is more interested in linear time-invariant filters that can be easily implemented, and the goal of Section IV is therefore devoted to the filtering problem for nonlinear systems but with linear filters. It will be shown that the solvability of such a problem is dependent on the feasibility of certain second-order inequalities.

IV. Nonlinear H_∞ Filtering With Linear Filters

For the purpose of practical applications, this section is devoted to the study of linear H_∞ filters for nonlinear system (1).

The linear time-invariant filter under consideration is of the following structure

$$\begin{cases} \hat{x}_{k+1} = F\hat{x}_k + Gy_k \\ \hat{z}_k = M\hat{x}_k, \ \hat{x}_0 = 0 \end{cases} \tag{45}$$

where $\hat{x}_k$ is the state estimate, $\hat{z}_k$ is an estimate for z_k, and the constant matrices F, G and M are filter parameters to be determined. Similar to what we have done in Section II, we can obtain the following augmented system:

$$\begin{cases} \eta_{k+1} = f_e(\eta_k) + \eta_k + g_e(\eta_k)v_k + (h_e(\eta_k) + s_e(\eta_k)v_k)w_k \\ \tilde{z}_k := z_k - \hat{z}_k = m(x_k) - M\hat{x}_k \end{cases} \tag{46}$$

where

$$\begin{aligned} f_e(\eta_k) &= \begin{bmatrix} f(x_k) - x_k \\ (F-I)\hat{x}_k + r_k Gl(x_k) \end{bmatrix} \\ g_e(\eta_k) &= \begin{bmatrix} g(x_k) \\ r_k Gk(x_k) \end{bmatrix} \\ h_e(\eta_k) &= \begin{bmatrix} h(x_k) \\ 0 \end{bmatrix}, \quad s_e(\eta_k) = \begin{bmatrix} s(x_k) \\ 0 \end{bmatrix}. \end{aligned} \tag{47}$$

In virtue of Theorem 1, the following sufficient conditions for the filter parameters F, G and M to satisfy can be easily acquired.

Theorem 2: Given the disturbance attenuation level $\gamma > 0$ and the filter parameters F, G and M. If there exists a Lyapunov function $V(\eta) \in C^2(\mathbb{R}^{2n})$ such that the inequalities in (48), shown at the bottom of the page, hold where we have (49)–(51), as shown at the bottom of the page, then the stochastic H_∞ filtering problem for system (1) is solved by (45).

Proof: This proof is a straightforward consequence of that of Theorem 1 and is therefore omitted. ■

In order to have more simplified conditions for solving the stochastic H_∞ filtering problem with a linear filter, we set the Lyapunov function $V(\eta) = x^TPx + \hat{x}^TQ\hat{x}$ where P and Q are two positive definite matrices. Subsequently, the following corollary can be obtained.

Corollary 5: Given the disturbance attenuation level $\gamma > 0$ and the filter parameters F, G and M. If there exist a positive constant μ_2 and two positive definite matrices $P = P^T > 0$ and $Q = Q^T > 0$ satisfying the inequalities in (52), shown at the bottom of the next page, for any $x \in \mathbb{R}^n$, and the inequalities in (53) and (54), shown at the bottom of the next page, for any nonzero $x \in \mathbb{R}^n$, then the stochastic H_∞ filtering problem for system (1) can be solved by the linear filter (45).

$$\begin{cases} \mathbb{H}(\eta,\eta_\alpha) = \frac{1}{4}B(\eta,\eta_\alpha)A^{-1}(\eta,\eta_\alpha)B^T(\eta,\eta_\alpha) + \frac{\partial V^T}{\partial x}(\eta)(f(x)-x) + \frac{\partial V^T}{\partial \hat{x}}(\eta)(F-I)\hat{x} \\ \qquad + \beta\frac{\partial V^T}{\partial \hat{x}}(\eta)Gl(x) + D(\eta,\eta_\alpha) + \frac{1}{2}\theta^2 h^T(x)\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)h(x) + \|m(x) - M\hat{x}\|^2 < 0, \\ \qquad \text{for any } \eta \neq 0, \ \eta_\alpha \in \mathbb{R}^{2n}, \\ a(\|\eta\|) \le V(\eta), \ \text{with } a(r) \in CK, \\ A(\eta,\eta_\alpha) > 0, \ \text{for any } \eta, \ \eta_\alpha \in \mathbb{R}^{2n}, \ V(0) = 0 \end{cases} \tag{48}$$

$$\begin{aligned} A(\eta,\eta_\alpha) &= \gamma^2 I - \frac{1}{2}g^T(x)\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)g(x) - \frac{1}{2}\beta k^T(x)G^T\frac{\partial^2 V}{\partial \hat{x}^2}(\eta_\alpha)Gk(x) - \beta k^T(x)G^T\frac{\partial^2 V}{\partial x^T\partial \hat{x}}(\eta_\alpha) \\ &\quad \times g(x) - \frac{1}{2}\theta^2 s^T(x)\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)s(x) \end{aligned} \tag{49}$$

$$\begin{aligned} B(\eta,\eta_\alpha) &= \frac{\partial V^T}{\partial x}(\eta)g(x) + \beta\frac{\partial V^T}{\partial \hat{x}}(\eta)Gk(x) + \theta^2 h^T(x)\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)s(x) + (f(x)-x)^T\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)g(x) \\ &\quad + ((F-I)\hat{x} + \beta Gl(x))^T\frac{\partial^2 V}{\partial x^T\partial \hat{x}}(\eta_\alpha)g(x) + \beta(f(x)-x)^T\frac{\partial^2 V}{\partial \hat{x}^T\partial x}(\eta_\alpha)Gk(x) \\ &\quad + \beta\hat{x}^T(F-I)^T\frac{\partial^2 V}{\partial \hat{x}^2}(\eta_\alpha)Gk(x) + \beta l^T(x)G^T\frac{\partial^2 V}{\partial \hat{x}^2}(\eta_\alpha)Gk(x) \end{aligned} \tag{50}$$

$$\begin{aligned} D(\eta,\eta_\alpha) &= \frac{1}{2}(f(x_k)-x_k)^T\frac{\partial^2 V}{\partial x^2}(\eta_\alpha)(f(x)-x) + (f(x)-x)^T\frac{\partial^2 V}{\partial \hat{x}^T\partial x}(\eta_\alpha)((F-I)\hat{x} + \beta Gl(x_k)) \\ &\quad + \frac{1}{2}\hat{x}^T(F-I)^T\frac{\partial^2 V}{\partial \hat{x}^2}(\eta_\alpha)(F-I)\hat{x} + \beta\hat{x}^T(F-I)^T\frac{\partial^2 V}{\partial \hat{x}^2}(\eta_\alpha)Gl(x) \\ &\quad + \frac{1}{2}\beta l^T(x)G^T\frac{\partial^2 V}{\partial \hat{x}^2}(\eta_\alpha)Gl(x) \end{aligned} \tag{51}$$

Proof: When $\hat{f}(\hat{x})$, $\hat{g}(\hat{x})$ and $\hat{m}(\hat{x})$ are replaced by $F\hat{x}$, G and $M\hat{x}$, respectively, it can be easily known that (43) implies (54). In addition, if μ_1 is taken as $2\lambda_{\max}(G^TQG)$, (40)–(42) imply (52)–(53). Therefore, the rest of the proof follows from that of Corollary 4 immediately. ■

Remark 6: As we know, the filter (45) is easy to be implemented in practice owing to its linear structure. Nevertheless, it might be difficult to verify the condition of Corollary 5 since nonlinear functions are involved in the inequalities (52) and (53).

Remark 7: Let the nonlinear functions $f(x), g(x), h(x), s(x), l(x)$, and $m(x)$ take the linear form as $f(x) = A_1x$, $g(x) = G_1$, $h(x) = A_2x$, $s(x) = G_2$, $l(x) = Lx$, and $m(x) = Nx$. In such a special case, the inequalities (52)–(54) can be reduced to a set of LMIs which can be easily solved by resorting to the Matlab LMI Toolbox. Therefore, the design problem of H_∞ filters for a linear discrete-time stochastic system with missing measurement can be readily dealt with based on the main results in this section.

Remark 8: Up until now, a series of criteria have been given for the filter analysis of nonlinear stochastic systems with missing measurements. Specifically, a filter of very general form is first designed such that the filtering process is stochastically stable and the filtering error satisfies H_∞ performance constraint for all admissible missing observations and nonzero exogenous disturbances under the zero-initial condition. The existence conditions of the desired filter are then described in terms of a second-order nonlinear inequality. Such an inequality can be decoupled into some auxiliary ones that can be solved independently by taking special form of the Lyapunov functionals. As a consequence, a linear time-invariant filter design problem is discussed for the benefit of practical applications, and some simplified conditions are obtained. In Section V, two numerical simulation examples will be given to illustrate the main results of this paper.

V. Illustrative Examples

In this section, we demonstrate the theory presented in this paper by means of two numerical examples, respectively, for the addressed H_∞ filtering problems with nonlinear and linear filters.

1) Example 1: H_∞ filtering design with a nonlinear filter.

Consider a nonlinear discrete-time stochastic system with missing measurement as shown in (55), at the bottom of the page. Assuming that the variance of w_k is $\theta^2 = 0.25$, the disturbance attenuation level is prescribed as $\gamma = \sqrt{3.05}$ and $\mathrm{Prob}\{r_k = 1\} = \beta = 0.8$, we can construct a filter of the form

$$\begin{cases} \hat{x}_{k+1} = \frac{1}{3}\hat{x}_k^{(2/3)} \sin \hat{x}_k^{(1/3)} + y_k \\ \hat{z}_k = \frac{1}{4}\sin \hat{x}_k, \ \hat{x}_0 = 0 \end{cases} \tag{56}$$

and then an augmented system can be given in the form of (4) with

$$\begin{aligned} f_e(\eta_k) &= \begin{bmatrix} \frac{1}{6}\sin x_k - \frac{5}{6}x_k \\ \frac{1}{3}\hat{x}_k^{(2/3)} \sin \hat{x}_k^{(1/3)} + \frac{1}{5}r_k x_k \cos x_k - \hat{x}_k \end{bmatrix}, \\ g_e(\eta_k) &= \begin{bmatrix} 1 \\ r_k \end{bmatrix}, \\ h_e(\eta_k) &= \begin{bmatrix} \frac{\sqrt{10}}{6}(x_k - \sin x_k) \\ 0 \end{bmatrix}, \quad s_e(\eta_k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{57}$$

It is not difficult to verify that $\hat{f}(\hat{x})$, $\hat{g}(\hat{x})$, and $\hat{m}(\hat{x})$ satisfy the conditions of Corollary 4 with the Lyapunov functional being $V(x, \hat{x}) = x^2 + \hat{x}^2$. It follows from Corollary 4 that the filter of the form (56) is a desired state estimator that achieves the stochastic stability as well as the prescribed H_∞ performance constraint. Simulation results are shown in Figs. 1 and 2, where the trajectory and estimation of the state x_k of (55) is given in Fig. 1 and the estimation error $\tilde{z}_k$ is depicted in Fig. 2.

Remark 9: In general, the desired H_∞ filter is not unique. For example, $\hat{x}_{k+1} = (1/3)\hat{x}_k \sin \hat{x}_k + y_k$, $\hat{z}_k = (1/4)\hat{x}_k$ is also a feasible H_∞ filter for the stochastic system (55).

2) Example 2: H_∞ filtering design with a linear filter.

In this example, we consider the nonlinear discrete-time stochastic system with missing measurement, as shown in (58), at the bottom of the page. Let the probability $\beta = 0.75$, the variance $\theta^2 = 0.25$ and the disturbance attenuation level $\gamma = \sqrt{1.625}$. We adopt a linear filter as follows:

$$\begin{cases} \hat{x}_{k+1} = \frac{1}{3}\hat{x}_k + \frac{1}{\sqrt{2}}y_k \\ \hat{z}_k = \frac{1}{5}\hat{x}_k, \ \hat{x}_0 = 0. \end{cases} \tag{59}$$

$$\gamma^2 I - g^T(x)Pg(x) - \theta^2 s^T(x)Qs(x) > (\beta\lambda_{\max}(G^TQG) + \mu_2)I \tag{52}$$

$$\begin{aligned} &\frac{4}{\mu_2}\left(\|f^T(x)Pg(x)\|^2 + \theta^4\|h^T(x)Ps(x)\|^2\right) + \theta^2 h^T(x)Ph(x) + (f(x)+x)^T P(f(x)-x) \\ &\quad + \left(\frac{4\lambda_{\max}^2(G^TQG)\beta^2}{\mu_2} + (\lambda_{\max}(G^TQG)+1)\beta\right)\|l(x)\|^2 + 2\|m(x)\|^2 < 0 \end{aligned} \tag{53}$$

$$\left(\frac{4\beta^2}{\mu_2} + \beta\right)F^TQGG^TQF + (F+I)^TQ(F-I) + 2M^TM < 0 \tag{54}$$

$$\begin{cases} x_{k+1} = \frac{1}{6}(x_k + \sin x_k) + v_k + \left(\frac{\sqrt{10}}{6}(x_k - \sin x_k) + v_k\right)w_k \\ y_k = r_k\left(\frac{1}{5}x_k \cos x_k + v_k\right) \\ z_k = \frac{1}{4}\sin x_k. \end{cases} \tag{55}$$

$$\begin{cases} x_{k+1} = \frac{1}{4}(x_k \cos x_k + \sin x_k) + \frac{1}{2}v_k + (\frac{1}{3}x_k \cos x_k + \frac{1}{2}v_k)w_k \\ y_k = r_k\left(\frac{1}{8}(x_k - \sin 2x_k) + v_k\right) \\ z_k = \frac{1}{5}\sin x_k. \end{cases} \tag{58}$$

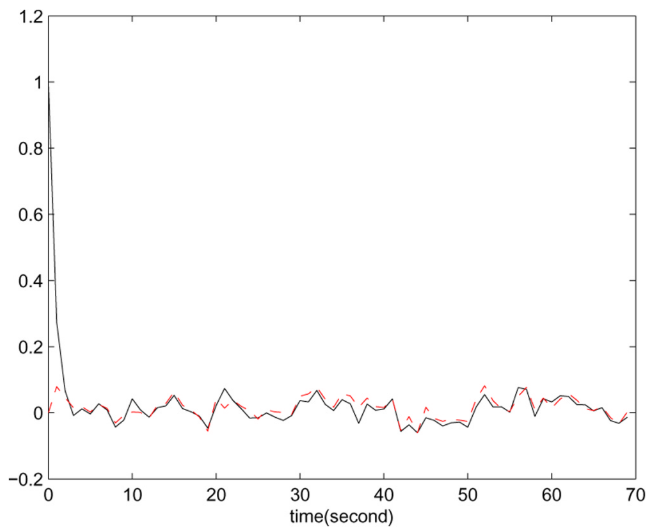

Fig. 1. x (solid) and $\hat{x}$ (dashed).

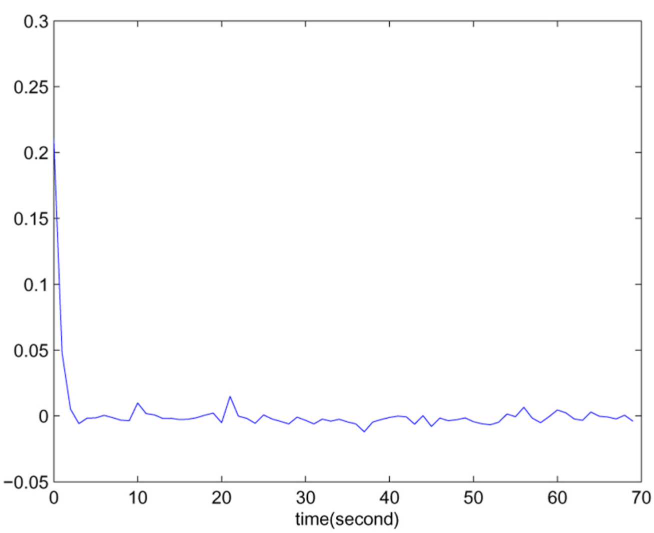

Fig. 2. Estimation error $\tilde{z}$ (solid).

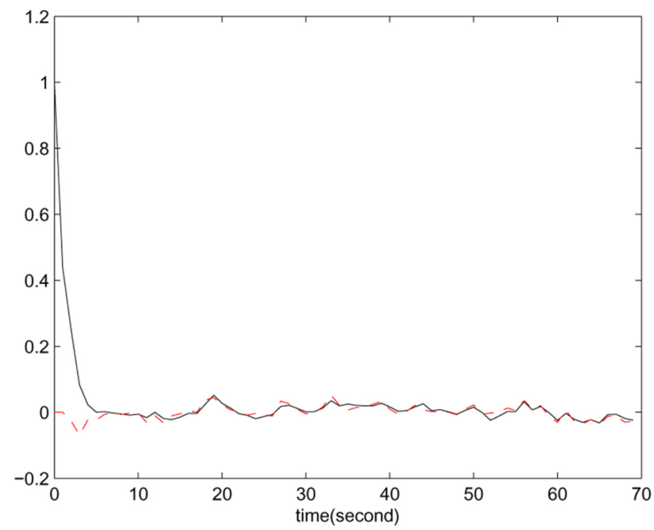

Fig. 3. x (solid) and $\hat{x}$ (dashed).

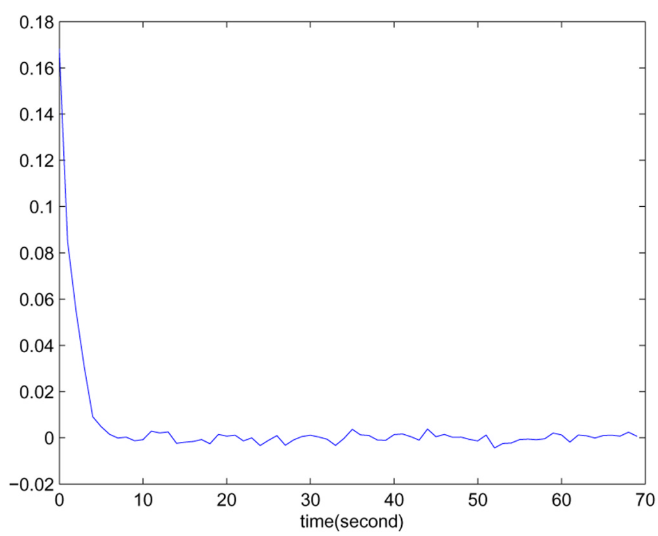

Fig. 4. Estimation error $\tilde{z}$ (solid).

According to Corollary 5, it can be seen that the filter of form (59) is a desired estimator for system (58) with the Lyapunov functional $V(x, \hat{x}) = x^2 + 2\hat{x}^2$. Figs. 3 and 4 show the simulation results which further confirm our theoretical analysis for the nonlinear H_∞ filtering problem with the given linear filter.

VI. Conclusions

In this paper, we have investigated a robust H_∞ filtering problem for a class of nonlinear discrete-time stochastic systems with missing measurements. The missing measurements are modeled by a Bernoulli distributed white sequence with a known conditional probability. A sufficient condition of the form of a second-order nonlinear inequality has been derived, which guarantees the augmented system is stochastically stable and the filtering error satisfies H_∞ performance constraint for all possible missing observations and all nonzero exogenous disturbances under the zero-initial condition. Subsequently, the second-order nonlinear inequality has been decoupled into two or more inequalities which can be solved independently. Then, we have obtained more simplified forms of the second-order nonlinear inequalities and some independent inequalities are deduced directly from the main results. Moreover, the nonlinear H_∞ filtering problem with a linear filter is investigated and some easy-to-verify criteria have been provided. The results of this paper have been demonstrated by two numerical simulation examples.

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Asymptotic Tracking for Uncertain Dynamic Systems Via a Multilayer Neural Network Feedforward and RISE Feedback Control Structure

Parag M. Patre, William MacKunis, Kent Kaiser, and Warren E. Dixon

***Abstract*—The use of a neural network (NN) as a feedforward control element to compensate for nonlinear system uncertainties has been investigated for over a decade. Typical NN-based controllers yield uniformly ultimately bounded (UUB) stability results due to residual functional reconstruction inaccuracies and an inability to compensate for some system disturbances. Several researchers have proposed discontinuous feedback controllers (e.g., variable structure or sliding mode controllers) to reject the residual errors and yield asymptotic results. The research in this paper describes how a recently developed continuous robust integral of the sign of the error (RISE) feedback term can be incorporated with a NN-based feedforward term to achieve semi-global asymptotic tracking. To achieve this result, the typical stability analysis for the RISE method is modified to enable the incorporation of the NN-based feedforward terms, and a projection algorithm is developed to guarantee bounded NN weight estimates.**

***Index Terms*—Adaptive control, asymptotic stability, Lyapunov methods, neural network, nonlinear systems, RISE feedback, robust control.**

I. Introduction

Control researchers have extensively investigated the use of neural networks (NNs) as a feedforward control element over the last fifteen years. The focus on NN-based control methods is spawned from the ramifications of the fact that NNs are universal approximators [1]. That is, NNs can be used as a black-box estimator for a general class of systems. Examples include: nonlinear systems with parametric uncertainty that do not satisfy the linear-in-the-parameters assumption required in most adaptive control methods; systems with deadzones or discontinuities; and systems with backlash. Typically, NN-based controllers yield global uniformly ultimately bounded (UUB) stability results (e.g., see [2]–[4] for examples and reviews of literature) due to residual functional reconstruction inaccuracies and an inability to compensate for some system disturbances. Motivated by the desire to eliminate the residual steady-state errors, several researchers have obtained asymptotic tracking results by combining the NN feedforward element with discontinuous feedback methods such as variable structure controllers (VSC) (e.g., [5] and [6]) or sliding mode (SM) controllers (e.g., [6] and [7]). A clever VSC-like controller was also proposed in [8], where the controller is not initially discontinuous, but exponentially becomes discontinuous as an exogenous control element exponentially vanishes. Well known limitations of VSC and SM controllers include a requirement for infinite control bandwidth and chattering. Unfortunately, ad hoc fixes for these effects result in a loss of asymptotic stability (i.e.,

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